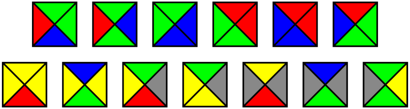
The structure of quasi-transitive graphs avoiding a minor with applications to the Domino Conjecture.

Louis Esperet*, Ugo Giocanti*, Clément Legrand-Duchesne◇

*Université Grenoble Alpes, Laboratoire G-SCOP, France
◇Université de Bordeaux, LaBRI, France

Séminaire GREYC, 2023

Wang tiling problem



Wang tiling problem

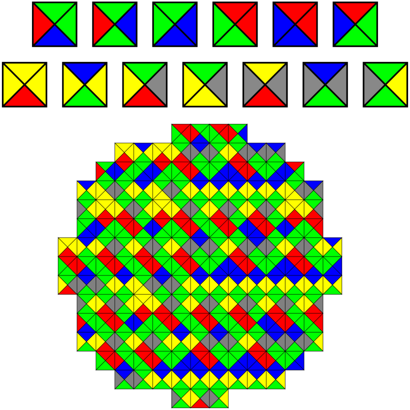
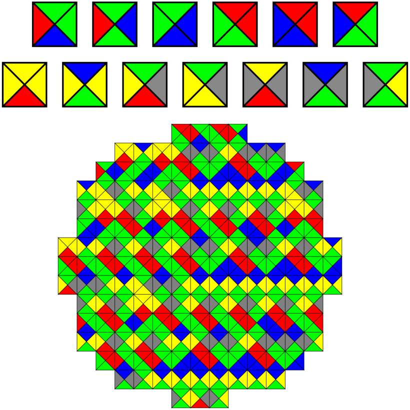


Image source:
https://commons.wikimedia.org/w/index.php?curid=12128873

Wang tiling problem



Theorem (Berger, '66)

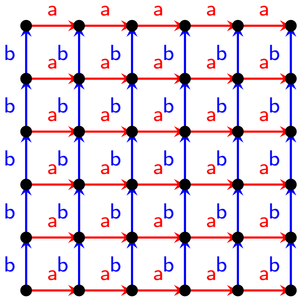
The Wang tiling problem is undecidable.

Cayley graphs

$\Gamma =< S >$: finitely generated group. Assume $S = S^{-1}$.

$\Gamma =< S >$: finitely generated group. Assume $S = S^{-1}$.$\mathrm{Cay}(\Gamma, S)$ is the labelled graph with vertex set Γ and adjacencies xy for every $x, y \in \Gamma$ such that $y \in x \cdot S$.

$\mathrm{Cay}(\mathbb{Z}^2, S)$,
with $S = \{(1,0), (-1,0), (0,1), (0,-1)\}$



Domino Problem on groups

Fix (Γ, S).
Pattern of $\mathrm{Cay}(\Gamma, S)$: coloring p of $\{1_\Gamma, s\}$ for some $s \in S$.
p appears in a vertex-coloring of $\mathrm{Cay}(\Gamma, S)$ if there is a pair $(w, w \cdot s)$ colored p.

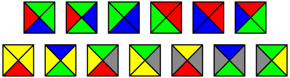
Domino Problem on groups

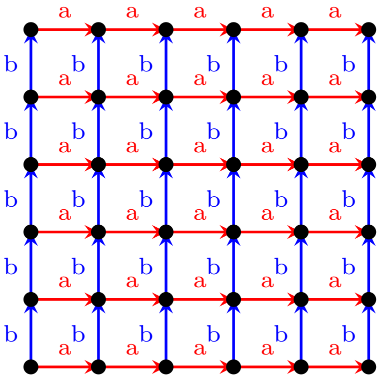
Fix (Γ, S).
Pattern of $\mathrm{Cay}(\Gamma, S)$: coloring p of $\{1_\Gamma, s\}$ for some $s \in S$.
p appears in a vertex-coloring of $\mathrm{Cay}(\Gamma, S)$ if there is a pair $(w, w \cdot s)$ colored p.
Domino problem on (Γ, S):
Input: a finite alphabet Σ and a finite set $\mathcal{F} = \{p_1, \dots, p_t\}$ of forbidden patterns.
Question: Is there a coloring $c : V(G) \to \Sigma$ avoiding $\mathcal{F}$?

Domino Problem on groups

Fix (Γ, S).
Pattern of $\mathrm{Cay}(\Gamma, S)$: coloring p of $\{1_\Gamma, s\}$ for some $s \in S$.
p appears in a vertex-coloring of $\mathrm{Cay}(\Gamma, S)$ if there is a pair $(w, w \cdot s)$ colored p.

