1. The Black-Scholes formula for a European Call option is given by

$$C(S,t) = S\Phi(d(S,t)) - Ee^{-r(T-t)}\Phi(d(S,t) - \sigma\sqrt{T-t}), \tag{1}$$

where $d(S,t) = \frac{\log(S/E)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$ and E, T, σ, and r denote the exercise price, expiry, volatility, and riskless interest rate, respectively.

(a) Compute **(give all details)** the following derivatives of the option price, the so-called Greeks,

$$\Delta = \frac{\partial C}{\partial S}, \; \Gamma = \frac{\partial^2 C}{\partial S^2}, \; \kappa = \frac{\partial C}{\partial \sigma}, \; \rho = \frac{\partial C}{\partial r}, \; \theta = \frac{\partial C}{\partial t}.$$

(b) Verify that C is a solution of the equation

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0, \quad S > 0, \; 0 < t < T.$$

Remark: $d(S,t) = d(S,t;\sigma,r) = \frac{\ln\frac{S}{E} + r(T-t) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} = \frac{\ln\frac{S}{E}}{\sigma}(T-t)^{-1/2} + \frac{(r+\frac{1}{2}\sigma^2)}{\sigma}(T-t)^{1/2}$

$$\Rightarrow \frac{\partial d}{\partial \sigma} = \frac{\cancel{(T-t)}\cdot\sigma\sqrt{T-t}}{\sigma^2\cancel{(T-t)}} - \frac{\left(\ln\frac{S}{E} + r(T-t) + \frac{1}{2}\sigma^2(T-t)\right)\cdot\sqrt{T-t}}{\sigma^2(T-t)} = \frac{\sqrt{T-t}}{\sigma} - \frac{d\cancel{\sqrt{T-t}}}{\sigma\cancel{\sqrt{T-t}}} = \frac{\sqrt{T-t} - d}{\sigma}$$

$$\Rightarrow \frac{\partial d}{\partial r} = \frac{\partial}{\partial r}\left(r\frac{\sqrt{T-t}}{\sigma}\right) = \frac{\sqrt{T-t}}{\sigma} \qquad \Rightarrow \frac{\partial d}{\partial t} = \frac{\ln\frac{S}{E}}{\sigma}\left(-\tfrac{1}{2}\right)(T-t)^{-3/2}\cdot(-1) + \left(\frac{r+\frac{1}{2}\sigma^2}{\sigma}\right)\cdot\frac{1}{2}(T-t)^{-1/2}\cdot(-1) = \frac{\ln\frac{S}{E}}{2\sigma(T-t)^{3/2}} - \frac{r+\frac{1}{2}\sigma^2}{2\sigma\sqrt{T-t}}$$

$$\Delta = \frac{\partial C}{\partial S} = \Phi(d) + S\cdot\Phi'(d)\frac{\partial d}{\partial S} - E\cdot e^{-r(T-t)}\Phi'(d-\sigma\sqrt{T-t})\cdot\frac{\partial d}{\partial S} = \Phi(d) + \frac{S}{\sqrt{2\pi}}e^{-\frac{d^2}{2}}\cdot\frac{1}{S\sigma\sqrt{T-t}} - E\cdot e^{-r(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{(d-\sigma\sqrt{T-t})^2}{2}}\frac{1}{S\sigma\sqrt{T-t}}$$

$$= \Phi(d) + \frac{1}{\sigma\sqrt{2\pi}\sqrt{T-t}}e^{-\frac{d^2}{2}} - \frac{E\cdot e^{-r(T-t)}}{S\sigma\sqrt{T-t}}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{d^2-2d\sigma\sqrt{T-t}+\sigma^2(T-t)}{2}} = \Phi(d) + \frac{1}{\sigma\sqrt{2\pi}\sqrt{T-t}}e^{-\frac{d^2}{2}} - \frac{E}{S}e^{-r(T-t)}\frac{1}{\sigma\sqrt{2\pi}\sqrt{T-t}}e^{-\frac{d^2}{2} - d\sigma\sqrt{T-t} + \frac{\sigma^2}{2}(T-t)}$$

$$= \Phi(d) + \frac{1}{\sigma\sqrt{2\pi}\sqrt{T-t}}e^{-\frac{d^2}{2}} - \frac{E}{S}e^{-r(T-t)}\frac{1}{\sigma\sqrt{2\pi}\sqrt{T-t}}e^{-\frac{d^2}{2} - \frac{\ln\frac{S}{E} + (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\sigma\sqrt{T-t} + \frac{\sigma^2}{2}(T-t)}$$

$$= \Phi(d) + \frac{1}{\sigma\sqrt{2\pi}\sqrt{T-t}}e^{-\frac{d^2}{2}} - \frac{E}{S}e^{-r(T-t)}\frac{1}{\sigma\sqrt{2\pi}\sqrt{T-t}}e^{-\frac{d^2}{2} + \ln\frac{S}{E} + r(T-t) + \frac{\sigma^2}{2}(T-t) - \frac{\sigma^2}{2}(T-t)}$$

$$= \Phi(d) + \frac{1}{\sigma\sqrt{2\pi}\sqrt{T-t}}e^{-\frac{d^2}{2}} - \frac{E}{S}\frac{1}{\sigma\sqrt{2\pi}\sqrt{T-t}}\cdot e^{-\frac{d^2}{2}}\cdot\frac{S}{E} \qquad = \Phi(d) + \frac{1}{\sigma\sqrt{2\pi}\sqrt{T-t}}e^{-\frac{d^2}{2}} - \frac{1}{\sigma\sqrt{2\pi}\sqrt{T-t}}e^{-\frac{d^2}{2}} = \Phi(d)$$

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{\partial \Delta}{\partial S} = \Phi'(d)\cdot\frac{\partial d}{\partial S} = \frac{1}{\sqrt{2\pi}}e^{-\frac{d^2}{2}}\cdot\frac{\partial}{\partial S}\left(\frac{\ln S}{\sigma\sqrt{T-t}}\right) = \frac{1}{S\sigma\sqrt{2\pi}\sqrt{T-t}}e^{-\frac{d^2}{2}}$$

$$\kappa = \frac{\partial C}{\partial \sigma} = S\cdot\Phi'(d)\frac{\partial d}{\partial \sigma} - E\cdot e^{-r(T-t)}\cdot\Phi'(d-\sigma\sqrt{T-t})\cdot\frac{\partial}{\partial\sigma}(d-\sigma\sqrt{T-t}) = \frac{S}{\sqrt{2\pi}}e^{-\frac{d^2}{2}}\cdot\left(\frac{\sqrt{T-t}-d}{\sigma}\right) = \frac{S(\sqrt{T-t}-d)}{\sigma\sqrt{2\pi}}e^{-\frac{d^2}{2}}$$

$$\rho = \frac{\partial C}{\partial r} = S\cdot\Phi'(d)\cdot\frac{\sqrt{T-t}}{r} - E\left[e^{-r(T-t)}\cdot(-1)(T-t)\,\Phi(d-\sigma\sqrt{T-t}) + e^{-r(T-t)}\Phi'(d-\sigma\sqrt{T-t})\frac{\sqrt{T-t}}{\sigma}\right]$$

$$= \frac{S\sqrt{T-t}}{\sigma\sqrt{2\pi}}e^{-\frac{d^2}{2}} - E\left[-(T-t)e^{-r(T-t)}\Phi(d-\sigma\sqrt{T-t}) + e^{-r(T-t)}\frac{\sqrt{T-t}}{\sigma\sqrt{2\pi}}e^{-\frac{(d-\sigma\sqrt{T-t})^2}{2}}\right]$$

