1) Conservation of Mass $\Rightarrow \frac{d}{dt}\int_a^b u(x,t)dt = q(a,t)-q(b,t) = -q(x,t)\Big|_{x=a}^{x=b} = -\int_a^b q_x(x,t)dx = -\int_a^b (q_{x,A}(x,t)+q_{x,D}(x,t))dx$

1) $\frac{d}{dt}\int_a^b u(x,t)dx = \int_a^b u_t(x,t)dt$

2) $q_A(x,t) = u(x,t)\cdot v(x,t) = (uv)(x,t) \Rightarrow q_{x,A}(x,t) = (uv)_x(x,t) \longrightarrow$ How do we deterministically say that $u\cdot v$ gives us the amount of advection?

3) $q_D(x,t) = -k\cdot u_x(x,t) \Rightarrow q_{x,D}(x,t) = -k\cdot\frac{\partial}{\partial x}u_x(x,t) = -k\cdot u_{xx}(x,t)$ for some $k>0$

$\Rightarrow \int_a^b u_t(x,t)dx = -\int_a^b [(uv)_x(x,t) - ku_{xx}(x,t)]dx \Rightarrow \int_a^b u_x(x,t)+(uv)_x(x,t)-ku_{xx}(x,t)=0$

Lemma: For any cts fn $f:\mathbb{R}\times(0,\infty)\to\mathbb{R}$, fixed $t_0\geq 0$ and $S\subseteq\mathbb{R}\times\{t_0\}$ we have $\int_{S\times\{t_0\}} f(x,t_0)dx = 0 \iff f(x,t_0)\equiv 0$ on S λ-a.e. (λ = Lebesgue measure)

Corollary: Since $u_t - ku_{xx} + (uv)_x$ is cts on $\mathbb{R}\times(0,\infty)$ we have $u_t - ku_{xx} + (uv)_x = 0$

Additional Case: Substance degradation over time with rate $\varphi > 0$

Remark: Change in the density of the substance over time is now affected also by an additional factor $\varphi\cdot u$ for some $\varphi\in(0,1)$

Let $d(x,t)$ be the density of the degraded substance at point $x\in\mathbb{R}$ and time $t\geq 0$. Then $d(x,0)=0$ for any $x\in\mathbb{R}$ and as $t\to\infty$ we have

$\int_{-\infty}^{\infty} d(x,t)dx \longrightarrow \int_{-\infty}^{\infty} u(x,t)dx$. Also, for any $(x,t)\in\mathbb{R}\times[0,\infty)$ the density of the pollutant is now given by $(u-d)$

Remark: $\int_{-\infty}^{\infty}|d(x,t)|\,dx \leq \int_{-\infty}^{\infty}|u(x,t)|dx < \infty$ for any $t\geq 0$

Remark: By conservation of mass, total mass of substances at time $t\geq 0$ is given by $M_{degr}(t) = \int_{-\infty}^{\infty}(u(x,t)+d(x,t))dx$

Prop: Let $\varphi>0$ be the degradation rate of a pollutant with density function $u\in C^2[\mathbb{R}\times[0,\infty),\mathbb{R}]$ and let $d\in C[\mathbb{R}\times[0,\infty),\mathbb{R}]$ be the density function of the degraded substance. Then $d_t(x,t) = \varphi u(x,t)$

Proof: $\int_{-\infty}^{\infty} d(x,t)dx := D(t)$ total mass of degraded substance at time $t\geq 0$. Then $\frac{d}{dt}D(t) = \frac{d}{dt}\int_{-\infty}^{\infty} d(x,t)dx = \int_{-\infty}^{\infty} d_t(x,t)dx$ is the increase rate of degraded substance.

Degradation rate $\varphi>0$ is defined to be the ratio of the loss of mass over the mass of degradable substance at any instance $t>0$. Then

$$\varphi = \frac{D'(t)}{M(t)} = \frac{\int_{-\infty}^{\infty} d_t(x,t)dx}{\int_{-\infty}^{\infty} u(x,t)dx} \Rightarrow \int_{-\infty}^{\infty} d_t(x,t)dx = \varphi\int_{-\infty}^{\infty} u(x,t)dx \Rightarrow \int_{-\infty}^{\infty}[d_t(x,t)-\varphi u(x,t)]dx = 0 \Rightarrow d_t = \varphi u \text{ on } \mathbb{R}\times(0,\infty). \quad \square$$

