Fourier Series

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August 24, 2020

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1.0 Introduction

Ancient Greeks believed that our Earth was at the center of the Universe and that the Sun and all the other planets moved around the Earth in circles. However, when the observational data disproved this theory, Ptolemy came along and theorized that the heavenly bodies must be not only moving in large circles around the Earth, but also in an additional smaller circles called epicycles at the same time. When observational data came and disproves this theory once again, they kept adding more smaller circles until it aligns with their observations. Figure 1 below shows Ptolemy's model.

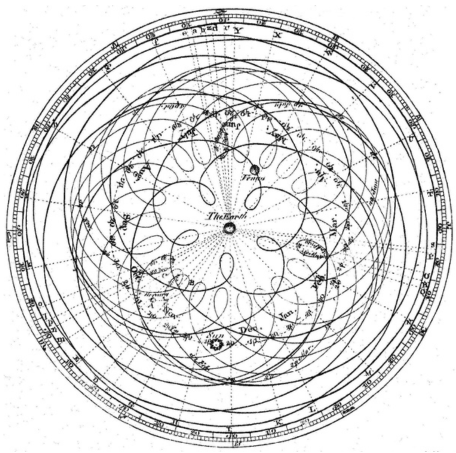

Figure 1 : Ptolemy's Universe [1]

This model was able to provide good predictions about the position of planets and this theory was widely regarded to be true for almost 1,300 years. Of course, this model was proven wrong afterwards but the fact that it was able to give good predictions for so many years is astounding! The only reason why this model worked was because of the nature of the relationships of the circles. We learned from this model that by adding up enough circles, we can approximate any orbit to some degree of accuracy.

This concept of adding circles up together can not only be used to approximate orbits, but also can be used to approximate signals! Radio signals, music waves, etc.! Figure below shows the usage of sine and cosine waves with different amplitudes and frequencies to approximate a square wave.

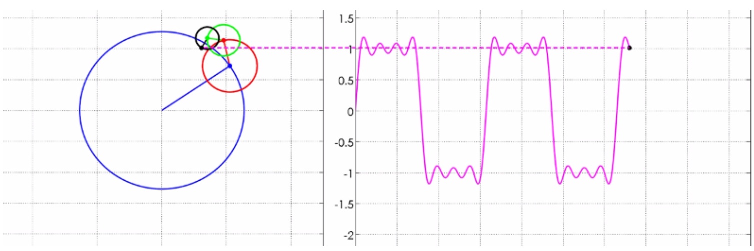

Figure 2 : Approximating a Square Wave [2]

2.0 Problem Statement

A function $f(t)$ is said to be periodic if the value of the function repeat at *regular intervals* (**i.e. period**) of the independent variable t. If T denotes the period, then

$$f(t + T) = f(t)$$

Joseph Fourier was looking for a way to describe how heat transfers in a metal plate and discovered that a periodic function can be approximated by using a series of cosine and sine functions. This can be proved via the Fourier Theorem which will not be discussed here. Thus, a periodic function f(t) with a period of T can be approximated using

$$\sum_{n=0}^{\infty}(a_n cos\ n\omega t + b_n sin\ n\omega t)$$

where $\omega = \frac{2\pi}{T}$. The ω is added to make sure the period of the functions at any value of $n \geq 1$, does not exceed T. It can be shorter than T but not longer. In other words, the functions must repeat itself at least once in T.

The intuition behind this form is that by adding sine and cosine functions of different amplitudes (a_n and b_n) and frequencies (n), the series converge to the *target*[1] periodic function. We shall take the more formal definition stated below.

Let f be a piecewise continous function on $[-\pi, \pi]$. Then the Fourier series of f is the series

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n cos\ nt + b_n sin\ nt) \tag{1}$$

NOTE : ω was not used since the period is 2π thus making $\omega = 1$. Also when $n\ =\ 0$,

$$\begin{aligned} &a_0 cos\ (0)\ +\ b_0 sin\ (0) \\ &= a_0(1)\ +\ 0 \\ &= a_0 \end{aligned}$$

The question that arises next is how to calculate the coefficients (a_n and b_n) of Fourier series?

[1] In this note, we'll call functions to be approximated with Fourier series as target functions.