Colors:

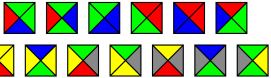




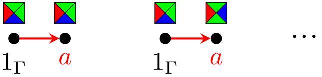
Domino Problem on groups

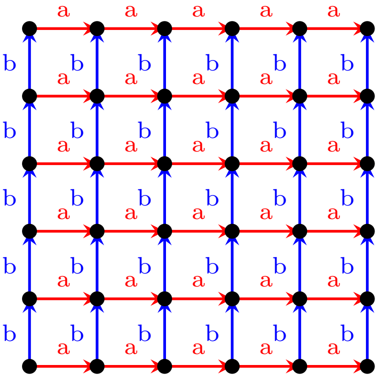
Fix (Γ, S).
Pattern of $\mathrm{Cay}(\Gamma, S)$: coloring p of $\{1_\Gamma, s\}$ for some $s \in S$.
p appears in a vertex-coloring of $\mathrm{Cay}(\Gamma, S)$ if there is a pair $(w, w \cdot s)$ colored p.

Colors:

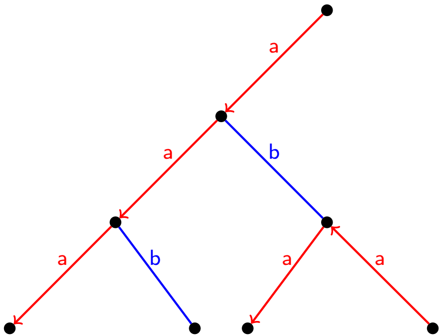


Forbidden patterns:





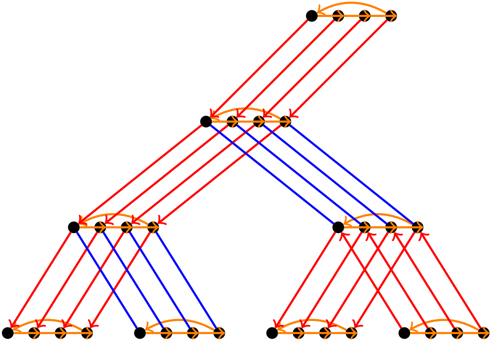
Free-groups: groups Γ that admit a tree as a Cayley graph.



Virtually-free groups

Free-groups: groups Γ that admit a tree as a Cayley graph.
Γ is virtually-free if it admits some subgroup of finite index which is free.

Free-groups: groups Γ that admit a tree as a Cayley graph.
Γ is virtually-free if it admits some subgroup of finite index which is free.



Virtually-free groups

Free-groups: groups Γ that admit a tree as a Cayley graph.
Γ is virtually-free if it admits some subgroup of finite index which is free.

Theorem (Karass, Pietrowski, Solitar '73)

Γ is virtually-free if and only if one/all its Cayley graphs have bounded treewidth.

Claim: If G has bounded degree, then G has bounded treewidth if and only if G is a subgraph of a k-blow up of a tree for some $k \geqslant 0$.

Virtually-free groups

Free-groups: groups Γ that admit a tree as a Cayley graph.
Γ is virtually-free if it admits some subgroup of finite index which is free.

Theorem (Karass, Pietrowski, Solitar '73)

Γ is virtually-free if and only if one/all its Cayley graphs have bounded treewidth.

Claim: If G has bounded degree, then G has bounded treewidth if and only if G is a subgraph of a k-blow up of a tree for some $k \geqslant 0$.

Conjecture (Ballier-Stein 2018)

The domino problem on Γ is decidable if and only if Γ is virtually-free.

Minors (reminder)

A graph H is a minor of G if H can be obtained from G after performing the following operations:

- vertex deletions;
- edge deletions;
- edge contractions.

Minor-excluded groups

A group is planar if one of its Cayley graphs is planar.

Minor-excluded groups

A group is planar if one of its Cayley graphs is planar.
[Maschke 1896] Exhaustive list of all the planar finite groups.

Minor-excluded groups

A group is planar if one of its Cayley graphs is planar.
[Maschke 1896] Exhaustive list of all the planar finite groups.
[Maskit 1965, Zieschang et al. 1980] Full characterization of planar groups.

Minor-excluded groups

A group is planar if one of its Cayley graphs is planar.
[Maschke 1896] Exhaustive list of all the planar finite groups.
[Maskit 1965, Zieschang et al. 1980] Full characterization of planar groups.
A group is minor excluded if one of its Cayley graphs excludes a (countable) minor.

Minor-excluded groups

A group is planar if one of its Cayley graphs is planar.

