ISL 323E Operations Research
Fall 2023

Assoc. Prof. Dr. Fuat Kosanoğlu

ISL 323E Operations Research

Week I

5 November 2023

Class Overview

- Lecture : Thursday 11:30-14:30, 14:30-17:30
- Office hours: By Appointment

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Coursework Details

- Grading:
  - 20% Homework's
  - 30% Midterm
  - 50% Final

Course Texts

1. **Operations Research: Applications and Algorithms,** Fourth Edition, Wayne L. Winston.
2. **Introduction to Operations Research**, Frederick Hillier and Gerald Lieberman, 8th Ed.
3. **Operations Research: An Introduction**, Eighth Edition Hamdy A. Taha.

Course Objectives

- The ability to write down an algebraic formulation of an optimization model that captures the main decision elements of practical problems.
- The ability to model a variety of basic problems as optimization models using Excel, and to solve them using Excel Solver.
- Basic experience in using an algebraic modeling language to model practical, large-scale problems.
- Understand the simplex method for linear programming

Course Objectives

- Understand the relationship between a linear program and its dual, including concepts such as complementary slackness and strong duality.
- Perform sensitivity analysis to understand how changes in the problem's input impact the optimal solution output.
- Modelling and solving Integer Programming problems
- Modelling transportation, assignment problems

Definition

- Operations Research: A set of mathematical tools for evaluating, optimizing, or making decisions about complex systems.
  - Particularly problems involving scarce resources and lots of combinations.
- Mathematical Model
  - Real life problems are a set of mathematical equations which involves to physical and logical relations.
  - Set of equations defines the problem.
    - Limited resources etc.

Sample OR Problems

- Dr. Atkins's Dilemma (Diet Problem)
  - I'm on a low-carb diet, and Dr. Atkins has recommended that I eat only the following foods:
    - Steak
    - Cheese
    - Chicken wings
    - Salad
    - Sugar-free fruit smoothies
  - But I also want to make sure that I get my USRDA of the following nutrients:
    - Calcium
    - Fiber
    - Vitamin D
    - Potassium

Dr. Atkins's Dilemma (Diet Problem)

- Each food has a certain amount of each nutrient. It also has a certain price.
- **Question:** Find the minimum-cost diet (amount of each food) that satisfies all of the nutrition requirements.
- Example:

Food	Amount	Cost	Calcium	Fiber	Vitamin D	Potassium
Meat	0	0	0%	0%	0%	0%
Cheese	1 kg	$10	90%	2%	64%	12%
Chicken	0	0	0%	0%	0%	0%
Salad	1,5 kg	$7.5	15%	75%	51%	16%
Fruit	0	0	0%	0%	0%	0%
Total		$17.5	105%	77%	115%	28%

Scheduling Problem

- A post office requires different numbers of full-time employees on different days of the week. The number of full-time employees required on each day is given in Table below. Union rules state that each full-time employee must work five consecutive days and then receive two days off. For example, an employee who works Monday to Friday must be off on Saturday and Sunday. The post office wants to meet its daily requirements using only fulltime employees. Formulate an LP that the post office can use to minimize the number of full-time employees who must be hired.

Day	# of Full-time Employee required
Monday	17
Tuesday	13
Wednesday	15
Thursday	19
Friday	14
Saturday	16
Sunday	11

OR Problems in Industry

- Telecommunications network design
- Distribution center location
- Petroleum blending
- Airline scheduling
- Inventory management
- Vehicle routing
- Factory layout
- Driving directions (Googlemap)
- Queue management (Disney World)

OR Headlines

- Industry applications
  - Crew scheduling saves American Airlines $20 million/year
  - Fleet scheduling (assigning aircraft to flight legs) saves Delta $100 million/year
  - Vehicle routing saves Yellow Freight $17 million/year
  - Integrated management of spare parts inventories saves IBM $20 million/year reduces inventory by $250 million
  - Hewlett-Packard increases revenue by $280 million/year by finding optimal sizes and locations of buffers in production lines

OR Headlines

- Public-sector applications:
  - Optimized water management policy in The Netherlands saves $15 million/year
  - San Francisco Police Dept. saves $11 million/year by optimally scheduling and deploying patrol officers
  - South African defense force saves $1.1 billion/year by redesigning size and shape of force and weapons
  - China saves $425 million/year by using OR to choose energy infrastructure projects

OR Headlines

- Find more success stories at:

  https://www.informs.org/Impact/O.R.-Analytics-Success-Stories

Career Opportunities

- Accounting
- Computer services
- Corporate planning
- Economic analysis
- Financial engineering
- Industrial engineering
- Investment analysis
- Logistics/supply chain mgmt.
- Manufacturing
- Market research
- Production engineering
- Transportation

History

- Origins
  - Began in WWII
  - Multi-disciplinary team of scientists trying to
    - Use new radar technology effectively
    - Improve deployment of aircraft
    - Find optimal patterns for searching for submarines and land mines
  - "Operations" = military operations
- 1940s, 1950s, 1960s
  - 1947: Simplex method for linear programs (George Dantzig)
  - Lots of new models
  - Algorithms to solve those models
  - Only small problems for now

History

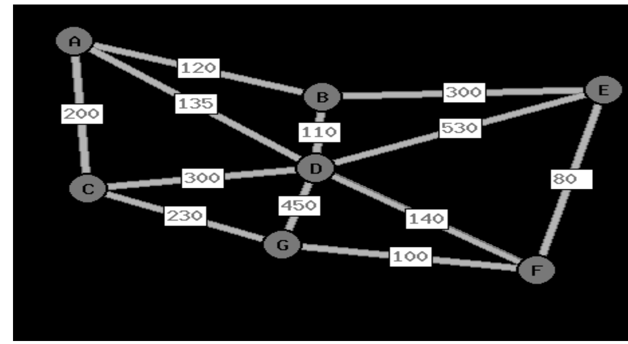
- 1970s
  - Reduced interest in OR
  - More realistic expectations
- 1980s
  - Increasing power of personal computers
  - Increased access to data
  - OR models start to become practical for meaningful problems
- 1990s
  - Rapid improvement in computing and OR technology
  - Makes it worthwhile to study OR algorithms again
  - OR tools become easier to use
    - Spreadsheet applications
    - Simulation packages
    - Modeling languages
    - etc.

History

- 2000s
  - Computing power and algorithm developments make huge problems solvable
  - Effect of e-business
    - Lots of data available
    - Need for automated decision-making
  - Tightening up of business operations (e.g., supply chain management)
  - Lots of opportunities for OR research and practice

What Does OR Analyst Do?

- Take a problem



- And turn it into mathematics

$$\max \sum_{j \in B} c_j x_j + \sum_{j \in I} c_j x_j + \sum_{j \in C} c_j x_j$$

- That can be solved on a computer

Typical Phases of an OR Project

1. Define the problem and gather data
2. Formulate a mathematical model to represent the problem
3. Choose an existing algorithm or develop a new one to solve the model by computer
4. Test the model and refine as necessary
5. Design decision-support system for on-going use
6. Implement

Step 1: Define Problem and Gather Data

- What problem is the organization facing?
  - Excess inventory
  - Bottlenecks in manufacturing process
  - Trucks inefficiently loaded and routed
  - Too much waste in paper-cutting operation
- What is the objective?
  - Reduce inventory
  - Minimize costs
  - Maximize revenues
  - Meet service commitment
- Gather data
  - Direct observation
  - Existing databases

Formulate Mathematical Model

- Always an approximation!
- Trade-off between accuracy and tractability
  - Modelling is an art and a science
- More about this later

Choose/Design Algorithm

- For linear programs, the Simplex method is usually the algorithm of choice
  - CPLEX/XPRESS or Excel/Solver
- For integer and non-linear programs, Simplex doesn't apply
  - "Off-the-shelf" algorithms may work
  - If not, design a "special-purpose" algorithm

Step 4: Test and Refine Model

- Run the model on a small example that you can verify by hand
- Find and correct errors (debugging)
- Test on “live” data
- Find and correct errors in data if results are unrealistic (model validation)
- Modify model if other features are desired and possible

Step 5: Design Decision-Support System

- You will not be running this model forever
- Sooner or later, some OR novice will be running it
- You have to make it easy for him or her to use
- Graphical User Interfaces (GUI)
  - VBA/Excel
  - Access
  - Web
  - Visual C++
- Data processing and report generation must be automated

Step 6: Implement

- Run the model on data you've gathered from the organization
- Present results
- Implement requested changes
- Hand it off to in-house analysts

Example: Wyndor Glass Co.

- Wyndor Glass Co.
  - Produces windows and doors
- Three plants
  - Plant 1: aluminium frames and hardware
  - Plant 2: wood frames
  - Plant 3: glass and assembly
- Two new products to be introduced
  - Product 1: 8-foot glass door with aluminium frame
  - Product 2: 4 × 6-foot double-hung wood-framed window
- Product 1 requires plants 1 and 3
- Product 2 requires plants 2 and 3
- Demand is unlimited
- Capacity is limited

Problem Definition

- The problem:
  - Choose production quantity of each product
  - To maximize total profit
  - Subject to capacity restrictions at each plant
- We might find:
  - It is optimal to produce both products
  - It is optimal to max out capacity with one product and not produce the other
  - It is optimal to produce neither product (can't turn a profit)
  - There is not enough capacity to produce either product (the problem is infeasible)

Data Requirements

- We need to know:
  - Number of hours of production time available at each plant per week (the available capacity)
  - Number of hours of production time required for one batch of each product at each plant
  - Profit per batch of each product produced

Data Requirements

- After months of meetings with Wyndor Glass managers, we find:

Plant	Production time hour/batch		Available hours
	Product 1	Product 2	
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	\$3000	\$5000	

Decision Variables

- Let
  - x_1= number of batches of product 1 produced per week
  - x_2= number of batches of product 2 produced per week
- We don't know the values of x_1 and x_2 yet
  - The model is supposed to decide
- These are called **decision variables**

Objective Function

- We want to maximize total profit
- Total profit (in \$1000s) is given by $3x_1 + 5x_2$

(recall: product 1 earns \$3000/batch, product 2 earns \$5000/batch)

- So we want to maximize $3x_1 + 5x_2$
- This is called the **objective function**

Constraints

- Each batch of product 1 requires 1 hour at plant 1
- Plant 1 has 4 hours available
- To avoid exceeding the capacity at plant 1, we have to say:
  - $x_1 \leq 4$
- Similarly, at plant 2:
  - $2x_2 \leq 12$
- And at plant 3:
  - $3x_1 + 2x_2 \leq 18$
- Also, the production amounts have to be non-negative:
  - $x_1 \geq 0, x_2 \geq 0$

The Lineer Program

$$
\begin{array}{llcll}
\text{maximize} & 3x_1 & + & 5x_2 & \\
\text{subject to} & x_1 & & & \leq 4 \\
 & & & 2x_2 & \leq 12 \\
 & 3x_1 & + & 2x_2 & \leq 18 \\
 & x_1 & & & \geq 0 \\
 & & & x_2 & \geq 0
\end{array}
$$

- Lineer Program (LP):
  - "Linear" because the objective function and constraints are all linear functions of the decision variables
  - No $x_1 x_2$, x_1^2, $\sqrt{x_1}$
  - "Program": Historically "plan"

The Lineer Program

- The layout of the LP should remind you of the data table:

$$
\begin{array}{lrcrl}
\text{maximize} & 3x_1 & + & 5x_2 & \\
\text{subject to} & 1x_1 & & & \le 4 \\
 & & & 2x_2 & \le 12 \\
 & 3x_1 & + & 2x_2 & \le 18 \\
 & x_1 & & & \ge 0 \\
 & & & x_2 & \ge 0
\end{array}
$$

Plant	Production time hour/batch		Available hours
	Product 1	Product 2	
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	\$3000	\$5000	

What Values of x_1 and x_2 are Allowed?