Remark: Since the pollutant is degrading, now we have $\int_a^b (u+d)_t(x,t)\,dx = q(a,t)-q(b,t)$ for any $(a,b)\subseteq\mathbb{R}$, $t>0$. Then

$$\int_a^b (u_t+\varphi u)(x,t)dx = -\int_a^b q_x(x,t)dx = -\int_a^b [(uv)_x - ku_{xx}](x,t)\,dx \Rightarrow \int_a^b [u_t+\varphi u - ku_{xx}+(uv)_x](x,t)dx = 0 \Rightarrow u_t + \varphi u - ku_{xx} + (uv)_x = 0. \quad \square$$

2) a) $L_j^n = \frac{u(x_j,t_{n+1})-u(x_j,t_n)}{\Delta t} + a\frac{u(x_j,t_n)-u(x_{j-1},t_n)}{h}$, $\left|\frac{a\Delta t}{h}\right| \le 1$.

Taylor's Thm For Functions Of Two Variables: Let $u(x,t)$ be twice ctsly diff fn on $\mathbb{R}\times(0,\infty)$. Then $\forall (x,t)\in\mathbb{R}\times(0,\infty)$ we have

$$u(x+h,t+\Delta t) = \sum_{r=0}^{1}\frac{1}{r!}(h\partial_x+\Delta t\partial_t)^r(u)(x,t) + \frac{1}{2!}(h\partial_x+\Delta t\partial_t)^2(u)(x+\lambda h,t+\theta\Delta t) \text{ for some } \lambda,\theta\in(0,1)$$

Remark: $u(x+h,t+\Delta t) = \left[\frac{1}{0!}\cdot u(x,t)+\frac{1}{1!}\cdot h\cdot u_x(x,t)+\frac{1}{1!}\cdot\Delta t\, u_t(x,t)\right] + \frac{1}{2!}\left[h^2\partial_x^2 + h\partial_x\Delta t\partial_t + \Delta t\partial_t h\partial_x + \Delta t^2\partial_t^2\right](u)(x+\lambda h, t+\theta\Delta t)$

$$= u(x,t)+hu_x(x,t)+\Delta t u_t(x,t)+\frac{1}{2}\left(h^2u_{xx}(x+\lambda h,t+\theta\Delta t)+h\Delta t\cdot u_{tx}(x+\lambda h,t+\theta\Delta t)+h\Delta t\, u_{xt}(x+\lambda h,t+\theta\Delta t)+\Delta t^2 u_{tt}(x+\lambda h,t+\theta\Delta t)\right)$$

Hence,

$\to u(x_j,t_{n+1}) = u(x_j,t_n)+0\cdot u_x(x_j,t_n)+\Delta t u_t(x_j,t_n)+\frac{1}{2}\Delta t^2 u_{tt}(x_j,t_n+\theta\Delta t)$ for some $\theta\in(0,1)$

$$= u(x_j,t_n)+\Delta t u_t(x_j,t_n)+\frac{1}{2}\Delta t^2 u_{tt}(x_j,t_n+\theta\Delta t)$$

$\to u(x_{j-1},t_n) = u(x_j,t_n)+(-h)u_x(x_j,t_n)+0\cdot u_t(x_j,t_n)+\frac{1}{2}h^2u_{xx}(x_j-\lambda h,t_n)$ for some $\lambda\in(0,1)$

$$= u(x_j,t_n)-hu_x(x_j,t_n)+\frac{1}{2}h^2u_{xx}(x_j-\lambda h,t_n)$$

$\Rightarrow \frac{u(x_j,t_{n+1})-u(x_j,t_n)}{\Delta t} = u_t(x_j,t_n)+\frac{\Delta t}{2}u_{tt}(x_j,t_n+\theta\Delta t)$ and $a\cdot\frac{u(x_j,t_n)-u(x_{j-1},t_n)}{h} = a\left(u_x(x_j,t_n)-\frac{h}{2}u_{xx}(x_j-\lambda h,t_n)\right)$

$$\Rightarrow L_j^n = \frac{u(x_j,t_{n+1})-u(x_j,t_n)}{\Delta t}+a\cdot\frac{u(x_j,t_n)-u(x_{j-1},t_n)}{h}$$

$$= u_t(x_j,t_n)+\frac{\Delta t}{2}u_{tt}(x_j,t_n+\theta\Delta t)+a\left(u_x(x_j,t_n)-\frac{h}{2}u_{xx}(x_j-\lambda h,t_n)\right)$$

$$= \underbrace{\left(u_t(x_j,t_n)+au_x(x_j,t_n)\right)}_{=0}+\left(\frac{\Delta t}{2}u_{tt}(x_j,t_n+\theta\Delta t)-\frac{ah}{2}u_{xx}(x_j-\lambda h,t_n)\right)$$

