Package ‘sfofr’

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Title Spatial Function-on-Function Regression Model

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Description Functions for implementing methods for spatial function-on-function regression model.

Contents

`data_generation` *Generate Spatially Correlated Functional Data*

Description

Simulates spatially correlated functional data for use in spatial function-on-function regression models. The function generates functional predictors, responses, spatial weight matrices, and the underlying regression and spatial autocorrelation structures, based on specified formulas and parameters.

Usage

```
data_generation(n, nphi = 10, gpy = NULL, gpx = NULL, W.type = c("ivw", "expw"),
              rf = 0.9, sd.error = 0.01, tol = 0.001, max_iter = 1000)
```

Arguments

`n`	Number of samples to generate.
`nphi`	Number of basis functions used to construct the functional predictor.
`gpy`	Grid points for the functional response. Defaults to 101 equidistant points over the interval [0,1] if not provided.
`gpx`	Grid points for the functional predictor. Defaults to 101 equidistant points over the interval [0,1] if not provided.
`W.type`	Type of spatial weight matrix. Options are "ivw" (inverse distance weighting) and "expw" (exponential decay weighting).
`rf`	Spatial dependence parameter α, which regulates the strength of spatial effects.
`sd.error`	Standard deviation of the error term added to the functional response.
`tol`	Tolerance for the Neumann series approximation used to compute the spatial effects.
`max_iter`	Maximum number of iterations for the Neumann series approximation.

Details

The function generates data based on the spatial function-on-function regression model:

$$Y_i(t) = \sum_{i'=1}^{n} w_{ii'} \int_0^1 Y_{i'}(u)\rho(u,t)du + \int_0^1 X_i(s)\beta(s,t)ds + \epsilon_i(t),$$

where:

- $w_{ii'}$ are elements of a spatial weight matrix $\boldsymbol{W}$ that encodes spatial relationships between locations.
- $\rho(u,t)$ is the spatial autocorrelation function defined as $\rho(u,t) = \alpha \frac{1+ut}{1+|u-t|}$, where α controls the spatial dependence strength.
- $\beta(s,t)$ is the bivariate regression coefficient function, specified as $\beta(s,t) = 2 + s + t + 0.5\sin(2\pi st)$.
- $\epsilon_i(t)$ is a random error term.

The functional predictor $X_i(s)$ is constructed as:

$$X_i(s) = \sum_{k=1}^{10} \frac{1}{k^{3/2}} \left(\nu_{i1,k}\sqrt{2}\cos(k\pi s) + \nu_{i2,k}\sqrt{2}\sin(k\pi s) \right),$$

where $\nu_{i1,k}$ and $\nu_{i2,k}$ are independent standard normal random variables.

Two types of spatial weight matrices ($\boldsymbol{W}$) are supported:

- *Inverse distance weighting (ivw)*: $w_{ii'} = \frac{1}{1+|i-i'|}$, normalized to ensure each row sums to 1.
- *Exponential decay weighting (expw)*: $w_{ii'} = \exp(-d|i-i'|)$, with a decay parameter $d = 0.5$, also normalized row-wise.

The functional response is computed iteratively using the Neumann series approximation until convergence, ensuring accurate spatial effects representation.

Value

Returns a list containing:

`Y`	Simulated functional response matrix $Y_i(t)$.
`Y_true`	Functional response matrix without the error term.
`X`	Simulated functional predictor matrix $X_i(s)$.
`W`	Spatial weight matrix.
`rho`	Spatial autocorrelation matrix $\rho(u, t)$.
`beta`	True regression coefficient matrix $\beta(s, t)$.

Note

Ensure that the inputs, particularly `W.type`, are specified correctly to avoid errors. The spatial dependence strength (`rf`) must satisfy the contraction condition $\|\rho\|_\infty < 1/\|\boldsymbol{W}\|_\infty$ to guarantee convergence.

Author(s)

Ufuk Beyaztas, Han Lin Shang, Gizel Bakicierler, Abhijit Mandal, Roger S. Zoh, and Carmen D. Tekwe

References

Hoshino, T. (2024), Functional spatial autoregressive models, Technical report, arXiv. URL: https://arxiv.org/pdf/2402.14763.

See Also

`getSPCA`, `sffreg`, `predict_sffreg`

Examples

```
## Example usage of the data_generation function
n <- 100
data <- data_generation(n, W.type = "ivw", rf = 0.8)
str(data)
```

---

getSPCA — *Spatial Principal Component Analysis for Functional Data*

Description

Performs Spatial Principal Component Analysis (SFPC) on functional data with a spatial structure. The method computes spatial functional principal components (SFPCs) and their corresponding scores, capturing both variability and spatial dependence in the data.