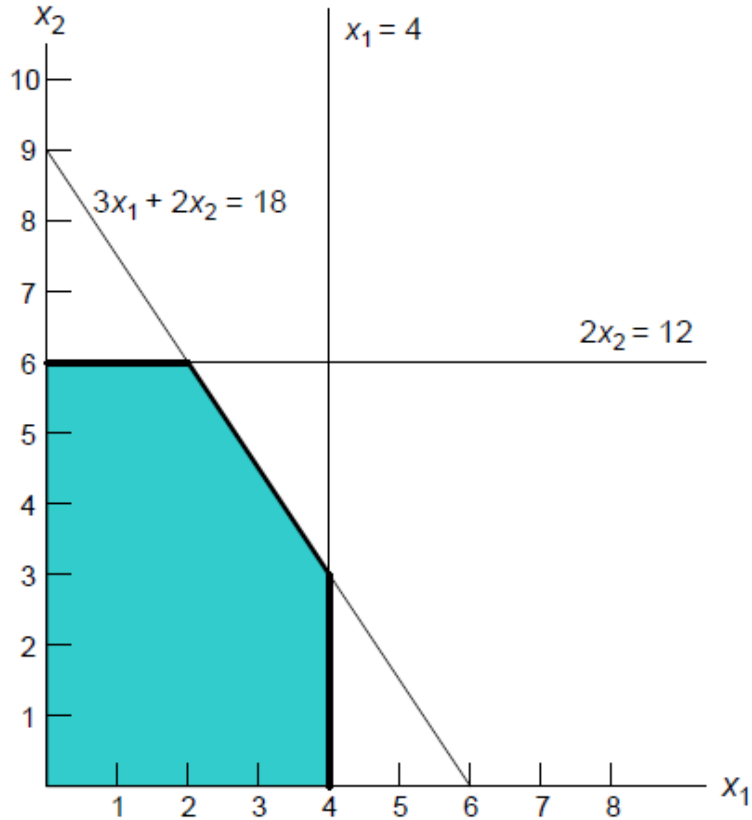
$x_1 \geq 0$

$x_2 \geq 0$

$x_1 \leq 4$

$2x_2 \leq 12$

$3x_1 + 2x_2 \leq 18$



The shaded area is the feasible region

The Optimal Solution

- Every solution (x_1, x_2) in the feasible region is a valid solution for our LP
  - "Solution" does not necessarily mean the final or best answer to a problem!
- Such solutions are called **feasible solutions**
- We want to find the feasible solution that maximizes the objective function
- The best feasible solution is called the **optimal solution**
- There may be many such solutions

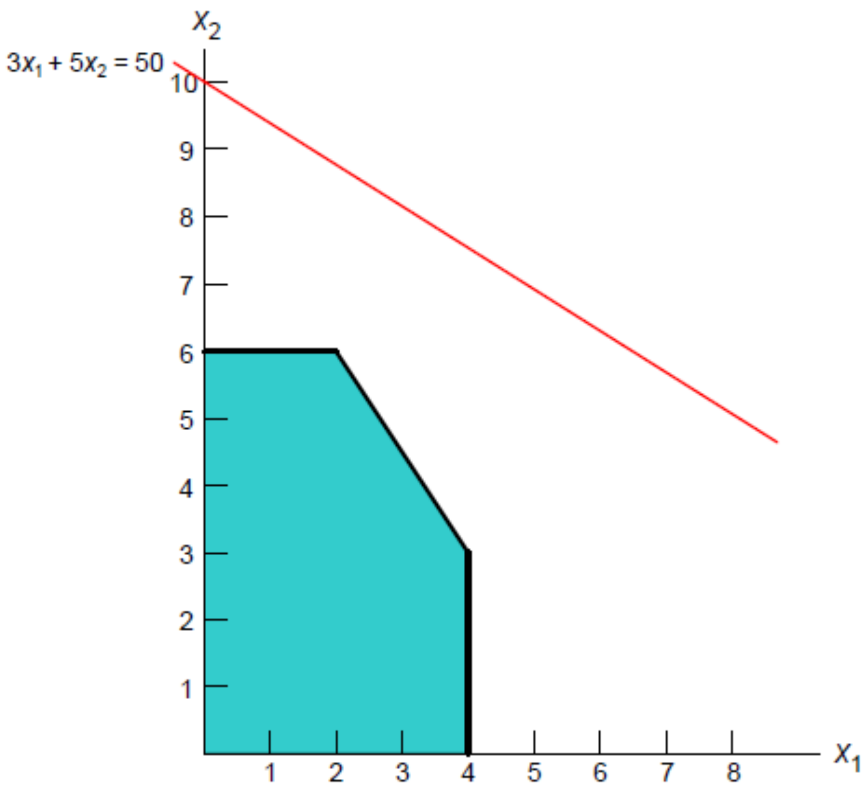
Lineer Program

$$
\begin{array}{llcl}
\max & 3x_1 + 5x_2 & & \\
\text{const.} & x_1 & \leq & 4 \\
 & 2x_2 & \leq & 12 \\
 & 3x_1 + 2x_2 & \leq & 18 \\
 & x_1 & \geq & 0 \\
 & x_2 & \geq & 0
\end{array}
$$

Graphical Method for Finding the Optimal Solution

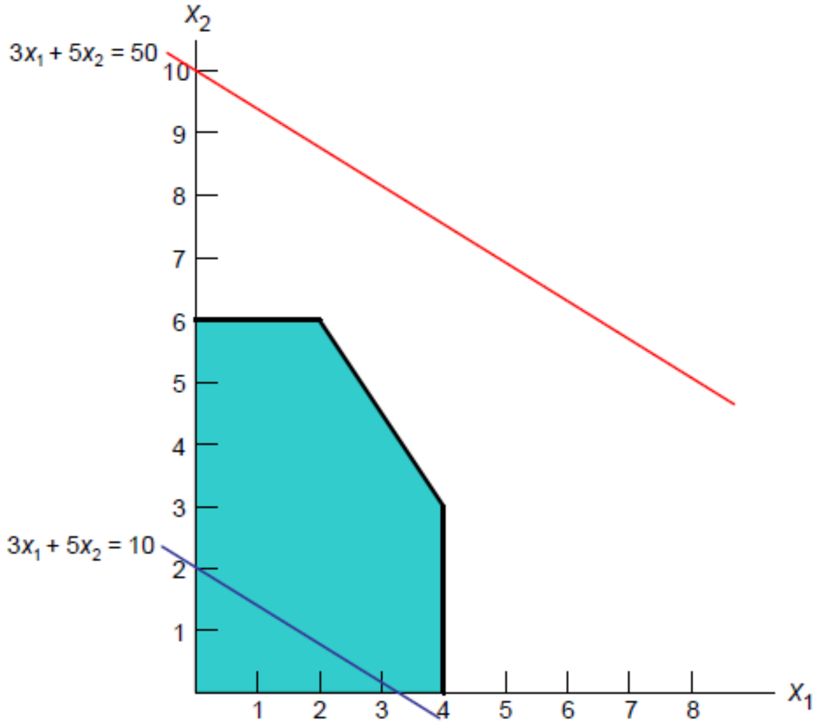
Is there a solution with the objective function equal to 50?

No, since the line $3x_1 + 5x_2 = 50$ does not intersect the feasible region.



Graphical Method for Finding the Optimal Solution

- How about equal to 10?

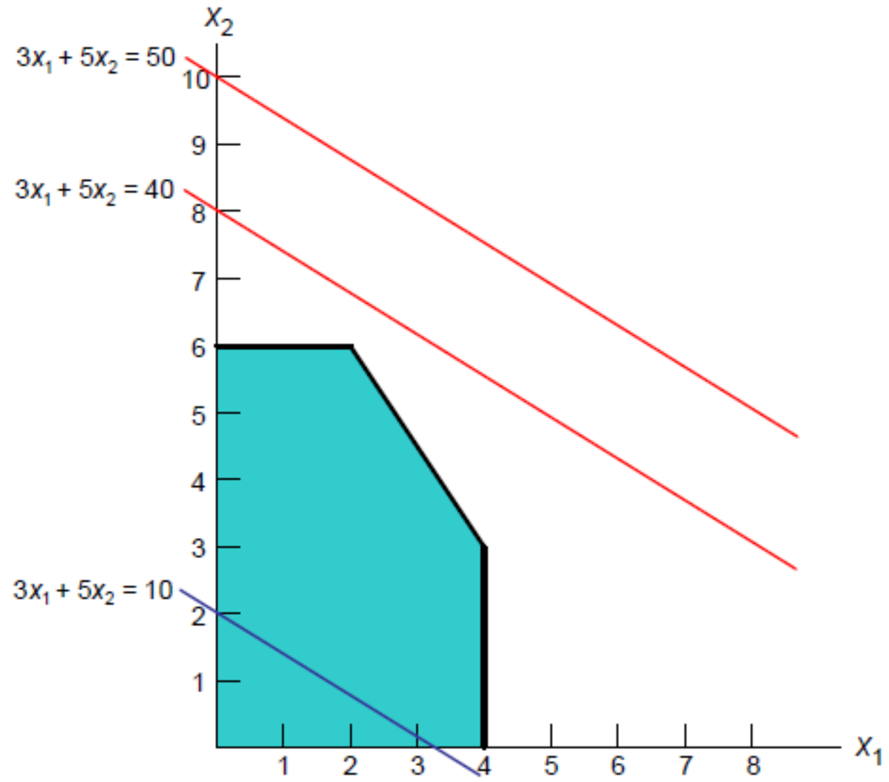


Yes, there are lots since the line $3x_1 + 5x_2$=10 goes through the feasible region.
But they aren't optimal

Graphical Method for Finding the Optimal Solution

Equal to 40?