$\Rightarrow |L_j^n| \le \frac{\Delta t}{2}\left|u_{tt}(x_j,t_n+\theta\Delta t)\right|+\frac{ah}{2}\left|u_{xx}(x_j-\lambda h,t_n)\right| \le \frac{\Delta t}{2}\sup_{\substack{x\in\mathbb{R}\\t\in[0,T]}}\left|u_{tt}(x,t)\right|+\frac{h}{2}\sup_{\substack{x\in\mathbb{R}\\t\in[0,T]}}a\left|u_{xx}(x,t)\right| < \infty$ since u is integrable

Prop: For all $j\in\mathbb{Z}$ and $n\in\mathbb{N}_0$ there exists $C_L\in\mathbb{R}$ s.t. $|L_j^n|\le C_L\cdot(\Delta t+h)$.

Set $\mathcal{M} = \max\left\{\sup_{\substack{x\in\mathbb{R}\\t\in[0,T]}}|u_{tt}(x,t)|, \sup_{\substack{x\in\mathbb{R}\\t\in[0,T]}}a|u_{xx}(x,t)|\right\}$. Then $\frac{\Delta t}{2}\sup_{\substack{x\in\mathbb{R}\\t\in[0,T]}}\left|u_{tt}(x,t)\right|+\frac{h}{2}\sup_{\substack{x\in\mathbb{R}\\t\in[0,T]}}a\left|u_{xx}(x,t)\right| \le \frac{\Delta t}{2}\mathcal{M}+\frac{h}{2}\mathcal{M} = \mathcal{M}\cdot\frac{h+\Delta t}{2}$

Hence, $C_L = \frac{1}{2}\cdot\max\left\{\sup_{\substack{x\in\mathbb{R}\\t\in[0,T]}}|u_{tt}(x,t)|, \sup_{\substack{x\in\mathbb{R}\\t\in[0,T]}}a|u_{xx}(x,t)|\right\}$ and for fixed $j\in\mathbb{Z}, n\in\mathbb{N}_0$ we have

$$|L_j^n| \le \frac{\Delta t}{2}\sup_{\substack{x\in\mathbb{R}\\t\in[0,T]}}\left|u_{tt}(x,t)\right|+\frac{h}{2}\sup_{\substack{x\in\mathbb{R}\\t\in[0,T]}}a\left|u_{xx}(x,t)\right| \le \Delta t\cdot\frac{\mathcal{M}}{2}+h\cdot\frac{\mathcal{M}}{2} = C_L(\Delta t+h).$$

$\square$

b) **Error:** $e_j^n := u(x_j,t_n) - u_j^n$ for all $n\in\mathbb{N}$. WTS $\frac{e_j^{n+1}-e_j^n}{\Delta t} + a\frac{e_j^n - e_{j-1}^n}{h} = L_j^n$ for all $n\in\mathbb{N}$. Fix $n\in\mathbb{N}$.

$$\Rightarrow \frac{e_j^{n+1}-e_j^n}{\Delta t} = \frac{(u(x_j,t_{n+1})-u_j^{n+1})-(u(x_j,t_n)-u_j^n)}{\Delta t} = \frac{(u(x_j,t_{n+1})-u(x_j,t_n))}{\Delta t} - \frac{(u_j^{n+1}-u_j)}{\Delta t}$$

$$\Rightarrow a\frac{e_j^n-e_{j-1}^n}{h} = a\frac{(u(x_j,t_n)-u_j^n)-(u(x_{j-1},t_n)-u_{j-1}^n)}{h} = a\frac{(u(x_j,t_n)-u(x_{j-1},t_n))}{h} - a\frac{u_j^n-u_{j-1}^n}{h}$$