$$= \frac{S\sqrt{T-t}}{\sigma\sqrt{2\pi}}e^{-\frac{d^2}{2}} + E(T-t)e^{-r(T-t)}\Phi(d-\sigma\sqrt{T-t}) - Ee^{-r(T-t)}\frac{\sqrt{T-t}}{\sigma\sqrt{2\pi}}e^{-\frac{d^2-2d\sigma\sqrt{T-t}+\sigma^2(T-t)}{2}}$$

$$= \frac{S\sqrt{T-t}}{\sigma\sqrt{2\pi}}e^{-\frac{d^2}{2}} + E(T-t)e^{-r(T-t)}\Phi(d-\sigma\sqrt{T-t}) - Ee^{-r(T-t)}\frac{\sqrt{T-t}}{\sigma\sqrt{2\pi}}e^{-\frac{d^2}{2}}e^{\ln\frac{S}{E}+(r+\frac{1}{2}\sigma^2)(T-t)}e^{-\frac{1}{2}\sigma^2(T-t)}$$

$$= \frac{S\sqrt{T-t}}{\sigma\sqrt{2\pi}}e^{-\frac{d^2}{2}} + E(T-t)e^{-r(T-t)}\Phi(d-\sigma\sqrt{T-t}) - S\cdot\frac{\sqrt{T-t}}{\sigma\sqrt{2\pi}}e^{-\frac{d^2}{2}} = E(T-t)e^{-r(T-t)}\Phi(d-\sigma\sqrt{T-t})$$

$$\theta = \frac{\partial C}{\partial t} = \frac{\partial S}{\partial t}\,\Phi(d) + S\cdot\Phi'(d)\frac{\partial d}{\partial t} - E\left[r\cdot e^{-r(T-t)}\Phi(d-\sigma\sqrt{T-t}) + e^{-r(T-t)}\Phi'(d-\sigma\sqrt{T-t})\cdot\frac{\partial}{\partial t}\left(d-\sigma(T-t)^{1/2}\right)\right]$$

$$= \frac{\partial S}{\partial t}\Phi(d) + \frac{S}{\sqrt{2\pi}}e^{-\frac{d^2}{2}}\frac{\partial d}{\partial t} - Ere^{-r(T-t)}\Phi(d-\sigma\sqrt{T-t}) - Ee^{-r(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{d^2}{2}+\ln\frac{S}{E}+r(T-t)+\frac{1}{2}\sigma^2(T-t)-\frac{\sigma^2(T-t)}{2}}\cdot\left(\frac{\partial d}{\partial t}-\sigma\cdot\frac{1}{2}(T-t)^{-\frac{1}{2}}\cdot(-1)\right)$$

$$= \frac{\partial S}{\partial t}\cdot\Phi(d) + \frac{S}{\sqrt{2\pi}}e^{-\frac{d^2}{2}}\frac{\partial d}{\partial t} - Ere^{-r(T-t)}\Phi(d-\sigma\sqrt{T-t}) - \frac{S}{\sqrt{2\pi}}e^{-\frac{d^2}{2}}\left(\frac{\partial d}{\partial t}+\frac{\sigma}{2\sqrt{T-t}}\right)$$

$=0$

$$= \frac{\partial S}{\partial t}\Phi(d) - \frac{\sigma S}{2\sqrt{2\pi}\sqrt{T-t}}e^{-\frac{d^2}{2}} - Er\cdot e^{-r(T-t)}\Phi(d-\sigma\sqrt{T-t})$$

b) Rewriting BSE in terms of greeks gives us:

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S_0^2\frac{\partial^2 C}{\partial S^2} - rC = \theta + rS\Delta + \frac{1}{2}\sigma^2 S_0^2\Gamma - rC$$

$$= \left(\frac{\partial S}{\partial t}\Phi(d) - \frac{\sigma S}{2\sqrt{2\pi}\sqrt{T-t}}e^{-\frac{d^2}{2}} - Ere^{-r(T-t)}\Phi(d-\sigma\sqrt{T-t})\right) + rS\Phi(d) + \frac{1}{2}\sigma^2 S^2\frac{1}{S\sigma\sqrt{2\pi}\sqrt{T-t}}e^{-\frac{d^2}{2}} - r\left(S\cdot\Phi(d) - E\cdot e^{-r(T-t)}\Phi(d-\sigma\sqrt{T-t})\right)$$

$=0$

$$= \frac{\partial S}{\partial t}\Phi(d) - \frac{S\cdot\sigma}{2\sqrt{2\pi}\sqrt{T-t}}e^{-\frac{d^2}{2}} - Ere^{-r(T-t)}\Phi(d-\sigma\sqrt{T-t})\ rS\Phi(d) + \frac{S\sigma}{2\sqrt{2\pi}\sqrt{T-t}}e^{-\frac{d^2}{2}} - rS\Phi(d) + Ere^{-r(T-t)}\Phi(d-\sigma\sqrt{T-t}) = 0$$

$=0$ $=0$

There is no $\frac{\partial S}{\partial t}$ term in the greeks except for θ. Hence, I believe we assume $\frac{\partial S}{\partial t} = 0$, so that the claim follows.

Hence, the claim follows.

2. Use a 3-step binomial tree to value a European Call Option with the following conditions:

- The Option expires 9 months from now and has a Exercise price of £44.
- One underlying share is currently worth £41 and the volatility and drift parameters take the following values: $\sigma = 30\%$ and $\mu = 0$.
- There is a risk free interest rate of 8%.

Ensure your solution includes **all working and appropriate tree diagrams** and give your final answer **correct to 2 decimal places**. [20 marks]

$\Delta t = \frac{1}{4}$, 3 steps. Current value $V_0 = £41$, Strike price at 9th month is $s = £44$, $r = 0{,}08$

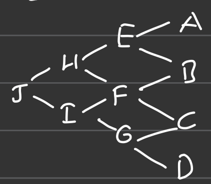
Binomial Model for $V_{\Delta t}$: $V_0 \xrightarrow{p=\frac{1}{2}} V_0 u$, $V_0 \xrightarrow{p=\frac{1}{2}} V_0 d$ $\Rightarrow \mathbb{E}[V_{\Delta t}] = V_0 \cdot \frac{u+d}{2}$, $Var(V_{\Delta t}) = \frac{1}{2} \cdot (V_0 u - V_0 \frac{u+d}{2})^2 + \frac{1}{2}(V_0 d - V_0 \frac{u+d}{2})^2 = \dots = V_0^2 \cdot \frac{(u-d)^2}{4}$

Log-Normal Model for $\frac{V_{\Delta t}}{V_0}$: $\frac{V_{\Delta t} - V_0}{V_0} \approx \ln(1 + \frac{V_{\Delta t} - V_0}{V_0}) = \ln(\frac{V_{\Delta t}}{V_0}) \sim \mathcal{N}(\mu \Delta t, \sigma^2 \Delta t) = \mathcal{N}(0, 0.0225) \Rightarrow \mathbb{E}[V_{\Delta t}] = V_0$, $Var(V_{\Delta t}) = V_0^2 \cdot Var(\frac{V_{\Delta t} - V_0}{V_0}) = V_0^2 \; 0{,}075$

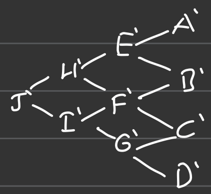
$\hookrightarrow \sigma^2 \Delta t = (0.3)^2 \cdot \frac{1}{4} = 0.0225$

With these assumptions we obtain:

$\frac{u+d}{2} = 1$, $\frac{u-d}{2} = (0{,}0225)^{1/2} \Rightarrow u = 2 - d$, $1 - d = (0{,}0225)^{1/2} \Rightarrow d = 1 - (0{,}0225)^{1/2}$; $u = 1 + (0{,}0225)^{1/2}$



(Stock Price)



(Call option price)

By Lemma 2.3.1 in lecture notes, the price of a call option C_{t_0} is the net present value of the portfolio with value $C_{t_0+\Delta t} - \delta V_0 \cdot u X$ at $t = t_0 + \Delta t$ where $X \in \{-1, 1\}$ with pmf $(\frac{1}{2}, \frac{1}{2})$ and $\delta > 0$ such that value of the portfolio is not affected by the underlying share price.