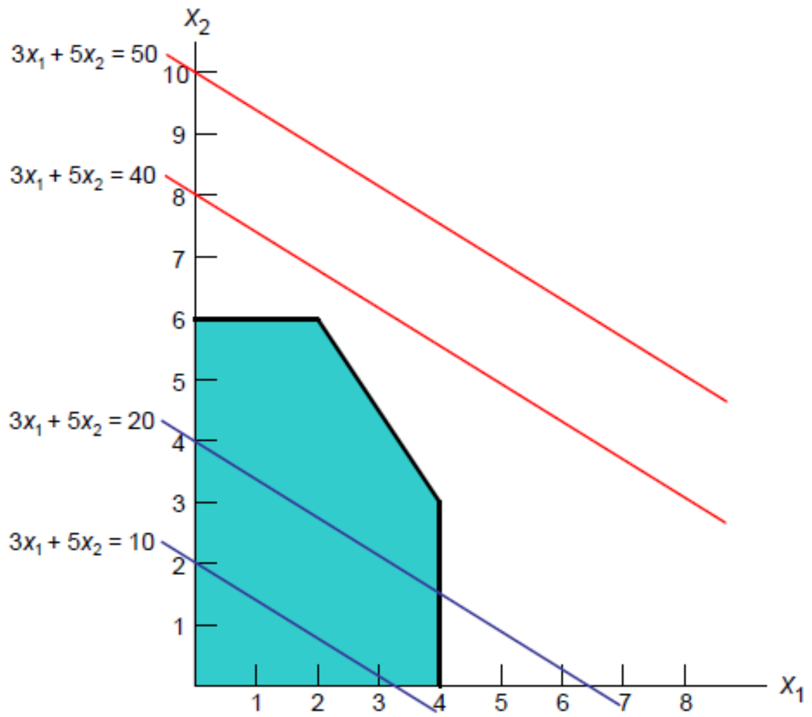
NO!



Graphical Method for Finding the Optimal Solution

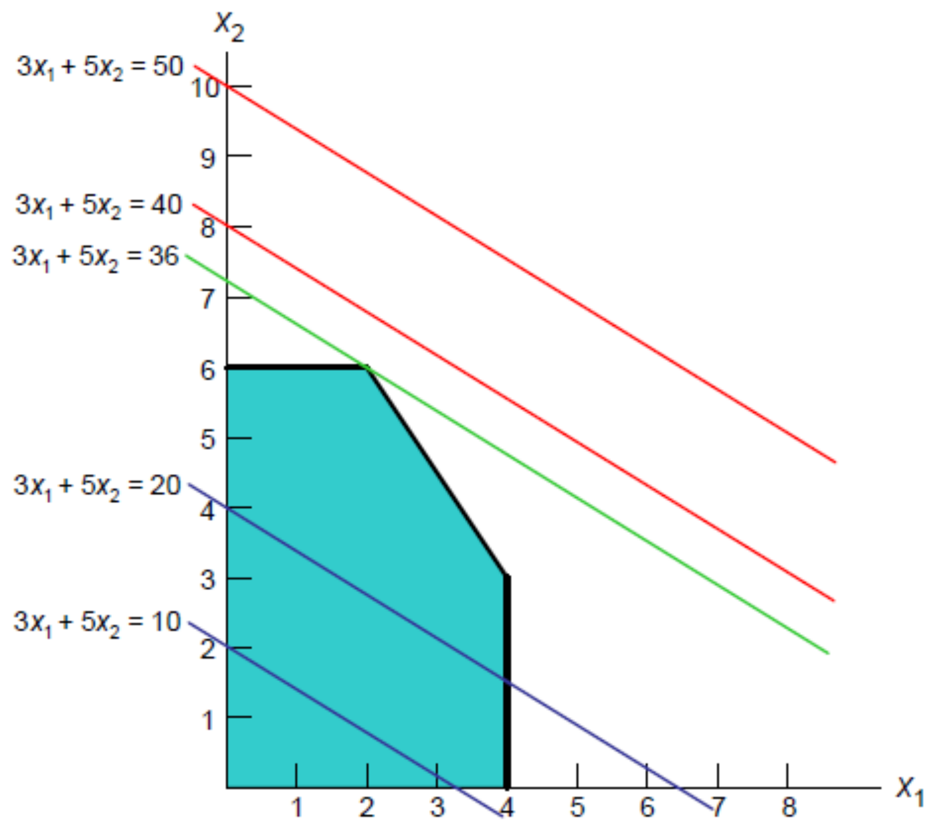
$3x_1 + 5x_2 = 20$?

YES



Grafik Metod Cozumuyle Optimum Sonucu Bulmak

$3x_1 + 5x_2$=36 ?



Yes, if x_1= 2 and $x_2 = 6$, then $3x_1 + 5x_2$=36 , and (2,6) is feasible. This is the optimal solution.

The Objective Function

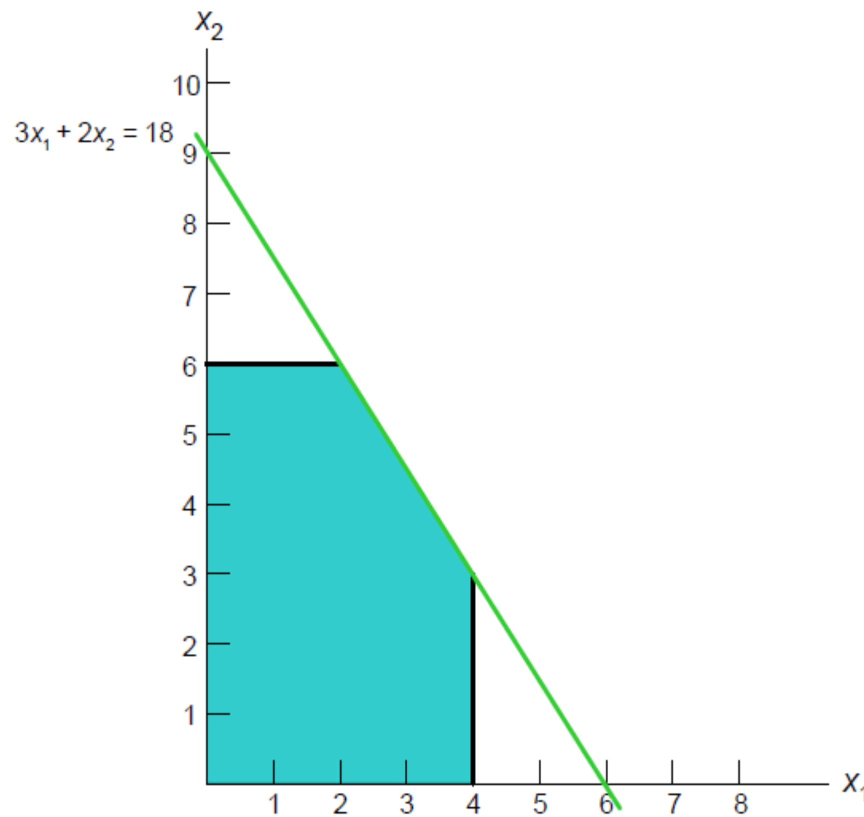
- All of the objective function lines are parallel
- They all have the same slope
- To see this, convert to slope-intercept form:

$$x_2 = -\frac{3}{5}x_1 + \frac{1}{5}Z$$

- Z is the "guess" objective function value

The Optimal Solution

- It's possible for there to be many optimal solutions.
- For ex., if the objective function were $3x_1 + 2x_2$



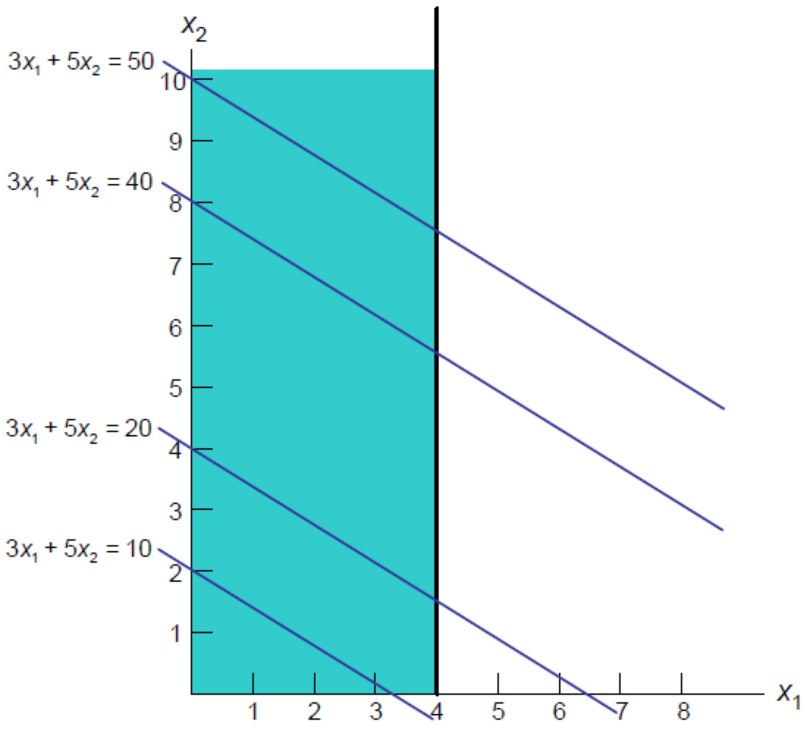
The Optimal Solution

- It's possible for there to be no optimal solution.
- For ex., if the constraint