Usage

```
getSPCA(data, nbasis, gp, W)
```

Arguments

`data`	A matrix of functional data, where rows correspond to observations and columns represent functional values evaluated at discrete grid points.
`nbasis`	Number of basis functions to be used for the B-spline basis expansion.
`gp`	Grid points for the functional data, representing the domain over which the functions are observed.
`W`	Spatial weight matrix, encoding the spatial relationships among the observations.

Details

This function implements Spatial Principal Component Analysis (SFPC), which extends the classical FPCA to account for spatial dependencies among functional data. The methodology is grounded in the eigen-decomposition of the covariance function, augmented with a spatial weight matrix to capture spatial autocorrelation.

The covariance function of a centered functional variable $x(t)$ is decomposed using the Karhunen-Loève theorem:

$$G(s,t) = \sum_{k=1}^{\infty} \delta_k \eta_k(s) \eta_k(t),$$

where $\eta_k(t)$ are the eigenfunctions and δ_k are the eigenvalues. In SFPC, these eigenfunctions are computed with additional constraints that maximize spatial autocorrelation as measured by Moran's I statistic:

$$\mathrm{Var}(\widetilde{\boldsymbol{\kappa}}) I(\widetilde{\boldsymbol{\kappa}}) = n^{-1} \widetilde{\boldsymbol{\chi}}^\top \boldsymbol{D}^\top \boldsymbol{W} \boldsymbol{D} \widetilde{\boldsymbol{\chi}},$$

where $\boldsymbol{W}$ is the spatial weight matrix and $\boldsymbol{D}$ is the matrix of basis expansion coefficients.

The eigenfunctions obtained from SFPC represent spatially smoothed principal components, and their corresponding scores summarize the joint variation and spatial structure of the functional data.

Value

A list with the following components:

`PCAcoef`	A `fd` object representing the estimated spatial principal components (eigenfunctions).
`PCAscore`	A matrix of SFPC scores for each observation.
`meanScore`	A `fd` object representing the mean function of the functional data.
`ncomp`	Number of components retained, based on the proportion of variance explained.
`bs_basis`	B-spline basis object used for the functional representation.
`evalbase`	Evaluated basis matrix for grid points `gp`.
`gp`	The grid points over which the functions are evaluated.
`W`	The spatial weight matrix used for SFPC analysis.

Author(s)

Ufuk Beyaztas, Han Lin Shang, Gizel Bakicierler, Abhijit Mandal, Roger S. Zoh, and Carmen D. Tekwe

References

Khoo, T. H., Pathmanathan, D. and Dabo-Niang, S. (2023), ‘Spatial autocorrelation of global stock exchanges using functional areal spatial principal component analysis’, Mathematics 11(2), 674.

See Also

data_generation, sffreg, predict_sffreg

Examples

```
n <- 100
data <- data_generation(n, W.type = "ivw", rf = 0.8)
sfpc <- getSPCA(data$Y, nbasis = 10, gp = seq(0, 1, length.out = 101), W = data$W)
```

---

predict_sffreg — *Prediction for Spatial Function-on-Function Regression Models*

Description

Generates predictions for new functional predictors using a fitted Spatial Function-on-Function Regression (SFoFR) model. The function supports both independent and spatial regression types and handles spatial weight matrices for test samples when spatial regression is used.

Usage

```
predict_sffreg(object, xnew, Wnew = NULL)
```

Arguments

`object`	An `sffreg` object containing a fitted Spatial Function-on-Function Regression model.
`xnew`	A matrix of new functional predictors for the test sample, where rows correspond to observations and columns represent function evaluations at discrete grid points.
`Wnew`	An optional spatial weight matrix for the test sample. Required if `model.type = "spatial"`.

Details

The function uses the fitted `sffreg` object to predict the functional response for new functional predictors based on the selected model type (`"independent"` or `"spatial"`).

- `"independent"`: Predicts the functional response by projecting the new functional predictor onto the eigenfunctions of the functional principal components (FPC) used in the original model.
- `"spatial"`: Incorporates spatial autocorrelation by utilizing the spatial weight matrix (`Wnew`) in conjunction with the spatial functional principal components (SFPC). The predictions are computed by solving a linear system involving the Kronecker product of `Wnew` and the spatial coefficients.

The function ensures that the predictions align with the dimensionality and structure of the original functional response.

Value

A matrix of predicted functional responses, where rows correspond to observations in the test sample and columns represent predicted function values at the evaluation grid points.

