Network Models

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Network Models

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Network Models

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Graph

Definition

Graph

Given a list of locations, a graph is a structured representation of the locations and the relationships between them.

Network Flow

Definition

Network flow

Network flow is the sending of a certain amount of assets from one location to another on the graph.

Mathematical Programming

Definition

Mathematical programming

Mathematical programming is the optimization of problems formulated as minimization (or maximization) of an objective function subject to a set of constraints.

Combinatorial Optimization

Definition

Combinatorial optimization

Combinatorial optimization is a class of mathematical programming, where optimization is performed over a discrete set of feasible solutions.

Network Flow Problem

Definition

Network flow problem

Network flow problems are mathematical programming problems that can be converted into combinatorial optimization problems dealing with network flows.

Mathematical Optimization

Mathematical Optimization

- Linear programming
  - Simplex algorithm
  - Duality
- Decomposition methods
  - Dantzig-Wolfe (complicating constraints, column(extreme point) generation, duality gap between upper and lower bounds)
  - Benders (complicating variables, cut generation, duality gap between upper and lower bounds)
- Mixed-integer programming
  - Branch-and-bound (BaB)
  - BaB + Cutting planes = Branch-and-cut
  - BaB + Column(variable for pricing, extreme point for decomposition) generation = Branch-and-price
  - BaB + Cutting planes + Column generation = Branch-price-and-cut

Mathematical Optimization

Mathematical Optimization

- Constraint programming
  - Constraint propagation
  - Domain reduction
- Combinatorial optimization
  - Some problems are easy to solve
    - ★ Special fast algorithms
  - Some problems are hard to solve
    - ★ Mixed-integer programming
    - ★ Heuristics

Motivations

Network Flow Problems

- Network flow problems
  - Combinatorial optimization
  - Wide application area in Operations Research
  - Special fast algorithms suitable for large problem instances
  - Network flow problem as an embedded subproblem

Graph

Definition

Graph [1]

A graph $G(V, E)$ consists of a set of vertices V and edges E. Edges are used to model the relationship between vertices.

Graph

Definition

Graph [2]

A graph $G(N, A)$ consists of a set of nodes N and arcs A. Arcs are used to model the relationship between nodes.

Graph

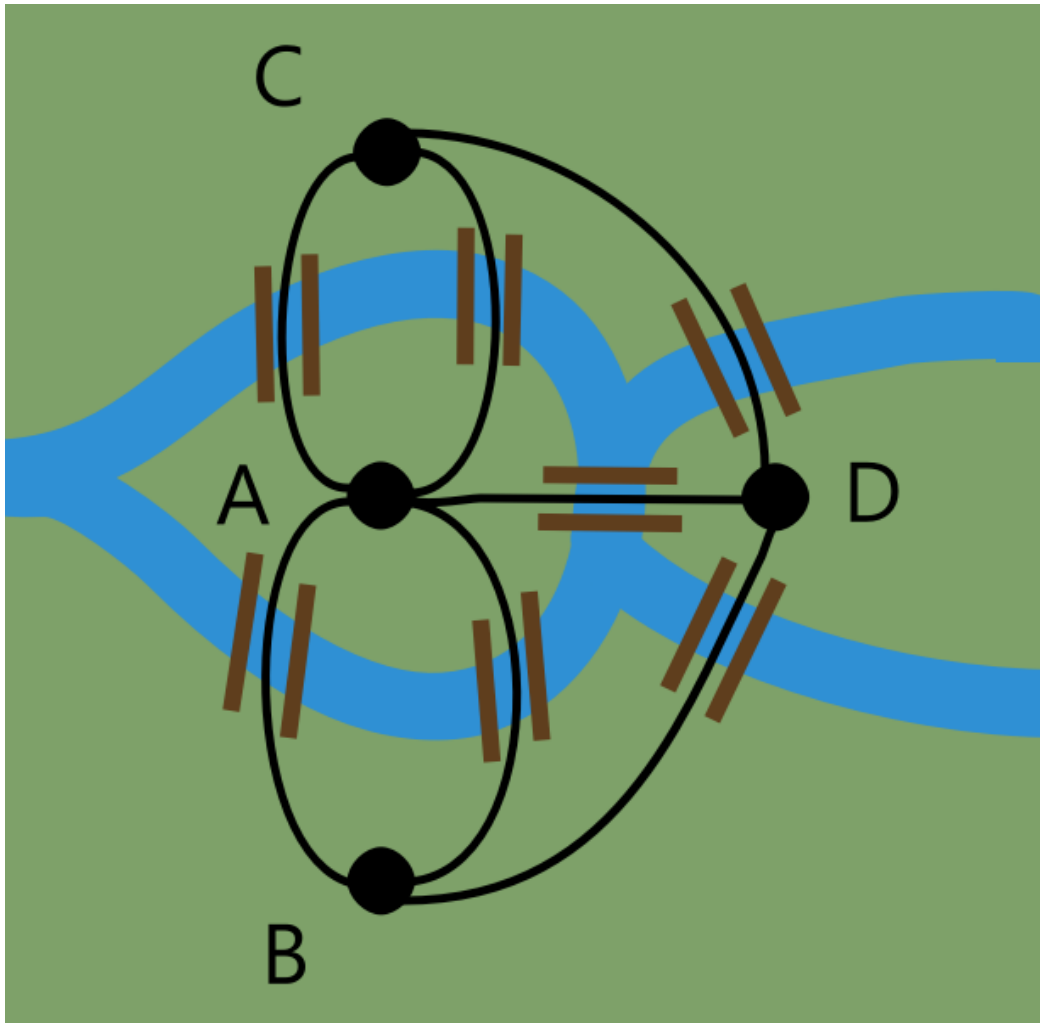
Definition

Subgraph

A graph $G'(V', E')$ is a subgraph of $G(V, E)$ if $V' \subset V$ and $E' \subset E$.

Graph

Example



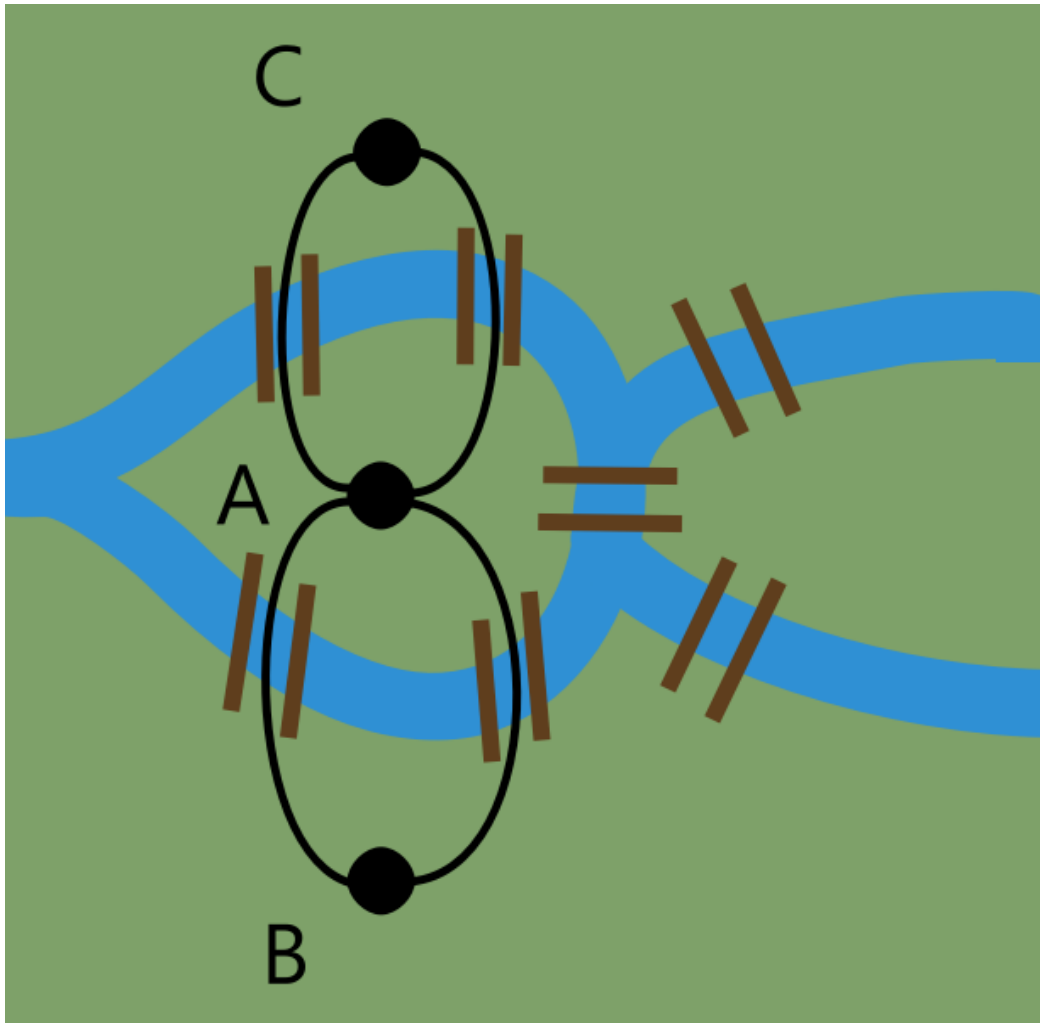
Graph

The Euler's problem

- Is it possible to start from a vertex, move along all edges, traversing every edge only once, and finally return to the starting vertex?

Graph

Example



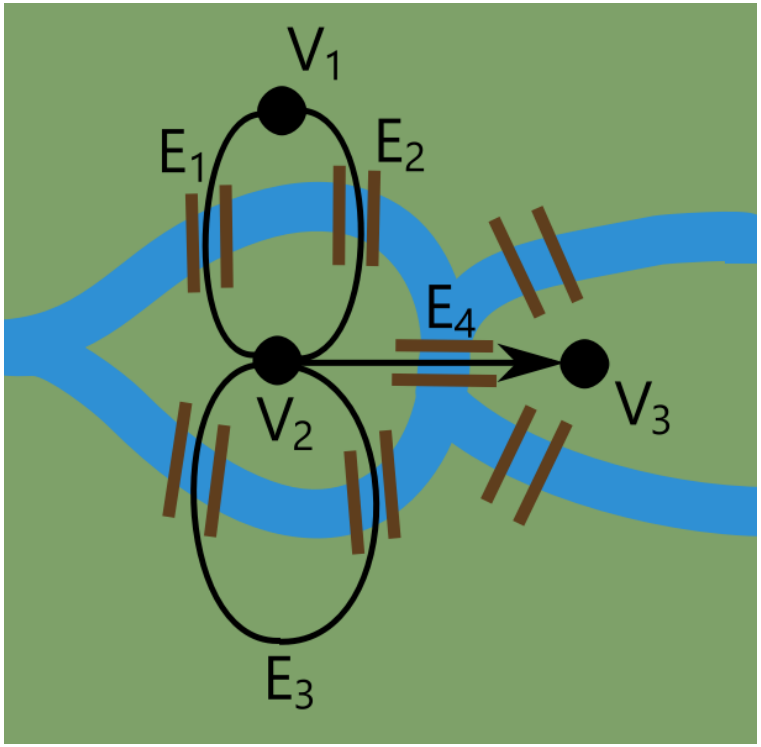
Graph

The Hamilton's problem

- Is it possible to start from a vertex, visit each of all vertices exactly once, and finally return to the starting vertex?

Graph

Directed edges, multiple edges and loops



- E_1 and E_2 are multiple edges
- E_3 is a loop
- E_4 is a directed edge
- V_2(tail) and V_3(head) are the endpoints of the edge(arc) E_4.

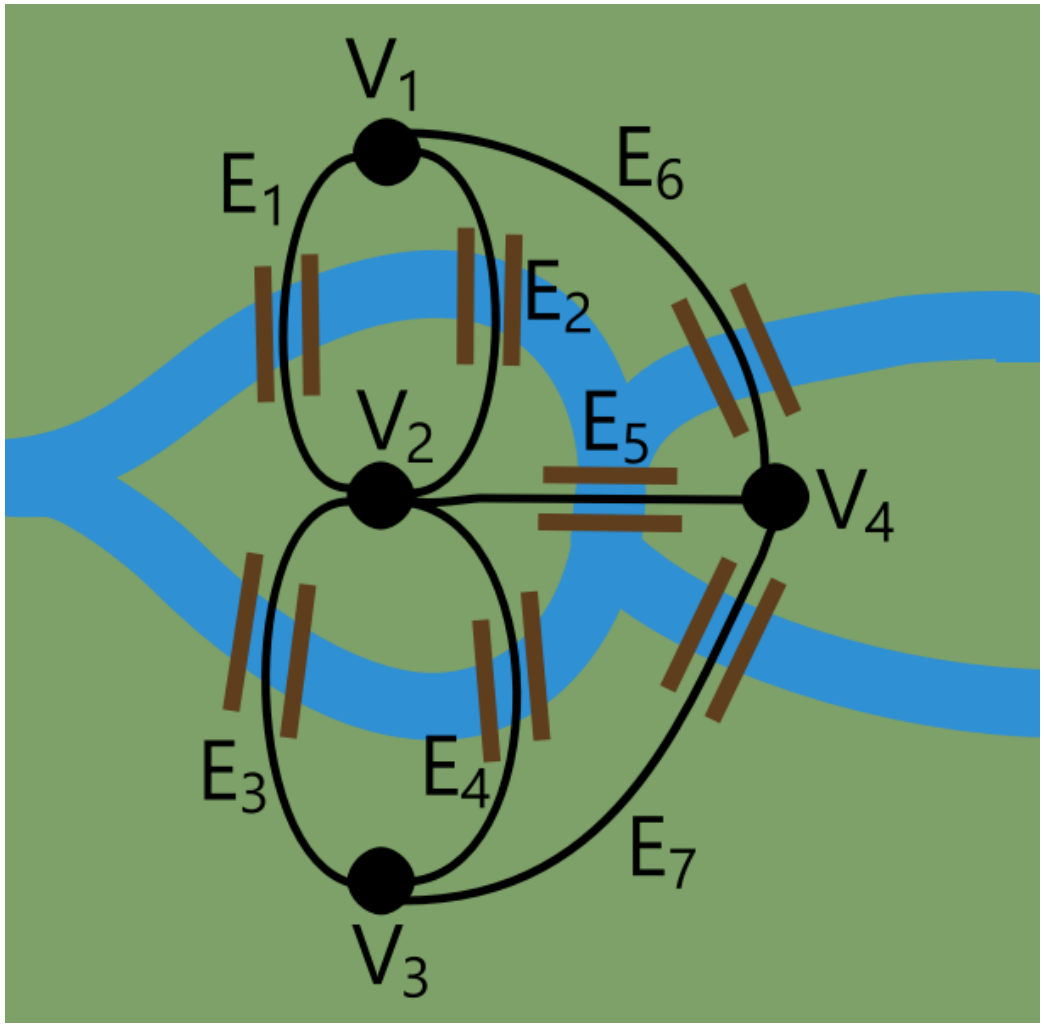
Graph

Graph types [1]

Type	Edges	Multiple edges	Loops
Simple graph	Undirected	✘	✘
Multigraph	Undirected	✔	✘
Pseudograph	Undirected	✔	✔
Simple directed graph	Directed	✘	✘
Directed multigraph	Directed	✔	✔
Mixed graph	Directed and undirected	✔	✔

Graph

A multigraph



Graph

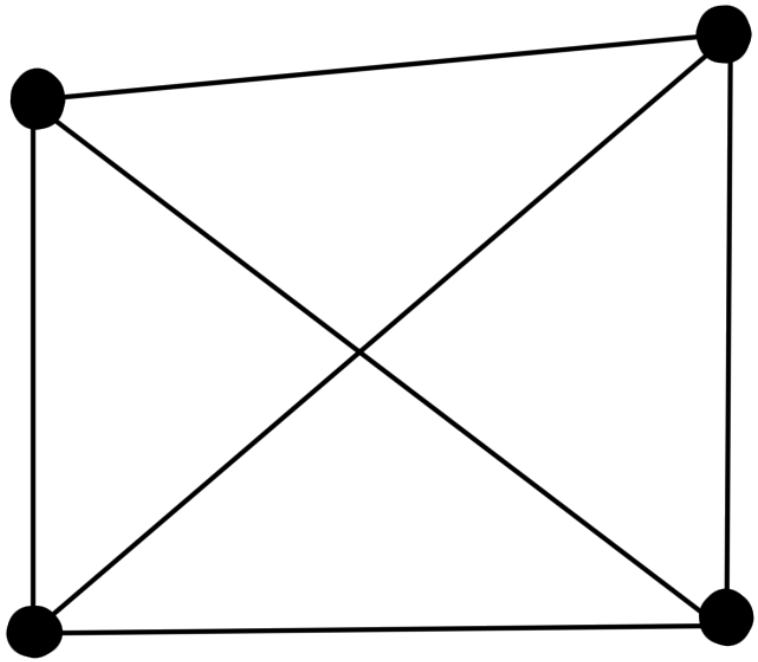
Definitions

Complete graph [1]

Complete graph is a simple graph where each pairs of distinct vertices are connected.

Graph

A complete graph

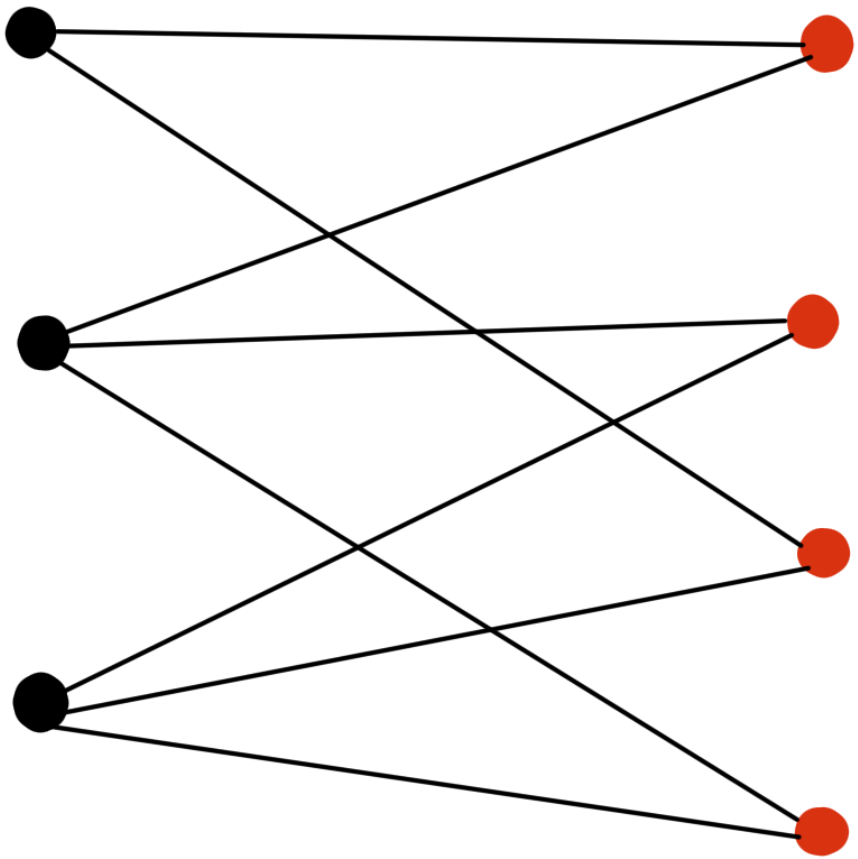


Bipartite simple graph [1]

A simple graph $G(V, E)$ is bipartite if $\exists V_1, V_2 : V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$ such that every edge in E connects a vertex in V_1 to a vertex in V_2.