$$\begin{aligned} 3x_1 + 2x_2 &\leq 18 \\ 2x_2 &\leq 12 \end{aligned}$$

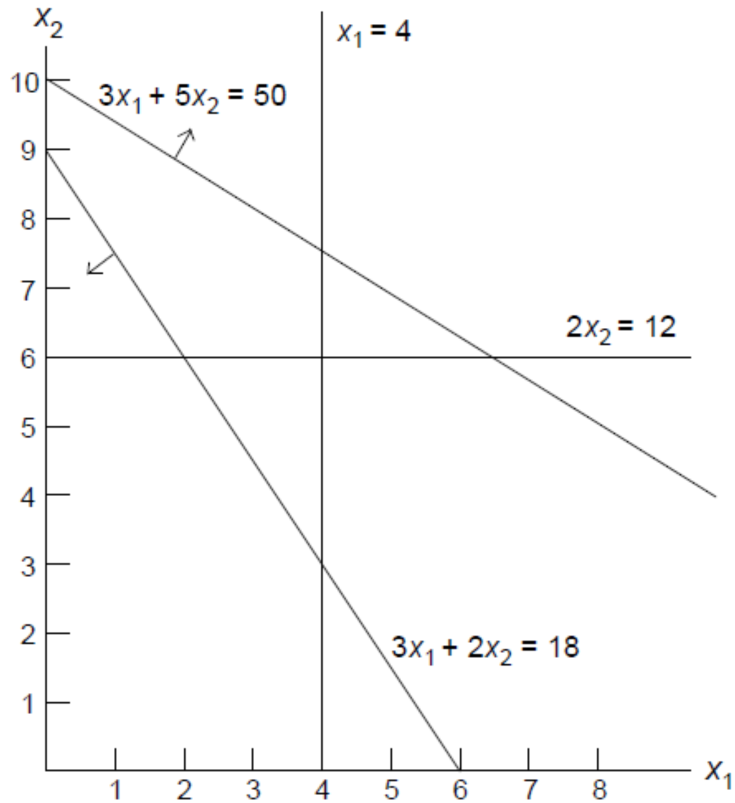
weren't there.

  - In this case LP is unbounded



The Optimal Solution

- It's possible for there to be no feasible solutions.
- For ex., if there were also a constraint
  $3x_1 + 5x_2 \geq 50$



Finding an optimal solution in two dimensions: Summary

- The optimal solution (if one exists) occurs at a "corner point" of the feasible region.
- In two dimensions with all inequality constraints, a corner point is a solution at which two (or more) constraints are binding.
- There is always an optimal solution that is a corner point solution (if a feasible solution exists).
- More than one solution may be optimal in some situations
- In *n* dimensions, one cannot evaluate the solution value of every extreme point efficiently. (There are too many.)
- The simplex method finds the best solution by a neighborhood search technique.
- Two feasible corner points are said to be "*adjacent*" if they have one binding constraint in common.

Data for the GTC Problem

	Wrench	Plier	Available
Steel	1.5	1.0	15,000 pounds
Molding Machine	1.0	1.0	12,000 hrs
Assembly Machine	.4	.5	5,000 hrs
Demand Limit	8,000	10,000	
Contribution ($ per unit)	$.40	$.30	

Want to determine the number of wrench and plier to produce given the available raw materials, machine hours and demand.

Formulating the GTC Problem

P = number of pliers manufactured

W = number of wrenches manufactured

Maximize Profit = $.4\,W + .3\,P$

Steel:	$1.5\,W + P \leq 15{,}000$
Molding:	$W + P \leq 12{,}000$
Assembly:	$0.4\,W + 0.5\,P \leq 5{,}000$
Wrench Demand:	$W \leq 8{,}000$
Pliers Demand:	$P \leq 10{,}000$
Non-negativity:	$P, W \geq 0$

Reformulation

P = number of 1000s of pliers manufactured

W = number of 1000s of wrenches manufactured

Maximize Profit = $400\,W + 300\,P$

Steel: $1.5\,W + P \leq 15$

Molding: $W + P \leq 12$

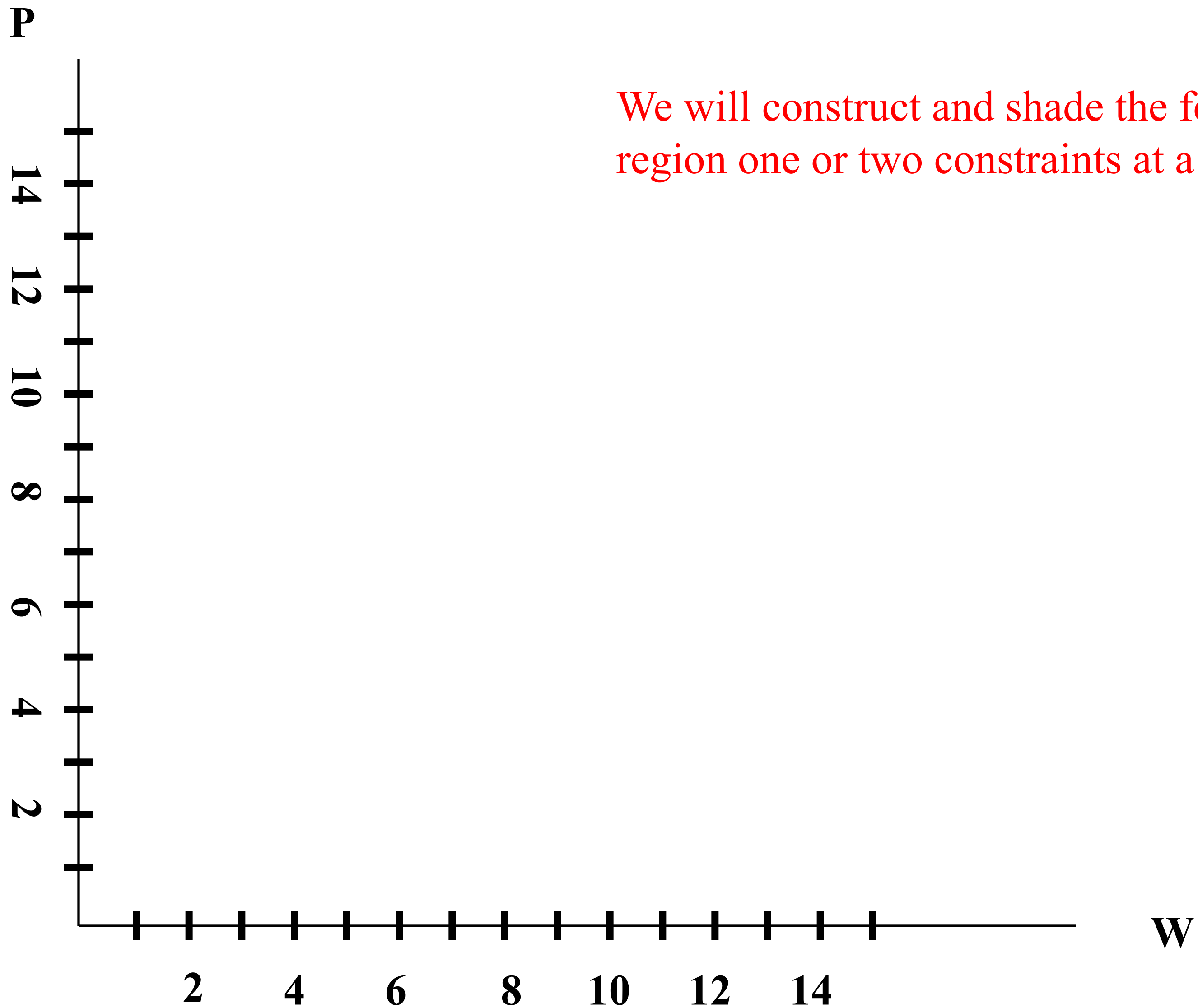
Assembly: $0.4\,W + 0.5\,P \leq 5$

Wrench Demand: $W \leq 8$

Pliers Demand: $P \leq 10$

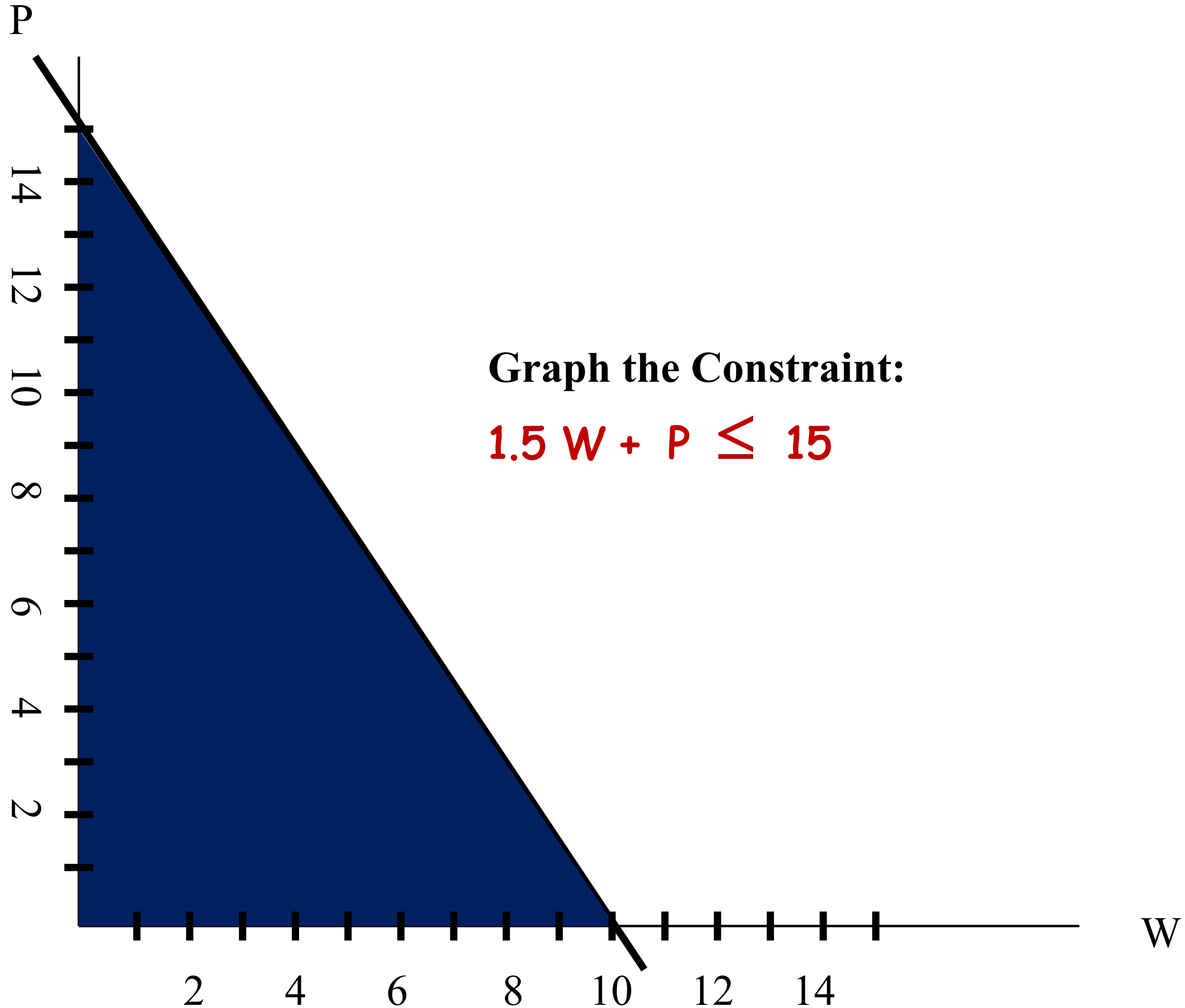
Non-negativity: $P, W \geq 0$

Graphing the Feasible Region



We will construct and shade the feasible region one or two constraints at a time.