Stock price at $t = 0$ is $J = £41$. Then

$A = £41 \cdot (1 + (0{,}0225)^{1/2})^3 \Rightarrow$, $A' = \max(A - £44, 0) \approx 18{,}36$

$B = £41 \cdot (1 + (0{,}0225)^{1/2})^2 (1 - (0{,}0225)^{1/2})$ $\quad B' = \max(B - £44, 0) \approx 2{,}09$

$C = £41 \cdot (1 + (0{,}0225)^{1/2}) (1 - (0{,}0225)^{1/2})^2$ $\quad C' = \max(C - £44, 0) = 0$

$D = £41 (1 - (0{,}0225)^{1/2})^3$ $\quad D' = \max(D - £44, 0) = 0$

$\Leftrightarrow E' = \frac{A' - B'}{u - d} + \frac{uB' - dA'}{e^{0.02}(u-d)} = \frac{18{,}36 - 2{,}09}{0{,}3} + \frac{1{,}15 \cdot 2{,}09 - 0{,}85 \cdot 18{,}36}{e^{0.02} \cdot 0{,}3} \approx 54{,}24 - 42{,}71 = 11{,}53$ pounds

$F' = \frac{B' - C'}{u - d} + \frac{uC' - dB'}{e^{0.02}(u-d)} = \frac{2{,}09}{0.3} + \frac{1{,}15 \cdot 0 - 0{,}85 \cdot 2{,}09}{e^{0.02} \cdot 0{,}3} \approx 6{,}97 - 5{,}80 = 1{,}17$ pounds

$G' = 0$

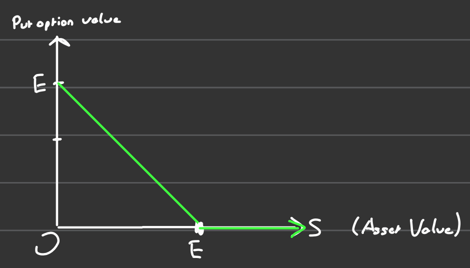
$H' = \frac{E' - F'}{0.3} + \frac{uF' - dE'}{e^{0.02} \cdot 0{,}3} = \frac{11{,}53 - 1{,}17}{0.3} + \frac{1{,}15 \cdot 1{,}17 - 0{,}85 \cdot 11{,}53}{e^{0.02} \cdot 0{,}3} \approx 34{,}53 - 27{,}63 = 6{,}90$ pounds

$I' = \frac{F' - G'}{0.3} + \frac{uG' - dF'}{e^{0.02} \cdot 0{,}3} = \frac{1{,}17}{0{,}3} - \frac{0{,}85 \cdot 1{,}17}{e^{0.02} \cdot 0{,}3} \approx 3{,}90 - 3{,}25 = 0{,}65$ pounds

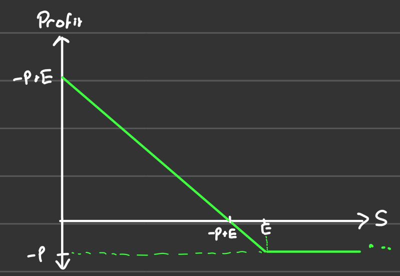
And finally, $J' = \frac{H' - I'}{0.3} - \frac{1{,}15 \cdot I' - 0{,}85 \cdot H'}{e^{0.02} \cdot 0{,}3} = \frac{6{,}90 - 0{,}65}{0{,}3} - \frac{1{,}15 \cdot 0{,}65 - 0{,}85 \cdot 6{,}90}{e^{0.02} \cdot 0{,}3} \approx 20{,}84 - 16{,}72 = 4{,}12$ pounds

3. Derive the Black-Scholes formula for the European Put option. Proceed along the lines of the corresponding calculations for the European Call option presented in the lectures, starting out from the Black-Scholes PDE with the payoff for the European Put option as final condition. **Show all steps and working.** [30 marks]

Say we bought a put option for £P_0. Then we can sell an asset of price S with exercise price E. If S is lower than E, we will sell the stock for E. If S is higher, then we can sell the option for S instead of E. For a fair trade, the amount of exchange at maturity time should be zero. Hence, $P_t = \max(E - S(t), 0)$;



$\Rightarrow$



Let S_t, $t \in [0,T]$, $T \geq 0$ represent the value of an asset which follows the assumption $\ln\left(\frac{S_{\Delta t}}{S_0}\right) \sim \mathcal{N}(\mu\Delta t, \sigma^2 \Delta t)$ where $\mu\Delta t \in \mathbb{R}$ is the drift rate, $\sigma\sqrt{\Delta t} \in (0,\infty)$ is the volatility.

For the value of a put option, we can use binomial model on $S_{\Delta t}$ to discretize the domain of P_0;

$u := 1 + \mu\Delta t + \sigma\sqrt{\Delta t}$ and $d = 1 + \mu\Delta t - \sigma\sqrt{\Delta t}$ are the chosen coefficients for $S_{\Delta t}$: $S_{\Delta t} \in \{S_0 \cdot u, S_0 \cdot d\}$ with pmf $\mathbb{P}(S_{\Delta t} = S_0 u) = \mathbb{P}(S_{\Delta t} = S_0 d) = \frac{1}{2}$. The derivation for (u,d) in the parameter space $(0,\infty)^2$ comes from approximating $\ln\left(1 + \frac{S_{\Delta t} - S_0}{S_0}\right) \approx \frac{S_{\Delta t} - S_0}{S_0}$ over small time intervals by the help of Taylor expansion, and then solving the equations between the means and variances of binomial and log-normal return models.

1-Step Binomial Tree for Put Option Price

P_0 $(t=0)$ → $P^+ = \max(E - S_0 u, 0)$, $P^- = \max(E - S_0 d, 0)$ $(t = \Delta t)$

(Put option)

$\Rightarrow$ $P_0 + \delta S_0$ $(t=0)$ → $P^+ + \delta S_0 u$, $P^- + \delta S_0 d$ $(t = \Delta t)$

(Value at hand)

We buy a put option and buy stocks of amount δ so that the net present value of this portfolio at $t = \Delta t$ is not affected by ΔS.

Self Thought (If S goes up, then the long position should cover the cost of the put option. If S goes down, then the return from exercising the put option should cover the value loss in the long position.)

Assume that we have risk-free interest rate r. Then the present value of any asset X over k periods is $X \cdot e^{-k \cdot r\Delta t}$

$$\Rightarrow P_0 + \delta S_0 = e^{-r\Delta t}(P^+ + \delta S_0 u) = e^{-r\Delta t}(P^- - \delta S_0 d)$$

$$\Rightarrow P^+ + \delta S_0 u = P^- + \delta S_0 d \quad \Rightarrow P^+ - P^- = \delta S_0 (d-u) \Rightarrow \delta = -\frac{P^+ - P^-}{S_0(u-d)}$$

$$\Rightarrow P_0 + \delta S_0 = P_0 - \frac{P^+ - P^-}{u-d} = e^{-r\Delta t}\left(P^+ - \frac{P^+ - P^-}{S_0(u-d)} S_0 u\right) = e^{-r\Delta t}\left(P^+ - \frac{uP^+ - uP^-}{u-d}\right) = \frac{P^+u - P^+d - uP^+ + uP^-}{e^{r\Delta t}(u-d)} = \frac{uP^+ - dP^-}{e^{r\Delta t}(u-d)}$$

$$\Rightarrow P_0 = \frac{P^+ - P^-}{u-d} + \frac{uP^- - dP^+}{e^{r\Delta t}(u-d)}$$

Now, fix $(S_0, t_0) \in [0,\infty) \times [0,T]$ and rewrite P^+ and P^- in terms of (S_0, t_0).