[Maschke 1896] Exhaustive list of all the planar finite groups.

[Maskit 1965, Zieschang et al. 1980] Full characterization of planar groups.

A group is minor excluded if one of its Cayley graphs excludes a (countable) minor.

Remark: G minor-excluded $\Leftrightarrow$ G is K_∞-minor free.

Decidable on virtually-free groups;

[Berger 1966] Undecidable on $\mathbb{Z}^2$;

[ABM 2019] Undecidable on fundamental groups of surfaces.

Domino Problem

Decidable on virtually-free groups;

[Berger 1966] Undecidable on $\mathbb{Z}^2$;

[ABM 2019] Undecidable on fundamental groups of surfaces.

Theorem

The conjecture is true for planar groups and more generally for minor-excluding groups.

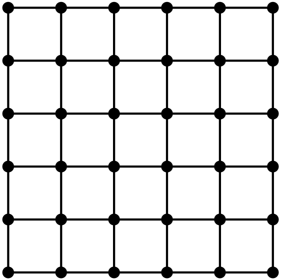
G: (connected) graph, countable vertex set, locally finite.

Quasi-transitive graphs

G: (connected) graph, countable vertex set, locally finite.
G transitive (resp. quasi-transitive) if the action of $\mathrm{Aut}(G)$ on $V(G)$ has one (resp. a finite number of) orbit.

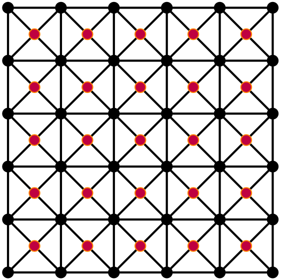
Quasi-transitive graphs

G: (connected) graph, countable vertex set, locally finite.
G transitive (resp. quasi-transitive) if the action of $\mathrm{Aut}(G)$ on $V(G)$ has one (resp. a finite number of) orbit.



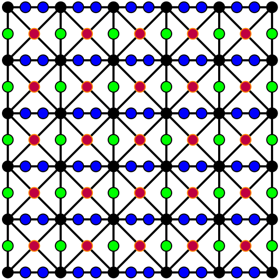
Quasi-transitive graphs

G: (connected) graph, countable vertex set, locally finite.
G transitive (resp. quasi-transitive) if the action of $\mathrm{Aut}(G)$ on $V(G)$ has one (resp. a finite number of) orbit.

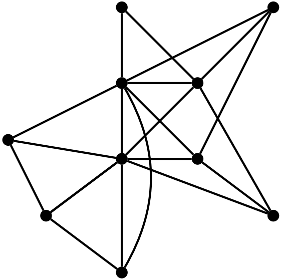


Quasi-transitive graphs

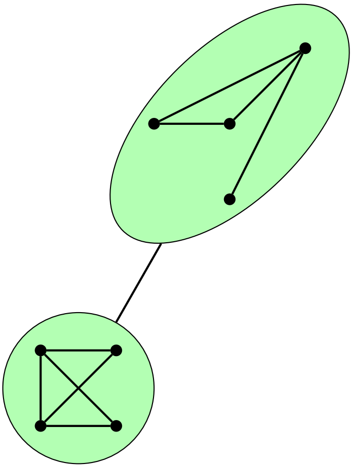
G: (connected) graph, countable vertex set, locally finite.
G transitive (resp. quasi-transitive) if the action of $\mathrm{Aut}(G)$ on $V(G)$ has one (resp. a finite number of) orbit.