3.0 Finding Fourier Coefficients

3.1 Finding a_n

An even function with $T = 2\pi$ can be represented as a sum of cosines of various frequencies via the equation

$$X_e(t) = \sum_{n=0}^{\infty} a_n cos(nt) \tag{2}$$

Multiplying both sides by $cos(mt)$ where $m \in \mathbb{Z}$ and integrating them (integrating them both would yield the same result as both are equivalent functions) over one period (it can be from -π to π or from 0 to 2π or from any arbitrary value, z to z+2π) gives us

$$\int_{-\pi}^{\pi} X_e(t)cos(mt)\ dt = \int_{-\pi}^{\pi} \sum_{n=0}^{\infty} a_n cos(nt)cos(mt)\ dt$$

Since

$$cos(m)cos(n) = \frac{1}{2}[cos(m+n) + cos(m-n)],$$

$$\begin{aligned}\int_{-\pi}^{\pi} X_e(t)cos(mt)\ dt &= \frac{1}{2}\int_{-\pi}^{\pi} \sum_{n=0}^{\infty} a_n\ [cos((n+m)t) + cos((n-m)t)]\ dt \\ &= \frac{1}{2}\sum_{n=0}^{\infty} a_n\ \int_{-\pi}^{\pi} [cos((n+m)t) + cos((n-m)t)]\ dt \\ &= \frac{1}{2}\sum_{n=0}^{\infty} a_n\ \int_{-\pi}^{\pi} cos((n+m)t)\ dt + \frac{1}{2}\sum_{n=0}^{\infty} a_n \int_{-\pi}^{\pi} cos((n-m)t)\ dt\end{aligned}$$

When $m \geq 0$,

$$\begin{aligned}\int_{-\pi}^{\pi} cos((n+m)t)\ dt &= [-sin((n+m)t)]_{-\pi}^{\pi} \\ &= -sin((n+m)(\pi)) + sin((n+m)(-\pi))\end{aligned}$$

Since $sin(\text{a}\pi) = 0$ for $a \in \mathbb{Z}$,

$$\begin{aligned}\int_{-\pi}^{\pi} cos((n+m)t)\ dt &= -sin((n+m)(\pi)) + sin((n+m)(-\pi)) \\ &= 0 + 0 \\ &= 0\end{aligned}$$

When $m \neq n$,

$$\int_{-\pi}^{\pi} cos((n-m)t)\ dt = [-sin((n-m)t)]_{-\pi}^{\pi}$$
$$= -sin((n-m)(\pi)) + sin((n-m)(-\pi))$$

Since $sin(\mathrm{a}\pi) = 0$ for $a \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} cos((n-m)t)\ dt = -sin((n-m)(\pi)) + sin((n-m)(-\pi)) = 0 + 0 = 0$$

Finally, when $m = n$,

$$\begin{aligned}\int_{-\pi}^{\pi} cos((n-m)t)\ dt &= \int_{-\pi}^{\pi} cos(0)\ dt \\ &= \int_{-\pi}^{\pi} 1dt \\ &= [t]_{-\pi}^{\pi} \\ &= \pi + \pi \\ &= 2\pi\end{aligned}$$

Considering the three cases above, it can be seen that for every value of m except when $m = n$, $\int_{-\pi}^{\pi} X_e(\mathrm{t})\ \cos(mt)\ dt = 0$. When $m = n$ (we'll replace all m to n due to this),

$$\int_{-\pi}^{\pi} X_e(t)cos(nt)\ dt = \frac{1}{2}(a_n)(2\pi) = a_n\pi \tag{3}$$

Therefore,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} X_e(t)cos(nt)\ dt \tag{4}$$

Consider the special case when $m = n = 0$.

$$\begin{aligned}\int_{-\pi}^{\pi} X_e(t)cos(mt)\ dt &= \int_{-\pi}^{\pi}\sum_{n=0}^{\infty} a_n cos(nt)cos(mt)\ dt \\ \int_{-\pi}^{\pi} X_e(t)cos(0)\ dt &= \int_{-\pi}^{\pi}\sum_{n=0}^{\infty} a_n cos(nt)cos(0)\ dt \\ \int_{-\pi}^{\pi} X_e(t)\ dt &= \int_{-\pi}^{\pi}\sum_{n=0}^{\infty} a_n cos(nt)\ dt \\ &= a_0\int_{-\pi}^{\pi} cos(0)\ dt \\ &= 2\pi a_0\end{aligned}$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_e(t)\ dt \tag{5}$$