$\Rightarrow P^+ = P(S_0 u, t_0 + \Delta t) = P(S_0 + (u-1)S_0, t_0 + \Delta t)$

$\Rightarrow P^- = P(S_0 d, t_0 + \Delta t) = P(S_0 + (d-1)S_0, t_0 + \Delta t)$

For small Δt and under smoothness assumptions, Taylor Expansion is applicable to P:

$$P^+ = P(S_0 + (u-1)S_0, t_0 + \Delta t) = P(S_0, t_0) + \frac{\partial P}{\partial S}(u-1)S_0 + \frac{\partial P}{\partial t}\Delta t + \frac{1}{2!}\frac{\partial^2 P}{\partial S^2}(u-1)^2 S_0^2 + O(\Delta t^2)$$

$$= P(S_0, t_0) + \frac{\partial P}{\partial S}(\mu\Delta t + \sigma\sqrt{\Delta t})S_0 + \frac{\partial P}{\partial t}\Delta t + \frac{1}{2!}\frac{\partial^2 P}{\partial S^2}(\mu^2\Delta t^2 + 2\mu\Delta t\sigma\sqrt{\Delta t} + \sigma^2\Delta t)S_0^2 + O(\Delta t^2)$$

$$P^- = P(S_0 + (d-1)S_0, t_0 + \Delta t) = P(S_0, t_0) + \frac{\partial P}{\partial S}(\mu\Delta t - \sigma\sqrt{\Delta t})S_0 + \frac{\partial P}{\partial t}\Delta t + \frac{1}{2!}\frac{\partial^2 P}{\partial S^2}(\mu^2\Delta t^2 - 2\mu\Delta t\sigma\sqrt{\Delta t} + \sigma^2\Delta t)S_0^2 + O(\Delta t^2)$$

Now the derivation of P_0 will be done by using the equation $P_0 = \frac{P^+ - P^-}{u-d} + \frac{uP^- - dP^+}{e^{r\Delta t}(u-d)}$: $e^{r\Delta t}P_0 = e^{r\Delta t}\frac{P^+ - P^-}{u-d} + \frac{uP^- - dP^+}{u-d}$

Since Δt is set to be small and $e^x \in C^\infty(\mathbb{R})$, $e^{r\Delta t} \approx 1 + r\Delta t$ via Taylor expansion. Hence, $(1+r\Delta t)P_0 = (1+r\Delta t)\frac{P^+ - P^-}{2\sigma\sqrt{\Delta t}} + \frac{uP^- - dP^+}{2\sigma\sqrt{\Delta t}}$.
Rearrangement of this equation will yield the BSE for Put option at $t=0$, i.e. $P_0 = P(S_0, t_0)$

i)
$$P^+ - P^- = \left(P(S_0, t_0) + \frac{\partial P}{\partial S}(\mu\Delta t + \sigma\sqrt{\Delta t})S_0 + \frac{\partial P}{\partial t}\Delta t + \frac{1}{2!}\frac{\partial^2 P}{\partial S^2}(\mu^2\Delta t^2 + 2\mu\Delta t\sigma\sqrt{\Delta t} + \sigma^2\Delta t)S_0^2 + O(\Delta t^2)\right)$$

$$- \left(P(S_0, t_0) + \frac{\partial P}{\partial S}(\mu\Delta t - \sigma\sqrt{\Delta t})S_0 + \frac{\partial P}{\partial t}\Delta t + \frac{1}{2!}\frac{\partial^2 P}{\partial S^2}(\mu^2\Delta t^2 - 2\mu\Delta t\sigma\sqrt{\Delta t} + \sigma^2\Delta t)S_0^2 + O(\Delta t^2)\right)$$

$$P^+ - P^- = 2\sigma\sqrt{\Delta t}\, S_0 \cdot \frac{\partial P}{\partial S} + \frac{1}{2!}S_0^2 \cdot 4\mu\sigma(\sqrt{\Delta t})^3\frac{\partial^2 P}{\partial S^2}$$

$$\Rightarrow \frac{P^+ - P^-}{u-d} = \frac{P^+ - P^-}{2\sigma\sqrt{\Delta t}} = S_0\frac{\partial P}{\partial S} + S_0^2\mu\Delta t\frac{\partial^2 P}{\partial S^2}$$

$$\Rightarrow (1+r\Delta t)\frac{P^+ - P^-}{u-d} = S_0\frac{\partial P}{\partial S} + S_0^2\mu\Delta t\frac{\partial^2 P}{\partial S^2} + rS_0\Delta t\frac{\partial P}{\partial S} + rS_0^2\mu(\Delta t)^2\frac{\partial^2 P}{\partial S^2}$$

Remark:
$$(1+\mu\Delta t + \sigma\sqrt{\Delta t})(\mu\Delta t - \sigma\sqrt{\Delta t}) = \mu\Delta t - \sigma\sqrt{\Delta t} + \mu^2\Delta t^2 - \mu\sigma(\sqrt{\Delta t})^3 + \mu\sigma(\sqrt{\Delta t})^3 - \sigma^2\Delta t = (\mu\Delta t - \sigma\sqrt{\Delta t}) + (\mu^2\Delta t^2 - \sigma^2\Delta t)$$
$$- (1+\mu\Delta t - \sigma\sqrt{\Delta t})(\mu\Delta t + \sigma\sqrt{\Delta t}) = \mu\Delta t + \sigma\sqrt{\Delta t} + \mu^2\Delta t^2 + \mu\sigma(\sqrt{\Delta t})^3 - \mu\sigma(\sqrt{\Delta t})^3 - \sigma^2\Delta t = (\mu\Delta t + \sigma\sqrt{\Delta t}) + (\mu^2\Delta t^2 - \sigma^2\Delta t)$$
$$= -2\sigma\sqrt{\Delta t}$$

Remark:
$$(1+\mu\Delta t + \sigma\sqrt{\Delta t})(\mu^2\Delta t^2 - 2\mu\sigma(\sqrt{\Delta t})^3 + \sigma^2\Delta t) =$$
$$- (1+\mu\Delta t - \sigma\sqrt{\Delta t})(\mu^2\Delta t^2 + 2\mu\sigma(\sqrt{\Delta t})^3 + \sigma^2\Delta t)$$
$$-4\mu\sigma(\sqrt{\Delta t})^3 - 4\mu^2\sigma(\sqrt{\Delta t})^5 + 2\sigma\mu^2(\sqrt{\Delta t})^5 + 2\sigma^3(\sqrt{\Delta t})^3$$

ii)
$$uP^- = (1+\mu\Delta t + \sigma\sqrt{\Delta t})P(S_0, t_0) + (1+\mu\Delta t + \sigma\sqrt{\Delta t})(\mu\Delta t - \sigma\sqrt{\Delta t})S_0\frac{\partial P}{\partial S} + (1+\mu\Delta t + \sigma\sqrt{\Delta t})\Delta t\frac{\partial P}{\partial t}$$
$$+ (1+\mu\Delta t + \sigma\sqrt{\Delta t})(\mu^2\Delta t^2 - 2\mu\sigma(\sqrt{\Delta t})^3 + \sigma^2\Delta t)\frac{1}{2}S_0^2\frac{\partial^2 P}{\partial S^2} + O(\Delta t^2)$$