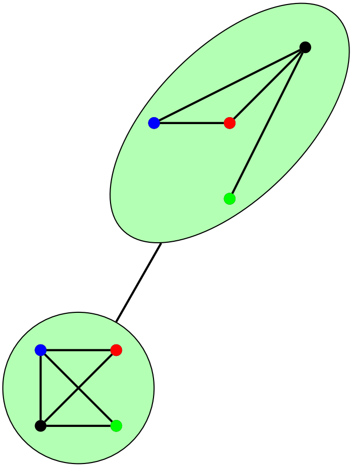
Canonical tree-decompositions



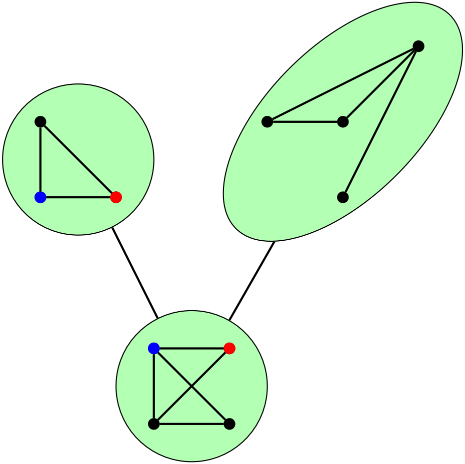
Canonical tree-decompositions



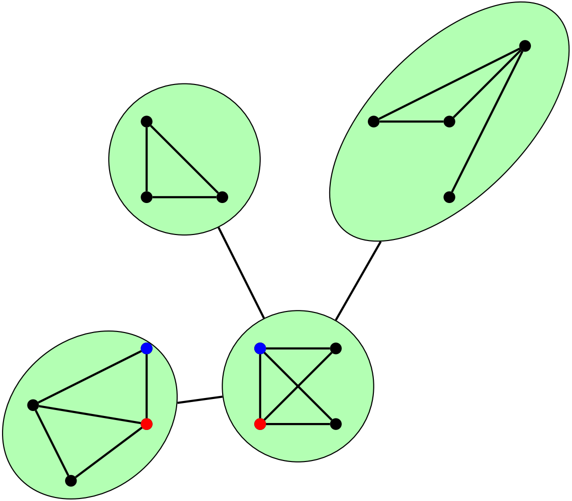
Canonical tree-decompositions



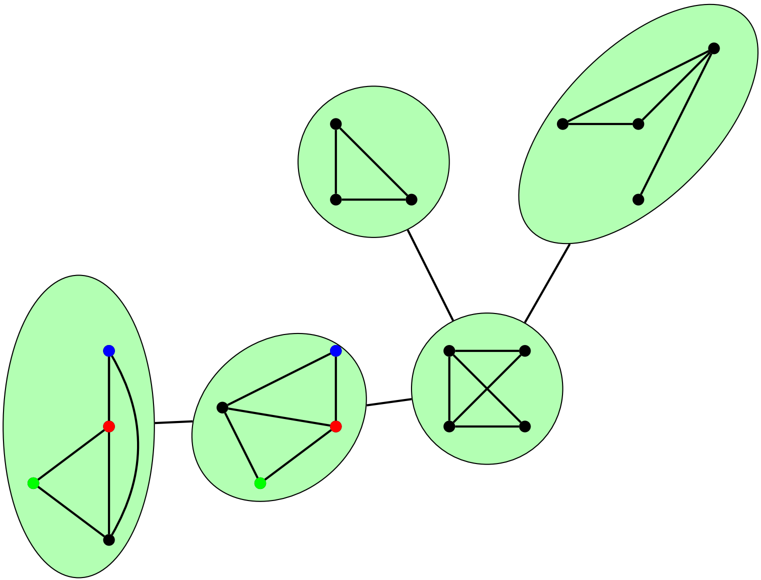
Canonical tree-decompositions



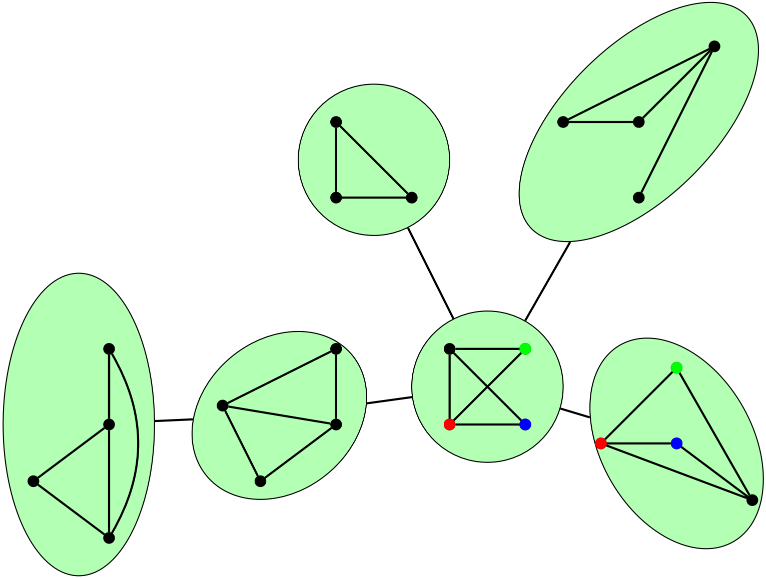
Canonical tree-decompositions



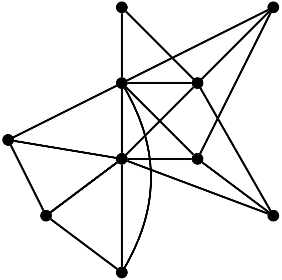
Canonical tree-decompositions



Canonical tree-decompositions

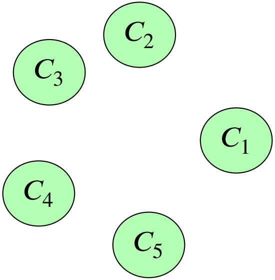


Canonical tree-decompositions



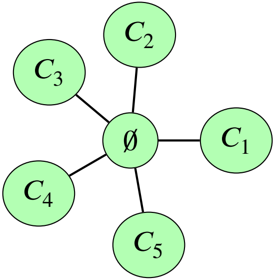
Decompositions in components of higher connectivity

G: any graph, components $C_1, C_2, \ldots$.