Graph

A bipartite simple graph



Matching [1]

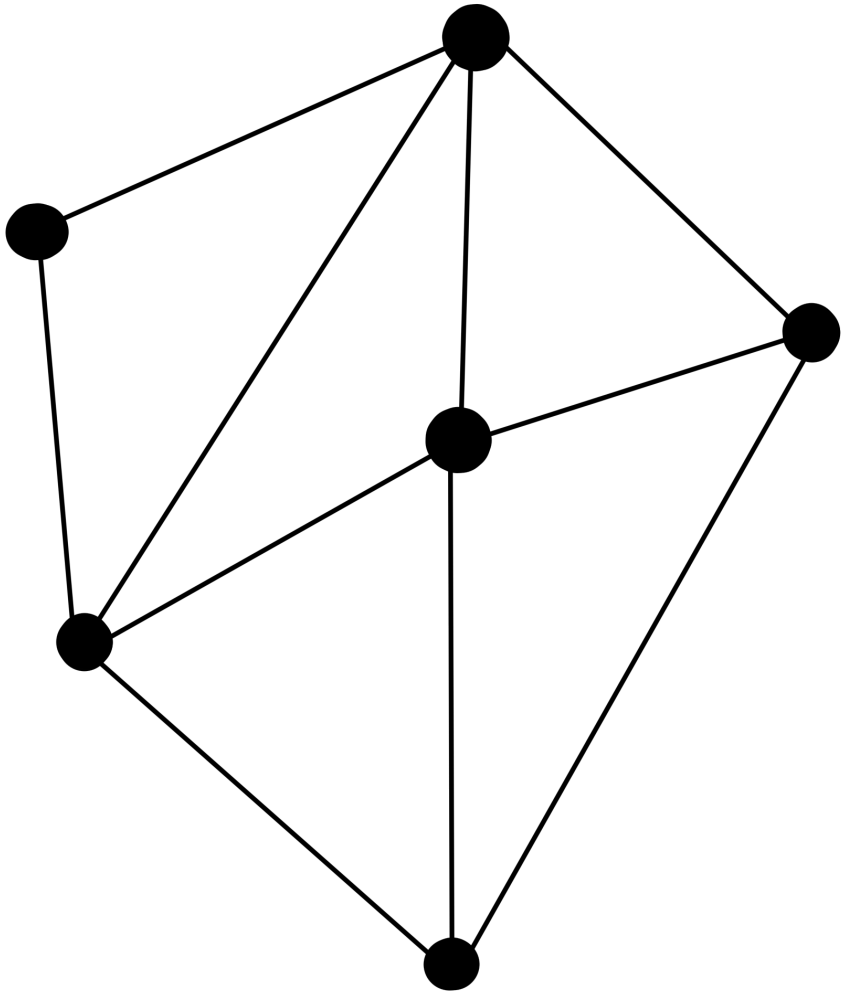
A matching M in a simple graph $G(V, E)$ is a subset of E, i.e. $M \subseteq E$ such that $\forall m, m' \in M$, all the endpoints of m and m' are distinct vertices.

Maximal matching

The maximal matching of G is the matching with the largest $|M|$.

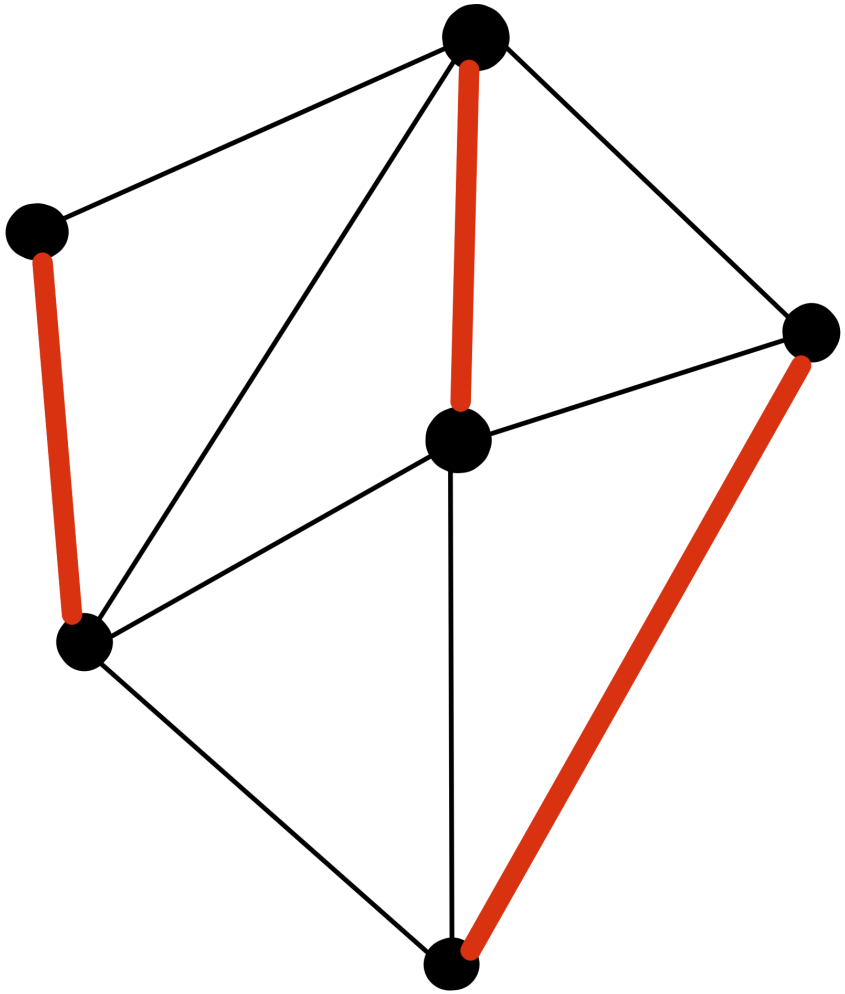
Graph

A simple graph



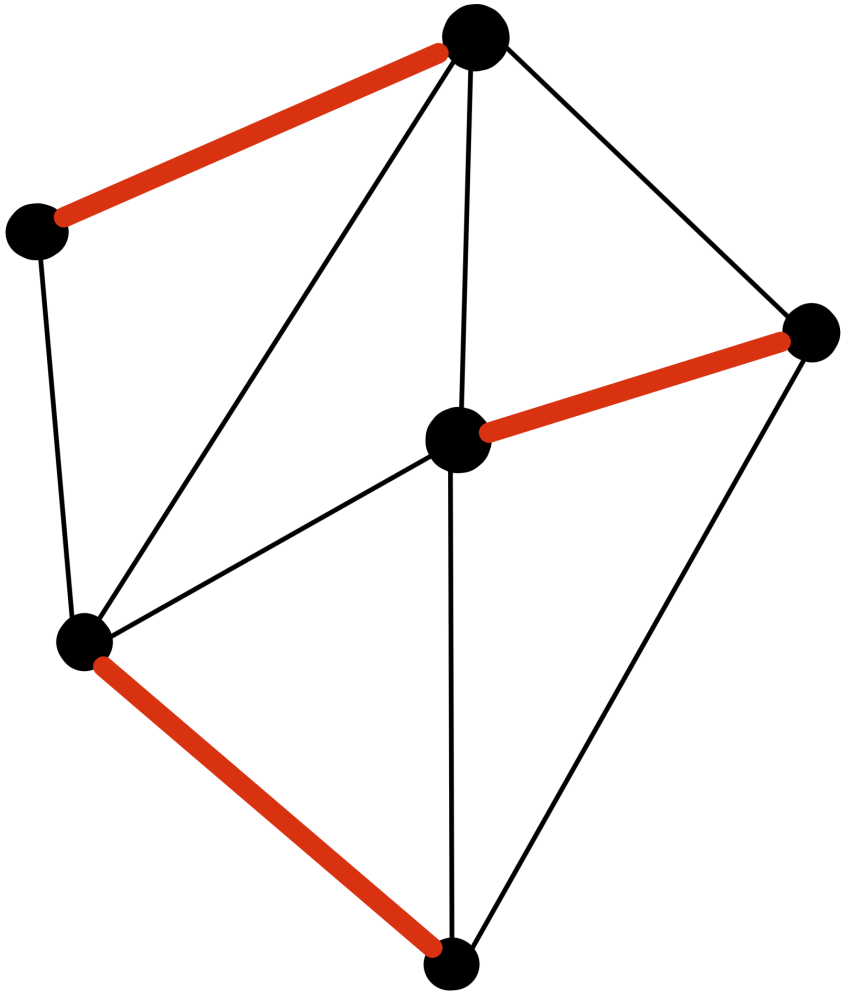
Graph

A maximal matching



Graph

Another maximal matching



Graph

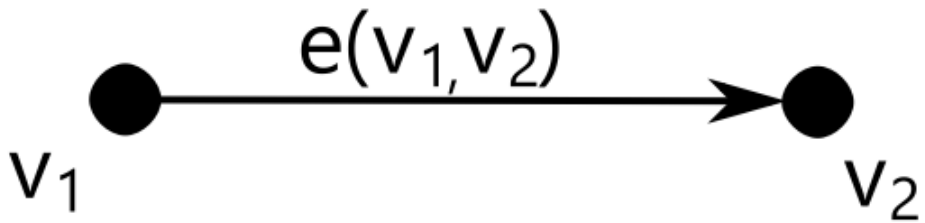
Definitions

Adjacent vertices in an undirected graph

Two vertices are adjacent in an undirected graph G if they are endpoints of an edge in G.

Adjacent vertices in a directed graph [1]

In a directed graph G, the vertex v_1 is adjacent to the vertex v_2 if they are endpoints of a directed edge $e(v_1, v_2)$ in G. Or, it can be said that v_2 is adjacent from v_1.



Graph

Definitions

An edge of an undirected graph G is incident with the vertices that are endpoints of this edge.

Graph

Definitions

Degree of a vertex in an undirected graph [1]

The degree of a vertex v in an undirected graph G, $deg(v)$ is equal to the number of edges incident with the vertex v, where a loop is equivalent to two edges.

Graph

Definitions

Given an undirected graph $G(V, E)$

$\sum_{v \in V} deg(v) = 2|E|$

Graph

Definitions

Degree of a vertex in a directed graph [1]

The indegree(outdegree) of a vertex v in a directed graph G, $deg^-(v)(deg^+(v))$ is equal to the number of edges with v as their terminal(initial) vertex.

Graph

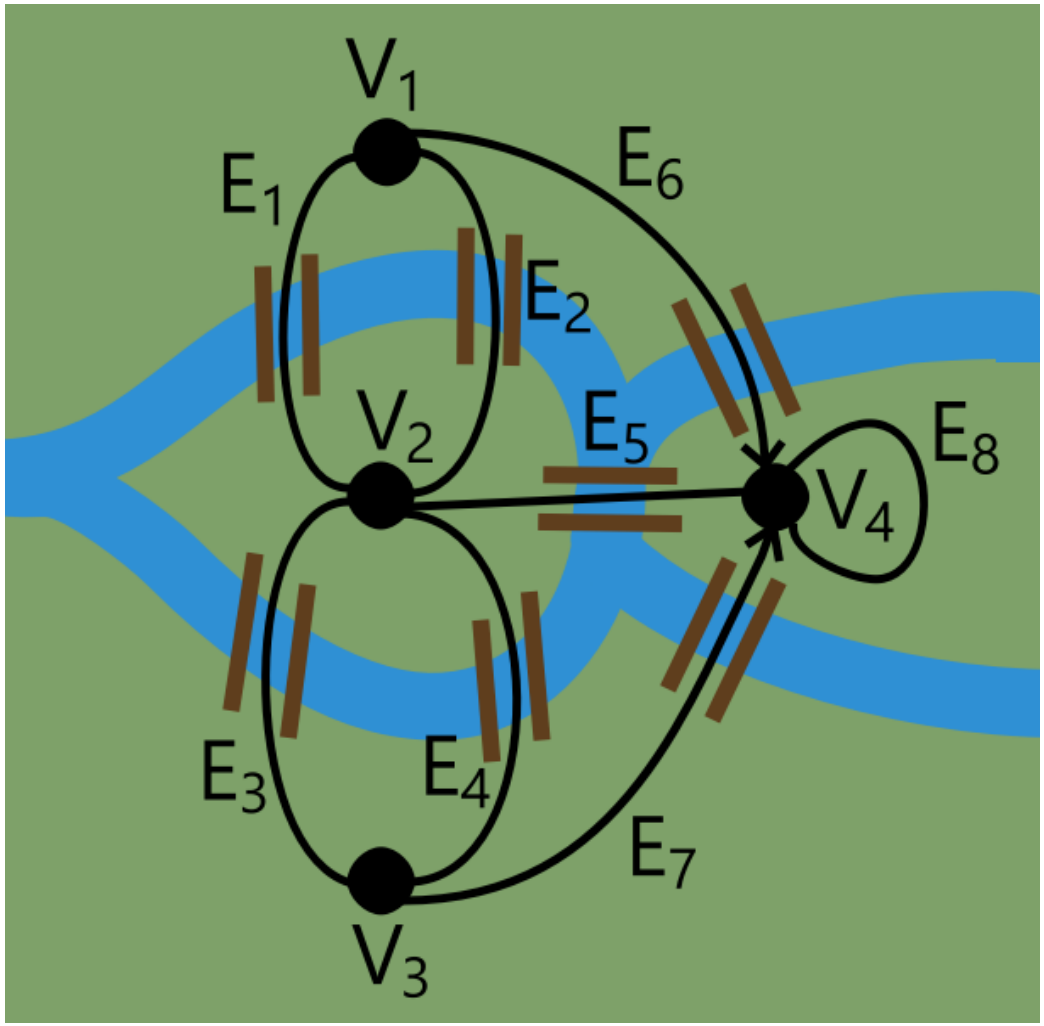
Definitions

Given a directed graph $G(V, E)$

$$\sum_{v \in V} deg^-(v) = \sum_{v \in V} deg^+(v) = |E|$$

Graph

A mixed graph



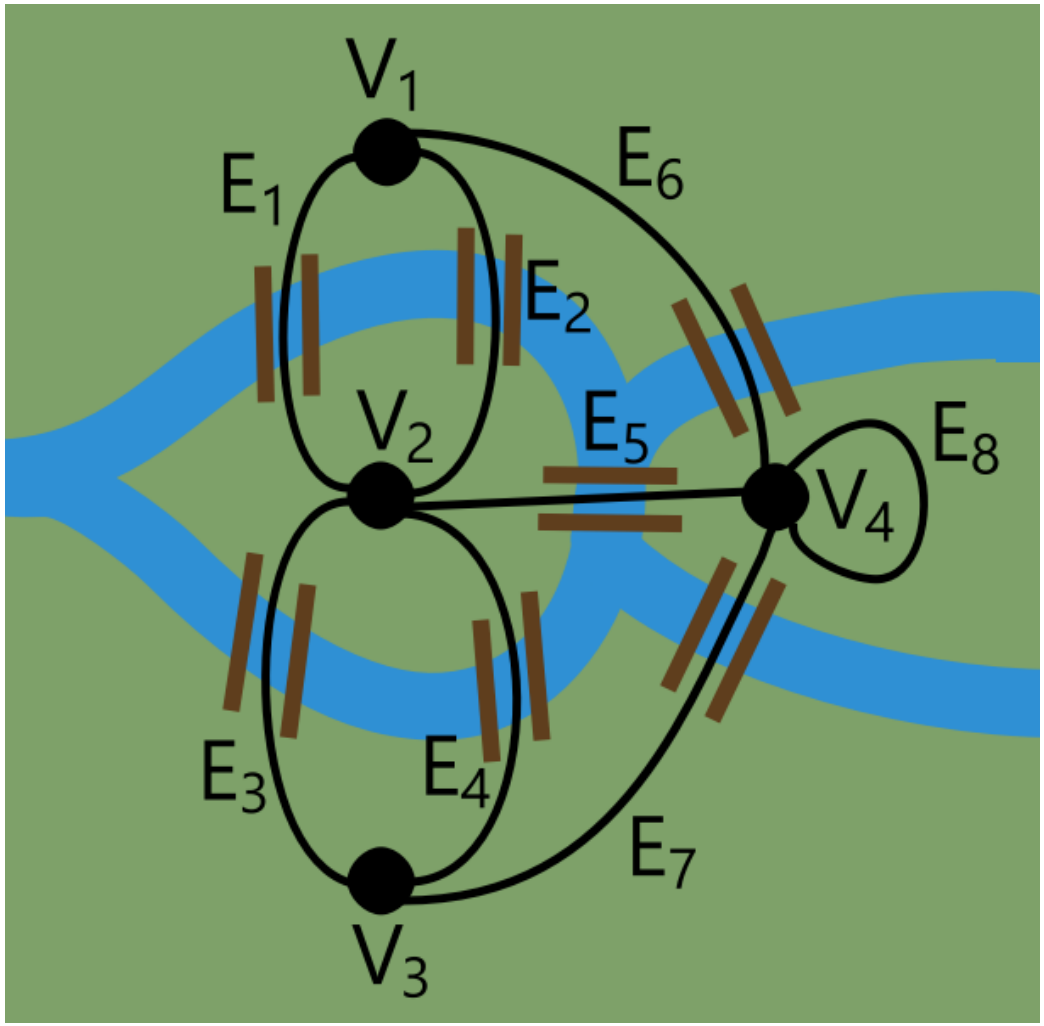
Graph

Adjacency matrix

	V_1	V_2	V_3	V_4
V_1	0	2	0	1
V_2	2	0	2	1
V_3	0	2	0	1
V_4	0	1	0	1

Graph

A pseudograph



Graph

Incidence matrix

	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8
V_1	1	1	0	0	0	1	0	0
V_2	1	1	1	1	1	0	0	0
V_3	0	0	1	1	0	0	1	0
V_4	0	0	0	0	1	1	1	1

Graph

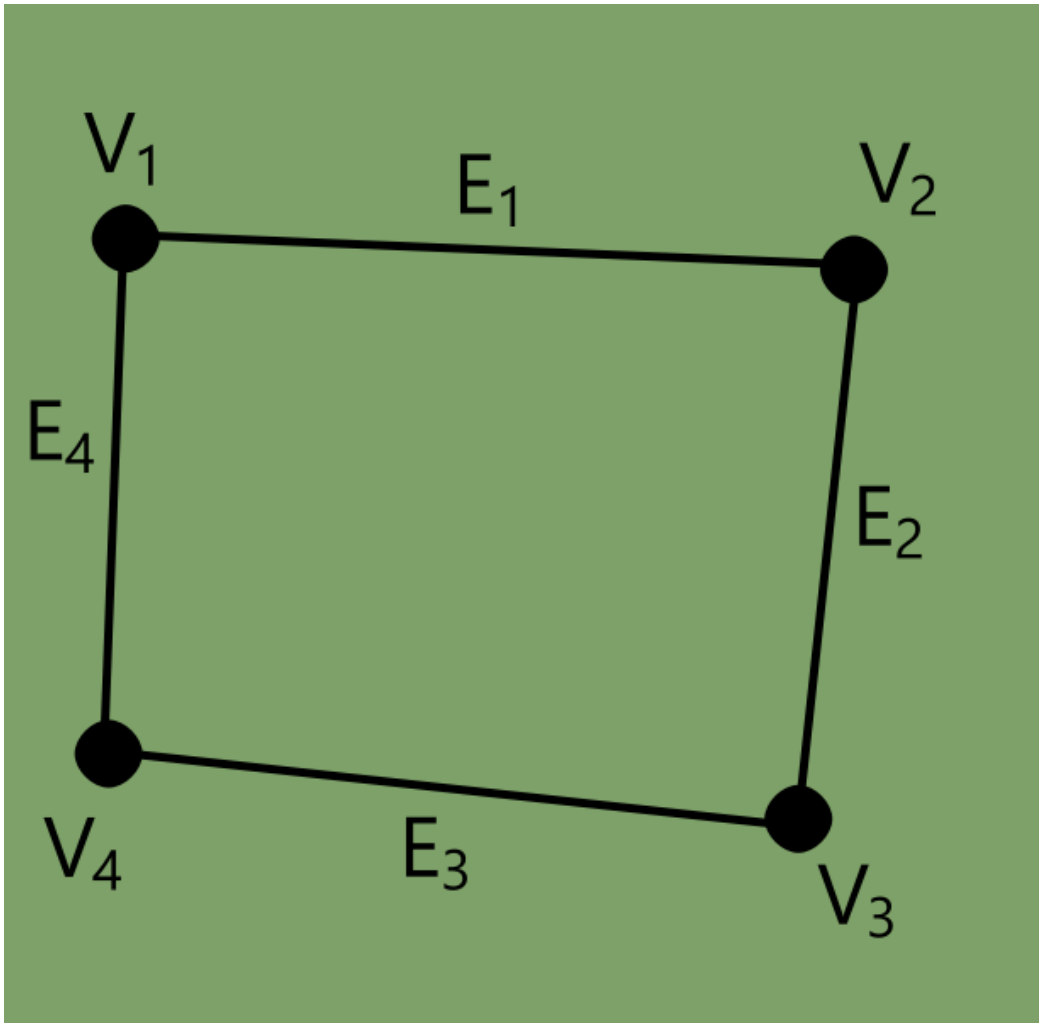
Definitions

Isomorphism of graphs [1]

Two simple graphs $G(V, E)$ and $G'(V', E')$ are isomorphic if and only if there exists a permutation of V', denoted as V'^p, leading to $G'^p(V'^p, E')$, where G and G'^p have the same adjacency matrix.