See Also

sffreg, data_generation

Examples

```
## Simulate data and fit SFoFR model
n <- 500
data <- data_generation(n, W.type = "ivw", rf = 0.8)
fit <- sffreg(data$Y, data$X, data$W, model.type = "spatial")

## Generate new functional predictors
data_test <- data_generation(n, W.type = "ivw", rf = 0.8)

## Predict using spatial model
Wnew <- data_test$W  # Use the same weight matrix for simplicity
predictions_spatial <- predict_sffreg(fit, data_test$X, Wnew = Wnew)
```

---

sffreg — *Spatial Function-on-Function Regression Model*

Description

Fits a Spatial Function-on-Function Regression (SFoFR) model, where a functional response is regressed on a functional predictor with spatial autocorrelation captured through a spatial weight matrix. The model leverages functional principal component (FPC) and spatial functional principal component (SFPC) decompositions for dimension reduction.

Usage

```
sffreg(Y, X, W, nbasis = NULL, gpy = NULL, gpx = NULL,
model.type = c("independent", "spatial"))
```

Arguments

Y	A matrix of functional responses, where rows correspond to observations and columns represent function evaluations at discrete grid points.
X	A matrix of functional predictors, structured similarly to Y.
W	A spatial weight matrix encoding the spatial relationships among observations.
nbasis	Number of basis functions for B-spline basis expansion. If NULL, a default value is used.
gpy	Grid points for the functional response. Defaults to 101 equally spaced points over $[0, 1]$.
gpx	Grid points for the functional predictor. Defaults to 101 equally spaced points over $[0, 1]$.
model.type	Specifies the regression model type. Options are: • "independent": Assumes no spatial autocorrelation. • "spatial": Includes spatial autocorrelation using SFPC-based techniques.

Details

The SFoFR model is formulated as:

$$Y_i(t) = \sum_{i'=1}^{n} w_{ii'} \int_0^1 Y_{i'}(u)\rho(u,t)du + \int_0^1 X_i(s)\beta(s,t)ds + \epsilon_i(t),$$

where:

- $w_{ii'}$ are elements of the spatial weight matrix $\boldsymbol{W}$.
- $\rho(u,t)$ is the spatial autocorrelation function.
- $\beta(s,t)$ is the bivariate regression coefficient function.
- $\epsilon_i(t)$ is the random functional error term.

Using SFPC decomposition, the infinite-dimensional SFoFR model is reduced to a multivariate spatial autoregressive (MSAR) model:

$$\widetilde{Y} = (\boldsymbol{\rho}^\top \otimes \boldsymbol{W})\widetilde{Y} + \boldsymbol{X}^{\top *}\widetilde{\beta} + \widetilde{e},$$

where $\widetilde{Y}$, $\widetilde{\beta}$, and $\widetilde{e}$ are vectorized representations of the response, regression coefficients, and residuals, respectively. The parameters $\boldsymbol{\rho}$ and $\boldsymbol{\beta}$ are estimated via a least-squares objective function, ensuring identifiability under specific regularity conditions.

The estimates of the spatial autocorrelation and regression coefficient functions are:

$$\widehat{\rho}(u,t) = \phi^\top(u)\widehat{\boldsymbol{\rho}}\phi(t), \quad \widehat{\beta}(s,t) = \psi^\top(s)\widehat{\boldsymbol{\beta}}\phi(t),$$

where $\phi(t)$ and $\psi(s)$ are SFPC eigenfunctions, respectively.

Value

A list with the following components:

`fitted.values`	Predicted functional response values.
`bhat`	Estimated regression coefficient matrix.
`rhohat`	Estimated spatial autocorrelation matrix.
`residuals`	Residuals of the fitted model.
`model_info`	A list containing model-specific details, including the type of model, basis functions, and principal components.

Author(s)

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References

Khoo, T. H., Pathmanathan, D. and Dabo-Niang, S. (2023), ‘Spatial autocorrelation of global stock exchanges using functional areal spatial principal component analysis’, Mathematics 11(2), 674.

Zhu, X., Cai, Z. and Ma, Y. (2022), ‘Network functional varying coefficient model’, Journal of the American Statistical Association: Theory and Methods 117(540), 2074–2085.

See Also

`data_generation`, `getSPCA`, `predict_sffreg`

Examples

```
## Generate data and fit SFoFR model
n <- 500
data <- data_generation(n, W.type = "expw", rf = 0.8)
fit <- sffreg(data$Y, data$X, data$W, model.type = "spatial")
```

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