$$dP^+ = (1+\mu\Delta t - \sigma\sqrt{\Delta t})P(S_0, t_0) + (1+\mu\Delta t - \sigma\sqrt{\Delta t})(\mu\Delta t + \sigma\sqrt{\Delta t})S_0\frac{\partial P}{\partial S} + (1+\mu\Delta t - \sigma\sqrt{\Delta t})\Delta t\frac{\partial P}{\partial t}$$
$$- \quad + (1+\mu\Delta t - \sigma\sqrt{\Delta t})(\mu^2\Delta t^2 + 2\mu\sigma(\sqrt{\Delta t})^3 + \sigma^2\Delta t)\frac{1}{2}S_0^2\frac{\partial^2 P}{\partial S^2} + O(\Delta t^2)$$

$$uP^- - dP^+ = 2\sigma\sqrt{\Delta t} \cdot P(S_0, t_0) - 2\sigma\sqrt{\Delta t}\, S_0\frac{\partial P}{\partial S} + 2\sigma(\sqrt{\Delta t})^3\frac{\partial P}{\partial t} + \left[-4\mu\sigma(\sqrt{\Delta t})^3 + 2\sigma^3(\sqrt{\Delta t})^3\right]\frac{1}{2}S_0^2\frac{\partial^2 P}{\partial S^2} + O((\sqrt{\Delta t})^3)$$

$$\Rightarrow \frac{uP^- - dP^+}{2\sigma\sqrt{\Delta t}} = P(S_0, t_0) - S_0\frac{\partial P}{\partial S} + \Delta t\frac{\partial P}{\partial t} + (-2\mu\sigma\Delta t + \sigma^2\Delta t)\frac{1}{2}S_0^2\frac{\partial^2 P}{\partial S^2} + O((\sqrt{\Delta t})^3)$$

$$\Rightarrow (1+r\Delta t)\, P_0 = (1+r\Delta t)\frac{P^+ - P^-}{2\sigma\sqrt{\Delta t}} + \frac{uP^- - dP^+}{2\sigma\sqrt{\Delta t}}$$

$$= S_0\frac{\partial P}{\partial S} + S_0^2\mu\sigma\Delta t\frac{\partial^2 P}{\partial S^2} + rS_0\Delta t\frac{\partial P}{\partial S} + rS_0^2\mu(\Delta t)^2\frac{\partial^2 P}{\partial S^2} + P_0 - S_0\frac{\partial P}{\partial S} + \Delta t\frac{\partial P}{\partial t} + \left(-\mu\sigma\Delta t + \frac{1}{2}\sigma^2\Delta t\right)S_0^2\frac{\partial^2 P}{\partial S^2} + O((\Delta t)^3)$$

$$\Rightarrow r\Delta t\cdot P_0 = \underbrace{S_0\frac{\partial P}{\partial S} - S_0\frac{\partial P}{\partial S}}_{=0} + \underbrace{\mu\sigma\Delta t\, S_0^2\frac{\partial^2 P}{\partial S^2} - \mu\sigma\Delta t\, S_0^2\frac{\partial^2 P}{\partial S^2}}_{=0} + rS_0\Delta t\frac{\partial P}{\partial S} + \Delta t\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S_0^2\Delta t\frac{\partial^2 P}{\partial S^2} + O((\Delta t)^{3/2})$$

$$\Rightarrow rP_0 = rS_0\frac{\partial P}{\partial S} + \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S_0^2\frac{\partial^2 P}{\partial S^2} + O(\sqrt{\Delta t})$$

As $\Delta t \to 0$, $\sqrt{\Delta t} \to 0$. Hence, $O(\sqrt{\Delta t})$ vanishes in the differential equation.

Hence, we obtained the Black-Scholes equation for a Put option with final condition

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0 \quad , \quad 0 < S < \infty,\ 0 < t < T$$

$$P(S(T), T) = \max(E - S(T), 0), \quad 0 < S(T) < \infty$$

4. Consider the UBS Discount Certificate on the Allianz share, with details given in the attached product information.

(a) Construct a portfolio consisting of options and underlying Allianz shares which represents (i.e. has equivalent cash flows at maturity) this Discount Certificate. What are the exercise price, initial Call option price, and maturity of the option(s)?

is this needed here?

(b) Plot the payoff diagram and the profit diagram of the Discount Certificate.

(c) On 20/03/14, the Discount Certificate and the Allianz share were traded for EUR 116.40 and EUR 121.30, respectively. How would you advise an investor (regarding opportunities and risks of the two investments) who is considering to either buy the Discount Certificate or invest in the Allianz share directly on this day?

1. Description of the Product

Information on Underlying

Underlying(s)	Initial Underlying Level	Cap Level	Conversion Ratio
Allianz SE Bloomberg: ALV GY / Valor: 322646	EUR 121.35	EUR 145.50 (119.90%)	1:1

Product Details

Security Numbers	Valor: 23638559 / ISIN: CH0236385594 / WKN: US3CDU
SIX Symbol	BLOALH
Issue Size	up to 8,650 Units (with reopening clause)
Issue Price	EUR 115.78 (unit quotation)
Settlement Currency	EUR
Discount	4.59%
Maximum Return	25.67% (31.65% p.a.)

Dates

Launch Date	03 February 2014
Pricing Date ("Pricing")	03 February 2014
First SIX Trading Date (anticipated)	12 February 2014
Payment Date (Issue Date)	12 February 2014
Last Trading Day/Time	28 November 2014 / 17:15 CET
Expiration Date ("Expiry")	28 November 2014 (subject to Market Disruption Event provisions)
Redemption Date	05 December 2014 (subject to Market Disruption Event provisions)

a) The UBS Discount Certificate, given above, is an agreement between the holder and the writer where the holder is promised to be given an Allianz SE stock iff the value of the stock is less than €145.50, and €145.50 iff the value of the stock is higher than €145.50.

Let A denote the value of one Allianz SE stock, $C = C(A(t), t)$ denote the Call option price for Allianz SE, $P = P(A(t), t)$ denote the Put option price for Allianz SE, E_C denote the exercise price for the Call option C, E_P the Put option counterpart.
Our aim is to find a portfolio $\alpha = (\alpha_A, \alpha_C, \alpha_P)$, $\alpha_A, \alpha_C, \alpha_P \geq 0$, comprised of Allianz SE shares, and call and put options for Allianz SE shares, in the given order, such that at the expiry date of the discount certificate, the portfolio α will result in the same amount of money flow with the exchange due to one discount certificate.

Remark: $t \in [0, T]$, $T \approx 28\text{-}11\text{-}2014$, $t \approx 0$ on $03\text{-}02\text{-}2014$

General Portfolio Scheme

i) Assume at maturity time T, $A(T) < E_C$. Then the Call option is not exercised. Hence, no change in the portfolio.
Otherwise, $E_C \leq A(T) \Rightarrow$ Call option exercised

ii) Assume at maturity time T, $A(T) < E_P$. Then the Put option is exercised.
Otherwise, $E_P \leq A(T) \Rightarrow$ Put option is not exercised. No change in the portfolio.

Case I: $E_c \le E_p$

E_c and E_p mark the breakpoints along the $A(T)$ axis.