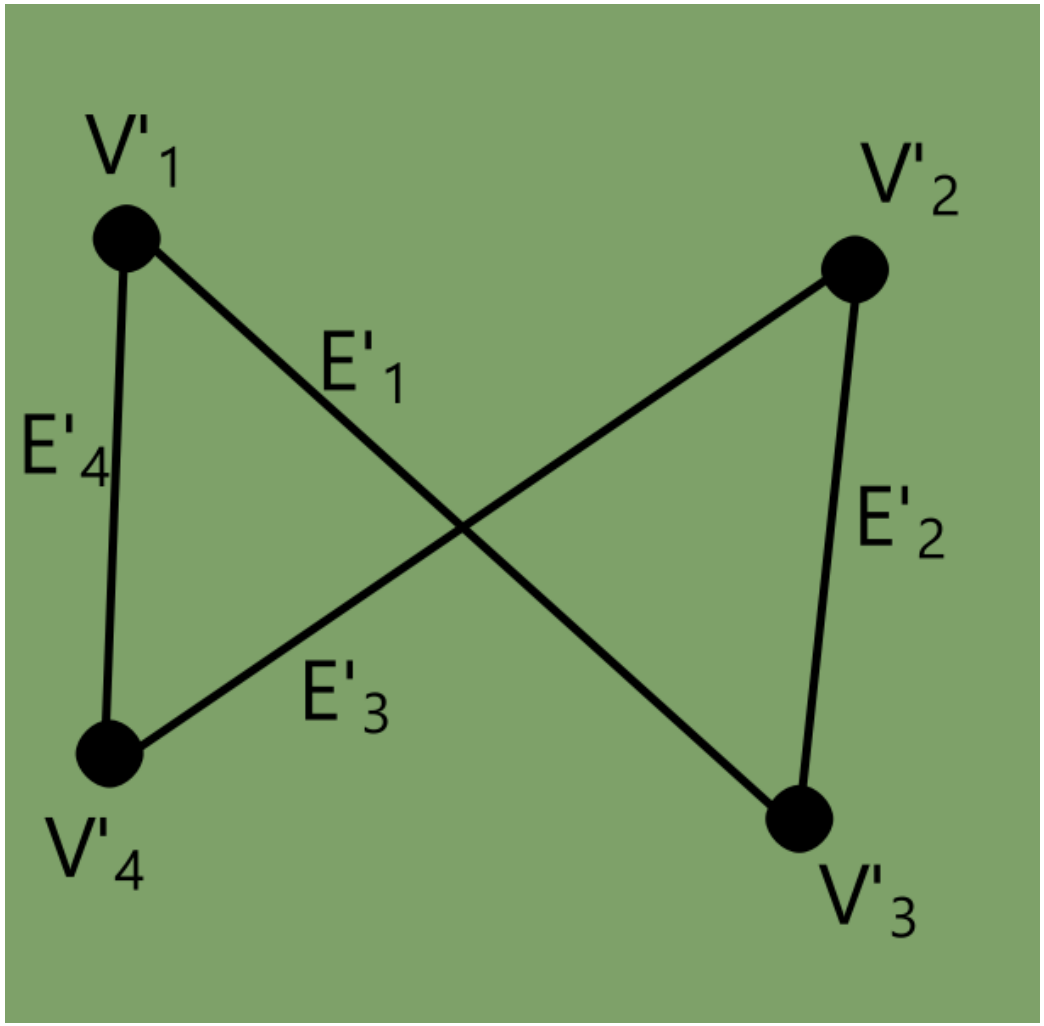
Graph

$G(V, E)$



Graph

$G'(V', E')$



Walk [2]

A walk is a series of vertices that are connected to each other by means of edges.

Graph

Definitions

Simple walk (trail) [1]

A simple walk (trail) is a walk that does not contain the same edge more than once.

Graph

Definitions

Directed walk [2]

A directed walk is a series of vertices that are connected to each other by means of edges in a way that respects the edge directions.

Graph

Definitions

Path [2], [1]

A path is a walk that visits each vertex in the walk only once. A path is also a trail.

Directed path [2]

A directed path is a directed walk that visits each vertex in the directed walk only once.

Graph

Definitions

Circuit [2], [1]

A circuit(closed walk) is a walk of length strictly positive that starts and ends at the same vertex. A simple circuit does not contain the same edge more than once.

Cycle [2]

A cycle is a closed path.

Directed circuit

A directed circuit(closed directed walk) is a directed walk of length strictly positive that starts and ends at the same vertex. A simple directed circuit does not contain the same edge more than once.

Directed cycle [2]

A directed cycle is a directed closed path.

Connected [1]

An undirected graph $G(V, E)$ is connected when a walk exists between each pair of vertices $v, v' \in V^2$ and $v \neq v'$.

Graph

Definitions

Connected [1]

An directed graph $G(V, E)$ is strongly connected when a directed walk exists between each pair of vertices $v, v' \in V^2$ and $v \neq v'$. Let $G'(V, E')$ be the undirected graph obtained by replacing directed edges of G with undirected edges. G is weakly connected if G' is connected.

Graph

Definitions

Network [2]

A network is a graph where vertices and edges have associated properties in the form of numerical values.

The length of a walk [1]

The length of a walk is equal to the sum of the weights of its edges.

The number of walks [1]

Let A be the adjacency matrix of a graph $G(V, E)$, then the cell with index (i, j) of the matrix A^d is equal to the number of walks of length $d \in \mathbb{Z}^+$ from v_i to v_j, where $v_i, v_j \in V^2$.

Graph

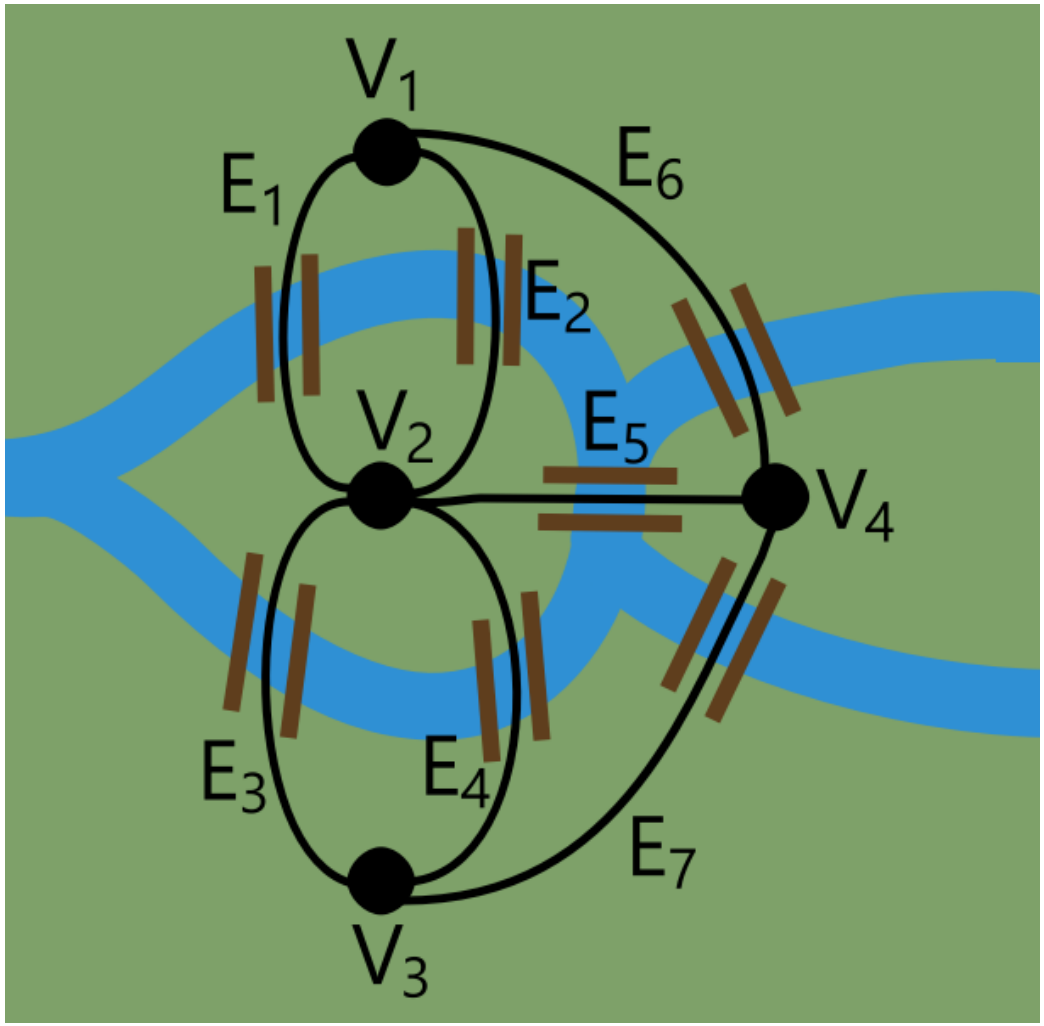
Definitions

Euler walk and circuit [1]

A simple circuit traversing all edges of a graph G is an Euler circuit.
Similarly, a simple walk traversing all edges of a graph G is an Euler walk.

Graph

Can you find an Euler circuit in this multigraph?



An Euler circuit exists..[1]

An Euler circuit exists in a connected multigraph $G(V, E)$ with $|V| \geq 2$ if and only if $\forall v \in V$, $deg(v) \equiv 0 \, (mod \, 2)$.

An Euler walk exists..[1]

An Euler walk but not an Euler circuit exists in a connected multigraph $G(V, E)$ if and only if $\exists\, v', v'' \in V^2$, $v' \neq v''$, $deg(v') \equiv 1\,(mod\,2)$, $deg(v'') \equiv 1\,(mod\,2)$, and $\forall v \in V \setminus \{v', v''\}$, $deg(v) \equiv 0\,(mod\,2)$.

Chinese postman (route inspection) problem

Chinese postman problem looks for the shortest circuit traversing every edge of a connected multigraph at least once.

Chinese postman problem

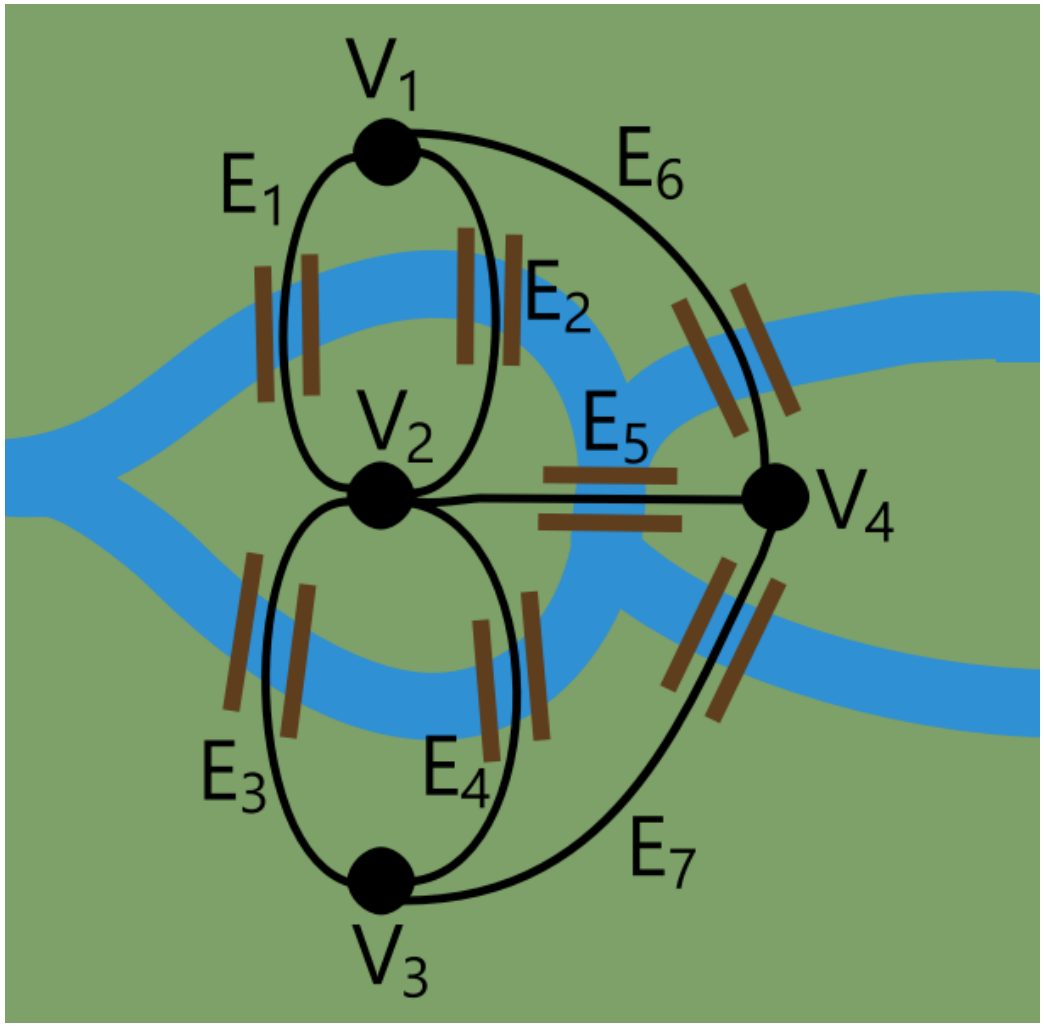
What if an Euler circuit exists in a connected multigraph?

Hamilton path and cycle [1]

A simple circuit visiting every vertex of a graph G exactly once is an Hamilton cycle. Similarly, a simple walk visiting every vertex of a graph G exactly once is an Hamilton path.

Graph

Can you find an Hamilton cycle in this multigraph?



Graph

Definitions (Dirac's theorem)

An Hamilton cycle exists..[1]

An Hamilton cycle exists in a graph $G(V, E)$ if G is a simple graph with $|V| \geq 3$ and $\forall v \in V$, $deg(v) \geq \frac{|V|}{2}$.

Traveling salesman problem

Traveling salesman problem looks for the shortest circuit visiting every vertex of a connected graph exactly once.

Traveling salesman problem

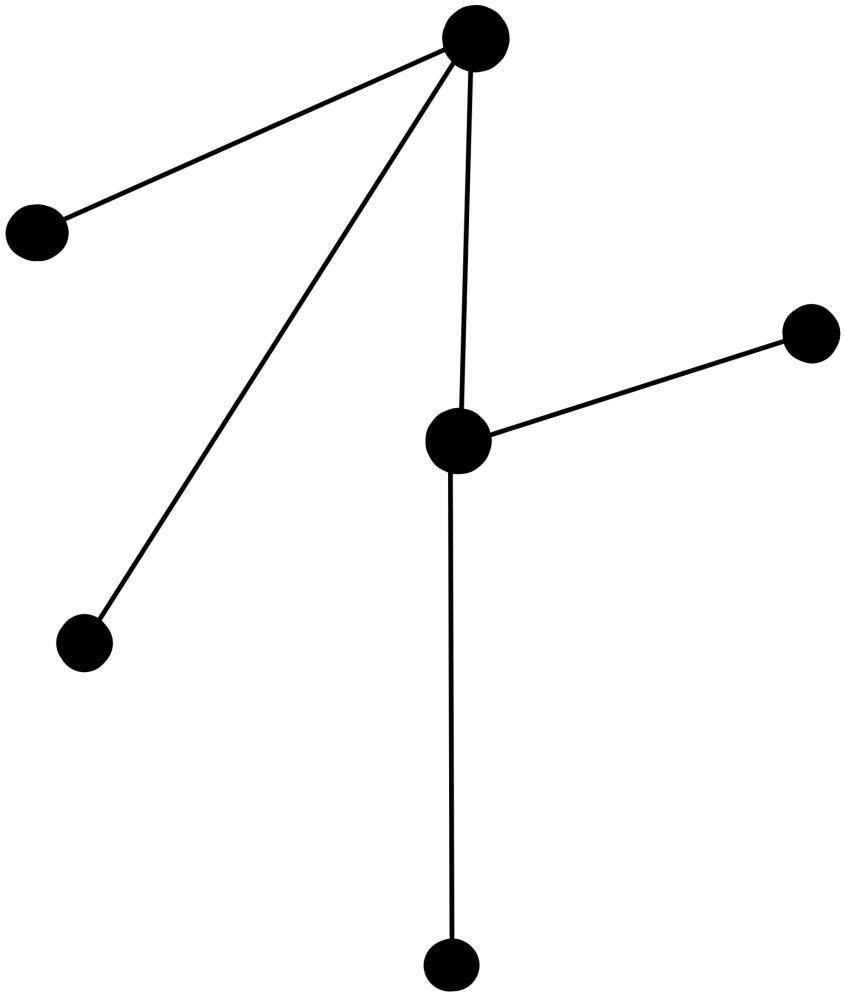
What about the feasible solutions of a traveling salesman problem if it is defined on a complete simple graph with more than 3 vertices? Is this problem feasible?

Tree [2]

A connected graph that contains no cycle is called tree.

Graph

A tree



Graph

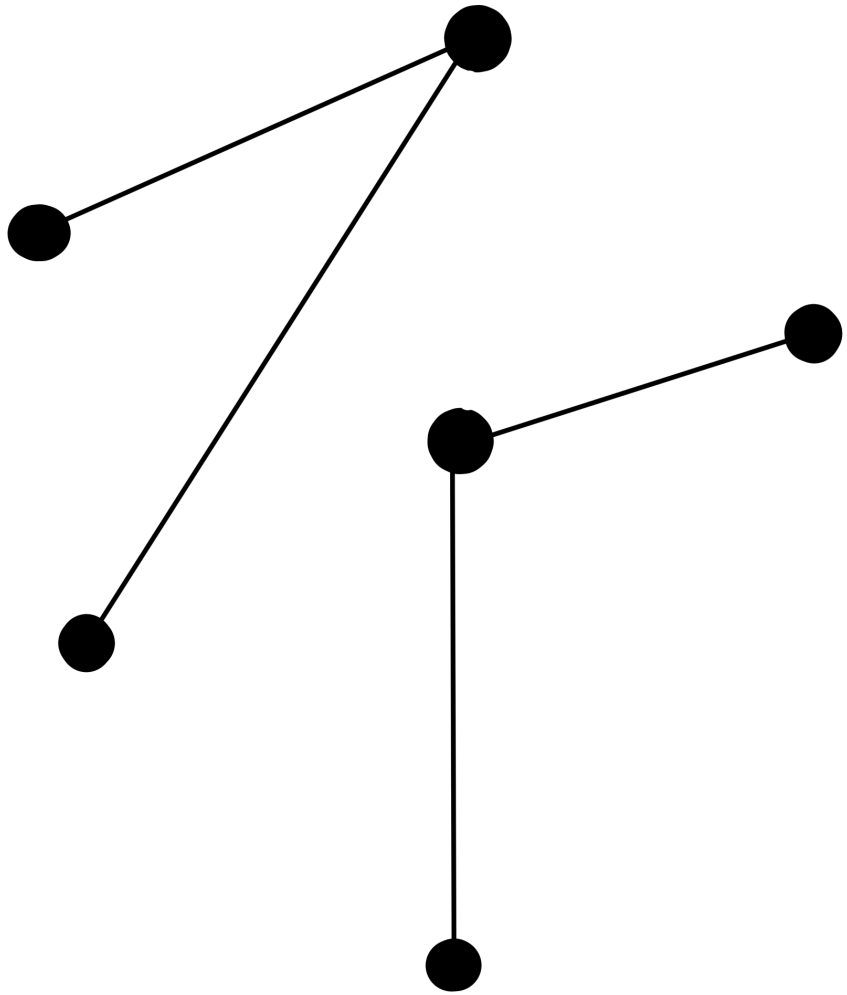
Definitions

Forest [2]

A collection of trees is called forest.