$$\Rightarrow \frac{e_j^{n+1}-e_j^n}{\Delta t} + a\frac{e_j^n-e_{j-1}^n}{h} = \left[\frac{(u(x_j,t_{n+1})-u(x_j,t_n))}{\Delta t} + a\frac{(u(x_j,t_n)-u(x_{j-1},t_n))}{h}\right] - \underbrace{\left[\frac{(u_j^{n+1}-u_j)}{\Delta t} + a\frac{u_j^n-u_{j-1}^n}{h}\right]}_{=0\text{ by the numerical scheme}}$$

$$= \frac{(u(x_j,t_{n+1})-u(x_j,t_n))}{\Delta t} + a\frac{(u(x_j,t_n)-u(x_{j-1},t_n))}{h} = L_j^n$$

$$\Rightarrow e_j^{n+1} - e_j^n = \Delta t\left(L_j^n - a\frac{e_j^n-e_{j-1}^n}{h}\right)$$

$$\Rightarrow e_j^{n+1} = e_j^n + \Delta t L_j^n - \frac{\Delta t a e_j^n}{h} + \frac{\Delta t a e_{j-1}^n}{h} = e_j^n\left(1-\frac{a\Delta t}{h}\right) + e_{j-1}^n\frac{a\Delta t}{h} + \Delta t\cdot L_j^n$$

Prop: $\sup_{j\in\mathbb{Z}} |e_j^n| \le n\Delta t\cdot C_L(\Delta t+h)$ for any $n\in\left\{0,1,2,\dots,\left\lfloor\frac{T}{\Delta t}\right\rfloor\right\}$

Remark: $n\Delta t = t_n$

Proof: By induction on n.

$\underline{n=0}$: $e_j^0 = u(x_j,0)-u_j^0 = u_0(x_j)-u_0(x_j) = 0 \le n\Delta t C_L(\Delta t+h) \Rightarrow \sup_{j\in\mathbb{Z}}|e_j^0| \le n\Delta t C_L(\Delta t+h)$

Now assume $\sup |e_j^k| \le k\Delta t C_L(\Delta t+h)$ for some $k\in\left\{0,1,2,\dots,\left\lfloor\frac{T}{\Delta t}\right\rfloor-1\right\}$. Then we have

$$e_j^{k+1} \le \left|e_j^k\left(1-\frac{a\Delta t}{h}\right) + e_{j-1}^k\frac{a\Delta t}{h} + \Delta t\cdot L_j^k\right| \le \left|e_j^k\left(1-\frac{a\Delta t}{h}\right)\right| + \left|e_{j-1}^k\frac{a\Delta t}{h}\right| + \left|\Delta t\cdot L_j^k\right| = |e_j^k|\cdot\left(1-\frac{a\Delta t}{h}\right) + |e_{j-1}^k|\frac{a\Delta t}{h} + \Delta t|L_j^k|$$

$\le \sup_{j\in\mathbb{Z}}|e_j^k|\cdot\left(1-\frac{a\Delta t}{h}\right) + \sup_{j\in\mathbb{Z}}|e_j^k|\frac{a\Delta t}{h} + \Delta t|L_j^k| = \sup_{j\in\mathbb{Z}}|e_j^k| + \Delta t|L_j^k|$. Previously we showed that $|L_j^k| \le C_L\cdot(h+\Delta t)$.

Also, by our induction assumption we have $\sup_{j\in\mathbb{Z}}|e_j^k| \le k\Delta t\cdot C_L(h+\Delta t)$. Then

$$|e_j^{k+1}| \le \sup_{j\in\mathbb{Z}}|e_j^k| + \Delta t|L_j^k| \le k\Delta t\cdot C_L(h+\Delta t) + \Delta t C_L(h+\Delta t) = (k+1)\Delta t C_L(h+\Delta t)$$

Hence, $\forall n\in\mathbb{N}$ $\sup_{j\in\mathbb{Z}}|e_j^n| \le n\Delta t C(h+\Delta t)$.

$\square$

3)a) Let $z:[0,\infty)\to\mathbb{R}$ be the characteristic of a flow passing through $x\in\mathbb{R}$ when $s=t$. Then z is the solution of the IVP given below:

$$\# \begin{cases} \dot{z}(s)=v(z(s)),\ s\in\mathbb{R} \\ z(t)=x \end{cases}$$

Remark: Velocity of the flow doesn't depend on t; $v\in C^1(\mathbb{R})$. Also $\forall x\in\mathbb{R}$ we have $v(x)\leq 0$ and $v'(x)\geq 0$.