Portfolio at $t=T$	$(\alpha_A - \alpha_p, \alpha_c, 0)$	$(\alpha_A + \alpha_c - \alpha_p, 0, 0)$	$(\alpha_A + \alpha_c, 0, \alpha_p)$
Profit	$\alpha_p \cdot E_p - \alpha_p \cdot A(T)$ $= \alpha_p(E_p - A(T))$	$(\alpha_c \cdot A(T) - \alpha_c E_c) + (\alpha_p \cdot E_p - \alpha_p A(T))$ $= \alpha_p E_p - \alpha_c E_c + (\alpha_c - \alpha_p) A(T)$	$\alpha_c A(T) - \alpha_c E_c$ $= \alpha_c (A(T) - E_c)$

Case II: $E_p < E_c$

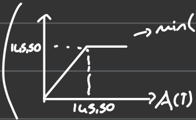
E_p and E_c mark the breakpoints along the $A(T)$ axis.

Portfolio at $t=T$	$(\alpha_A - \alpha_p, \alpha_c, 0)$	$(\alpha_A, \alpha_c, \alpha_p)$	$(\alpha_A + \alpha_c, 0, \alpha_p)$
Value Change	$\alpha_p(E_p - A(T))$	0	$\alpha_c(A(T) - E_c)$

$\Rightarrow$ For a bountiful trade, one needs high E_p and low E_c.

Our aim is to investigate the discount certificate via equivalency with a portfolio $(\alpha_A, \alpha_c, \alpha_p)$. Ownership of a certificate guarantees that stocks will be returned if $A(T) \le €145.50$ and otherwise, each certificate is worth €145.50.

$$\Rightarrow 1 \text{ Certificate} = \begin{cases} 1 \text{ share}, & A(T) \le €145.50 \\ €145.50, & \text{o/w} \end{cases}$$



$\rightarrow \min(145.50, A(T)) = -\max(-145.50, -A(T))$

With one Call option, $A(T) \le E_c \Rightarrow$ Holder doesn't exercise the call option

$E_c < A(T) \Rightarrow$ Holder exercises the option $\Rightarrow$ Writer obtains E_c and gives one share to the holder

Hence, the owner of one certificate can be considered as writer of a Call option with exercise price €145.50:

$\rightarrow A(T) < 145.50 \Rightarrow$ Holder doesn't exercise the Call option $\Rightarrow$ Holder doesn't buy the share $\Rightarrow$ Writer has one share and since having a share is equivalent to being given one, case follows

$\rightarrow A(T) \ge 145.50 \Rightarrow$ Holder exercises the Call option $\Rightarrow$ Holder buys one share for €145,50 and gets one share $\Rightarrow$ Writer gives one share and obtains €145,50.

Hence, owning one certificate is equivalent to writing a Call option for one's long position. In other words,

$$(\alpha_A, \alpha_c, \alpha_p) = (1, -1, 0)$$

Buying one certificate, then, is of equivalent value with buying one asset and selling one Call option. Set $U = U(t)$ to be the value of one certificate at time $t \in [0, T]$. By the equivalency, we have

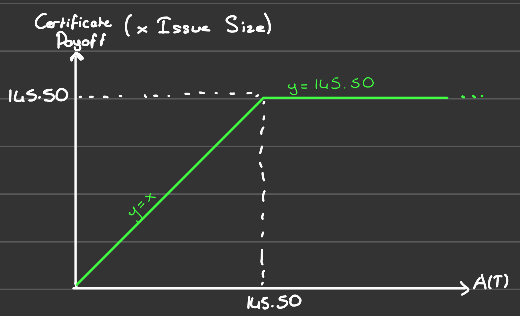
$$\forall t \in [0,T] \quad U(t) = A(t) - C(A(t),t) \implies C(A(t),t) = A(t) - U(t) \quad \text{for all } t \in [0,T].$$

Hence, to find the initial Call option value, we will plug in $t=0$:

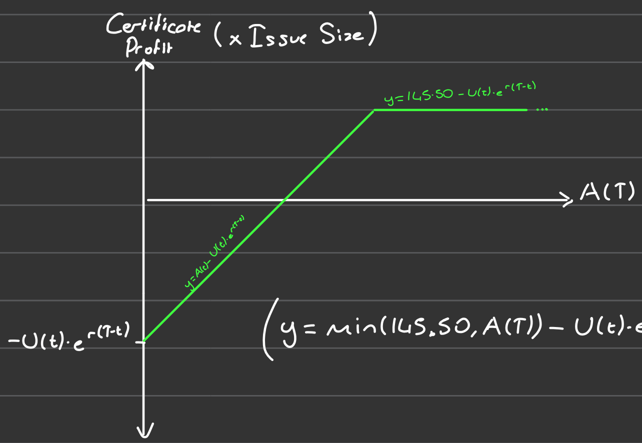
$$C(A(0),0) = A(0) - U(0) = 121{,}35 - 115{,}78 = 5{,}57$$

b) Payoff is exactly as given in the Redemption subsection of the first section.

To calculate the profit we have to be careful. Instead of putting $U(t)$ into risk-free bonds or into a bank account at time t, we spend the money to buy the certificate. So, at time T, when the payoff and profit is decided, the value of $U(t)$ becomes $U(t) \cdot e^{r(T-t)}$ at time T.



$$\left(y = \min(145.50, A(T))\right)$$



$$\left(y = \min(145.50, A(T)) - U(t) \cdot e^{r(T-t)}\right)$$

c) For $t=33$, we are given that $U(t) = 116.40$ and $A(t) = 121.30$. From the certificate we read $U(0) = 115.78$ and $A(0) = 121.35$.

Remark: $T = 205$ (206 Business days from $t=0$ to $t=T$)

From day 0 to day 33, the entry price $U(t)$ increased and the stock value $A(t)$ fell down. The change in the potential profit from the certificate is then

$$-\left(U(33)e^{r(205-33)} - U(0)e^{r \cdot 205}\right) = -e^{r \cdot 205}\left(116.40 - 115.78 \cdot e^{-33r}\right)$$

assuming that r is given to be interest rate per work day.

The change in the value of A from $t=0$ to $t=33$ seems too incremental, but still could be useful for a projection. Say every 33 days, the value drops by 0.05. The value will drop by 0.05 six times until the expiration date, and some days will remain until the expiry date comes. Say the value drops once more in those days, to be pedantic. Then the projection tells us that

$$A(T) = 121{,}40 - 7 \cdot 0{,}05 = 121{,}05 > 116.40 = U(33)$$

The potential profit is lower, but still prominent; return of the certificate is at least $\frac{121,05 - 116,40}{116,40} = 0,04 = 4\%$ and at most $\frac{145,50 - 116,40}{116,40} = 0,25 = 25\%$

The act of buying a stock may increase the value of the discount certificate

Also, as $t \to T$, $U(t)$ has to converge to $A(T)$. $U(T) \leq A(T)$ is clear; $U(t)$ is the value of a discount certificate at the end. If $U(T) < A(T)$, then an investor would buy the certificates to exercise them immediately, contradicting the no-arbitrage principle.