Graph

A forest



Graph

Definitions

The number of edges in a tree

If the graph $G(V, E)$ is a tree than $|E| = |V| - 1$

Graph

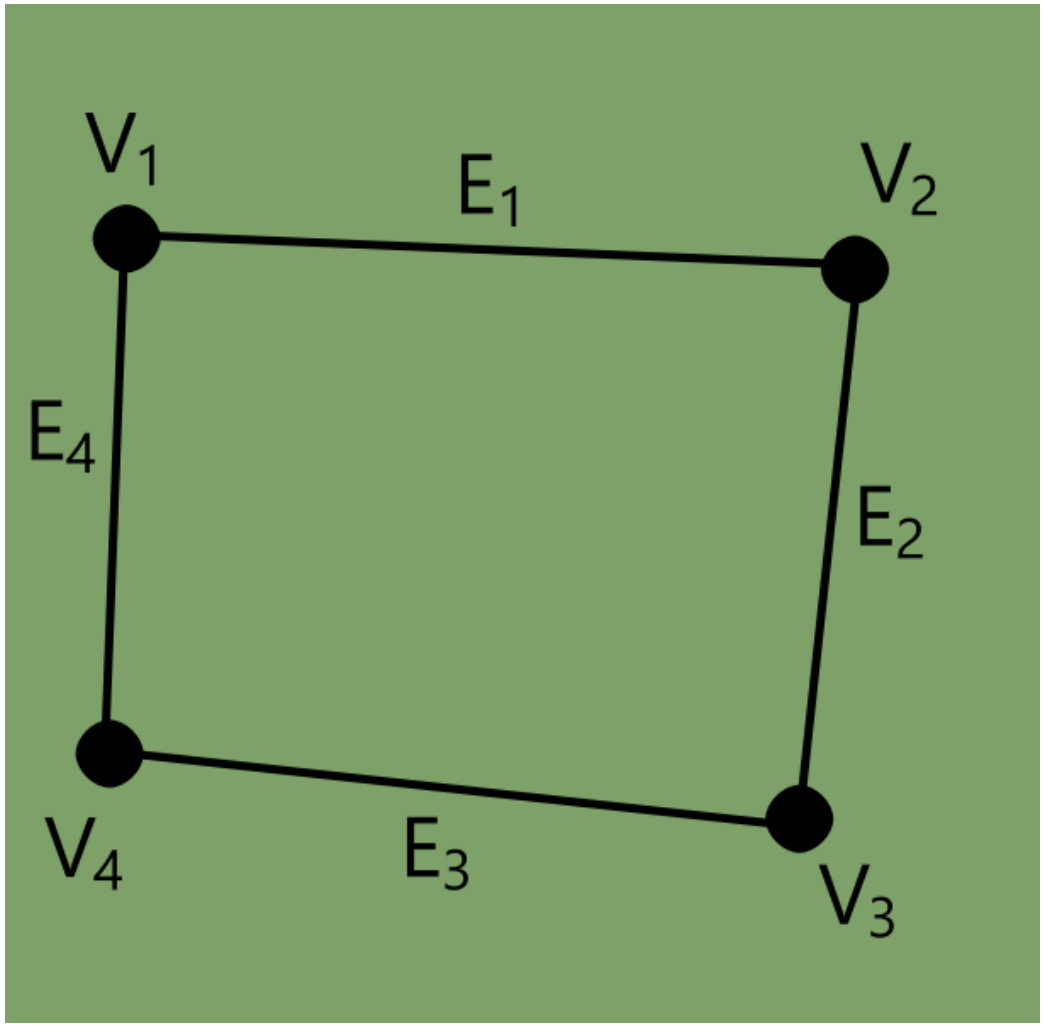
Definitions

Planar graph [1]

A planar graph can be drawn in two dimensions without any edges intersecting each other.

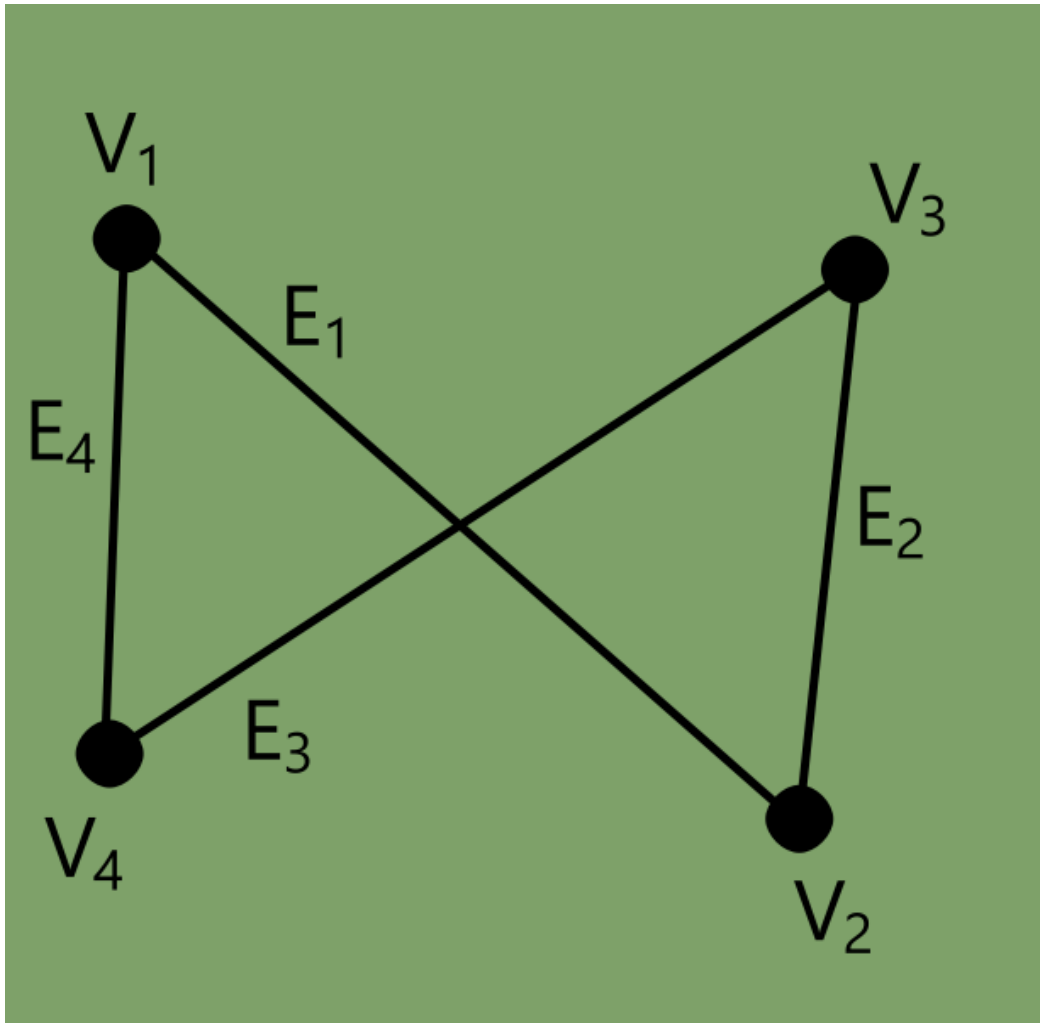
Graph

Planar representation of a planar graph



Graph

Non-planar representation of a planar graph



Graph

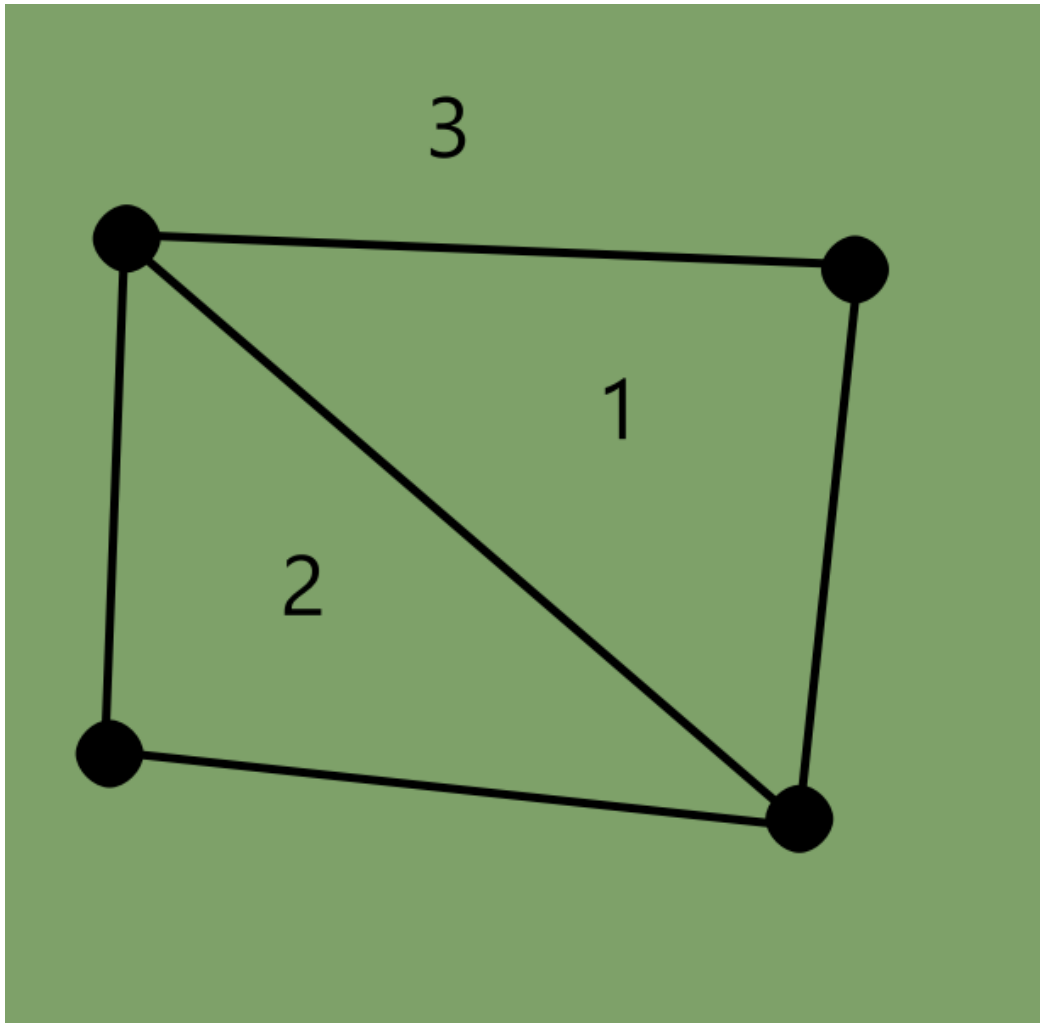
Definitions

Euler's formula [1]

A connected planar simple graph $G(V, E)$ has $|E| - |V| + 2$ regions.

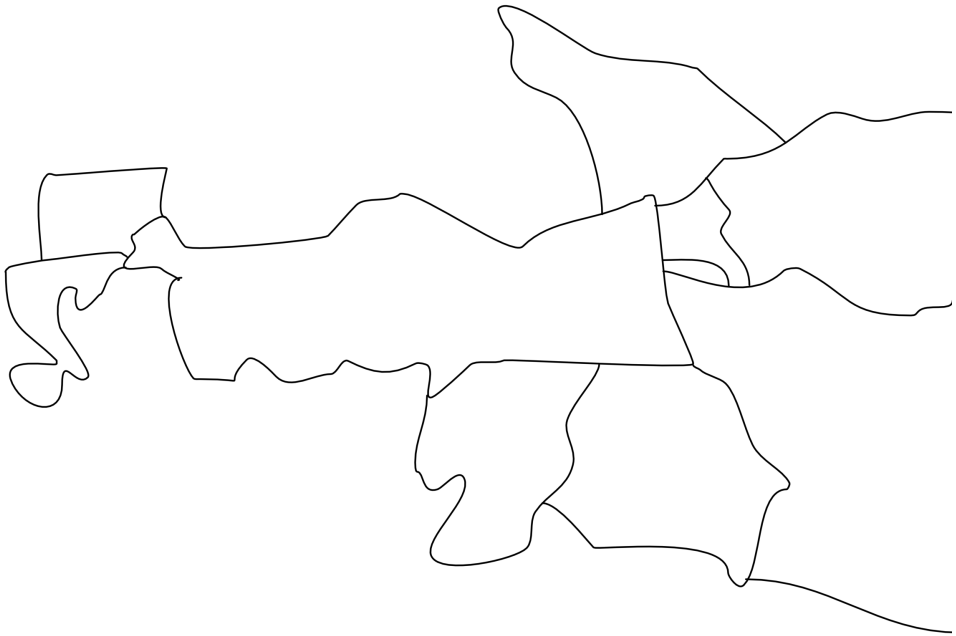
Graph

$3(= 5 - 4 + 2)$ regions of a planar graph



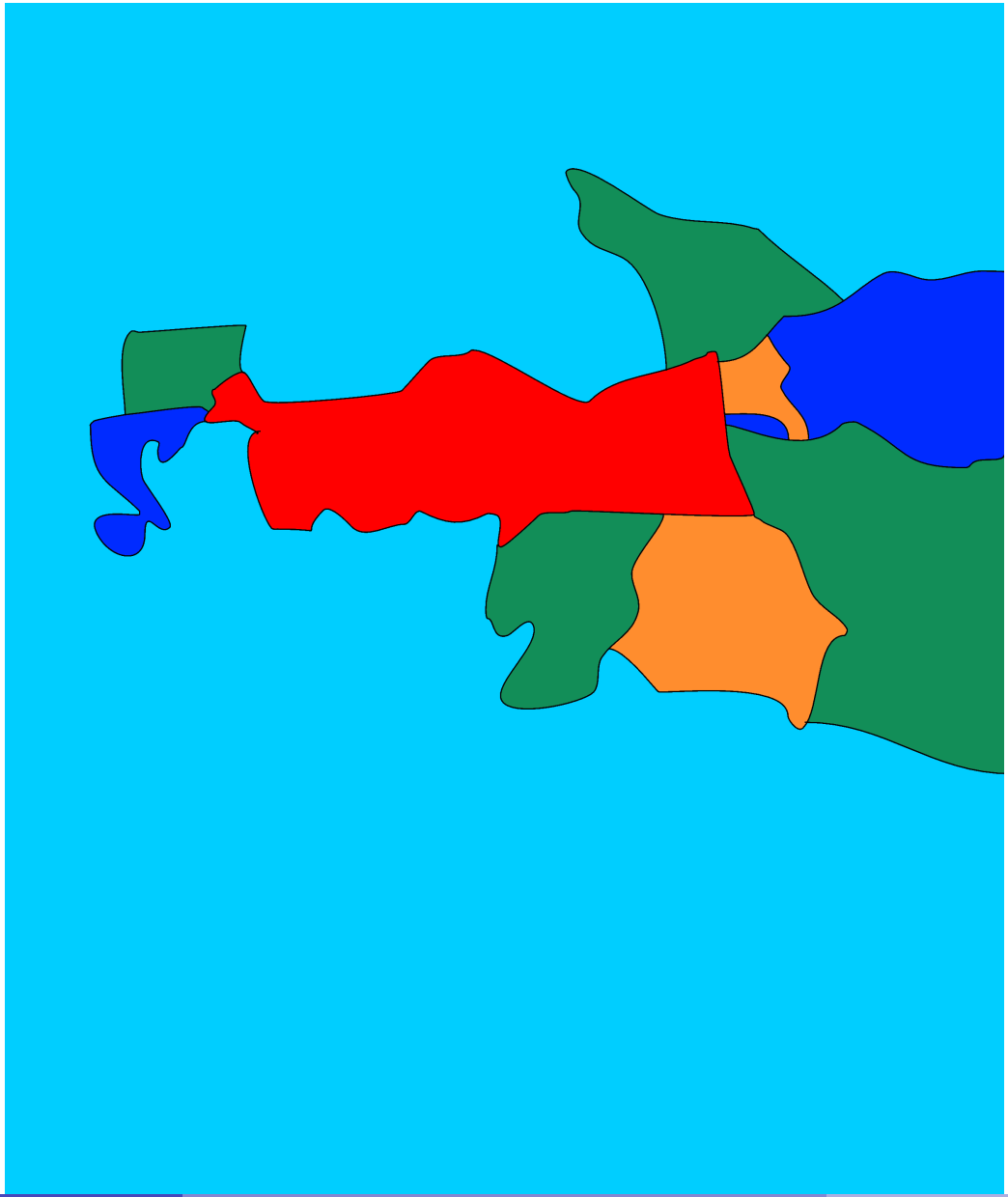
Graph

Map coloring example



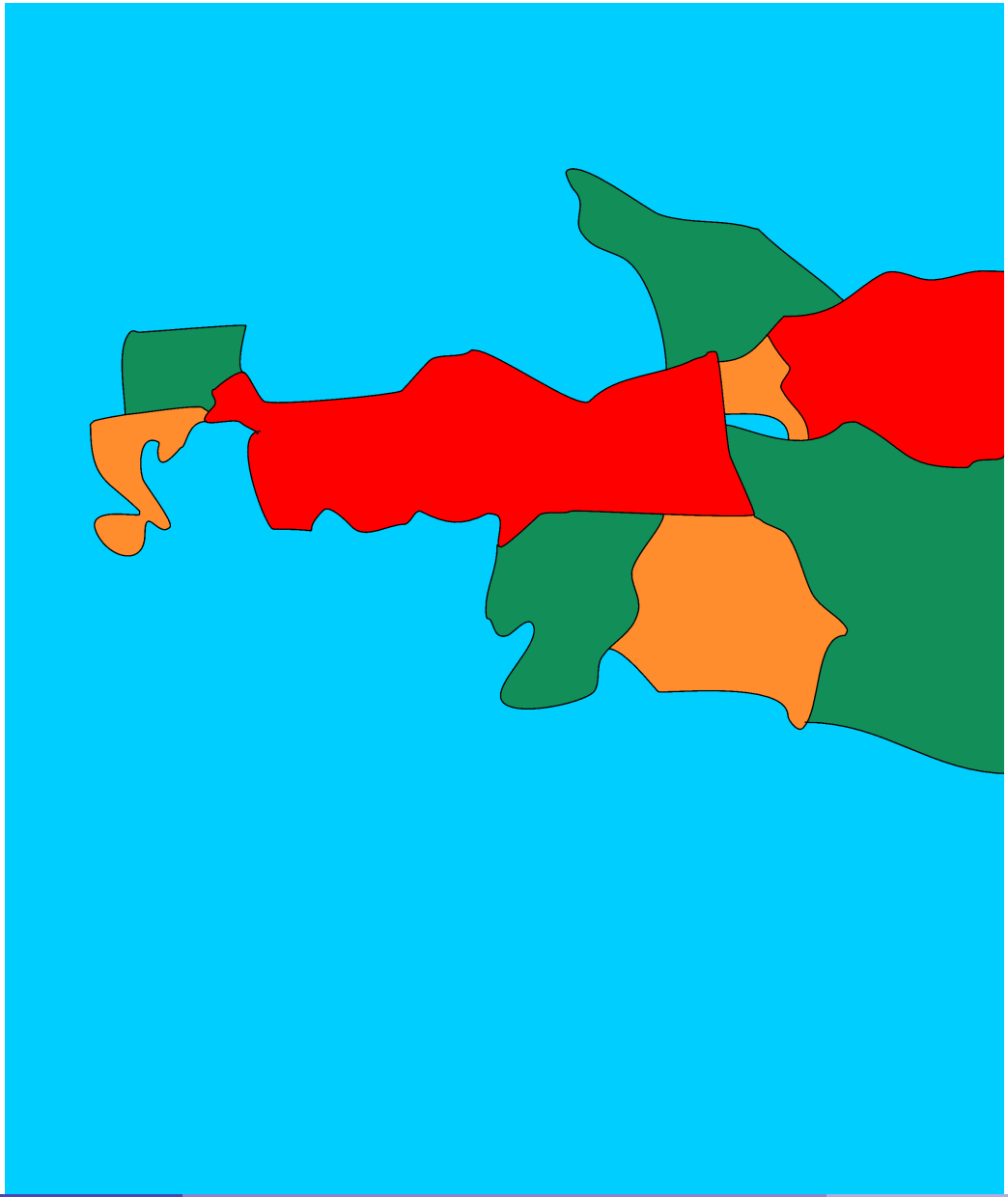
Graph

Map coloring example I (5 colors)