3.2 Finding b_n

An odd function with $T = 2\pi$ can be represented as a sum of sines of various frequencies via the equation

$$X_o(t) = \sum_{n=1}^{\infty} b_n sin(nt) \tag{6}$$

Multiplying both sides by $sin(mt)$ where $m \in \mathbb{Z}$ and integrating them over one period gives us

$$\int_{-\pi}^{\pi} X_o(t) sin(mt)\ dt = \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} b_n sin(nt) sin(mt)\ dt$$

Since

$$sin(m)sin(n) = \frac{1}{2}[cos(m-n) - cos(m+n)],$$

$$\begin{aligned}
\int_{-\pi}^{\pi} X_o(t) sin(mt)\ dt &= \frac{1}{2} \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} b_n\ [cos((n-m)t) - cos((n+m)t)]\ dt \\
&= \frac{1}{2} \sum_{n=1}^{\infty} b_n\ \int_{-\pi}^{\pi} [cos((n-m)t) - cos((n+m)t)]\ dt \\
&= \frac{1}{2} \sum_{n=1}^{\infty} b_n\ \int_{-\pi}^{\pi} cos((n-m)t)\ dt - \frac{1}{2} \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} cos((n+m)t)\ dt
\end{aligned}$$

When $m \geq 0$,

$$\begin{aligned}
\int_{-\pi}^{\pi} cos((n+m)t)\ dt &= [-sin((n+m)t)]_{-\pi}^{\pi} \\
&= -sin((n+m)\pi) + sin((n+m)(-\pi))
\end{aligned}$$

Since $sin(\mathrm{a}\pi) = 0$ for $a \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} cos((n+m)t)\ dt = -sin((n+m)(\pi)) + sin((n+m)(-\pi)) = 0 + 0 = 0$$

When $m \neq n$,

$$\begin{aligned}
\int_{-\pi}^{\pi} cos((n-m)t)\ dt &= [-sin((n-m)t)]_{-\pi}^{\pi} \\
&= -sin((n-m)(\pi)) + sin((n-m)(-\pi))
\end{aligned}$$

Since $sin(\mathrm{a}\pi) = 0$ for $a \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} cos((n-m)t)\ dt = -sin((n-m)(\pi)) + sin((n-m)(-\pi)) = 0 + 0 = 0$$

Finally, when $m = n$,

$$\begin{aligned}\int_{-\pi}^{\pi} cos((n-m)t)\ dt &= \int_{-\pi}^{\pi} cos(0)\ dt \\ &= \int_{-\pi}^{\pi} 1dt \\ &= [t]_{-\pi}^{\pi} \\ &= \pi + \pi \\ &= 2\pi\end{aligned}$$

Considering the three cases above, it can be seen that for every value of m except when $m = n$, $\int_{-\pi}^{\pi} X_o(\mathrm{t})\ \sin(mt)\ dt = 0$. When $m = n$ (we'll replace all m to n due to this),

$$\int_{-\pi}^{\pi} X_o(t) sin(mt)\ dt = \frac{1}{2} b_n(2\pi) = b_n\pi \tag{7}$$

Therefore,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} X_o(t) sin(nt)\ dt \tag{8}$$

NOTE: There is no b_0 as $\sin(0) = 0$.

3.3 Fourier Coefficients

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} X_e(t)\ dt \tag{9}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} X_e(t) cos(nt)\ dt \tag{10}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} X_o(t) sin(nt)\ dt \tag{11}$$

4.0 Complex Fourier Series

4.1 Fourier Series in Exponential Form

We have seen that Fourier series in trigonometric form is

$$a_0 + \sum_{n=1}^{\infty}(a_n cos\ nt + b_n sin\ nt)$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} X_e(t)\ dt$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} X_e(t)cos(nt)\ dt$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} X_o(t)sin(nt)\ dt$$

NOTE: X_e and X_o are even and odd functions (*target* functions) respectively. Note that (refer to A for more information)

$$cos(x) = \frac{e^{ix} + e^{-ix}}{2} \tag{12}$$

and

$$isin(x) = \frac{e^{ix} - e^{-ix}}{2} \quad or \quad sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