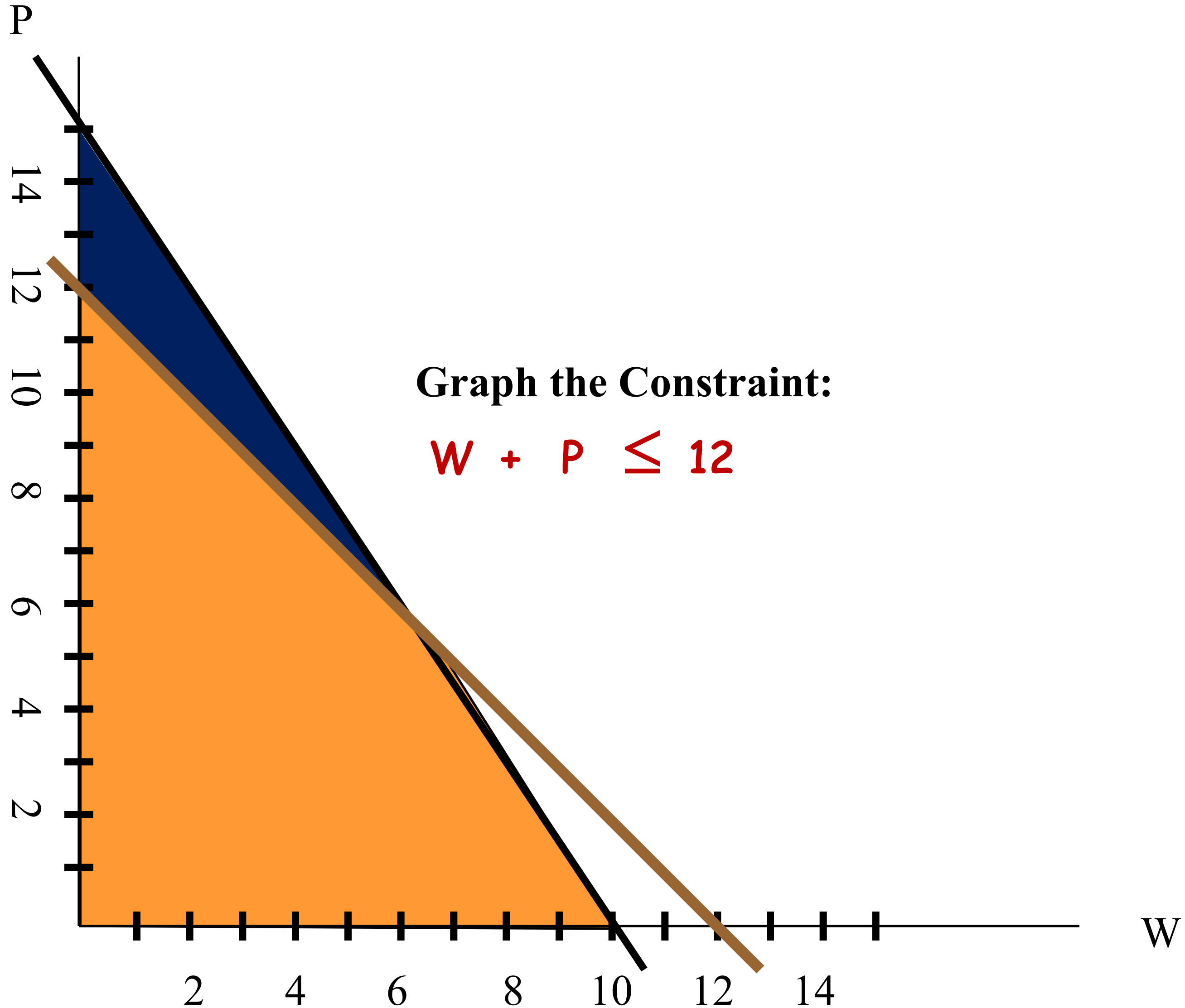
Graphing the Feasible Region



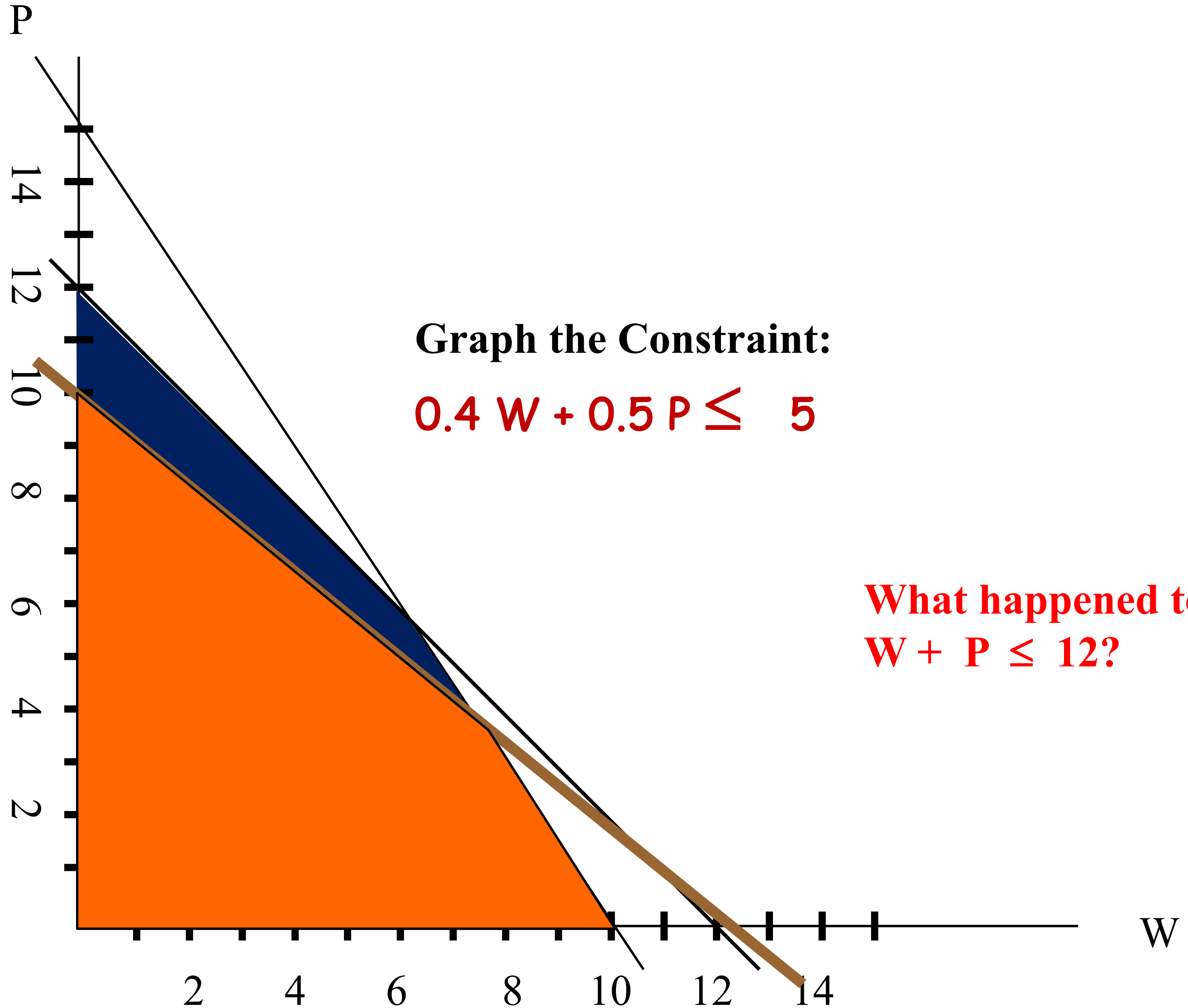
Graph the Constraint:

$1.5\,W + P \leq 15$

Graphing the Feasible Region



Graphing the Feasible Region

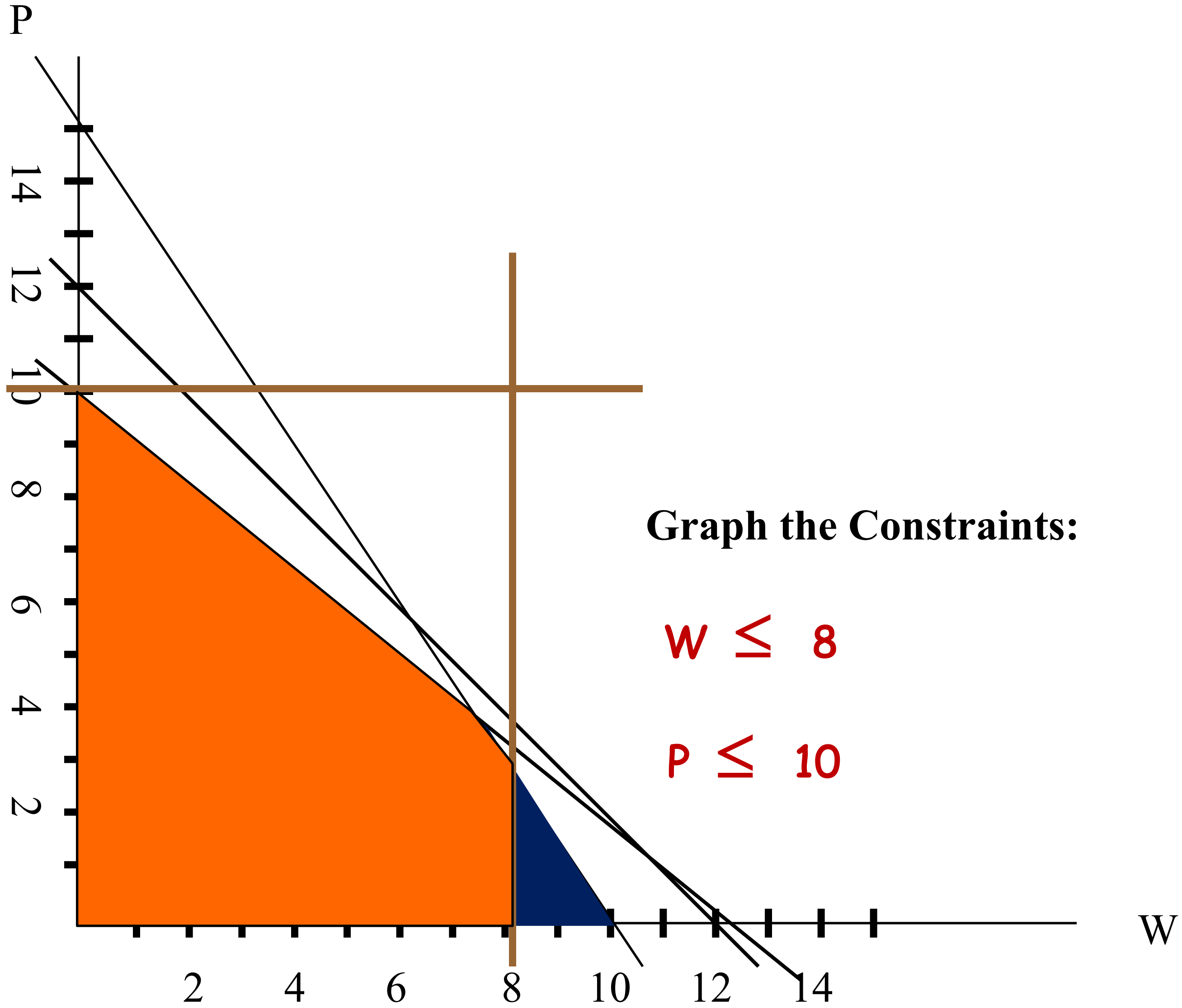


Graph the Constraint:

$0.4\,W + 0.5\,P \leq 5$

What happened to the constraint :
$W + P \leq 12$?

Graphing the Feasible Region

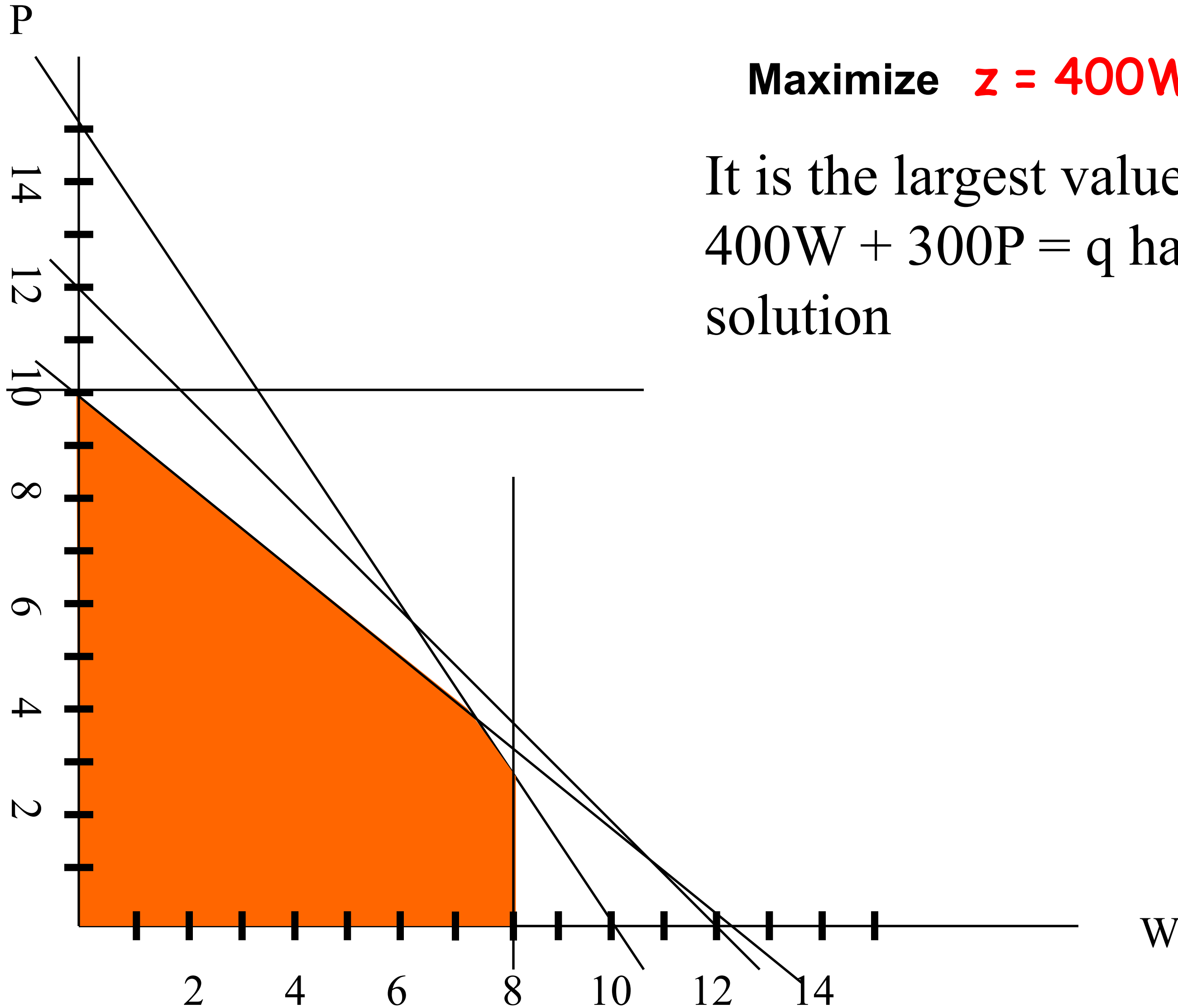


Graph the Constraints:

$W \leq 8$

$P \leq 10$

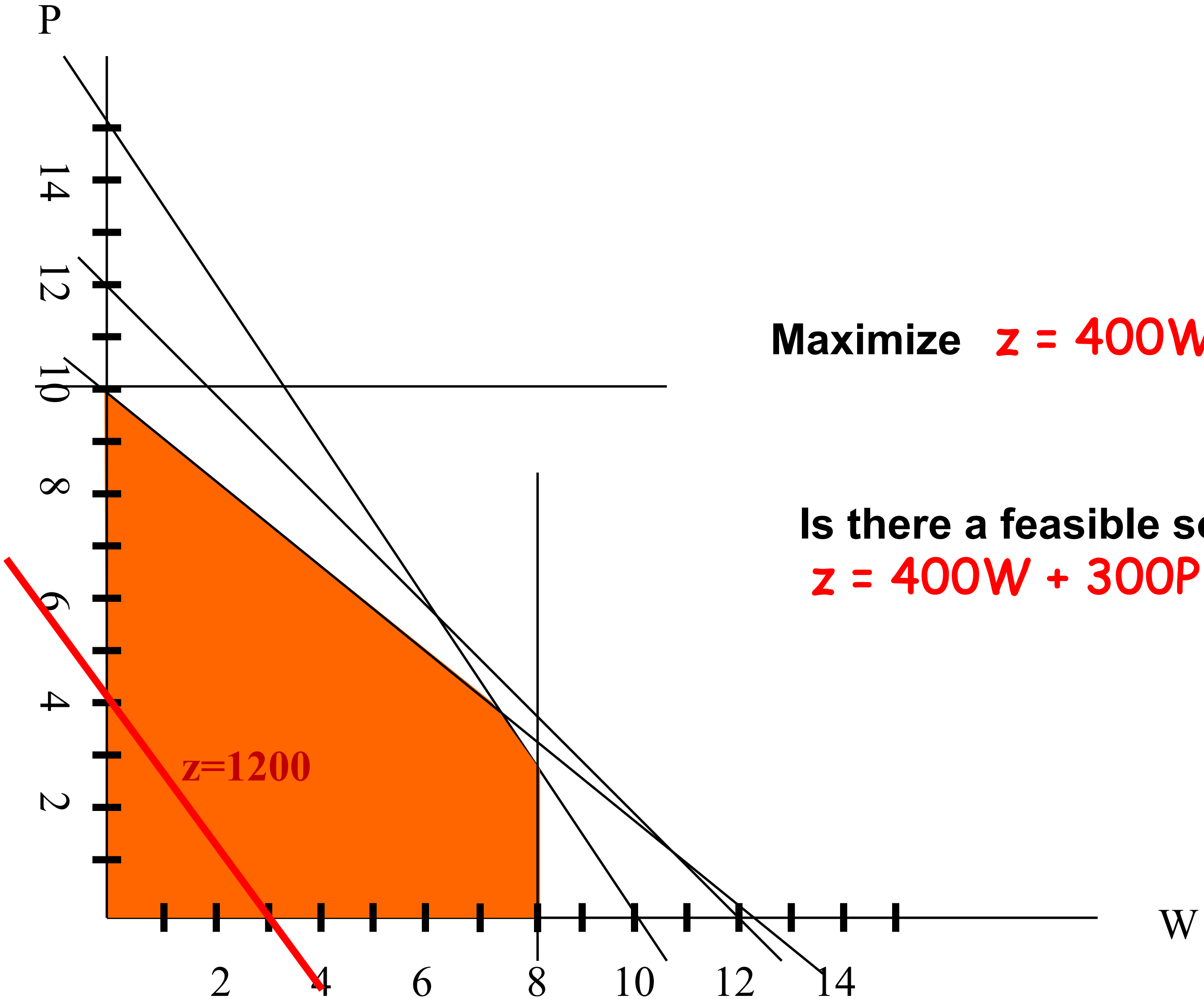
How do we find an optimal solution?