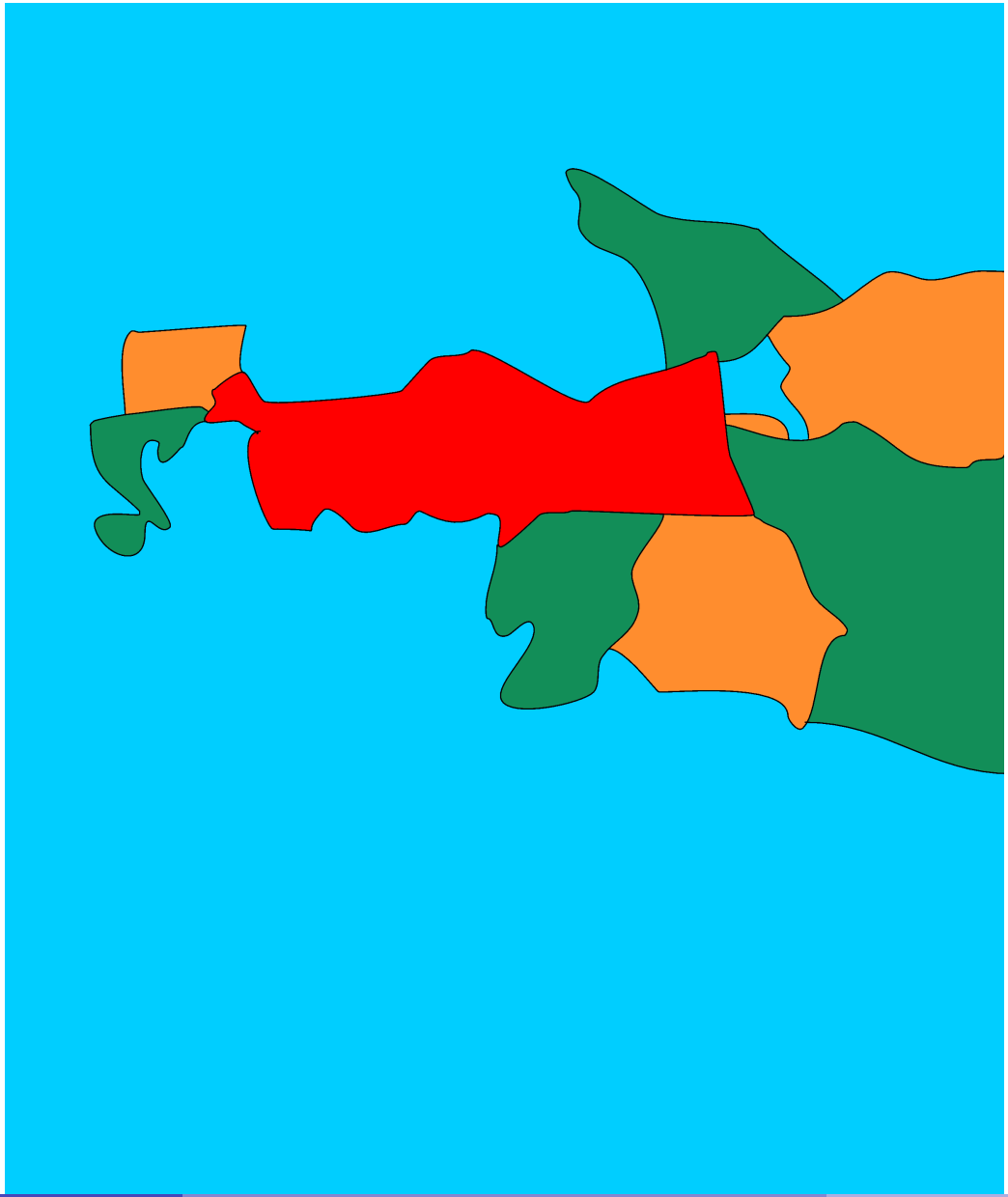
Graph

Map coloring example II (4 colors)



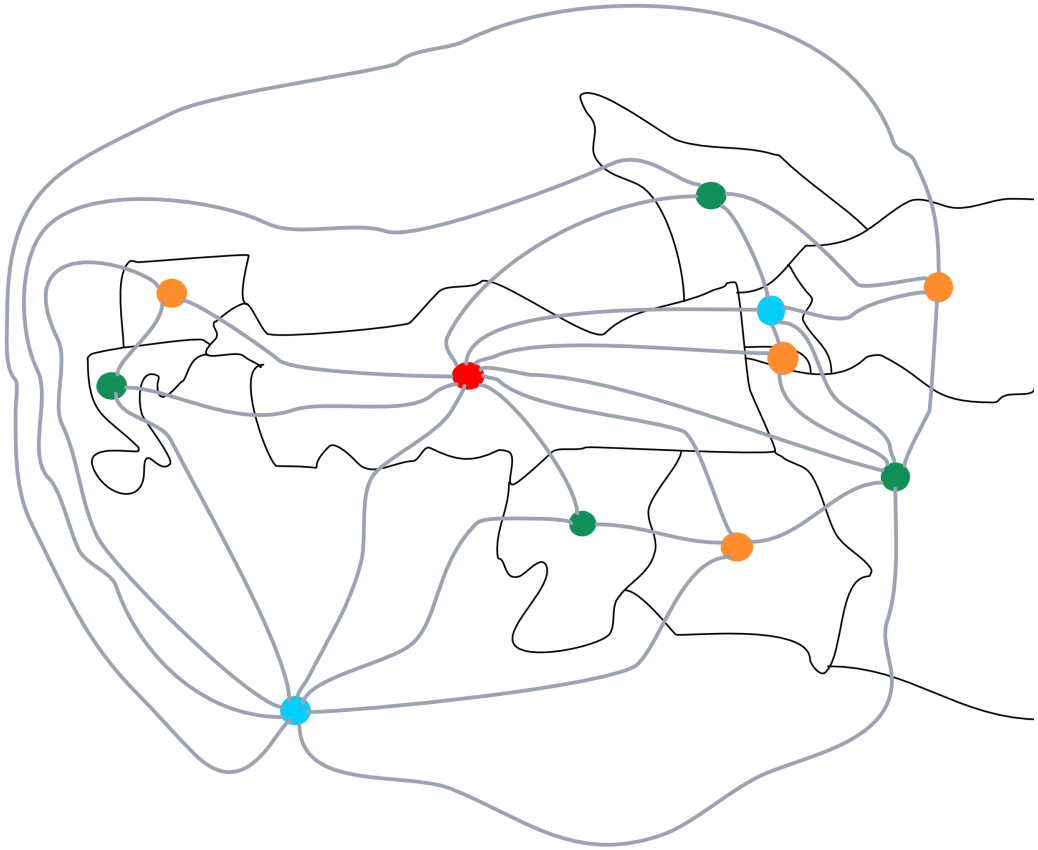
Graph

Map coloring example III (4 colors)



Graph

Dual graph (III) (4 colors)



Graph

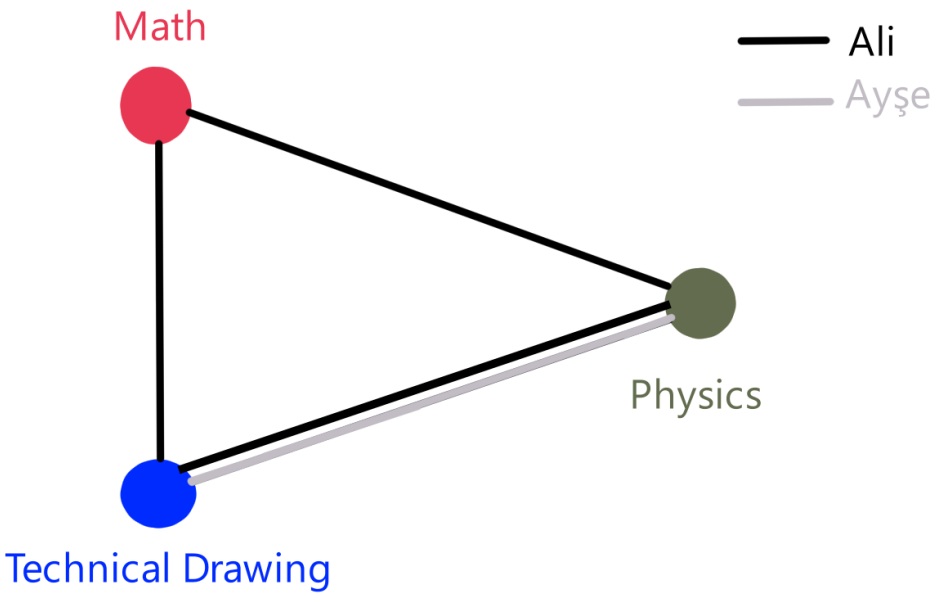
Definitions

The four color theorem [1]

The chromatic number (minimum number of colors) of a planar simple graph ≤ 4.

Graph

Graph coloring example



$G_{ind}(G, V')$, the subgraph of $G(V, E)$ induced by $V' \subseteq V$

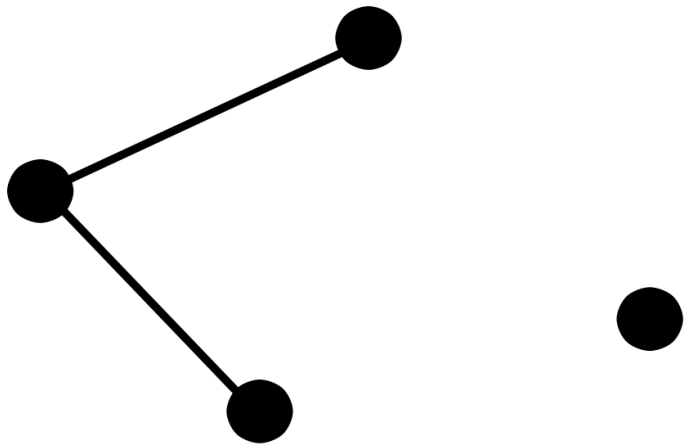
Assume that $G(V, E)$ is a simple undirected graph. Let $G_{ind}(G, V')$ be the subgraph of G formed by the vertex set $V' \subseteq V$ and the edge set $\{vv' : vv' \in E, v \in V', v' \in V'\}$.

$\overline{G}$, the complement(inverse) graph of $G(V, E)$

Assume that $G(V, E)$ is a simple undirected graph. Let $\overline{G}$ be the complement graph of G formed by the vertex set V and the edge set $\{vv' : vv' \notin E, v \in V, v' \in V, v \neq v'\}$.

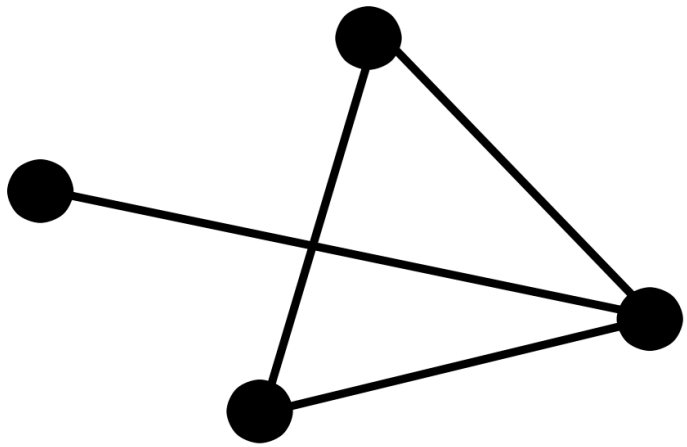
Graph

$G(V, E)$



Graph

$\overline{G}(V, \overline{E})$



Independent(stable) set

An independent set s of a simple undirected graph $G(V, E)$ is $V' \subseteq V : \forall v, v' \in V', vv' \notin E$. Let
$IS(G) = \{s | s \text{is an independent set of } G\}$.

Maximal independent set

An independent set s of a simple undirected graph $G(V, E)$ is maximal if and only if $s \not\subset s', \forall s' \in IS(G)$. Let
$IS_M(G) = \{s | s \text{is a maximal independent set of } G\}$.

Graph

Definitions

Clique

A clique s of a simple undirected graph $G(V, E)$ is $V' \subseteq V : G_{ind}(G, V')$ is complete. Let $CL(G) = \{s | s \text{ is a clique of } G\}$.

Maximal clique

A clique s of a simple undirected graph $G(V, E)$ is maximal if and only if $s \not\subset s', \forall s' \in CL(G)$. Let $CL_M(G) = \{s | s \text{ is a maximal clique of } G\}$.

Vertex cover

A vertex cover s of a simple undirected graph $G(V, E)$ is $V' \subseteq V : \forall vv' \in E, v \in V' \vee v' \in V'$. Let $VC(G) = \{s | s \text{ is a vertex cover of } G\}$.

Minimal vertex cover

A vertex cover s of a simple undirected graph $G(V, E)$ is minimal if and only if $s \subseteq s'$, $\forall s' \in VC(G)$. Let $VC_M(G) = \{s | s \text{ is a minimal vertex cover of } G\}$.

Graph

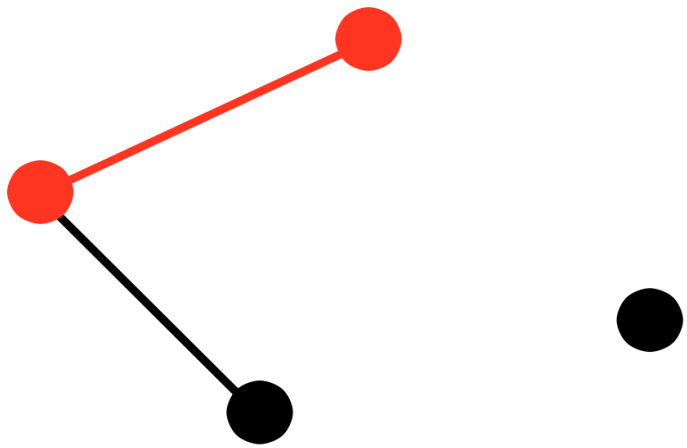
Definitions

Relationship between independent set, clique and vertex cover [3]

$G(V, E)$ is a simple undirected graph. s is a clique of $G \equiv s$ is an independent set of $\overline{G} \equiv V \setminus s$ is a vertex cover of $\overline{G}$.

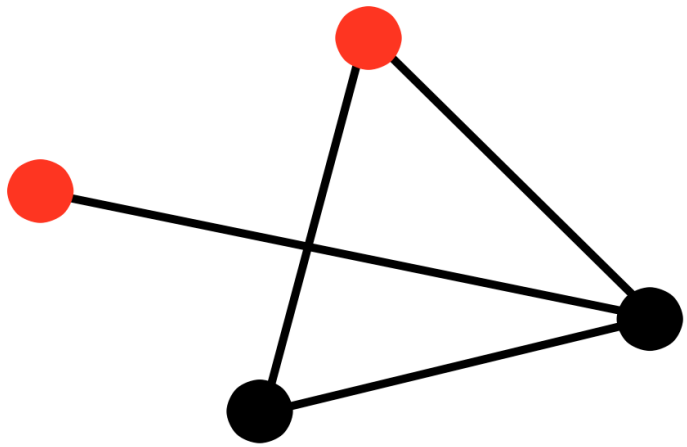
Graph

A clique in G



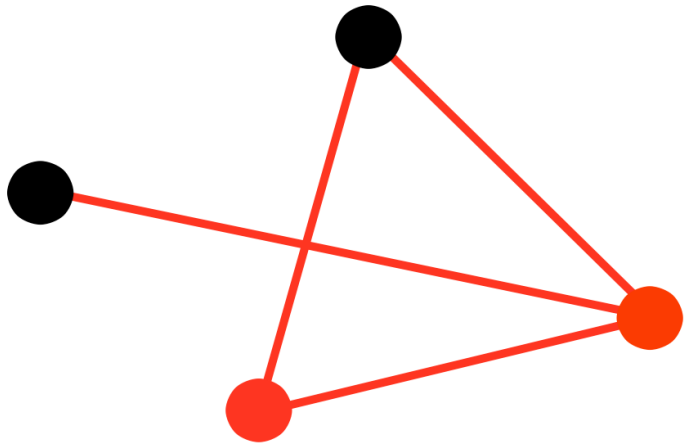
Graph

An independent set in $\overline{G}$



Graph

A vertex cover in $\overline{G}$



Clique, independent set, vertex cover

Can you find another clique, independent set and vertex cover in G, $\overline{G}$ and $\overline{G}$, respectively?

Graph

Maximum vertex weighted independent set problem

Maximum vertex weighted independent set problem in $G(V, E)$ [4]

$$
\begin{aligned}
& max \sum_{v \in V} w_v x_v & & \\
s.t. \quad & x_v + x_{v'} \leq 1 & & \forall vv' \in E \\
& x_v \in \{0, 1\} & & \forall v \in V
\end{aligned}
$$

Here, w_v is the weight of vertex v. The maximum vertex weighted independent set corresponds to x_v^* with value 1.

Graph

Maximum vertex weighted clique problem

Maximum vertex weighted clique problem in $G(V, E)$ [3]

$$
\begin{aligned}
& max \sum_{v \in V} w_v x_v & & \\
s.t. \quad & x_v + x_{v'} \leq 1 & & \forall vv' \in \overline{E} \\
& x_v \in \{0, 1\} & & \forall v \in V
\end{aligned}
$$

Here, w_v is the weight of vertex v. The maximum vertex weighted clique corresponds to x_v^* with value 1.

Graph

Minimum vertex weighted vertex cover problem

Minimum vertex weighted vertex cover problem in $G(V, E)$

$$
\begin{aligned}
& max \sum_{v \in V} w_v x_v & & \\
s.t. \quad & x_v + x_{v'} \leq 1 & & \forall vv' \in E \\
& x_v \in \{0, 1\} & & \forall v \in V
\end{aligned}
$$

Here, w_v is the weight of vertex v. The minimum vertex weighted vertex cover corresponds to x_v^* with value 0.

Minimum coloring problem [4]

$$min \sum_{s \in IS_M(G)} x_s$$

$$s.t. \sum_{s \in IS_M(G): v \in s} x_s \geq 1 \qquad \forall v \in V$$

$$x_s \in \{0, 1\} \qquad \forall s \in IS_M(G)$$

Relaxed and restricted minimum coloring problem [4]

$$
\begin{aligned}
min \quad & \sum_{s \in \tilde{IS}_M(G)} x_s & & \\
s.t. \quad & \sum_{s \in \tilde{IS}_M(G): v \in s} x_s \geq 1 & & \forall v \in V \\
& x_s \geq 0 & & \forall s \in \tilde{IS}_M(G)
\end{aligned}
$$

where $\tilde{IS}_M(G)$ is the restricted maximal independent set.

Branching operations that preserve graph structure [4]

$\exists\ s' \in \tilde{IS}_M(G)$ and $s'' \in \tilde{IS}_M(G) : x_{s'} \vee x_{s''} \notin \{0,1\}$, $\exists\ v \in s' \cap s''$ and $v' \in s' \setminus s''$, then the branching for forcing integrality is as follows:

- Samecolor(v, v'): merge the vertices v and v'
- Differentcolor(v, v'): add an edge between the vertices v and v'

A branch-and-price algorithm [4]

At each node,

- A: This node can be fathomed if the minimum possible feasible color number (size of the maximum cardinality clique or size of a maximal clique for convenience) is greater (worse) or equal than the upper bound (incumbent). Otherwise, go to step B.
- B: Solve the relaxed and restricted minimum coloring problem. In the optimal solution, call the dual variable corresponding to the vertex v w_v.
- C: For the pricing problem, solve the maximum vertex weighted independent set problem using w_v as the weight of vertex v. If the objective function value is greater than 1 in the optimal solution of the pricing problem, then expand the set $\tilde{IS}_M(G)$ by adding a new maximal independent set corresponding to the x_v^* with value 1.
- ...

A branch-and-price algorithm [4]

- ...
- D: Repeat the steps B and C until there is no new independent set that can expand the set $\tilde{IS}_M(G)$ according to the pricing problem.
- E: If the optimal solution of the relaxed and restricted minimum coloring problem is greater (worse) or equal than the upper bound (incumbent), then this node can be pruned. Otherwise, if the optimal solution is not integral, then create two new sub-problems using the procedures "Samecolor" and "Differentcolor". If the optimal solution is integral, upgrade the upper bound if required.

Minimum coloring problem [5]

$$
\begin{aligned}
& min \sum_{c \in C} x_c & & \\
& s.t. \sum_{c \in C} y_{vc} = 1 & & \forall v \in V \\
& \quad y_{vc} + y_{v'c} \leq x_c & & \forall vv' \in E, \forall c \in C \\
& \quad x_c \leq \sum_{v \in V} y_{vc} & & \forall c \in C \\
& \quad x_c \geq x_{c+1} & & \forall c \in C \setminus \{\text{last color}\} \\
& \quad x_c \in \{0,1\} & & \forall c \in C \\
& \quad y_{vc} \in \{0,1\} & & \forall v \in V, \forall c \in C
\end{aligned}
$$

where C is the color set. And, $|C| = |V|$.

Network Problems

Minimum cost flow problem

Minimum cost flow problem

Let $G(V, E)$ be a directed graph with costs $c_{vv'}$ and capacities $u_{vv'}$ defined on edges $vv' = e \in E$, where $v \neq v'$, v and $v' \in V$. Let $b_v > 0$ be the supply and $b_v < 0$ be the demand associated with each vertex $v \in V$. Moreover, $x_{vv'}$ denotes the amount of flow from a vertex v to another vertex v'. Then, minimum cost flow problem minimizes the total cost incurred from all flows in G satisfying both flow conservation constraints and flow limits.