$\Rightarrow \forall s\in\mathbb{R}$ $\dot{z}(s)=v(z(s))\leq 0 \Rightarrow z$ is a monotone non-increasing function

$\Rightarrow$ particles in the flow move in the negative direction.

Remark: $\ddot{z}(s)=\frac{d}{ds}\dot{z}(s)=\frac{d}{ds}v(z(s))=v'(z(s))\cdot\dot{z}(s)=v'(z(s))\cdot v(z(s))\leq 0$ for all $s\in\mathbb{R}$

$\Rightarrow z(s)$ approaches $-\infty$ faster as s increases

$$\frac{d}{ds}(u(z(s),s))=u_x(z(s),s)\cdot\dot{z}(s)+u_t(z(s),s)\cdot 1=u_x(z(s),s)\cdot v(z(s))+u_t(z(s),s)=\underbrace{u_t(z(s),s)+(uv)_x(z(s),s)}_{=0}-u(z(s),s)\cdot v'(z(s))$$
$$=-u(z(s),s)\cdot v'(z(s))$$

$$\Rightarrow \ln(u(z(t),t))-\ln(u(z(0),0))=\ln(u(x,t))-\ln(u_0(z(0)))=\int_0^t -v'(z(s))ds$$

$$\Rightarrow u(x,t)=u_0(z(0))\cdot\exp\left(-\int_0^t v'(z(s))ds\right)\leq u_0(z(0))=u_0(z(0))$$

Remark: $\forall x\in\mathbb{R}$ $v'(x)\geq 0 \Rightarrow -\int_0^t v'(z(s))ds\leq\int_0^t 0\,ds=0$ for all $t\in[0,\infty)$

Remark: For any $(x,t)\in\mathbb{R}\times(0,\infty)$ we have $u(x,t)dx\geq 0$ by definition.

$\Rightarrow \forall(x,t)\in\mathbb{R}\times(0,\infty)$ $u(x,t)\in[0,u(z(0),0)]$

Prop: $\sup_{x\in\mathbb{R}}|u(x,t)|\leq\sup_{x\in\mathbb{R}}|u_0(x)|$ for all t

Proof: Fix $t\in[0,\infty)$. Then $\sup_{x\in\mathbb{R}}|u(x,t)|\leq u_0(z(0))\leq\sup_{x\in\mathbb{R}}|u_0(x)|$. $\square$

b) Assume that the scheme below is stable:

$$⊛\begin{cases} \frac{U_j^{n+1}-U_j^n}{\Delta t}+V_j\cdot\frac{U_j^n-U_{j-1}^n}{h}+U_j^n\cdot\frac{V_j-V_{j-1}}{h}=0,\ j\in\mathbb{Z},\ n\in\mathbb{N}_0 \\ U_j^0=u_0(x_j),\ j\in\mathbb{Z}. \end{cases}$$

Remark: A scheme is stable iff for any fixed $n\in\mathbb{N}_0$ we have $\sup_{j\in\mathbb{Z}}|U_j^n|\leq\sup_{j\in\mathbb{Z}}|U_j^0|$ and

Given: $\frac{U_j^{n+1}-U_j^n}{\Delta t}+V_j\cdot\frac{U_j^n-U_{j-1}^n}{h}+U_j^n\cdot\frac{V_j-V_{j-1}}{h}=0 \Rightarrow \frac{U_j^{n+1}-U_j^n}{\Delta t}=-\left(V_j\cdot\frac{U_j^n-U_{j-1}^n}{h}+U_j^n\cdot\frac{V_j-V_{j-1}}{h}\right)$

$$\Rightarrow U_j^{n+1}=U_j^n-\Delta t\cdot V_j\cdot\frac{U_j^n-U_{j-1}^n}{h}-\Delta t\cdot U_j^n\cdot\frac{V_j-V_{j-1}}{h}=U_j^n\left[1-\Delta t\cdot\frac{V_j-V_{j-1}}{h}\right]+\frac{\Delta t V_j}{h}(U_{j-1}^n-U_j^n)$$

$$= u_j^n\left[1-\Delta t\,\frac{v_j^n-v_{j-1}^n}{h}-\frac{\Delta t\, v_j^n}{h}\right]+u_{j-1}^n\cdot\frac{\Delta t\, v_j^n}{h} = u_j^n\left(1-\frac{\Delta t}{h}.(2v_j^n-v_{j-1}^n)\right)+u_{j-1}^n\,\frac{\Delta t}{h}\,v_j^n$$