Maximize $z = 400W + 300P$

It is the largest value of q such that 400W + 300P = q has a feasible solution

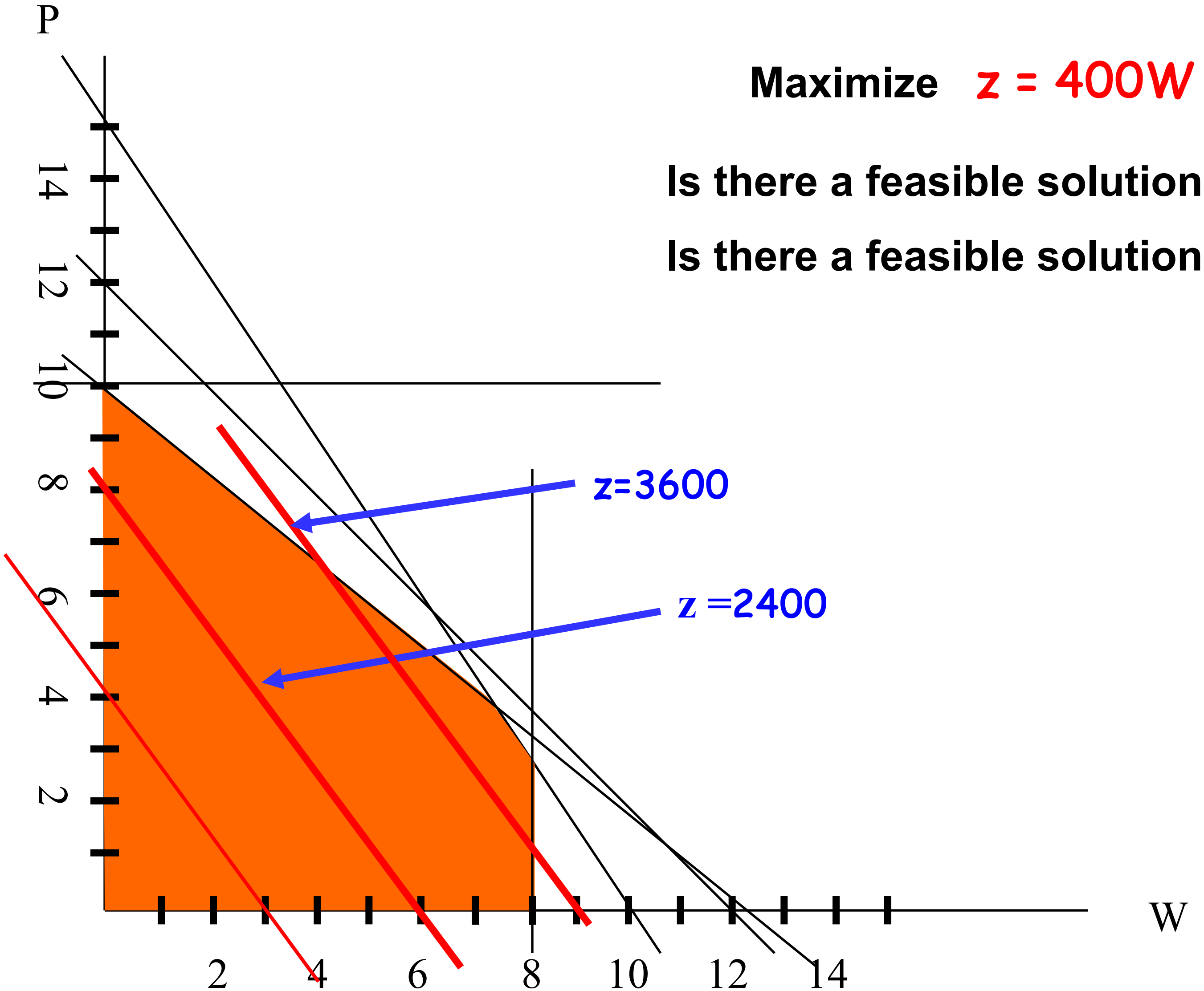
How do we find an optimal solution?



Maximize $z = 400W + 300P$

Is there a feasible solution with $z = 400W + 300P = 1200$**?**

How do we find an optimal solution?

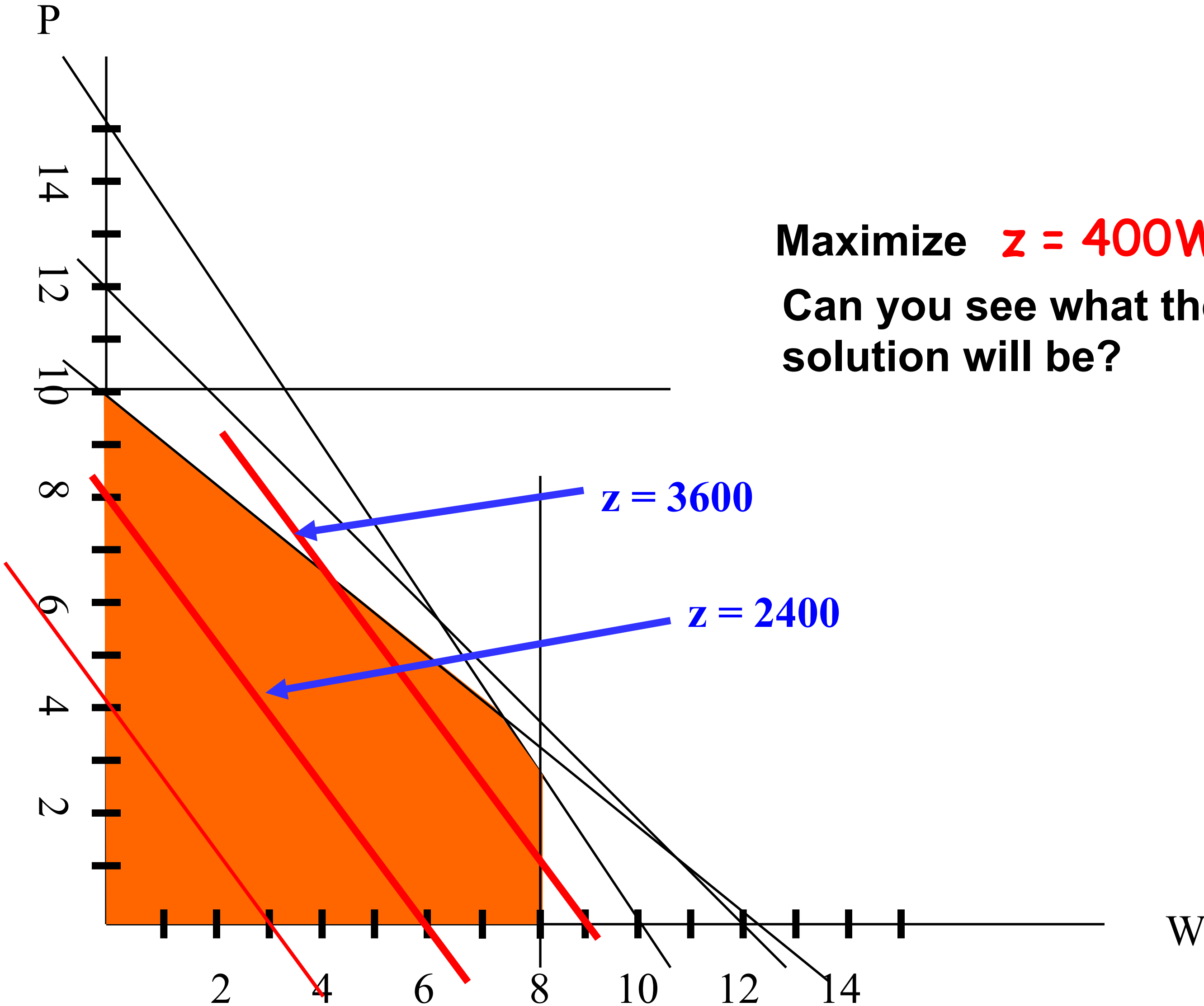


Maximize $z = 400W + 300P$

Is there a feasible solution with $z = 2400$?

Is there a feasible solution with $z = 3600$?

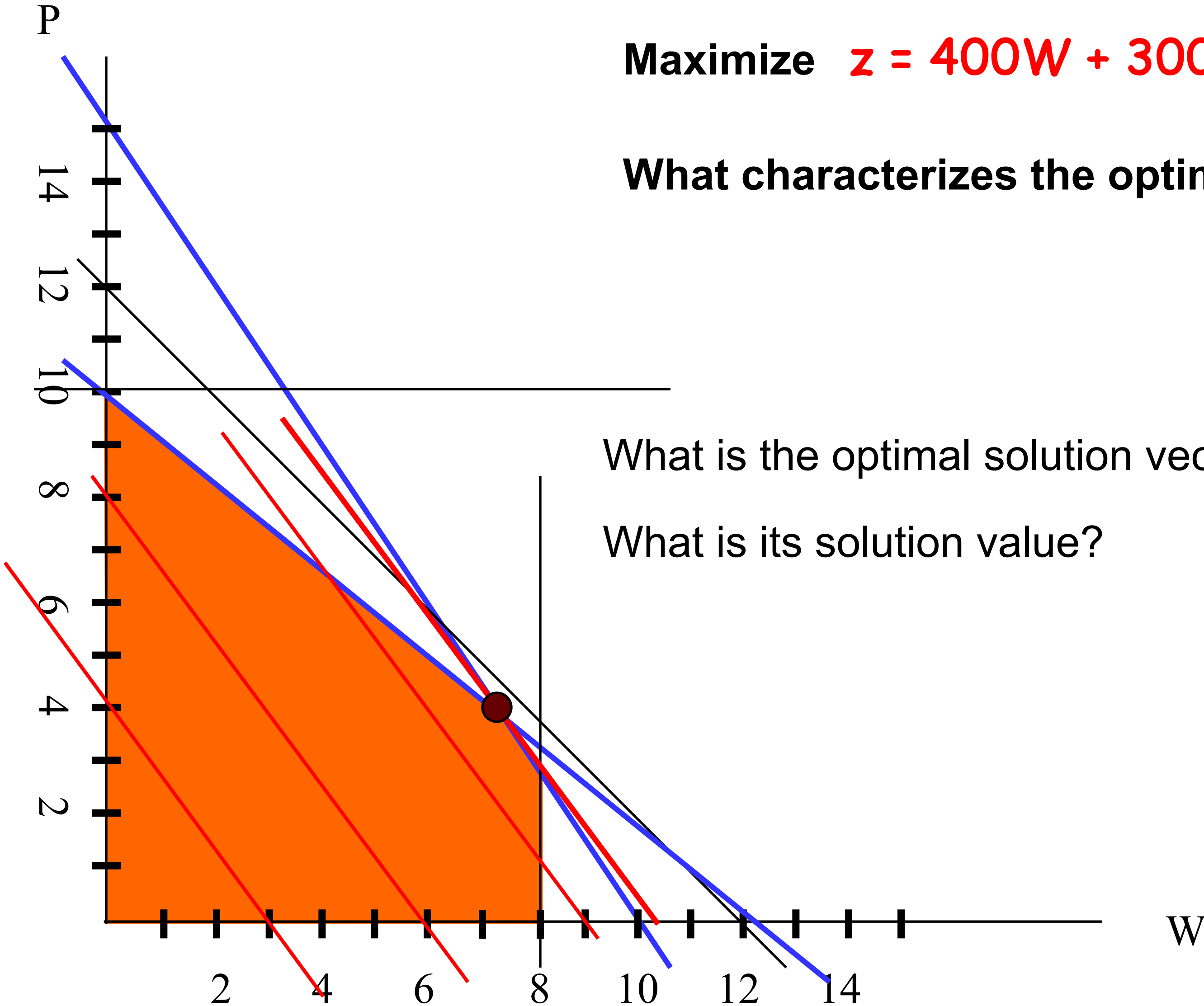
How do we find an optimal solution?