Network Problems

Minimum cost flow problem

Minimum cost flow problem

$$min \sum_{vv' \in E} c_{vv'} x_{vv'}$$

$$s.t. \sum_{v':vv' \in E} x_{vv'} - \sum_{v':v'v \in E} x_{v'v} = b_v \qquad \forall v \in V$$

$$0 \leq x_{vv'} \leq u_{vv'} \qquad \forall vv' \in E$$

Network Problems

Minimum cost flow problem

Assumptions [2]

- $\forall e \in E,\ c_e \in \mathcal{Z}_0^+$
- $\forall v \in V,\ b_v \in \mathcal{Z}$ and $\sum_v b_v = 0$
- $\forall e \in E,\ u_e \in \mathcal{Z}_0^+$
- $\forall v, v' \in V^2, \exists$ an uncapaciated directed path from v to v'

Network Problems

Definitions

Polynomial time algorithm

A polynomial time algorithm has a running time polynomial in the length (number of bits) of the input.

Weakly polynomial time algorithm [2]

There is a polynomial function $f_p(|V|, |E|, log(C), log(U))$ as an upper bound for the worst-case complexity of a weakly polynomial time algorithm.

Strongly polynomial time algorithm [2]

There is a polynomial function $f_p(|V|, |E|)$ as an upper bound for the worst-case complexity of a strongly polynomial time algorithm.

Network Problems

Definitions

Pseudo-polynomial time algorithm

A pseudo-polynomial time algorithm has a running time polynomial in the numeric value (largest value) of the input.

Worst-case complexity

There is a polynomial function $f_p(|V|, |E|, C, U)$ as an upper bound for the worst-case complexity of a pseudo-polynomial time algorithm [2].

Subclass of exponential-time algorithms

Pseudo-polynomial time algorithms are a subclass of exponential-time algorithms [2].

Network Problems

Minimum cost flow problem

Polynomial time algorithms [2]

- Capacity scaling with $\mathcal{O}(\,(|E|log(U))(|E|+|V|log(|V|))\,)$ running time
- Cost scaling with $\mathcal{O}(|V|^3 log(|V|C))$ running time
- Double scaling with $\mathcal{O}(|V||E|log(U)log(|V|C))$ running time
- Minimum mean cycle-canceling with $\mathcal{O}(|V|^2|E|^3 log(|V|))$ running time
- Repeated capacity scaling with $\mathcal{O}(\,(|E|^2 log(|V|))(|E|+|V|log(|V|))\,)$ running time
- Enhanced capacity scaling algorithm $\mathcal{O}(\,(|E|log(|V|))(|E|+|V|log(|V|))\,)$ running time

where, $c_e \leq C$, $\forall e \in E$ and $u_e \leq U$, $\forall e \in E$

Network Problems

Minimum cost flow problem

Pseudo-polynomial time algorithms [2]

- Cycle-canceling with $\mathcal{O}(|E|CU)$ iterations
- Successive shortest path with $\mathcal{O}(|V|U)$ iterations
- Primal-dual algorithm with $\mathcal{O}(min(|V|U, |V|C))$ iterations
- Out-of-kilter with $\mathcal{O}(|V|U)$ iterations
- Relaxation

Network Problems

Minimum cost flow problem

Complexity of some minimum cost flow algorithms [6]

- Ford and Fulkerson, $\mathcal{O}(|V|^4 CU)$
- Out-of-kilter, $\mathcal{O}(|E|^3 U)$
- Successive shortest path, $\mathcal{O}(|V|^2|E|U)$
- Cycle-cancelling, $\mathcal{O}(|V||E|^2 CU)$
- Cost-scaling (generic), $\mathcal{O}(|V|^2|E|log(|V|C))$
- Cancel-and-tighten (dynamic trees), $\mathcal{O}(|V||E|log(|V|)min(log(|V|C, |E|log(|V|))))$
- Primal network simplex (dynamic trees), $\mathcal{O}(|V||E|log(|V|)min(log(|V|C, |E|log(|V|))))$
- Dual network simplex (Orlin), $\mathcal{O}(|E|(|E| + |V|log|V|)min(log(|E|U), |E|log(|V|)))$
- Dual network simplex (Armstrong and Jin), $\mathcal{O}(|V||E|log|V|(|E| + |V|log|V|))$

Network Problems

Minimum cost flow problem

Study of minimum cost flow algorithms [6]

Cost-scaling and primal network simplex were both efficient and robust.

Network Problems

Minimum cost flow problem

Study of seven state-of-the-art algorithms [7]

- Simple cycle canceling
- Minimum mean cycle canceling
- Cancel and tighten
- Successive shortest path
- Capacity scaling
- Network simplex
- Cost scaling

where, network simplex was the fastest algorithm in $\approx 75\%$ of the studied cases

Maximum flow problem

Let $G(V, E)$ be a directed graph with capacities $u_{vv'} \geq 0$ defined on edges $vv' = e \in E$, where $v \neq v'$, v and $v' \in V$. Let $b_v > 0$ be the supply and $b_v < 0$ be the demand associated with each vertex $v \in V$. Moreover, $x_{vv'}$ denotes the amount of flow from a vertex v to another vertex v'. Then, maximum flow problem maximizes the amount of flow from the source vertex $s \in V$ to the sink vertex $t \in V$, $s \neq t$, and all flows in G satisfy both flow conservation constraints and flow limits.

Maximum flow problem

$$max \quad \alpha$$

$$s.t. \quad \sum_{v':vv' \in E} x_{vv'} - \sum_{v':v'v \in E} x_{v'v} = \begin{cases} \alpha & for \quad v = s \\ 0 & \forall v \in V \setminus \{s, t\} \\ -\alpha & for \quad v = t \end{cases}$$

$$0 \leq x_{vv'} \leq u_{vv'} \qquad \forall vv' \in E$$

Network Problems

Maximum flow problem

Special case of minimum cost flow problem

- Maximum flow problem from s to t on $G(V, E)$
- Set $b_v = 0, \forall v \in V$
- Set $c_e = 0, \forall e \in E$
- Add a new edge ts with $c_{ts} = -1$ and $u_{ts} = \infty$
- $E' = E \cup \{ts\}$
- Minimum cost flow problem on $G'(V, E') \equiv$ Maximum flow problem on $G(V, E)$

Network Problems

Maximum flow problem

Assumptions [2]

- $\forall e \in E,\ u_e \in \mathcal{Z}_0^+$
- $\nexists$ an uncapaciated directed path from s to t
- If $vv' \in E$ than $v'v \in E$
- No multiple edges

Network Problems

Maximum flow problem

Running times of maximum flow algorithms [2]

- Labeling, $\mathcal{O}(|V||E|U)$
- Capacity scaling, $\mathcal{O}(|V||E|log(U))$
- Successive shortest path, $\mathcal{O}(|V|^2|E|)$
- Generic preflow-push, $\mathcal{O}(|V|^2|E|)$
- FIFO preflow-push, $\mathcal{O}(|V|^3)$
- Highest-label preflow-push, $\mathcal{O}(|V|^2\sqrt{|E|})$
- Excess scaling, $\mathcal{O}(|V||E| + |V|^2log(U))$

Network Problems

Minimum cost flow and maximum flow problems

Running time of an almost linear time algorithm [8] for minimum cost flows and maximum flows

- Demands, costs and capacities are bounded polynomially
- Demands, costs and capacities are integral
- Runs in $m^{1+\mathcal{O}(1)}$ time

Network Problems

Maximum flow problem

Feasible flow problem

$$\sum_{v':vv' \in E} x_{vv'} - \sum_{v':v'v \in E} x_{v'v} = b_v \qquad \forall v$$

$$0 \leq x_{vv'} \leq u_{vv'} \qquad \forall vv' \in E$$

Network Problems

Maximum flow problem

Procedure to create a transformed network $G'(V', E')$ [2]

- Add the vertex s
- $\forall v \in V$ with $b_v > 0$, add the edges sv with $u_{sv} = b_v$
- Add the vertex t
- $\forall v \in V$ with $b_v < 0$, add the edges vt with $u_{vt} = -b_v$
- $V' = V \cup \{s, t\}$
- $E' = E \cup \{sv : v \in V,\ b_v > 0\} \cup \{vt : v \in V,\ b_v < 0\}$

Network Problems

Maximum flow problem

Maximum flow problem on the transformed network $G'(V', E')$

$$max \quad \alpha$$

$$s.t. \quad \sum_{v':vv' \in E'} x_{vv'} - \sum_{v':v'v \in E'} x_{v'v} = \begin{cases} \alpha & for \quad v = s \\ 0 & \forall v \in V' \setminus \{s, t\} \\ -\alpha & for \quad v = t \end{cases}$$

$$0 \leq x_{vv'} \leq u_{vv'} \qquad \forall vv' \in E'$$

Network Problems

Maximum flow problem

Feasible flow problem

If α^* of the maximum flow problem on the transformed network $G'(V', E')$ is equal to $\sum_{v \in V,\, b_v > 0} b_v$ than the flow problem is feasible.

Network Problems

Maximum flow problem

Maximum flow problem with lower bounds on $G(V, E)$

$$max \quad \alpha$$

$$s.t. \quad \sum_{v':vv' \in E} x_{vv'} - \sum_{v':v'v \in E} x_{v'v} = \begin{cases} \alpha & for \quad v = s \\ 0 & \forall v \in V \setminus \{s, t\} \\ -\alpha & for \quad v = t \end{cases}$$

$$l_{vv'} \leq x_{vv'} \leq u_{vv'} \qquad \forall vv' \in E$$

Network Problems

Maximum flow problem

Procedure to create a circulation network $G^c(V, E^c)$ [2]

- Add the edge ts with $u_{ts} = \infty$
- $E^c = E \cup \{ts\}$

so that it is possible to send the flow from s to t back to s from t by using the edge ts with $u_{ts} = \infty$.

Network Problems

Maximum flow problem

Circulation problem (a feasible flow of the maximum flow problem with lower bounds) [2]

$$\sum_{v':vv' \in E^c} x_{vv'} - \sum_{v':v'v \in E^c} x_{v'v} = 0 \qquad \forall v \in V$$

$$l_{vv'} \leq x_{vv'} \leq u_{vv'} \qquad \forall vv' \in E^c$$

Network Problems

Maximum flow problem

Transformed ($x_{vv'} = x'_{vv'} + l_{vv'}$) circulation problem [2]

$$\sum_{v':vv' \in E^c} x'_{vv'} - \sum_{v':v'v \in E^c} x'_{v'v} = b_v \qquad \forall v \in V$$

$$0 \leq x'_{vv'} \leq u_{vv'} - l_{vv'} \qquad \forall vv' \in E^c$$

where $b_v = \sum_{v':v'v \in E^c} l_{v'v} - \sum_{v':vv' \in E^c} l_{vv'} \qquad \forall v \in V$

Network Problems

Maximum flow problem

Feasible flow problem

A feasible flow can be found by solving a maximum flow problem on the transformed network $G^{c\prime}(V', E^{c\prime})$.

Network Problems

Maximum flow problem

Residual capacities on $G(V, E)$ [2]

A residual capacity of an edge vv' is denoted as $r_{vv'} = (u_{vv'} - x_{vv'}) + (x_{v'v} - l_{v'v})$, where $x_{vv'}$'s and $x_{v'v}$'s are the feasible flows found in the previous step.

Maximum flow problem with residual capacities on $G(V, E)$

Solve the maximum flow problem with residual capacities on $G(V, E)$. Note that the residual capacity $r_{vv'}$ denotes the maximum possible increase in flow for the edge vv'.

Find the solution of the maximum flow problem with lower bounds

Find the solution of the maximum flow problem with lower bounds on $G(V, E)$ by increasing feasible flows found in the feasible flow problem by values from the maximum flow problem with residual capacities.

Network Problems

Maximum flow problem

Minimum value problem [2] with lower bounds on $G(V, E)$

$$min \quad \alpha$$

$$s.t. \quad \sum_{v':vv' \in E} x_{vv'} - \sum_{v':v'v \in E} x_{v'v} = \begin{cases} \alpha & for \quad v = s \\ 0 & \forall v \in V \setminus \{s, t\} \\ -\alpha & for \quad v = t \end{cases}$$

$$l_{vv'} \leq x_{vv'} \leq u_{vv'} \qquad \forall vv' \in E$$

Solution method for minimum value problem

First find a feasible flow. Than solve the maximum flow problem, where capacities $r^{inv}_{vv'}$ are equal to $(x_{vv'} - l_{vv'}) + (u_{v'v} - x_{v'v})$. Note that the capacity $r^{inv}_{vv'}$ denotes the maximum possible decrease in flow for the edge vv'. Finally, the solution of the minimum value problem with lower bounds on $G(V, E)$ can be found by decreasing feasible flows by values from the maximum flow problem with capacities $r^{inv}_{vv'}$.

Shortest path problem

Let $G(V, E)$ be a directed graph with costs $c_{vv'}$ defined on edges $vv' = e \in E$, where $v \neq v'$, v and $v' \in V$. Let $b_v > 0$ be the supply and $b_v < 0$ be the demand associated with each vertex $v \in V$. Moreover, $x_{vv'}$ denotes the amount of flow from a vertex v to another vertex v'. Then, shortest path problem minimizes the lengths of directed paths from a vertex s to all other vertices $t \in V$, $t \neq s$. Equivalently, shortest path problem minimizes the cost of sending an amount of unit flows from vertex s to all other vertices $t \in V$, $t \neq s$, where all flows in G are positive and satisfy the flow conservation constraints.

Network Problems

Shortest path problem

Shortest path problem

$$min \quad \sum_{vv' \in E} c_{vv'} x_{vv'}$$

$$s.t. \quad \sum_{v':vv' \in E} x_{vv'} - \sum_{v':v'v \in E} x_{v'v} = \begin{cases} |V| - 1 & for \quad v = s \\ -1 & \forall v \in V \setminus \{s\} \end{cases}$$

$$0 \leq x_{vv'} \qquad \forall vv' \in E$$

Network Problems

Shortest path problem

Special case of minimum cost flow problem

- Shortest path problem from vertex s to other vertices on $G(V, E)$
- Add $u_e = \infty$, $\forall e \in E$
- Minimum cost flow problem (with u_e) on $G(V, E) \equiv$ Shortest path problem from vertex s to other vertices on $G(V, E)$

Network Problems

Shortest path problem

Assumptions [2]

- $\forall e \in E,\ c_e \in \mathcal{Z}$
- $\exists$ a directed path from vertex s to any vertex t, $t \in V$, $t \neq s$
- $\nexists$ a negative cycle

Network Problems

Shortest path problem

Label-setting algorithms

- Once labels are set they are not allowed to be changed

Network Problems

Shortest path problem

Some graph features for label-setting algorithms [2]

- $G(V, E)$ is a directed acyclic (does not contain any directed cycle) network with possibly negative c_e's, $e \in E$
- or $G(V, E)$ is a network with $c_e \geq 0$, $e \in E$

Network Problems

Shortest path problem

Label-correcting algorithms

- Less restrictive problem formulations
- Less efficient than label-setting algorithms

Network Problems

Shortest path problem

Breadth-First Search

- It is a label-setting algorithm
- $\forall e \in E,\ c_e = 1$
- Runs in $\mathcal{O}(|V| + |E|)$ time [9]

Network Problems

Shortest path problem

Directed-acyclic graph algorithm

- It is a label-setting algorithm
- Runs in $\mathcal{O}(|V| + |E|)$ time [9]

Network Problems

Shortest path problem

Dijkstra's algorithm

- It is a label-setting algorithm
- $\forall e \in E,\ c_e \geq 0$
- Original implementation runs in $\mathcal{O}(|V|^2)$ time [2]

Network Problems

Shortest path problem

Running times of variants [2] of Dijkstra's algorithm

- Dial, $\mathcal{O}(|E| + |V|C)$
- d-Heap, $\mathcal{O}(|E| log_d(|V|))$, $d = \frac{|E|}{|V|}$
- Fibonacci heap implementation, $\mathcal{O}(|E| + |V| log(|V|))$
- Radix heap implementation, $\mathcal{O}(|E| + |V| log(|V|C))$

Network Problems

Shortest path problem

Bellman-Ford-Moore algorithm

- It is a label-correcting algorithm
- $\exists e \in E,\ c_e < 0$ (arbitrary)
- FIFO implementation runs in $\mathcal{O}(|V||E|)$ time [10]

Network Problems

Shortest path problem

Running times of label-correcting algorithms [2]

- Generic, $\mathcal{O}(min(|V|^2|E|C, |E|2^{|V|}))$
- Modified, $\mathcal{O}(min(|V||E|C, |E|2^{|V|}))$
- Modified FIFO, $\mathcal{O}(|V||E|)$
- Modified Dequeue, $\mathcal{O}(min(|V||E|C, |E|2^{|V|}))$

Network Problems

Shortest path problem

A shortest path simplex algorithm [11]

- Pseudo permanent labels
- Multiple pivot rule
- Runs in $\mathcal{O}(|V||E|)$ time

Network Problems

Shortest path problem

Floyd-Warshall algorithm

- It is an all-pairs (not only from one vertex s) label-correcting algorithm [2]
- Runs in $\mathcal{O}(|V|^3)$ time [2]

Network Problems

Shortest path problem

Johnson's algorithm

- It is an all-pairs (not only from one vertex s) label-correcting algorithm [9]
- Runs in $\mathcal{O}(|V|^2 log(|V|) + |V||E|)$ time [9]

Network Problems

Longest path problem

Longest path problem

- NP-hard (non-deterministic polynomial-time)

Network Problems

Longest path problem

Longest path problem

- $G(V, E)$ is a directed acyclic graph
- Let $E' = E$
- $\forall e' \in E'$, $c_{e'} = -c_e$
- Shortest path problem on $G'(V, E') \equiv$ longest path problem on $G(V, E)$

Network Problems

Matching problem

Matching

Let $G(V, E)$ be an undirected graph. A matching $G'(V', E')$ is a subgraph of G and furthermore G' satisfies the following condition: $\forall v \in G'$, $deg(v) \leq 1$.

Network Problems

Matching problem

Bipartite (cardinality) matching problem

Let $G(V, E)$ be a bipartite undirected graph. Bipartite matching problem in G looks for a matching that has the maximum cardinality.

Network Problems

Matching problem

Bipartite matching as maximum flow problem [2]

- $G(V, E)$ is a bipartite undirected graph
- V_1 and V_2 are a partition of V
- $V' = V \cup \{s, t\}$
- $E' = \{vv' : v \in V_1, v' \in V_2, vv' \in E\} \cup \{sv : v \in V_1\} \cup \{vt : v \in V_2\}$
- $\forall e \in E',\ u_e = 1$
- $G'(V', E')$ is a directed graph
- Bipartite matching problem on $G \equiv$ maximum flow problem on G'
- Solvable with the unit capacity flow algorithm in $\mathcal{O}(\sqrt{|V|}|E|)$ time

Network Problems

Matching problem

HopcroftKarp algorithm [12]

- Solves the bipartite matching problem
- Runs in $\mathcal{O}(|V|^{\frac{5}{2}})$ time

Network Problems

Matching problem

Bipartite weighted matching problem

Let $G(V, E)$ be a bipartite directed graph with weights c_e, $e \in E$. Moreover $\forall vv' \in E$, $v \in V_1$ and $v' \in V_2$. Bipartite weighted matching problem in G looks for a matching that has minimum weight.

Network Problems

Matching problem

Bipartite weighted matching (assignment) problem

$$
\begin{aligned}
min \quad & \sum_{vv' \in E} c_{vv'} x_{vv'} \\
s.t. \quad & \sum_{v':vv' \in E} x_{vv'} = 1 \qquad \forall v \in V_1 \\
& \sum_{v':v'v \in E} x_{v'v} = 1 \qquad \forall v \in V_2 \\
& 0 \leq x_{vv'} \qquad \forall vv' \in E
\end{aligned}
$$

Network Problems

Matching problem

Running times of algorithms for bipartite weighted matching problem [2]

- Successive shortest path, $\mathcal{O}(|V_1|S(|V|,|E|,C))$
- Hungarian (primal-dual), $\mathcal{O}(|V_1|S(|V|,|E|,C))$
- Relaxation, $\mathcal{O}(|V_1|S(|V|,|E|,C))$
- Cost scaling, $\mathcal{O}(|V||E|log(|V|C))$
- Modified cost scaling, $\mathcal{O}(\sqrt{|V_1|}|E|log(|V|C))$

where $S(|V|,|E|,C)$ is the running time of the shortest path problem with $c_e \geq 0, \forall e \in E$.