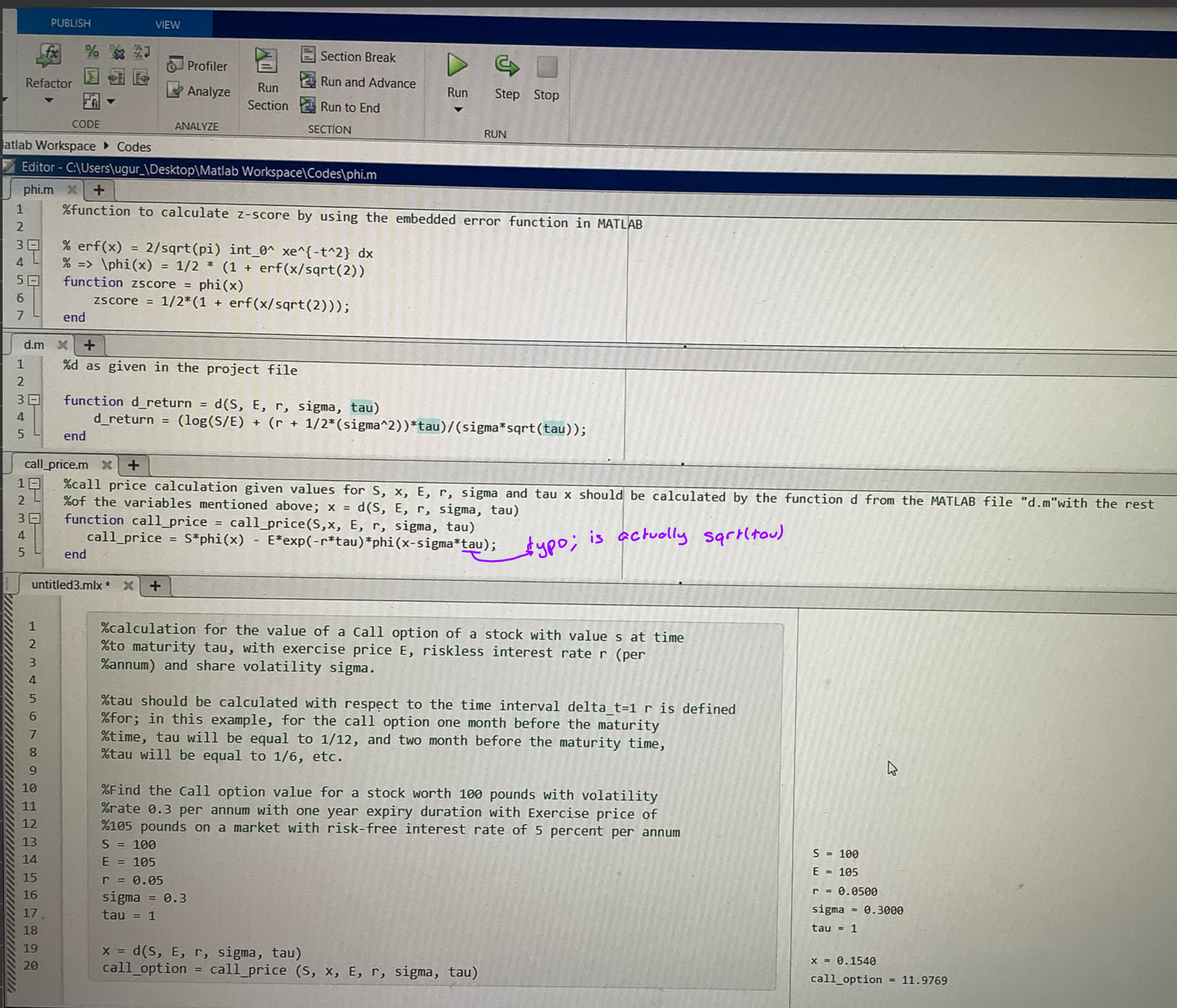
In light of these information, it makes sense to buy discount certificates instead of buying the assets, unless there is strong evidence for $A(T) > 145.50$.

5. (a) Write a MATLAB program to evaluate the Black-Scholes formula (1) for a European Call option using appropriate special functions of MATLAB. **Your code should be fully explained with comments.**

(b) Produce three plots, each showing three curves that compare how changes to parameters affect option prices. **Your plots should be clear, with appropriate titles, labels, legends etc.** Use the following combinations of parameters:

- $E = 35$, $E = 55$, $E = 75$, with $\sigma = 0.15$, $r = 4\%$ (per annum), and time four months to maturity.
- $\sigma = 0.05$, $\sigma = 0.15$ and $\sigma = 0.3$, with $E = 55$, $r = 4\%$ (per annum), and time four months to maturity.
- $r = 1\%$, $r = 4\%$ and $r = 10\%$, with $E = 55$, $\sigma = 0.15$, and time four months to maturity.

(c) Describe how the option price changes in each of these cases. Give a financial interpretation and a mathematical explanation for the changes. [1



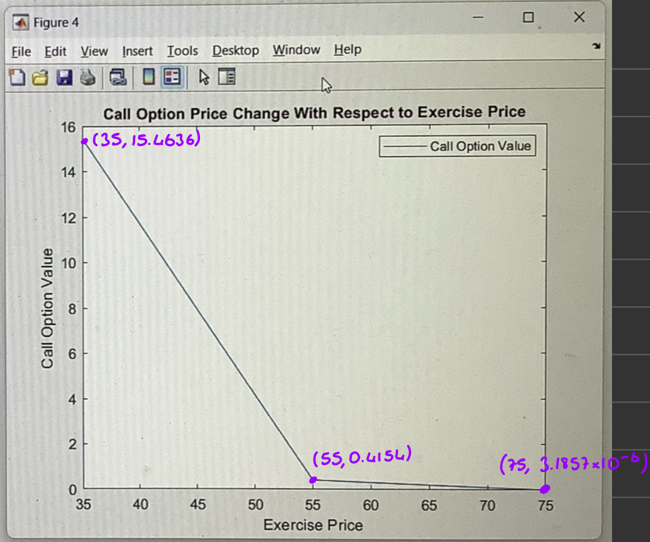
plots_live_script.mlx | plots_script.m | plot1_script.m | plot2_script.m | call_option_live_script.mlx | plot3_script.m | call_price.m | d.m

```matlab
S=50;
E=35;
r=0.04;
sigma=0.15;
tau=1;

%plot 1

%call_price necessary
C_E_1 = call_price(S,35, 0.04, 0.15, 1/3);
C_E_2 = call_price(S,55, 0.04, 0.15, 1/3);
C_E_3 = call_price(S,75, 0.04, 0.15, 1/3);

E_1 = [35 55 75];
E_2 = [C_E_1 C_E_2 C_E_3];
plot(E_1, E_2)
title("Call Option Price Change With Respect to Exercise Price")
xlabel("Exercise Price")
ylabel("Call Option Value")
legend("Call Option Value")
```



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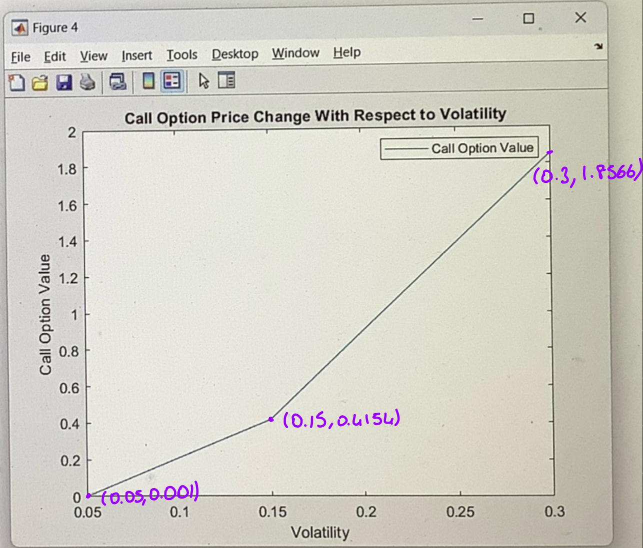
plots_live_script.mlx | plots_script.m | plot1_script.m | plot2_script.m | call_option_live_script.mlx | plot3_script.m | call_price.m | d.m

```matlab
S=50;
E=35;
r=0.04;
sigma=0.15;
tau=1;


%plot 2
C_sigma_1 = call_price(S,55, 0.04, 0.05, 1/3);
C_sigma_2 = call_price(S,55, 0.04, 0.15, 1/3);
C_sigma_3 = call_price(S,55, 0.04, 0.3, 1/3);

sigma_1 = [0.05 0.15 0.3];
sigma_2 = [C_sigma_1 C_sigma_2 C_sigma_3];
plot(sigma_1, sigma_2)
title("Call Option Price Change With Respect to Volatility")
xlabel("Volatility")
ylabel("Call Option Value")
legend("Call Option Value")
```