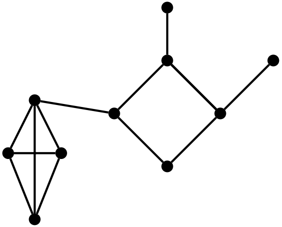
Decompositions in components of higher connectivity

G: any graph, components $C_1, C_2, \ldots$.



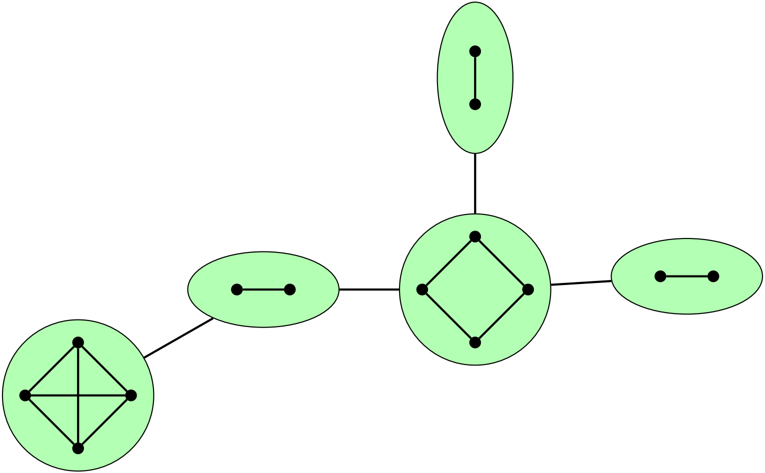
Decompositions in components of higher connectivity

G: connected graph.



Decompositions in components of higher connectivity

G: connected graph.



Robertson-Seymour structure theorem

[Robertson-Seymour '03] "If a finite graph G excludes some minor H, then there exists some $g_H \geqslant 0$ such that G has a tree-decomposition where each torso almost embeds in a surface of genus g_H."

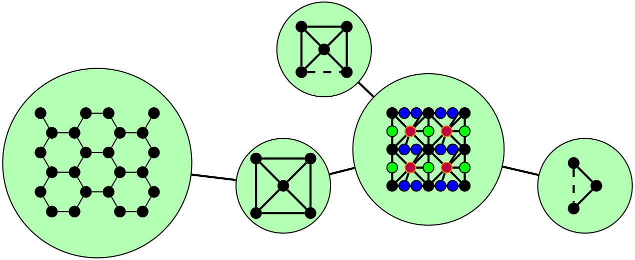
Robertson-Seymour structure theorem

[Robertson-Seymour '03] "If a finite graph G excludes some minor H, then there exists some $g_H \geqslant 0$ such that G has a tree-decomposition where each torso almost embeds in a surface of genus g_H."
[Diestel-Thomas '99]: "Extends to infinite graphs excluding some finite minor."

Main result

Theorem (finite/planar)

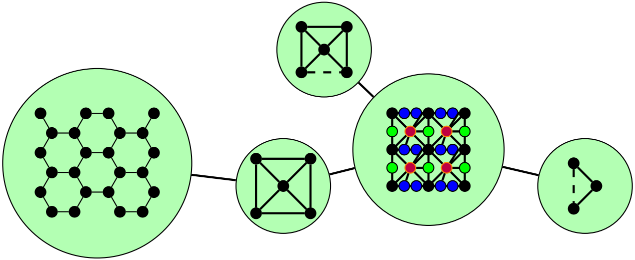
Let G be a quasi-transitive locally finite graph excluding K_∞ as a minor. Then there is an integer k such that G admits a canonical tree-decomposition $(T, \mathcal{V})$, of adhesion at most k whose torsos are either finite or quasi-transitive 3-connected planar minors of G.