$$= u_j^n\left[1-\frac{(2v_j^n-v_{j-1}^n)\Delta t}{h}\right]+u_{j-1}^n\cdot\frac{v_j\Delta t}{h} \leq \sup_{j\in\mathbb{Z}}|u_j^n|.\left(1-\frac{(2v_j^n-v_{j-1}^n)\Delta t}{h}\right)+\sup_{j\in\mathbb{Z}}|u_{j-1}^n|\cdot v_j^n\frac{\Delta t}{h}$$

$$\leq \sup_{j\in\mathbb{Z}}|u_j^0|\cdot\left(1-\frac{(2v_j^n-v_{j-1}^n)\Delta t}{h}+v_j^n\frac{\Delta t}{h}\right)=\sup_{j\in\mathbb{Z}}|u_j^0|\left(1-\frac{\Delta t}{h}(v_j^n-v_{j-1}^n)\right)$$

$\cdot\ \left(1-\frac{\Delta t}{h}(v_j^n-v_{j-1}^n)\right)\leq 1 \iff \frac{\Delta t}{h}(v_j-v_{j-1}^n)\geq 0 \iff v_j^n-v_{j-1}^n\geq\frac{h}{\Delta t}$

Prop: Let $u_0\in C[\mathbb{R}]$ with $\sup_{x\in\mathbb{R}}|u_0(x)|<\infty$. For the scheme $*$, assume that $v_j^n-v_{j-1}^n\geq\frac{h}{\Delta t}$. Then $\sup_{j\in\mathbb{Z}}|u_j^n|\leq\sup_{j\in\mathbb{Z}}|u_j^0|$ for any $n\in\mathbb{N}_0$, i.e. the scheme is stable.

Proof: For any $j\in\mathbb{Z}, n\in\mathbb{N}_0$ we have $u_j^n\leq\sup_{j\in\mathbb{Z}}|u_j^0|\left(1-\frac{\Delta t}{h}(v_j^n-v_{j-1}^n)\right)\leq\sup_{j\in\mathbb{Z}}|u_j^0|$. Then $\sup_{j\in\mathbb{Z}}|u_j^n|\leq\sup_{j\in\mathbb{Z}}|u_j^0|$ for any $n\in\mathbb{N}_0$.

$\square$

Check:

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+v_j.\frac{u_j^n-u_{j-1}^n}{h}+u_j^n.\frac{v_j-v_{j-1}}{h}=0$$

$$\Rightarrow \frac{u_j^{n+1}-u_j^n}{\Delta t}=-v_j\cdot\frac{u_j^n-u_{j-1}^n}{h}-u_j^n\frac{v_j-v_{j-1}}{h}=\frac{-v_j\cdot u_j^n}{h}+\frac{v_j\cdot u_{j-1}^n}{h}-u_j^n\frac{v_j-v_{j-1}}{h}=u_j^n\left[\frac{-v_j}{h}+\frac{v_{j-1}-v_j}{h}\right]+u_{j-1}^n\cdot\frac{v_j}{h}$$

$$\Rightarrow u_j^{n+1}=u_j^n+\Delta t\left[u_j^n\left(\frac{-v_j}{h}+\frac{v_{j-1}-v_j}{h}\right)+u_{j-1}^n\cdot\frac{v_j}{h}\right]=\left(1-\frac{\Delta t(2v_j-v_{j-1})}{h}\right)u_j^n+\frac{v_j\Delta t}{h}\cdot u_{j-1}^n$$

4) a) We want to solve $u_t - k u_{xx} + a \cdot u_x + \varphi u = 0$ with a transformation. Let $\tilde{u} = f(t) \cdot u$. Then for all $(x,t) \in \mathbb{R} \times [0,T]$ we have

i) $\tilde{u}_t(x,t) = \frac{\partial \tilde{u}}{\partial t}(x,t) = \frac{df}{dt}(t) \cdot u(x,t) + f(t) \cdot \frac{\partial u}{\partial t}(x,t) = f'(t) \cdot u(x,t) + f(t) \cdot u_t(x,t)$

ii) $\tilde{u}_x(x,t) = \frac{\partial \tilde{u}}{\partial x}(x,t) = \frac{df}{dx}(t) \cdot u(x,t) + f(t) \cdot u_x(x,t) = f(t) \cdot u_x(x,t)$

iii) $\tilde{u}_{xx}(x,t) = \frac{\partial \tilde{u}_x}{\partial x}(x,t) = \frac{df}{dx}(t) \cdot \tilde{u}_x(x,t) + f(t) \cdot \frac{\partial u_x}{\partial x} = f(t) \cdot u_{xx}(x,t).$