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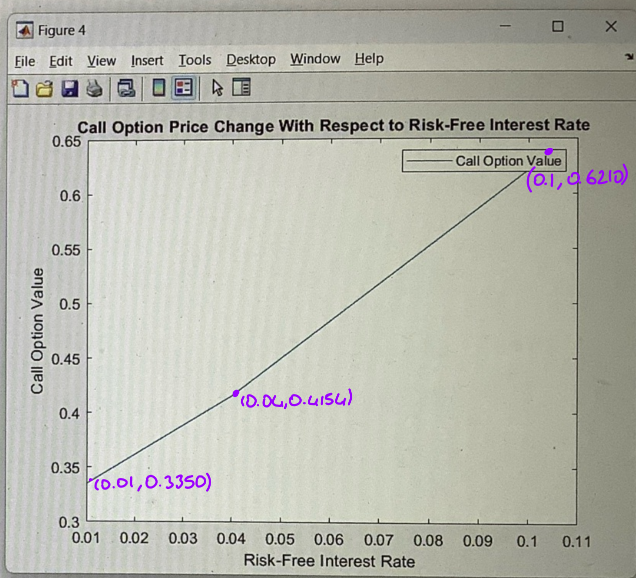
plots_live_script.mlx | plots_script.m | plot1_script.m | plot2_script.m | call_option_live_script.mlx | plot3_script.m | call_price.m | d.m

```matlab
S=50;
E=35;
r=0.04;
sigma=0.15;
tau=1;

%plot 3
C_r_1 = call_price(S,55, 0.01, 0.15, 1/3);
C_r_2 = call_price(S,55, 0.04, 0.15, 1/3);
C_r_3 = call_price(S,55, 0.1, 0.15, 1/3);


r_1 = [0.01 0.04 0.1];
r_2 = [C_r_1 C_r_2 C_r_3];

plot(r_1, r_2)
title("Call Option Price Change With Respect to Risk-Free Interest Rate")
xlabel("Risk-Free Interest Rate")
ylabel("Call Option Value")
legend("Call Option Value")
```



c) The effect of change can be obtained by simply looking at partial derivatives of the Call option formula.

<u>Recall:</u> $C(S,t;E,r,\sigma,T) = S\cdot\Phi(d(S,t)) - E\cdot e^{-r(T-t)}\Phi(d(S,t)-\sigma\sqrt{T-t})$, $d(S,t) = \dfrac{\ln\left(\frac{S}{E}\right) + \left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$

$$\Rightarrow \frac{\partial d}{\partial E} = \frac{-1}{E\sigma\sqrt{T-t}}$$

$$\Rightarrow \frac{\partial C}{\partial E} = S\cdot\Phi'(d)\frac{\partial d}{\partial E} - e^{-r(T-t)}\left[\Phi(d-\sigma\sqrt{T-t}) + E\cdot\Phi'(d-\sigma\sqrt{T-t})\frac{\partial d}{\partial E}\right]$$

$$= -\frac{S}{E}\frac{1}{\sigma\sqrt{2\pi}\sqrt{T-t}}e^{-\frac{d^2}{2}} - e^{-r(T-t)}\Phi(d-\sigma\sqrt{T-t}) + \frac{1}{\sigma\sqrt{2\pi}\sqrt{T-t}}e^{-\frac{d^2}{2}+\ln\frac{S}{E}+\left(r+\frac{\sigma^2}{2}\right)(T-t)-\frac{\sigma^2}{2}(T-t)}$$

$$= -\frac{S}{E}\frac{1}{\sigma\sqrt{2\pi}\sqrt{T-t}}e^{-\frac{d^2}{2}} - e^{-r(T-t)}\Phi(d-\sigma\sqrt{T-t}) + \frac{S}{E}\frac{1}{\sigma\sqrt{2\pi}\sqrt{T-t}}e^{-\frac{d^2}{2}+r(T-t)}$$

$$= \overset{>0}{\frac{S}{E}}\underbrace{\frac{1}{\sigma\sqrt{2\pi}\sqrt{T-t}}}_{>0}\underbrace{e^{-\frac{d^2}{2}}}_{>0}\underbrace{\Big(\underbrace{\overbrace{e^{r(T-t)}}^{>0}}_{>1}-1\Big)}_{>0} - \underbrace{e^{-r(T-t)}\Phi(d-\sigma\sqrt{T-t})}_{>0}$$

In plot 1, we see that as E increased, the value of the call option fell down drastically at first, and then a smaller decrease happened. A factor of $\frac{1}{E}$ is present in the positive term $\frac{S}{E}\cdot\frac{\left(e^{r(T-t)}-1\right)}{\sigma\sqrt{2\pi}\sqrt{T-t}}e^{-\frac{d^2}{2}}$, hinting a logarithmic curve

$$d^2 = \left(\frac{\ln S - \ln E + \left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)^2 = \frac{\left((-\ln E) + \left(\ln S + \left(r+\frac{1}{2}\sigma^2\right)(T-t)\right)\right)^2}{\sigma^2(T-t)} = \frac{(\ln E)^2 - 2\ln E\cdot\left(\ln S + \left(r+\frac{1}{2}\sigma^2\right)(T-t)\right) + \left(\ln S + \left(r+\frac{1}{2}\sigma^2\right)(T-t)\right)^2}{\sigma^2(T-t)}$$

$\approx \underbrace{\ln E\cdot(\ln E - 2)}_{>0 \text{ for } E>e^2 \text{ and } E=35,55,75}$, so E increases $\Rightarrow$ d^2 increases $\Rightarrow$ $\frac{-d^2}{2}$ decreases $\Rightarrow$ $e^{-\frac{d^2}{2}}$ decreases ()

Hence, as E increases, $\frac{S}{E}\cdot\frac{\left(e^{r(T-t)}-1\right)}{\sigma\sqrt{2\pi}\sqrt{T-t}}e^{-\frac{d^2}{2}}$ decreases. Similarly, E increases $\Rightarrow$ d decreases $\Rightarrow$ $\Phi(d-\sigma\sqrt{T-t})$ decreases

Financially, the picture is clear. For the same stock value $S>E$, an investor obviously prefers a lower exercise price; the payoff is exactly $S-E$ in this scenario, and higher profit is the aim.
For the payoff function, $S>E$ implies $\frac{\partial P}{\partial E}(E) = -1$. This automatically effects the call option; if I'm going to have higher exercise price for the same asset, at least the initial investment, i.e. the value of the Call option should be lower.

$$\frac{\partial C}{\partial \sigma} = \frac{S(\sqrt{T-t}-d)}{\sigma\sqrt{2\pi}}e^{-\frac{d^2}{2}}$$

The Call option is a security because it protects the owner from loss when the underlying share's value drops below a certain price, i.e. the exercise price. High volatility in a stock value implies higher possibility for bigger changes in the value, upwards and downward. Hence, higher volatility implies potentially higher profits with zero cost, given that other variables are fixed

$\frac{\partial C}{\partial r} = E(T-t)e^{-r(T-t)}\Phi(d-\sigma\sqrt{T-t}) > 0$. As r increases, d increases, so $\Phi(d-\sigma\sqrt{T-t})$ increases, but $e^{-r(T-t)}$ decreases.

Financially, if only the interest rate is higher, the return of putting the money into my bank account is higher. So, without even taking in the effect of the interest rate on S, it is clear that the Call option should have more value when every other variable is fixed.