Main result

Theorem (finite/planar)

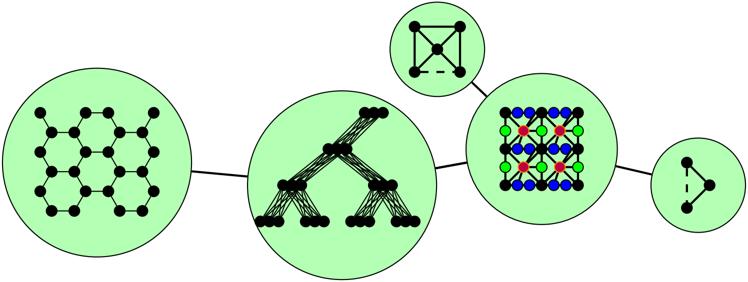
Let G be a quasi-transitive locally finite graph excluding K_∞ as a minor. Then there is an integer k such that G admits a canonical tree-decomposition $(T, \mathcal{V})$, of adhesion at most k whose torsos are either finite or quasi-transitive 3-connected planar minors of G. Moreover, $E(T)$ has finitely many $\mathrm{Aut}(G)$-orbits.



Main result

Theorem (finite treewidth/planar)

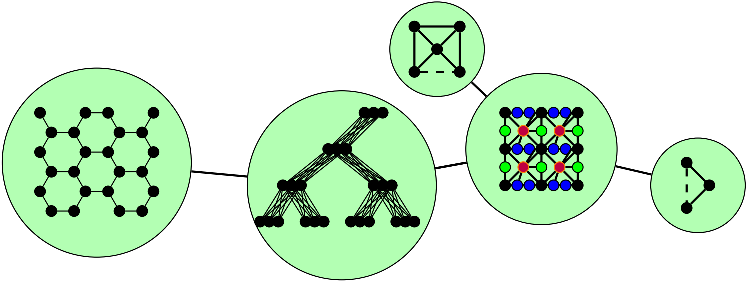
Let G be a quasi-transitive locally finite graph excluding K_∞ as a minor. Then there is an integer k such that G admits a canonical tree-decomposition $(T, \mathcal{V})$, of adhesion at most 3 whose torsos are quasi-transitive minors of G and have either treewidth at most k or are 3-connected planar.



Main result

Theorem (finite treewidth/planar)

Let G be a quasi-transitive locally finite graph excluding K_∞ as a minor. Then there is an integer k such that G admits a canonical tree-decomposition $(T, \mathcal{V})$, of adhesion at most 3 whose torsos are quasi-transitive minors of G and have either treewidth at most k or are 3-connected planar. Moreover, $E(T)$ has finitely many $\mathrm{Aut}(G)$-orbits.



A graph application

Corollary

For every locally finite quasi-transitive graph G avoiding K_∞ as a minor, there is an integer k such that G is K_k-minor-free.

Generalizes [Thomassen '92] dealing with the 4-connected case.

Conclusion

- Prove results on groups by working in the more general world of quasi-transitive graphs.
- Key tool: canonicity (allows to do induction in the context of tree-decompositions).

Conclusion

- Prove results on groups by working in the more general world of quasi-transitive graphs.
- Key tool: canonicity (allows to do induction in the context of tree-decompositions).

Questions:

- A quasi-transitive graphical reformulation of Domino's conjecture?
- If G is quasi-transitive, is there a proper colouring of G with a finite number of colours such that the colored graph G is quasi-transitive?

Conclusion

- Prove results on groups by working in the more general world of quasi-transitive graphs.
- Key tool: canonicity (allows to do induction in the context of tree-decompositions).

Questions:

- A quasi-transitive graphical reformulation of Domino's conjecture?
- If G is quasi-transitive, is there a proper colouring of G with a finite number of colours such that the colored graph G is quasi-transitive?