Assume $\tilde{u}_t - k\tilde{u}_{xx} + a\tilde{u}_x = 0.$

$\Rightarrow f(t) \cdot u_t(x,t) - k \cdot f(t) \cdot u_{xx}(x,t) + a \cdot f(t) \cdot u_x + f'(t) \cdot u(x,t) = 0$

$\Rightarrow f(t)\left[u_t(x,t) - k u_{xx}(x,t) + a u_x\right] + f'(t) \cdot u(x,t) = 0$

$\Rightarrow f(t) \cdot (-\varphi \cdot u(x,t)) + f'(t) \cdot u(x,t) = 0$

$\Rightarrow u(x,t) \cdot \left[-\varphi f(t) + f'(t)\right] = 0 \Rightarrow$ Either $u(x,t) = 0$ or $f'(t) - \varphi f(t) = 0$. $u(x,t) = 0$ is a trivial case, so we assume the other case.

$\rightarrow f'(t) = \varphi \cdot f(t) \Rightarrow \int_0^t \frac{1}{f(\tau)} \cdot f'(\tau) d = \int_0^t \varphi d\tau \Rightarrow \ln f(t) - \ln f(0) = \varphi t$ for all $t \in [0,T] \Rightarrow f(t) = f(0) \cdot e^{\varphi t}$ for all $t \in [0,T]$, $f(0) \in (0,\infty)$

Check: $f'(t) = f(0) \cdot \varphi e^{\varphi t} = \varphi f(t)$

Check: $\left[\tilde{u}_t - k\tilde{u}_{xx} + a\tilde{u}_x\right](x,t) = f(0)\varphi e^{\varphi t} \cdot u(x,t) + f(0) e^{\varphi t} u_t(x,t) - k f(0) e^{\varphi t} \cdot u_{xx}(x,t) + a \cdot f(0) e^{\varphi t} u_x(x,t)$

$= f(0) \cdot e^{\varphi t}\left[\varphi u + u_t - k u_{xx} + a u_x\right](x,t) = 0$

Hence, $\tilde{u}(x,t) = f(t) \cdot u(x,t) = f(0) \cdot e^{\varphi t} u(x,t)$ for all $(x,t) \in \mathbb{R} \times [0,T]$

b) Let $\tilde{u}(x+at,t) = w(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} w_0(y) \cdot e^{-\frac{(x-y)^2}{4kt}} dy$

Remark: $\tilde{u}_0(x) := \tilde{u}(x,0) = f(0) e^{\varphi \cdot 0} u(x,0) = f(0) \cdot u_0(x)$ for all $x \in \mathbb{R}$ $\Rightarrow w_0(x) = w(x,0) = \tilde{u}(x + a \cdot 0, 0) = \tilde{u}_0(x) = f(0) u_0(x)$ for all $x \in \mathbb{R}$

$\Rightarrow \tilde{u}(x,t) = \tilde{u}((x-at)+at, t) = w(x-at,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} w_0(y) e^{\frac{-(x-at-y)^2}{4kt}} dy = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} \tilde{u}_0(y) \cdot \exp\left(\frac{-(x-at-y)^2}{4kt}\right) dy$

c) Known: 1) $\tilde{u}_0(x) = \tilde{u}(x,0) = f(0) \cdot e^{\varphi \cdot 0} u(x,0) = f(0) u_0(x)$ $\forall x \in \mathbb{R}$

2) $\tilde{u}(x,t) = f(t) \cdot u(x,t) = f(0) e^{\varphi t} u(x,t)$ $\forall (x,t) \in \mathbb{R} \times (0,\infty)$

$\Rightarrow u(x,t) = \frac{e^{-\varphi t}}{f(0)} \tilde{u}(x,t) = \frac{e^{-\varphi t}}{f(0)\sqrt{4\pi k t}} \int_{-\infty}^{\infty} f(t) u_0(y) \cdot \exp\left(\frac{-(x-at-y)^2}{4kt}\right) dy = \frac{f(t) e^{-\varphi t}}{f(0)} \cdot \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} u_0(y) \exp\left(\frac{-(x-at-y)^2}{4kt}\right) dy$