Substituting these values in the Fourier series yields

$$\begin{aligned}
& a_0 + \sum_{n=1}^{\infty}(a_n\frac{e^{int} + e^{-int}}{2} + b_n\frac{e^{int} - e^{-int}}{2i}) \\
&= a_0 + \sum_{n=1}^{\infty}(\frac{a_ne^{int} + a_ne^{-int}}{2} + \frac{b_ne^{int} - b_ne^{-int}}{2i}) \\
&= a_0 + \sum_{n=1}^{\infty}(\frac{a_ne^{int} + a_ne^{-int}}{2} + \frac{(b_ne^{int} - b_ne^{-int})i}{(2i)i}) \\
&= a_0 + \sum_{n=1}^{\infty}(\frac{a_ne^{int} + a_ne^{-int}}{2} - \frac{(ib_ne^{int} - ib_ne^{-int})}{2}) \\
&= a_0 + \sum_{n=1}^{\infty}(\frac{a_ne^{int} + a_ne^{-int}}{2} + \frac{ib_ne^{-int} - ib_ne^{int}}{2}) \\
&= a_0 + \sum_{n=1}^{\infty}(\frac{a_n}{2} - \frac{ib_n}{2})e^{int} + \sum_{n=1}^{\infty}(\frac{a_n}{2} + \frac{ib_n}{2})e^{-int} \\
&= a_0 + \sum_{n=1}^{\infty}(\frac{a_n}{2} - \frac{ib_n}{2})e^{int} + \sum_{n=-1}^{-\infty}(\frac{a_{|n|}}{2} + \frac{ib_{|n|}}{2})e^{int}
\end{aligned}$$

Let

$$c_n = \begin{cases} \frac{1}{2}(a_n - ib_n) & n \geq 1 \\ \frac{1}{2}a_0 & n = 0 \\ \frac{1}{2}(a_{|n|} - ib_{|n|}) & n \leq 1 \end{cases}$$

then

$$a_0 + \sum_{n=1}^{\infty}(a_n cos\ nt + b_n sin\ nt) = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

Therefore, a periodic function (*target function*), f(t) can be approximated using complex Fourier series as shown below.

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int} \tag{13}$$

4.2 Finding c$_n$

We know that c_n is just a representation of the piecewise function as was shown in the previous subsection. However, we can also represent c_n in terms of the *target* function.

Multiplying both sides of (13) by e^{-imt} where $m \in \mathbb{Z}$, and integrating both sides by one period gives us

$$\begin{aligned}\int_{-\pi}^{\pi} f(t)e^{-imt}\ dt &= \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} c_n e^{int} e^{-imt}\ dt \\ &= \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} c_n e^{i(n-m)t}\ dt\end{aligned}$$

Since

$$\int_{-\pi}^{\pi} c_n e^{i(n-m)t}\ dt = \int_{-\pi}^{\pi} (cos((n-m)t) + isin((n-m)t))\ dt$$

When $m \neq n$,

$$\begin{aligned}\int_{-\pi}^{\pi} cos((n-m)t)\ dt &= [-sin((n-m)t)]_{-\pi}^{\pi} \\ &= -sin((n-m)\pi) + sin((n-m)(-\pi))\end{aligned}$$

Since sin(aπ) = 0 for $a \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} cos((n-m)t)\ dt = -sin((n-m)\pi) + sin((n-m)(-\pi)) = 0 + 0 = 0$$

Similarly,

$$\begin{aligned}i\int_{-\pi}^{\pi} sin((n-m)t)\ dt &= [cos((n-m)t)]_{-\pi}^{\pi} \\ &= cos((n-m)\pi) - cos((n-m)(-\pi))\end{aligned}$$

Since cosine is an even function, $cos(-x) = cos(x)$. Let $cos((n-m)\pi) =$ j.