Thanks

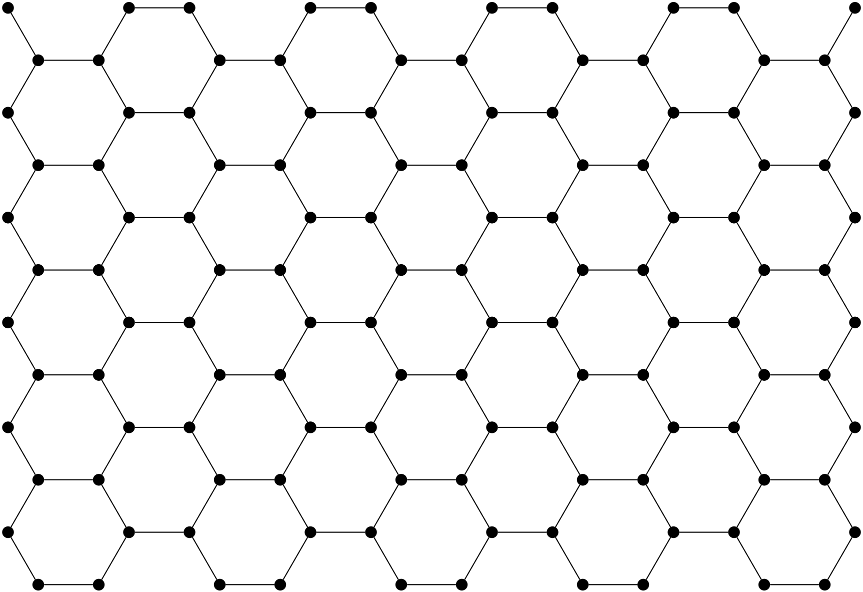
Proof idea

G is $k+1$-connected if $|V| \geqslant k+1$ and for every set X of at most k vertices, $G \setminus X$ is connected.
G is quasi-4-connected if it is 3-connected and the only vertex-cuts of order 3 separate exactly 2 components, and one of them have size 1.

Proof idea

G is $k+1$-connected if $|V| \geqslant k+1$ and for every set X of at most k vertices, $G \setminus X$ is connected.
G is quasi-4-connected if it is 3-connected and the only vertex-cuts of order 3 separate exactly 2 components, and one of them have size 1.
4-connected $\Rightarrow$ quasi-4-connected $\Rightarrow$ 3-connected $\Rightarrow$ 2-connected

Proof idea



Proof idea

G is $k+1$-connected if $|V| \geqslant k+1$ and for every set X of at most k vertices, $G \setminus X$ is connected.
G is quasi-4-connected if it is 3-connected and the only vertex-cuts of order 3 separate exactly 2 components, and one of them have size 1.
Try to combine the following two results:

Theorem (Thomassen '92)

Let G be a quasi-transitive, quasi-4-connected, locally finite graph which excludes K_∞ as a minor. Then G is planar or has finite treewidth.

Proof idea

G is $k+1$-connected if $|V| \geqslant k+1$ and for every set X of at most k vertices, $G \setminus X$ is connected.
G is quasi-4-connected if it is 3-connected and the only vertex-cuts of order 3 separate exactly 2 components, and one of them have size 1.
Try to combine the following two results:

Theorem (Thomassen '92)

Let G be a quasi-transitive, quasi-4-connected, locally finite graph which excludes K_∞ as a minor. Then G is planar or has finite treewidth.

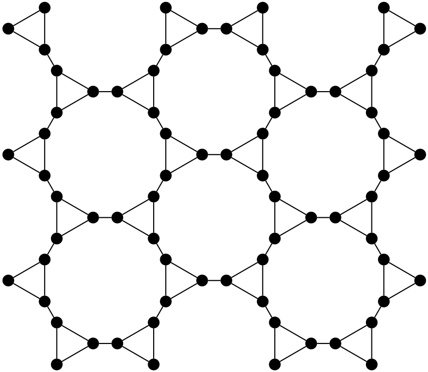
In this case there is nothing to decompose!

Grohe's decomposition

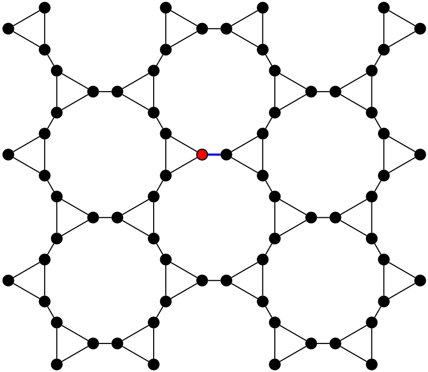
Theorem (Grohe '16)

Every finite 3-connected graph G has a tree-decomposition of adhesion at most 3 whose torsos are minor of G and are complete graphs on at most 4 vertices or quasi-4-connected graphs.