Network Problems

Matching problem

Karp algorithm [13]

- Solves the bipartite weighted matching problem
- Runs in $\mathcal{O}(|V||E|log(|V|))$ time

Network Problems

Matching problem

Stable marriage problem [2]

Stable marriage problem is defined on a directed bipartite graph $G(V, E)$, where $|V_1| = |V_2|$, $\forall v \in V_1$ and $\forall v' \in V_2$, $c_{vv'} \in \{1, ..., |V_1|\}$ and $c_{v'v} \in \{1, ..., |V_1|\}$. In addition, $\forall v \in V_1$, if $v' \neq v''$ than $c_{vv'} \neq c_{vv''}$. Furthermore, $\forall v \in V_2$, if $v' \neq v''$ than $c_{vv'} \neq c_{vv''}$. In other words, both $|V_1|$ men and $|V_2|$ women give distinct ranks to their potential mates. An unstable situation arises when an unmarried couple chooses each other over their current spouse.

Network Problems

Matching problem

The propose-and-reject algorithm [2]

- Solves stable marriage problem in $\mathcal{O}(|V_1|^2)$ time
- $\exists$ a stable matching for any set of rankings
- Man-optimal solution if man proposes first

Network Problems

Matching problem

Nonbipartite (cardinality) matching problem

Let $G(V, E)$ be an undirected graph. Nonbipartite matching problem in G looks for a matching that has the maximum cardinality.

Network Problems

Matching problem

Nonbipartite matching algorithm [2]

- Runs in $\mathcal{O}(|V|^3)$ time

However,

Bipartite matching algorithm [2]

- Runs in $\mathcal{O}(|V||E|)$ time
- Slower than the unit capacity flow algorithm which runs in $\mathcal{O}(\sqrt{|V|}|E|)$ time

Network Problems

Matching problem

Edmonds(Gabow) algorithm [14]

- Solves the maximum weight nonbipartite matching problem
- Runs in $\mathcal{O}(|V|^3)$ time

Network Problems

Chinese postman problem

A procedure that modifies a multigraph $G(V, E)$ to enable Euler tours [15]

- A: Let $V' = \{v' \in V : deg(v') \equiv 1 \, (mod \, 2)$ in $G\}$.
- B: Construct a complete graph $G'(V', E')$ such that each $e' \in E'$ corresponds to the shortest path in G between the vertices that are endpoints of e'.
- ...

Network Problems

Chinese postman problem

A procedure that modifies a multigraph $G(V, E)$ to enable Euler tours [15]

- ...
- C: Solve the minimum weight nonbipartite matching problem in G', where the weight of each $e' \in E'$ is equal to the length of the shortest path in G between the vertices that are endpoints of e'. Then, let $E^{extra} = \{e^{extra} \in E : e^{extra}$ is an element of any shortest path in G that is associated with a optimal matching of the minimum weight nonbipartite matching problem$\}$.
- D: Modify the graph G by adding all extra edges in E^{extra}.

Edmonds(Gabow) algorithm can be used to solve the minimum weight nonbipartite matching problem.

Network Problems

Chinese postman problem

Next-node algorithm [15] for finding an existing Euler tour in a multigraph $G(V, E)$

- A: Set $E_{used} = \emptyset$ and $E_{unused} = E$. Set $L_v = \emptyset$, $\forall v \in V$. Select any $v \in V$. Then, $startvertex := v$ and $currentvertex := v$. Select any $e \in E_{unused}$: e is incident with $currentvertex$. Then, $currentedge := e$.
- B: Select $v' \in V \setminus \{currentvertex\}$: v' is incident with $currentedge$. Then, $L_{v'} = L_{v'} \cup \{(currentvertex, |L_{v'}| + 1)\}$. Set $E_{used} = E_{used} \cup \{currentedge\}$ and $E_{unused} = E_{unused} \setminus \{currentedge\}$. Then, $nextvertex := v'$. If $\exists e \in E_{unused}$: e is incident with $nextvertex$, go to Step C. Otherwise, check that $nextvertex = startvertex$ and go to Step D.
- ...

Network Problems

Chinese postman problem

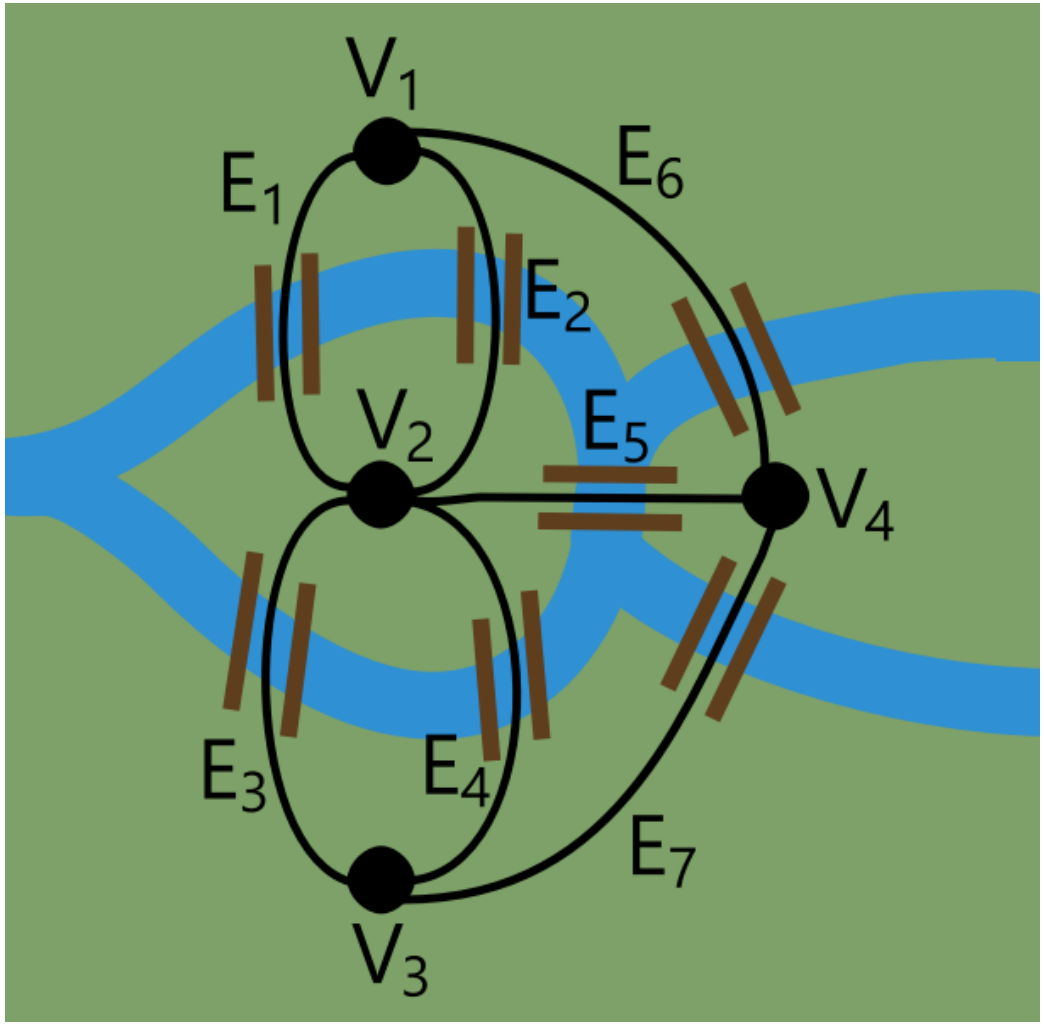
Next-node algorithm [15] for finding an existing Euler tour in a multigraph $G(V, E)$

- ...
- C: *currentvertex* $:=$ *nextvertex*. Select any $e \in E_{unused}$: e is incident with *currentvertex*. Go to Step B.
- D: If $\exists v \in V : \exists e \in E_{unused}$: e is incident with v and $\exists e' \in E_{used}$: e' is incident with v then, *startvertex* $:= v$ and *currentvertex* $:= v$. Then, *currentedge* $:= e$ and go to Step B. Otherwise, terminate the algorithm.

To find an Euler tour, follow L_v in reverse order. Where, i in $(\cdot, i)$ specifies the order.

Network Problems

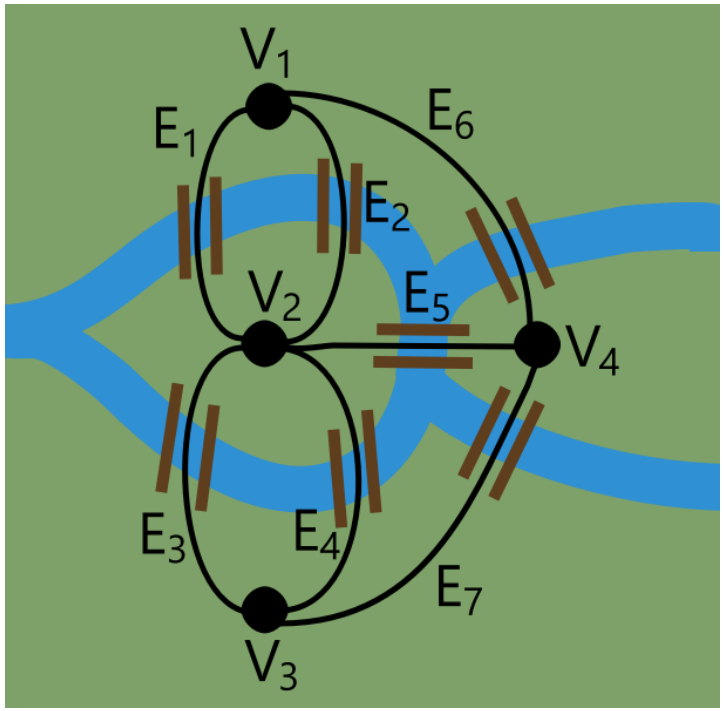
Chinese postman problem



Here, without loss of generality, all edges have equal weights.

Network Problems

Chinese postman problem



$deg(V_1) = 3, deg(V_2) = 5, deg(V_3) = 3, deg(V_4) = 3$

$E^{extra} = \{E_2^{extra}, E_7^{extra}\}$

$L_1 = \{(V_2, 1), (V_4, 2)\}$, $L_2 = \{(V_3, 1), (V_1, 2), (V_1, 3)\}$,

$L_3 = \{(V_2, 1), (V_4, 2)\}$, $L_4 = \{(V_3, 1), (V_2, 2)\}$

An Euler tour: $E_3 \rightarrow E_1 \rightarrow E_2 \rightarrow E_4 \rightarrow E_7 \rightarrow E_6 \rightarrow E_2 \rightarrow E_5 \rightarrow E_7$

Network Problems

Cut

Cut

A cut is a partition ($V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$) of the vertices of a directed graph $G(V, E)$. In particular, a cut is called an $s-t$ cut if $s \in V_1$ and $t \in V_2$.

Network Problems

Capacity of an $s-t$ cut

Capacity of an $s-t$ cut [2]

The capacity of an $s-t$ cut is equal to the maximum possible amount of net flow from V_1 to V_2, where $s \in V_1$ and $t \in V_2$:

$$\sum_{vv':v \in V_1,\, v' \in V_2} u_{vv'} - \sum_{v'v:v \in V_1,\, v' \in V_2} l_{v'v}$$

Network Problems

Minimum $s-t$ cut

Minimum $s-t$ cut [2]

A minimum $s-t$ cut has the minimum capacity among all possible partitions of the vertices of a directed graph $G(V, E)$ such that $s \in V_1$ and $t \in V_2$.

Network Problems

Generalized max-flow min-cut theorem

Generalized max-flow min-cut theorem [2]

The maximum amount of flow from s to t is equal to the capacity of the minimum $s - t$ cut.

Network Problems

Minimum spanning tree problem

Spanning forest

A spanning forest of an undirected graph $G(V, E)$ is an acyclic subgraph of G, denoted as $G'(V, E')$, where $|E'| < |V| - 1$.

Network Problems

Minimum spanning tree problem

Spanning tree

A spanning tree of an undirected graph $G(V, E)$ is a connected acyclic subgraph of G, denoted as $G'(V, E')$, where $|E'| = |V| - 1$.

Network Problems

Minimum spanning tree problem

Minimum spanning tree problem

Minimum spanning tree problem in an undirected graph G looks for a spanning tree that has the minimum total weight.

Network Problems

Minimum spanning tree problem

Minimum spanning tree problem [2]

$$
\begin{aligned}
min \quad & \sum_{e \in E} c_e x_e \\
s.t. \quad & \sum_{e \in E} x_e = |V| - 1 \\
& \sum_{e \in E' = \{e = vv' : v \in V' \text{ and } v' \in V'\}} x_e \leq |V'| - 1 \qquad \forall V' \subseteq V \\
& x_e \in \{0, 1\} \qquad \forall e \in E
\end{aligned}
$$

Cut optimality conditions [2]

A spanning tree in $G(V, E)$ is a minimum spanning tree denoted as $G'(V, E') \Leftrightarrow \forall e \in E'$, $\exists$ a unique cut that can be obtained by removing only edge e from $G'(V, E')$ such that
$\forall (v - v') : v \in V_1,\ v' \in V_2,\ (v - v') \in E;\ c_e \leq c_{(v-v')}$.

Path optimality conditions [2]

A spanning tree in $G(V, E)$ is a minimum spanning tree denoted as $G'(V, E') \Leftrightarrow \forall e = (i - j) \in E \setminus E'$, $\exists$ a unique path connecting vertices i and j, denoted as $p(e) = (i - v_0 - v_1 - v_2 ... j)$ whose elements(edges) are in E'; than $\forall e' \in p(e)$, $c_{e'} \leq c_e$.

Network Problems

Minimum spanning tree problem

Running times of algorithms for minimum spanning tree problem [2]

- Kruskal (based on path optimality conditions), $\mathcal{O}(|E| + |V|log(|V|)) + Sort(|E|)$
- Prim (based on cut optimality conditions), $\mathcal{O}(|E| + |V|log(|V|))$
- Sollin (based on cut optimality conditions), $\mathcal{O}(|E|log(|V|))$

Network Problems

All-pairs minimax path problem

All-pairs minimax path problem [2]

The all-pairs minimax path problem wants to determine a path for each pair of vertices such that the maximum edge weights on these paths are minimized. The solution of this problem corresponds to a minimum spanning tree.

Network Problems

Minimum spanning branching problem

Branching

In a directed graph $G(V, E)$, a branching is a directed forest, denoted as $G'(V', E')$ where the indegree (the number of edges with v as their terminal vertex) of a vertex v, $deg^-(v) \leq 1$ for all $v \in V'$.

Network Problems

Minimum spanning arborescence problem

(Rooted) Arborescence

In a directed graph $G(V, E)$, an (rooted) arborescence is a directed tree, denoted as $G'(V', E')$ where all edges are directed away from the root vertex. For an arborescence, the indegree (the number of edges with v as their terminal vertex) of a vertex v, $deg^-(v) \leq 1$ for all $v \in V'$.

Network Problems

Minimum spanning branching problem

Minimum(maximum) spanning branching problem

The minimum(maximum) spanning branching problem in a directed graph $G(V, E)$ looks for a branching with minimum(maximum) total weight on the edges, denoted as $G'(V, E')$, where $|E'| \leq |V| - 1$.

Network Problems

Minimum spanning arborescence problem

Minimum(maximum) spanning arborescence problem

Given a root vertex, the minimum(maximum) spanning arborescence problem in a directed graph $G(V, E)$ looks for an arborescence with minimum(maximum) total weight on the edges, denoted as $G'(V, E')$, where $|E'| = |V| - 1$.

Network Problems

Network simplex algorithm

Reduced costs

For each edge $e = (v - v')$ in $G(V, E)$ with $v \in V$, $v' \in V$, $v \neq v'$, the reduced cost $c^{\pi}_{(v-v')} = c_{(v-v')} - \pi(v) + \pi(v')$, where $\pi(v)$ and $\pi(v')$ are the node potentials associated with nodes(vertices) v and v', respectively. Let x_e's be the arc(edge) flows, $\forall e \in E$. Then, $\sum_{e \in E} c^{\pi}_e x_e = (\sum_{e \in E} c_e x_e) - \boldsymbol{\pi}^{\top} \mathbf{b}$, where $\boldsymbol{\pi}$ and $\mathbf{b}$ are column vectors associated with the node potentials and supplies/demands, respectively [2]. Furthermore, $\sum_{e \in Cyc} c^{\pi}_e = \sum_{e \in Cyc} c_e$, where Cyc is for any directed cycle and $\boldsymbol{\pi}$ is for any node potentials [2].

Network Problems

Network simplex algorithm

Free and restricted edges(arcs) [2]

An edge e with a feasible flow in $G(V, E)$ is called restricted if its amount of flow is equal to its lower or upper bound. Otherwise, it is called a free edge.

Cycle free (feasible) solution [2]

A cycle free solution is one with no cycle consisting of only free arcs. Hence, there is at least one restricted edge in an augmenting cycle associated with a cycle free solution. It is then possible to augment flow in only a single direction due to the presence of some restricted edges.

Network Problems

Network simplex algorithm

Cycle free property [2]

The optimal solution of a minimum cost flow problem with a bounded objective function over the feasible region is cycle free.

Network Problems

Network simplex algorithm

Spanning tree solution [2]

A spanning tree solution is where a feasible solution is associated with a spanning tree, and furthermore, every non-tree edge is restricted while tree edges are allowed to be free or restricted.

Network Problems

Network simplex algorithm

Spanning tree property [2]

A spanning tree can always be constructed from the cycle free optimal solution (if required by adding some restricted edges to the spanning forest resulting from the cycle free solution) of a minimum cost flow problem with a bounded objective function over the feasible region.

Network Problems

Network simplex algorithm

Spanning tree structure (T,L,U) [2]

- T: tree edges
- L: non-tree edges at their lower bounds
- U: non-tree edges at their upper bounds

Network Problems

Network simplex algorithm

Non-degenerate spanning tree [2]

If all tree edges are free than the spanning tree is non-degenerate, otherwise it is called degenerate.

Network Problems

Network simplex algorithm

Optimality conditions [2]

$\exists \boldsymbol{\pi}$,

- $\forall e \in T,\ c_e^\pi = 0$
- $\forall e \in L,\ c_e^\pi \geq 0$
- $\forall e \in U,\ c_e^\pi \leq 0$

Network simplex algorithm [2]

Find an edge $e \in \{L \cup U\}$ violating the optimality conditions

- add this edge to T in order to obtain a negative cycle
- send maximum amount of flow in this cycle
- remove an edge reaching its bound after augmenting flow (in the direction of this negative cycle) from this cycle

Network Problems

Network simplex algorithm

Calculating the node potentials given (T,L,U) [2]

- set $\pi(v) = 0$ for some node $v \in V$
- given the fact that $\forall e = (v - v') \in T$,
  $0 = c^{\pi}_{v-v'} = c_{v-v'} - \pi(v) + \pi(v')$, calculate $\pi(v') = \pi(v) - c_{v-v'}$ iteratively

Network Problems

Network simplex algorithm

Calculating the edge flows given (T,L,U) [2]

- for an edge $e \in L$, x_e is equal to its lower bound
- for an edge $e \in U$, x_e is equal to its upper bound
- for edges $e \in T$, x_e's are calculated starting from the leaf nodes by taking into account the node supplies/demands

Strongly feasible spanning tree [2]

Given a spanning tree structure (T,L,U) hanging from the root node, this spanning tree is strongly feasible if $\forall e \in T$, if edge e has zero flow than it points towards the root node and if edge e has flow quantity at its capacity than it points away from the root node. By iterating over the adjacent strongly feasible spanning tree structures, the network simplex algorithm runs in a finite number of steps.

Network Problems

Network simplex algorithm

Some variants

- Generalized network simplex algorithm
- Minimum cost proportional flow problem with disconnected subnetworks [16]

Network Problems

A few libraries

LEMON Graph Library (C++)

Library for Efficient Modeling and Optimization in Networks

NetworkX

A Python library for graphs and networks

Compressed sparse graph routines (scipy.sparse.csgraph)

Fast graph algorithms

MIP

MIP

Mixed integer programming (MIP)

Mixed integer programming (MIP) is a mathematical framework with many applications in the optimization of industrial systems.