Then,

$$i\int_{-\pi}^{\pi} sin((n-m)t)\ dt = i(cos((n-m)\pi) - cos((n-m)(-\pi))) = i(j-j) = i(0) = 0$$

When $m = n$,

$$e^{i(n-m)t} = e^0 = 1$$

From the two cases above, it can be seen that for every value of m except when $m = n$, $\int_{-\pi}^{\pi} c_n e^{i(n-m)t}\ dt = 0$. Therefore (replacing all m to n),

$$\begin{aligned}\int_{-\pi}^{\pi} f(t)e^{-int}\ dt &= \int_{-\pi}^{\pi} c_n(1)\ dt \\ &= c_n \int_{-\pi}^{\pi} 1\ dt \\ &= c_n 2\pi\end{aligned}$$

This gives us

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\ dt \tag{14}$$

Finally, the complex/exponential form of Fourier series of a periodic function, $f(t)$ is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int} \tag{15}$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\ dt \tag{16}$$

5.0 Example

5.1 Square Wave Approximation

Question : Find the Fourier coefficients and Fourier series of the square-wave function f defined by

$$f(x) = \begin{cases} 0 & -\pi \le x < 0 \\ 1 & 0 \le x < \pi \end{cases}$$

Solution using Trigonometric Series
Solution : Plotting the above piece-wise function for 2 periods would result in a graph as shown in Figure 3.

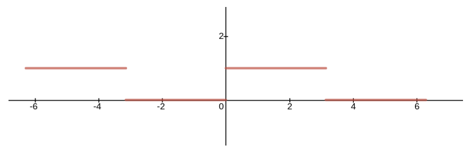

Figure 3 : Square Wave

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{0} 0dx + \frac{1}{2\pi}\int_{0}^{\pi} 1dx \\
&= 0 + \frac{1}{2\pi}(\pi) \\
&= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(nx)dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} (0)cos(nx)dx + \frac{1}{\pi}\int_{0}^{\pi} (1)cos(nx)dx \\
&= \frac{1}{\pi}\left[\frac{sin(nx)}{n}\right]_0^{\pi} \\
&= \frac{1}{n\pi}(sin(n\pi) - sin(0)) \\
&= \frac{1}{n\pi}sin(n\pi) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(nx)dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} (0)sin(nx)dx + \frac{1}{\pi}\int_{0}^{\pi} (1)sin(nx)dx \\
&= \frac{1}{\pi}\left[\frac{-cos(nx)}{n}\right]_0^{\pi} \\
&= -\frac{1}{n\pi}(cos(n\pi) - cos(0)) \\
&= -\frac{1}{n\pi}(cos(n\pi) - 1) \\
&= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{2}{n\pi} & \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$

Since odd numbers can be represented with (2n -1), the Fourier series for $f(x)$:

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(2n-1)\pi} sin((2n-1)x)$$

The above function (from n=1 to n=50) would result in a graph as shown in figure 4 below.

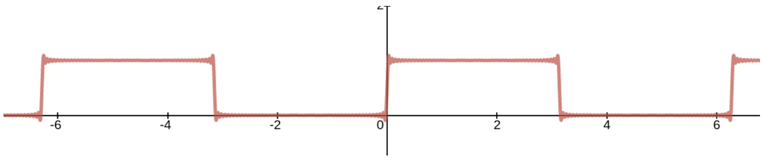

Figure 4 : Approximation using the first 50 nonzero terms.

Solution using Complex Fourier Series Solution :
The fourier series of a function $f(x)$ in complex form is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-inx} \, dx$$

The coefficient c_n is

$$\begin{aligned}
c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-inx} \, dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{0} (0) e^{-inx} \, dx + \frac{1}{2\pi} \int_{0}^{\pi} (1) e^{-inx} \, dx \\
&= \frac{1}{2\pi} \int_{0}^{\pi} e^{-inx} \, dx \\
&= \frac{1}{2\pi} \int_{0}^{\pi} [cos(nx) - isin(nx)] \, dx \\
&= \frac{1}{2\pi} \left[\int_{0}^{\pi} cos(nx) dx - \int_{0}^{\pi} isin(nx) dx \right] \\
&= \frac{1}{2\pi} \left[\frac{[sin(nx)]_0^{\pi}}{n} - i \frac{[-cos(nx)]_0^{\pi}}{n} \right] \\
&= \frac{1}{2\pi} \left[\frac{(sin(\pi n) - sin(0))}{n} - i \frac{(-cos(\pi n) + cos(0))}{n} \right] \\
&= \frac{1}{2\pi} \left[0 - i \frac{(-cos(\pi n) + 1)}{n} \right] \\
&= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{i}{n\pi} & \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$