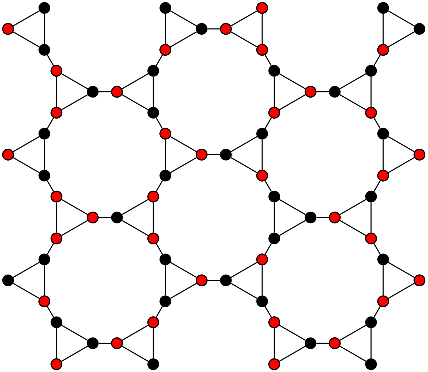
Grohe's decomposition



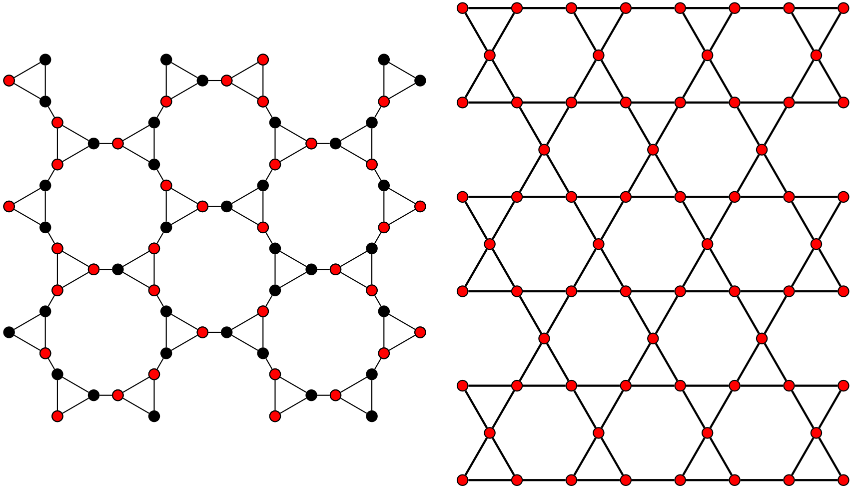
Grohe's decomposition



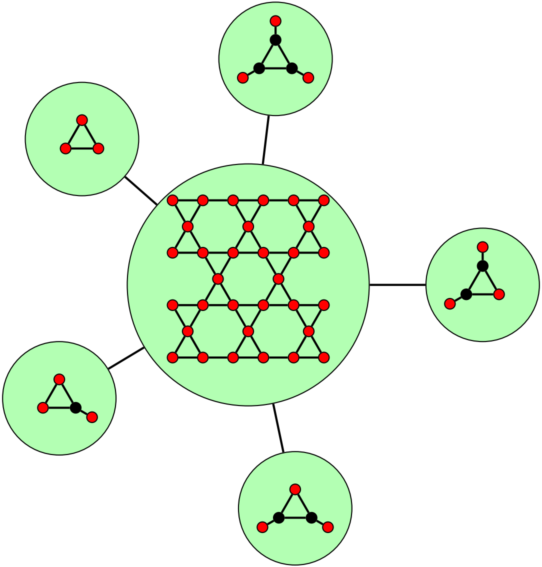
Grohe's decomposition



Grohe's decomposition



Grohe's decomposition



Grohe's decomposition

Theorem (Grohe '16)

Every finite 3-connected graph G has a tree-decomposition of adhesion at most 3 whose torsos are minor of G and are complete graphs on at most 4 vertices or quasi-4-connected graphs.

Bad news: only applies to finite graphs and no canonicity.

Application: Finite presentability.

Theorem (Droms '06)

Planar groups are finitely presented.

Application: Finite presentability.

Theorem (Droms '06)

Planar groups are finitely presented.

Corollary

Every minor-excluding finitely generated group Γ is finitely presented.

Proof based on the approach of [Hamann '18]

Application: accessibility

Accessibility: first defined in the context of groups.
A ray in a graph G is an infinite path $r = (x_1, x_2, x_3, \ldots)$.
$r \simeq r'$ iff for every finite $S \subseteq V(G)$, there is an inifinite component of G containing an infinite subpath of both r and r'.
An end ω is a class of equivalence of rays in a graph.
ω and ω' are k-distinguishable if there exist $S \subseteq V(G)$ of size at most k separating all their rays.

Application: accessibility

Accessibility: first defined in the context of groups.
A ray in a graph G is an infinite path $r = (x_1, x_2, x_3, \dots)$.
$r \simeq r'$ iff for every finite $S \subseteq V(G)$, there is an inifinite component of G containing an infinite subpath of both r and r'.
An end ω is a class of equivalence of rays in a graph.
ω and ω' are k-distinguishable if there exist $S \subseteq V(G)$ of size at most k separating all their rays.
G is accessible if there exists $k \in \mathbb{N}$ such that every two distinct ends are k-distinguishable.

Application: accessibility

[Woess '87] Locally finite quasi-transitive bounded treewidth graphs are accessible.

Application: accessibility

[Woess '87] Locally finite quasi-transitive bounded treewidth graphs are accessible.

[Dunwoody '07] Locally finite quasi-transitive planar graphs are accessible.

Application: accessibility

[Woess '87] Locally finite quasi-transitive bounded treewidth graphs are accessible.

[Dunwoody '07] Locally finite quasi-transitive planar graphs are accessible.

Corollary

Locally finite quasi-transitive graphs that exclude K_∞ as a minor are accessible.