MIP

Modeling MIP formulations

Modeling languages/tools

Modeling languages/tools are used to model and analyze MIP formulations. Usually these tools are not a standalone solver.

MIP

Modeling MIP formulations

Examples for modeling languages/tools

- AMPL (algebraic modeling language)
- GAMS (general algebraic modeling system)
- LINGO (for building and solving MIP formulations)
- MiniZinc (constraint modeling language)
- CVXPY (Python-embedded modeling language for convex optimization problems)
- COIN-OR (computational infrastructure for operations research) PuLP (Python library for modeling LP formulations)
- COIN-OR Python MIP (Python tools for modeling MIP formulations)

MIP

Solving MIP formulations

MIP solvers

MIP solvers are used to solve MIP formulations.

MIP

Solving MIP formulations

Examples for MIP solvers

- IBM CPLEX
- Gurobi
- SCIP
- MOSEK
- FICO Xpress
- LINDO
- HiGHS
- Cbc (COIN-OR branch and cut)
- GLPK (GNU Linear Programming Kit)

MIP

Integration with industrial systems

Application programming interfaces (API)

- Python API
- C++ API
- Java API
- ...

MIP

Integration with industrial systems

Deployment

- AWS
- Azure
- Google Cloud
- ...

MIP

File formats

File formats for problem exchange

- LP
- MPS (fixed or free)
- ...

Example I

A machine is used to manufacture the required components of a product "z". This machine can work 23 hours a day. To produce product "z", three components "x" and two components "y" are needed. It takes 0.2 hours to produce one product "x", and 0.25 hours to produce one product "y" on this machine. It is not possible to produce "x" and "y" components at the same time. At the end of the day, the machine must be empty so that a 1-hour planned maintenance can be performed. Find the maximum amount of "z" that can be sustainably produced in a day.

LP file format for Example I

Maximize
objective: z
Subject To
constraint1: 2 x - 3 y = 0
constraint2: x - 3 z = 0
constraint3: 0.2 x + 0.25 y <= 23
General
x y z
End

Use of CVXPY and SCIP for Example I

```
import cvxpy as cp
x = cp.Variable(1, integer=True)
y = cp.Variable(1, integer=True)
z = cp.Variable(1, integer=True)
objective = z
constraints = []
constraints += [2*x - 3*y == 0]
constraints += [x - 3*z == 0]
constraints += [0.2*x + 0.25*y <= 23]
problem = cp.Problem(cp.Maximize(objective), constraints)
problem.solve(solver=cp.SCIP, verbose=False)
```

Example II

In a production line, two models, namely "x" and "y" are produced with a profit margin 3 and 2, respectively. Each model has its own setup requirements, so the setup costs of models "x" and "y" are 15 and 25 respectively. The production quantity for model "X" is either 0 or must be between 50 and 250. The production quantity for model "Y" is either 0 or must be between 100 and 300. Total production quantity for models "x" and "y" should equal 300 units.

LP file format for Example II

```
Maximize
objective: 3 x + 2 y - 15 xs - 25 ys
Subject To
constraint1: x + y = 300
constraint2: 50 xs - x <= 0
constraint3: x - 250 xs <= 0
constraint4: 100 ys - y <= 0
constraint5: y - 300 ys <= 0
General
x  y
Binary
xs  ys
End
```

Use of CVXPY and SCIP for Example II

```
import cvxpy as cp
x = cp.Variable(1, integer=True)
y = cp.Variable(1, integer=True)
xs = cp.Variable(1, boolean = True)
ys = cp.Variable(1, boolean = True)
objective = 3*x + 2*y - 15*xs - 25*ys
constraints = []
constraints += [x + y == 300]
constraints += [50*xs - x <= 0]
constraints += [x - 250*xs <= 0]
constraints += [100*ys - y <= 0]
constraints += [y - 300*ys <= 0]
problem = cp.Problem(cp.Maximize(objective), constraints)
problem.solve(solver=cp.SCIP, verbose=False)
```

MIP

Traveling salesman problem

Traveling salesman problem

The optimal solution of the traveling salesman problem in a connected graph $G(V, E)$ is the shortest circuit visiting every vertex of $G(V, E)$ exactly once or the shortest Hamilton cycle.

MIP

Traveling salesman problem

Symmetric traveling salesman problem (sTSP)

When the traveling salesman problem is defined on an undirected graph $G(V, E)$, it is called symmetric traveling salesman problem.

MIP

Traveling salesman problem

A MIP formulation for symmetric traveling salesman problem (|V|>2) [17], [18]

$$min \sum_{vv' \in V^2,\, v<v'} c_{vv'} x_{vv'}$$

$$s.t. \sum_{v':v' \in V,\, v'<v} x_{v'v} + \sum_{v':v' \in V,\, v'>v} x_{vv'} = 2 \qquad \forall v \in V$$

$$\sum_{vv':vv' \in \mathcal{S}^2,\, v<v'} x_{vv'} \leq |\mathcal{S}| - 1 \qquad \forall \mathcal{S} \subset V:\ 3 \leq |\mathcal{S}| \leq |V| - 3$$

$$x_{vv'} \in \{0,1\} \qquad \forall vv' \in E:\ v < v'$$

MIP

Traveling salesman problem

Asymmetric traveling salesman problem (aTSP)

When the traveling salesman problem is defined on a directed graph $G(V, E)$, it is called asymmetric traveling salesman problem.

MIP

Traveling salesman problem

A MIP formulation for asymmetric traveling salesman problem [18]

$$
\begin{aligned}
min \quad & \sum_{vv' \in V^2} c_{vv'} x_{vv'} & & \\
s.t. \quad & \sum_{v' : v' \in V} x_{vv'} = 1 & & \forall v \in V \\
& \sum_{v' : v' \in V} x_{v'v} = 1 & & \forall v \in V \\
& \sum_{vv' : vv' \in \mathcal{S}^2} x_{vv'} \leq |\mathcal{S}| - 1 & & \forall \mathcal{S} \subset V : 2 \leq |\mathcal{S}| \leq |V| - 2 \\
& x_{vv'} \in \{0, 1\} & & \forall vv' \in E
\end{aligned}
$$

MIP

Traveling salesman problem

The Dantzig, Fulkerson and Johnson formulation of subtour elimination constraints for aTSP [17], [19]

$$\sum_{vv':vv' \in \mathcal{S}^2} x_{vv'} \leq |\mathcal{S}| - 1 \qquad \forall \mathcal{S} \subset V \setminus \{1\} : 2 \leq |\mathcal{S}| \leq |V| - 1$$

MIP

Traveling salesman problem

The Miller, Tucker and Zemlin formulation of subtour elimination constraints for aTSP [19]

$$u_v - u_{v'} + (|V| - 1)x_{vv'} \leq |V| - 2 \qquad \forall v \in V, \forall v' \in V \setminus \{1\}, v \neq v'$$

$$1 \leq u_v \leq |V| - 1 \qquad \forall v \in V \setminus \{1\}$$

where u_v's are auxiliary variables used to define the visiting order of vertex v

MIP

Traveling salesman problem

The Desrochers and Laporte formulation of subtour elimination constraints for aTSP [19]

$$u_v - u_{v'} + (|V| - 1)x_{vv'} + (|V| - 3)x_{v'v} \leq |V| - 2 \quad \forall v \in V, \forall v' \in V \setminus \{1\}, v \neq v'$$

$$1 + (|V| - 3)x_{v1} + \sum_{v' \in V \setminus \{1\}} x_{v'v} \leq u_v \quad \forall v \in V \setminus \{1\}$$

$$u_v \leq |V| - 1 - (|V| - 3)x_{1v} - \sum_{v' \in V \setminus \{1\}} x_{vv'} \quad \forall v \in V \setminus \{1\}$$

MIP

Traveling salesman problem

The Gavish and Graves (single commodity flow) formulation of subtour elimination constraints for aTSP [19]

$$\sum_{v':v' \in V} g_{v'v} - \sum_{v':v' \in V \setminus \{1\}} g_{vv'} = 1 \qquad \forall v \in V \setminus \{1\}$$

$$0 \leq g_{vv'} \leq (|V| - 1)x_{vv'} \qquad \forall v \in V, \forall v' \in V \setminus \{1\}$$

where $g_{vv'}$'s are auxiliary variables that indicate the number of edges remaining from vertex v' to the vertex 1 in order to complete the optimal tour.

MIP

Traveling salesman problem

Multiple traveling salesman problem (mTSP)

When the traveling salesman problem has more than one salesman, it is called multiple traveling salesman problem.

MIP

Traveling salesman problem

A MIP formulation for mTSP with m salesmen [20]

$$min \sum_{vv' \in V^2} c_{vv'} x_{vv'}$$

$$s.t. \sum_{v':v' \in V \setminus \{1\}} x_{1v'} = m$$

$$\sum_{v':v' \in V \setminus \{1\}} x_{v'1} = m$$

$$\sum_{v':v' \in V} x_{vv'} = 1 \qquad \forall v \in V \setminus \{1\}$$

$$\sum_{v':v' \in V} x_{v'v} = 1 \qquad \forall v \in V \setminus \{1\}$$

$$\dots$$

MIP

Traveling salesman problem

A MIP formulation for mTSP with m salesmen [20]

$$\sum_{vv':vv' \in \mathcal{S}^2} x_{vv'} \leq |\mathcal{S}| - 1 \qquad \forall \mathcal{S} \subseteq V \setminus \{1\}, \mathcal{S} \neq \emptyset$$

$$x_{vv'} \in \{0, 1\} \qquad \forall vv' \in E$$

MIP

Traveling salesman problem

The Miller, Tucker and Zemlin formulation of subtour elimination constraints for mTSP [20]

$$u_v - u_{v'} + \mu x_{vv'} \leq \mu - 1 \qquad \forall v \in V, \forall v' \in V \setminus \{1\},\ v \neq v'$$

$$1 \leq u_v \leq \mu \qquad \forall v \in V \setminus \{1\}$$

where u_v's are auxiliary variables used to define the visiting order of vertex v and μ is equal to the maximum number of vertices that can be visited by any salesman

MIP

Vehicle routing problem

Vehicle routing problem

The vehicle routing problem differs from mTSP in that it takes vehicle capacities into account.

MIP

Vehicle routing problem

Capaciated vehicle routing problem (CVRP)

Let $G(V, E)$ be a graph. The capacitated vehicle routing problem is defined on the graph G, where a fleet of identical vehicles with indices $m \in M$ and capacity Q based at one central depot deliver goods to meet the demand of customers at different vertices, namely d_v, $\forall v \in V$. If $\forall v, v' \in V$, $c_{vv'} = c_{v'v}$ then CVRP is symmetric (sCVRP), else it is asymmetric (aCVRP). For sCVRP, G is an undirected graph while aCVRP requires to be defined on a directed graph G.

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Vehicle routing problem

The two-index vehicle flow formulation for sCVRP [21], [22]

$$min \sum_{vv' \in V^2, v<v'} c_{vv'} x_{vv'}$$

$$s.t. \sum_{v':v' \in V, v'<v} x_{v'v} + \sum_{v':v' \in V, v'>v} x_{vv'} = 2 \qquad \forall v \in V \setminus \{1\}$$

$$\sum_{v \in V \setminus \{1\}} x_{1v} = 2|M|$$

$$\sum_{vv':vv' \in \mathcal{S} \times V \setminus \mathcal{S} \cup V \setminus \mathcal{S} \times \mathcal{S}, v<v'} x_{vv'} \geq 2\lceil \frac{\sum_{v \in \mathcal{S}} d_v}{Q} \rceil \qquad \forall \mathcal{S} \subseteq V \setminus \{1\}, |\mathcal{S}| \geq 2$$

$$x_{vv'} \in \{0,1\} \qquad \forall vv' \in E : v < v', v \neq 1$$

$$x_{1v} \in \{0,1,2\} \qquad \forall 1v \in E : 1 < v$$

MIP

Vehicle routing problem

The three-index vehicle flow formulation for sCVRP [22], [23]

$$min \sum_{vv' \in V^2,\, v<v'} c_{vv'} \sum_{m \in M} x^m_{vv'}$$

$$s.t. \sum_{v':v' \in V,\, v'<v} x^m_{v'v} + \sum_{v':v' \in V,\, v'>v} x^m_{vv'} = 2y^m_v \qquad \forall v \in V \setminus \{1\}, \forall m \in M$$

$$\sum_{vv':vv' \in \mathcal{S} \times V \setminus \mathcal{S} \cup V \setminus \mathcal{S} \times \mathcal{S}, v<v'} x^m_{vv'} \geq 2y^m_{v''} \qquad \forall \mathcal{S} \subseteq V \setminus \{1\}, |\mathcal{S}| \geq 2, \ldots$$

$$\forall v'' \in \mathcal{S}, \forall m \in M$$

$$\ldots$$

MIP

Vehicle routing problem

The three-index vehicle flow formulation for sCVRP [22], [23]

$$\sum_{m \in M} y_1^m = |M|$$

$$\sum_{m \in M} y_v^m = 1 \qquad \forall v \in V \setminus \{1\}$$

$$\sum_{v \in V \setminus \{1\}} d_v y_v^m \leq Q \qquad \forall m \in M$$

$$x_{vv'}^m \in \{0,1\} \qquad \forall vv' \in E : v < v', v \neq 1, \forall m \in M$$

$$x_{1v}^m \in \{0,1,2\} \qquad \forall 1v \in E : 1 < v, \forall m \in M$$

$$y_v^m \in \{0,1\} \qquad \forall v \in V, \forall m \in M$$

where y_v^m indicates whether vertex v is visited by vehicle m, and $x_{vv'}^m$ shows whether edge vv' is used by vehicle m.

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Vehicle routing problem

The three-index vehicle flow formulation for aCVRP [24]

Let $G(V, E)$ be a directed graph. We can extend G by adding another depot that is just a copy of the original to be used for returns. Let $\overline{V} = V \cup \{|V| + 1\}$ and $\overline{E} = E \cup_{v \in V, v' = |V|+1} \{vv'\}$. Moreover, let $c_{vv'} = c_{1v}$, $\forall v \in V \setminus \{1\}$ and $c_{1v'} = 0$, where $v' = |V| + 1$. Then, $\overline{G}(\overline{V}, \overline{E})$ is the extended graph of G.

MIP

Vehicle routing problem

The three-index vehicle flow formulation for aCVRP [24]

$$min \sum_{vv' \in \bar{E}} c_{vv'} \sum_{m \in M} x^m_{vv'}$$

$$s.t. \sum_{vv' \in \bar{E}, m \in M} x^m_{vv'} = 1 \qquad \forall v \in V \setminus \{1\}$$

$$\sum_{v'v \in \bar{E}} x^m_{v'v} - \sum_{vv' \in \bar{E}} x^m_{vv'} = 0 \qquad \forall v \in V \setminus \{1\}, \forall m \in M$$

$$\sum_{1v \in \bar{E}} x^m_{1v} = 1 \qquad \forall m \in M$$

$$\sum_{vv' \in \bar{E}} x^m_{vv'} = 1 \qquad \forall m \in M, v' = |V| + 1$$

$$\dots$$

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Vehicle routing problem

The three-index vehicle flow formulation for aCVRP [24]

$$\sum_{v \in V \setminus \{1\}} (d_v \sum_{vv' \in \overline{E}} x^m_{vv'}) \leq Q \quad \forall m \in M$$

$$x^m_{vv'} \in \{0, 1\} \quad \forall vv' \in \overline{E}, \forall m \in M$$

where $x^m_{vv'}$ shows whether edge vv' is used by vehicle m.

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Vehicle routing problem

The Miller, Tucker and Zemlin formulation of subtour elimination constraints for three-index aCVRP

$$u_v^m - u_{v'}^m + (|\overline{V}| - 1)x_{vv'}^m \leq |\overline{V}| - 2 \qquad \forall v \in \overline{V}, \forall v' \in \overline{V} \setminus \{1\}, v \neq v', \forall m \in M$$

$$1 \leq u_v^m \leq |\overline{V}| - 1 \qquad \forall v \in \overline{V} \setminus \{1\}, \forall m \in M$$

where u_v^m's are auxiliary variables used to determine the visiting order of vertex v by vehicle m

MIP

Vehicle routing problem

The time windows and subtour elimination constraints for three-index aCVRP [24]

$$u_v^m - u_{v'}^m + t_{vv'} \leq (1 - x_{vv'}^m)\, L \qquad \forall v \in \overline{V}, \forall v' \in \overline{V} \setminus \{1\}, v \neq v', \forall m \in M$$

$$s_v \leq u_v^m \leq f_v \qquad \forall v \in \overline{V} \setminus \{1\}, \forall m \in M$$

where u_v^m's are auxiliary variables used to define the visiting time of vertex v by vehicle m. For the vertex v, s_v and f_v denote the start and end of time window, respectively. $t_{vv'}$ is the time required to move from vertex v to vertex v'. Moreover,

$$L = \max_{v \in \overline{V}} f_v - \min_{v \in \overline{V} \setminus \{1\}} s_v$$

MIP

Vehicle routing problem

The two-commodity flow formulation for sCVRP [22], [25]

Let $G(V, E)$ be an undirected graph. We can extend G by adding another depot that is just a copy of the original to be used for returns. Let $\overline{V} = V \cup \{|V| + 1\}$ and $\overline{E} = E \cup_{v \in V \setminus \{1\}, v' = |V|+1} \{vv'\}$. Moreover, let $c_{vv'} = c_{1v}$, $\forall v \in V \setminus \{1\}$ where $v' = |V| + 1$. Then, $\overline{G}(\overline{V}, \overline{E})$ is the extended graph of G.

MIP

Vehicle routing problem

The two-commodity flow formulation for sCVRP [22], [25]

$$min \sum_{vv' \in \overline{E}} c_{vv'} x_{vv'}$$

$$s.t. \sum_{v' \in \overline{V}} (y_{v'v} - y_{vv'}) = 2d_v \qquad \forall v \in V \setminus \{1\}$$

$$\sum_{v' \in V \setminus \{1\}} y_{1v'} = \sum_{v \in V \setminus \{1\}} d_v$$

$$\sum_{v' \in V \setminus \{1\}} y_{v'1} = |M|Q - \sum_{v \in V \setminus \{1\}} d_v$$

$$\sum_{v' \in V \setminus \{1\}} y_{vv'} = |M|Q \qquad v = |V| + 1$$

$$\ldots$$

MIP

Vehicle routing problem

The two-commodity flow formulation for sCVRP [22], [25]

$$
\begin{aligned}
& y_{vv'} + y_{v'v} = Q x_{vv'} && \forall vv' \in \overline{E} \\
& \sum_{vv' \in \overline{E}} x_{vv'} = 2 && \forall v \in V \setminus \{1\} \\
& y_{vv'} \geq 0 && \forall vv' \in \overline{E} \\
& y_{v'v} \geq 0 && \forall vv' \in \overline{E} \\
& x_{vv'} \in \{0, 1\} && \forall vv' \in \overline{E}
\end{aligned}
$$

where $y_{vv'}$ denotes the flow of one commodity from v to v' while $y_{v'v}$ indicates the flow of other commodity from v' to v, and $x_{vv'}$ shows whether undirected edge vv' is used by any vehicle.

MIP

Vehicle routing problem

The set partitioning formulation for CVRP [22], [26]

$$
\begin{aligned}
min & \sum_{r \in R} c_r x_r \\
s.t. & \sum_{r \in R} a_{vr} x_r = 1 \quad \forall v \in V \setminus \{1\} \\
& \sum_{r \in R} x_r = |M| \\
& x_r \in \{0,1\} \qquad \forall r \in R
\end{aligned}
$$

where x_r indicates whether the route r is used by any vehicle, and a_{vr} is a parameter that is equal to 1 if the vertex v is included in the route r. The cost of using route r is equal to c_r.

MIP

Vehicle routing problem

The set covering formulation for CVRP [22], [27]

$$
\begin{aligned}
min & \sum_{r \in R} c_r x_r \\
s.t. & \sum_{r \in R} a_{vr} x_r \geq 1 \quad \forall v \in V \setminus \{1\} \\
& \sum_{r \in R} x_r = |M| \\
& x_r \in \{0,1\} \qquad \forall r \in R
\end{aligned}
$$

The set covering formulation can also be used if $\forall vv' \in E$, $c_{vv'}$ satisfies the triangle inequality [22].

MIP

Vehicle routing problem

Order batching problem (OBP) [28]

In OBP, customer orders are first grouped into batches and then collected by pickers following a specific routing policy.

MIP

Vehicle routing problem

Some routing policies for OBP [28]

- Traversal
- Return
- Midpoint
- Largest gap
- Composite
- Mixed
- Optimum

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