Substituting the coefficient in the original complex form yields

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{i}{(2n-1)\pi} e^{i(2n-1)x}$$
$$= \sum_{n=-\infty}^{\infty} \frac{i}{(2n-1)\pi} [cos((2n-1)x) + isin((2n-1)x)]$$

Since cosine is an even function and $cos(-x) = cos(x)$, when adding the cosine terms from $n = -\infty$ to $n = \infty$, the negative terms would be cancelled out with their corresponding positive terms. Therefore,

$$f(x) = -\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{(2n-1)} sin((2n-1)x)$$

It can be seen that even with the usage of the complex fourier series, when approximating a real valued function, the final result of the series is made up of real parts. The imaginary parts would cancel out in the process.

6.0 Conclusion and Limitations

We've seen that the trigonometric expression of Fourier Series for a periodic function $f(t)$ with a period of 2π is

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n cos\ nt + b_n sin\ nt)$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\ dt$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) cos(nt)\ dt$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) sin(nt)\ dt$$

It is obvious that a_0 determines whether the entire wave should be shifted upwards, downwards or neither relative to the x-axis. The other coefficients (a_n and b_n) represents the magnitude of the cosine or sine wave at a particular value of n. **NOTE** that during derivation of the coefficients, it was assumed that $f(t)$ was either an even or odd function. Here, we replaced X_e and X_o with f since f can either be an odd function, even function or neither. If f is an even function, only cosine waves are needed to approximate f and vice versa. b_n would be 0 if f is an even function and a_n would be 0 if f is an odd function (Refer to 5.1. Since the square wave was an odd function, a_n turned out to be 0).

As we have seen, another way of representing the Fourier Series is through the use of Euler number, e and imaginary number, i as such

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} \ dt$$

This complex form is just a simplified expression from the trigonometric form. If we use the fact that

$$e^{-i\theta} = cos(\theta) - isin(\theta)$$

and substitute this in the complex form, we'll see that the complex form and the trigonometric form are equivalent. One major advantage of using the complex form is that we only need to calculate one coefficient whereas in the trigonometric form, we're required to calculate three different coefficients.

We have seen the Fourier series both in trigonometric form and complex form when the period of the function is 2π. What happens to the expressions when the period is some arbitrary number, T? The only changes that will occur to the terms are, the integrations are done from -$\frac{T}{2}$ to $\frac{T}{2}$ or from 0 to T and angular frequency, ω which is defined as $\omega = \frac{2\pi}{T}$ will be added to the cosine/sine terms in the trigonometric series and in the exponential terms in the complex series as shown below.

Trigonometric Form :

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n cos \ n\omega t + b_n sin \ n\omega t)$$

where

$$a_0 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \ dt$$

$$a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) cos(n\omega t) \ dt$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)sin(n\omega t)\ dt$$

NOTE : Since the period is T instead of 2π, during the derivation of a_n and b_n, specifically at 3 and 7, the fraction would not get cancelled off and 2 is used at the left hand side to equate the equation instead.
Complex Form :

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}$$

where

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-in\omega t}\ dt$$

Fourier series converges for any periodic functions provided that the function, $f(x)$

- is $\int_{-\pi}^{\pi} f(x)\ dx < \infty$ **i.e** the function is not infinite over a finite interval,
- has finite discontinuities in one period, and
- has a finite number of maxima and minima in one period

Also, whenever there is a discontinuity in the function, the series converges to the midpoint of the discontinuity. In our previous example 5.1, at the discontinuities, the series connected the points with (almost) straight lines. That is because the midpoint of those discontinuities were at $y = 0.5$.

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[8] *Erik Cheever, S. (2020). Derivation of Fourier Series. Retrieved 21 June 2020, from https://lpsa.swarthmore.edu/Fourier/Series/DerFS.html*

A Equivalence of Exponential and Trigonometric Form

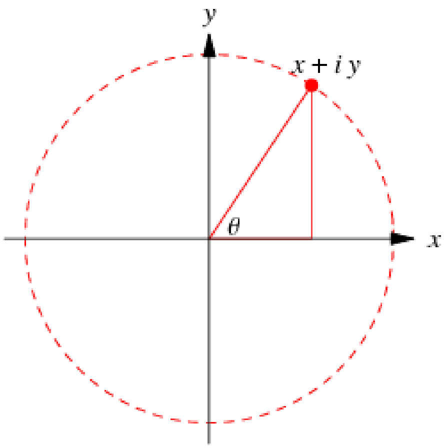

Figure 5 : Argand Diagram [7]

Figure 1 shows the Argand diagram where the x-axis represents the real numbers and y-axis represents the imaginary numbers. If the circle shown is a unit circle, then any point, z that falls on the circumference of the circle can be represented with

$$z = cos(\theta) + isin(\theta) = Cis(\theta) \tag{17}$$

Multiplying two points, z_1 and z_2 yields

$$\begin{aligned} z_1 z_2 &= (cos(\theta_1) + isin(\theta_1))(cos(\theta_2) + isin(\theta_2)) \\ &= (cos(\theta_1)cos(\theta_2) - sin(\theta_1)sin(\theta_2)) + i(sin(\theta_1)cos(\theta_2) + cos(\theta_1)sin(\theta_2)) \end{aligned}$$

Since

$$\begin{aligned} &cos(\theta_1 + \theta_2) = cos(\theta_1)cos(\theta_2) - sin(\theta_1)sin(\theta_2) \quad and \\ &sin(\theta_1 + \theta_2) = sin(\theta_1)cos(\theta_2) + cos(\theta_1)sin(\theta_2) \end{aligned}$$

$$\begin{aligned} z_1 z_2 &= (cos(\theta_1)cos(\theta_2) - sin(\theta_1)sin(\theta_2)) + i(sin(\theta_1)cos(\theta_2) + cos(\theta_1)sin(\theta_2)) \\ &= cos(\theta_1 + \theta_2) + isin(\theta_1 + \theta_2) \\ &= Cis(\theta_1 + \theta_2) \end{aligned}$$

Since exponents are also multiplied in the same way, $a^{\theta_1} a^{\theta_2} = a^{\theta_1 + \theta_2}$, we can represent the multiplication of two complex numbers as $Cis(\theta) = a^{\theta}$ where a is a constant.
Taking $a^{\theta} = e^{\theta(ln\ a)} = e^{A\theta}$ where $A = ln\ a$,

$$Cis(\theta) = a^{\theta} = e^{A\theta} = cos(\theta) + isin(\theta) \tag{18}$$

But what is A? We want to find the value of the constant A such that

$$e^{A\theta} = cos(\theta) + isin(\theta)$$

With Taylor Series, we know that

$$e^{A\theta} = 1 + A\theta + \frac{(A\theta)^2}{2!} + \frac{(A\theta)^3}{3!} + \frac{(A\theta)^4}{4!} + \frac{(A\theta)^5}{5!} + ... \tag{19}$$

$$cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - ... \tag{20}$$

$$sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - ... \tag{21}$$

From (19),

$$e^{A\theta} = (1 + \frac{(A\theta)^2}{2!} + \frac{(A\theta)^4}{4!} + \frac{(A\theta)^6}{6!} + ...) + (A\theta + \frac{(A\theta)^3}{3!} + \frac{(A\theta)^5}{5!} + \frac{(A\theta)^7}{7!} + ...)$$

By taking $A = i$,

$$\begin{aligned} e^{i\theta} &= (1 + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^6}{6!} + ...) + (i\theta + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^7}{7!} + ...) \\ &= (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + ...) + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + ...) \\ &= cos(\theta) + isin(\theta) \end{aligned}$$

Uusing the elimination technique on these two equations,

$$cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \tag{22}$$

$$isin(\theta) = \frac{1}{2}(e^{i\theta} - e^{-i\theta}) \tag{23}$$

the equation below can be derived.

$$e^{-i\theta} = cos(\theta) - isin(\theta) \tag{24}$$

It can be clearly seen that taking $A = i$ satisfies the equation. However, what this proof does not say is that i is the only value that can be substituted to satisfy the equation. In other words, we don't know if there are other values that can satisfy the equation as well. Fortunately, there are other proofs to show that i is the only value that satisfies the equation. One such proof can be found at `https://artofproblemsolving.com/wiki/index.php/Euler%27s_identity`.