Maximize $z = 400W + 300P$

Can you see what the optimal solution will be?

How do we find an optimal solution?



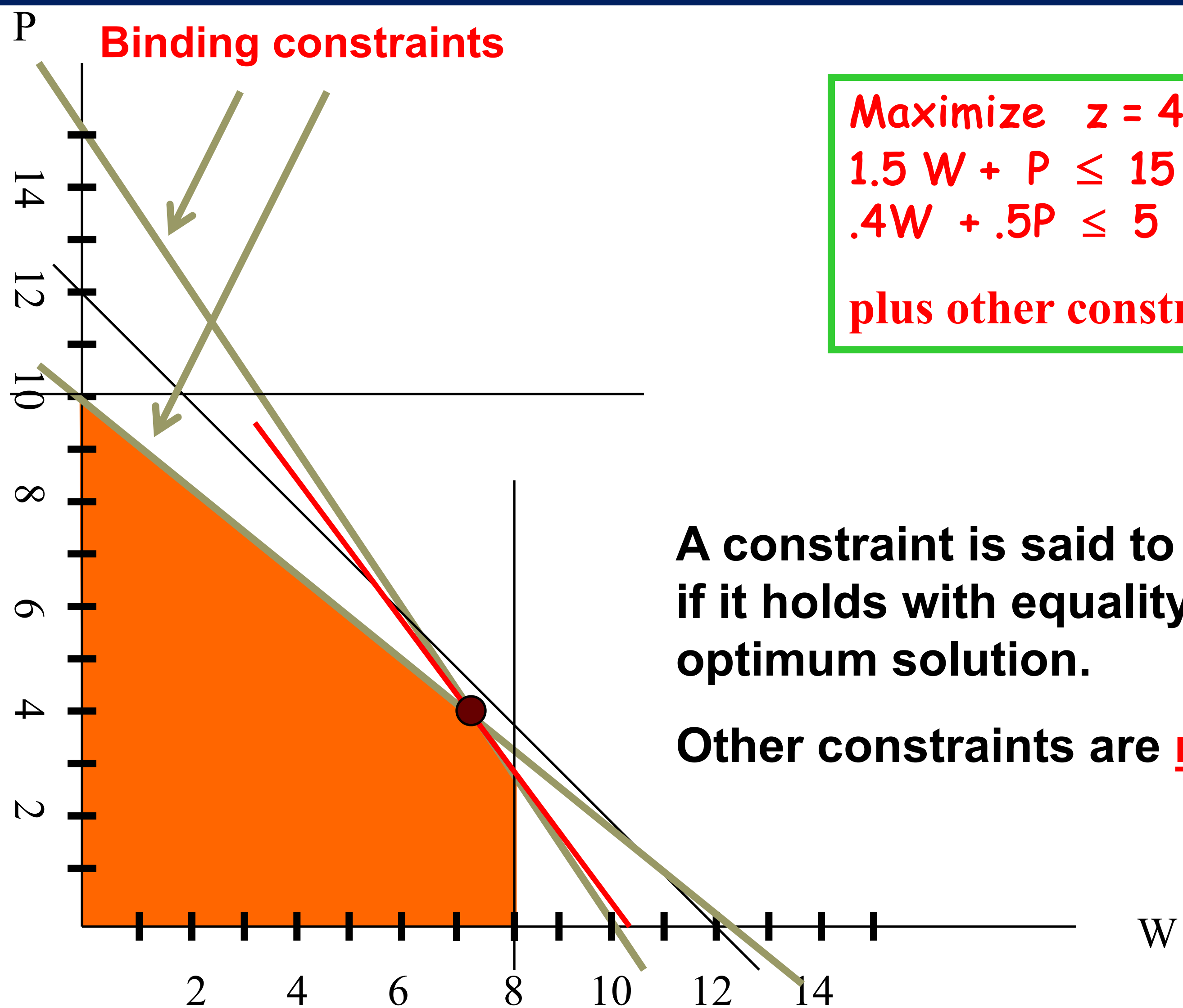
Maximize $z = 400W + 300P$

What characterizes the optimal solution?

What is the optimal solution vector? $W = ?$ $P = ?$

What is its solution value? $z = ?$

Optimal Solution Structure



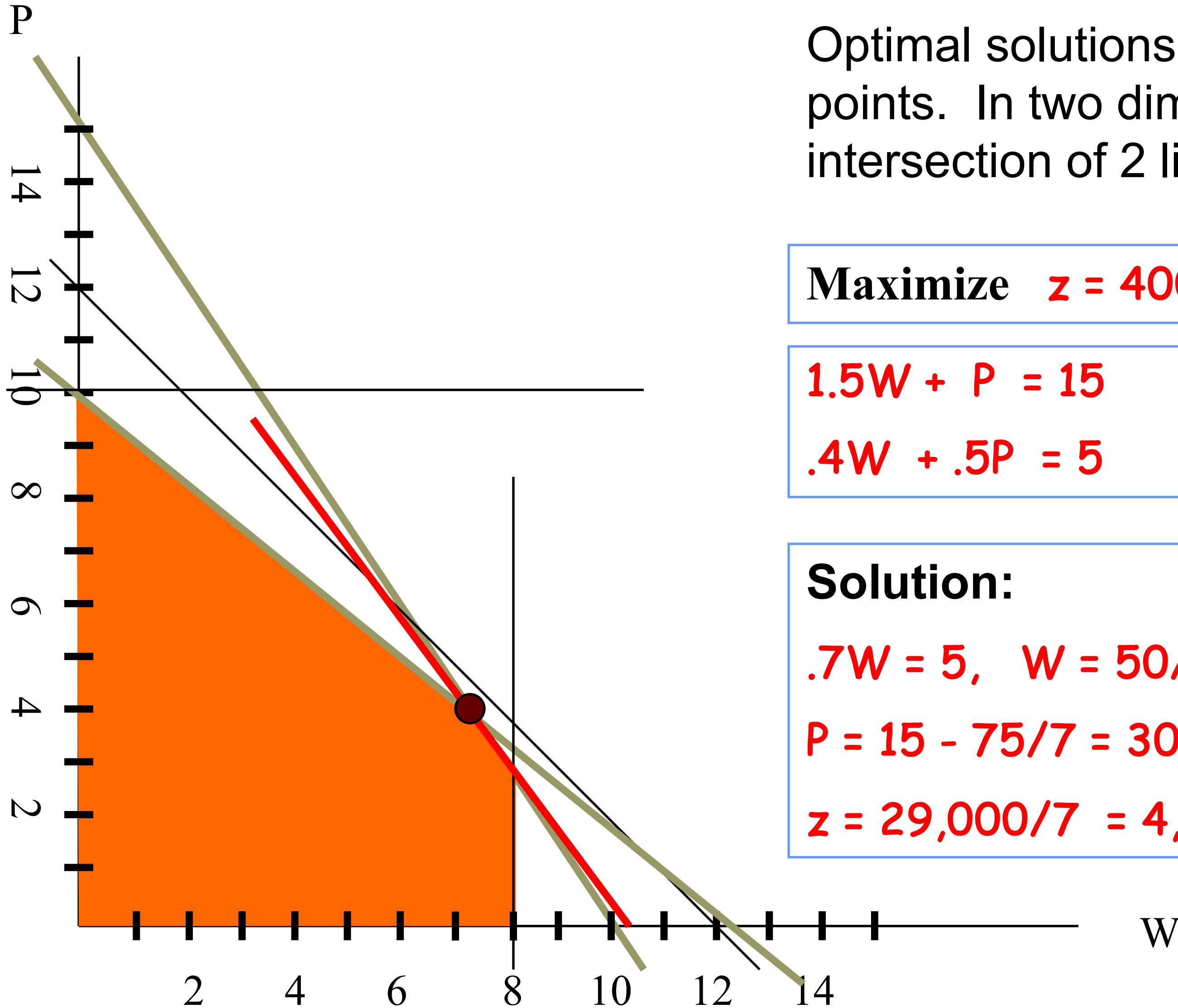
Maximize $z = 400W + 300P$
$1.5\,W + P \le 15$
$.4W + .5P \le 5$

plus other constraints

A constraint is said to be <u>binding</u> if it holds with equality at the optimum solution.

Other constraints are <u>non-binding</u>

How do we find an optimal solution?



Optimal solutions occur at corner points. In two dimensions, this is the intersection of 2 lines.

Maximize $z = 400W + 300P$

$1.5W + P = 15$

$.4W + .5P = 5$

Solution:

$.7W = 5, \quad W = 50/7$

$P = 15 - 75/7 = 30/7$

$z = 29{,}000/7 = 4{,}142\ 6/7$

LP Terminology

- In general, an LP involves setting the level of a number of activities to minimize or maximize some performance measure, with constraints on the resources.
  - For example, choose the production rate for each product to maximize profit subject to production capacities.
- Common notation:
  - n = number of activities
  - m = number of resources
  - Z = overall performance measure (objective function value)
  - x_j = level of activity j, j = 1, . . . , n
  - cj = increase in Z that would result from 1 unit increase in x_j (i.e., cost or revenue per unit)
  - b_i = amount of resource i available, i = 1, . . . ,m
  - a_{ij} = amount of resource i consumed by 1 unit of activity j

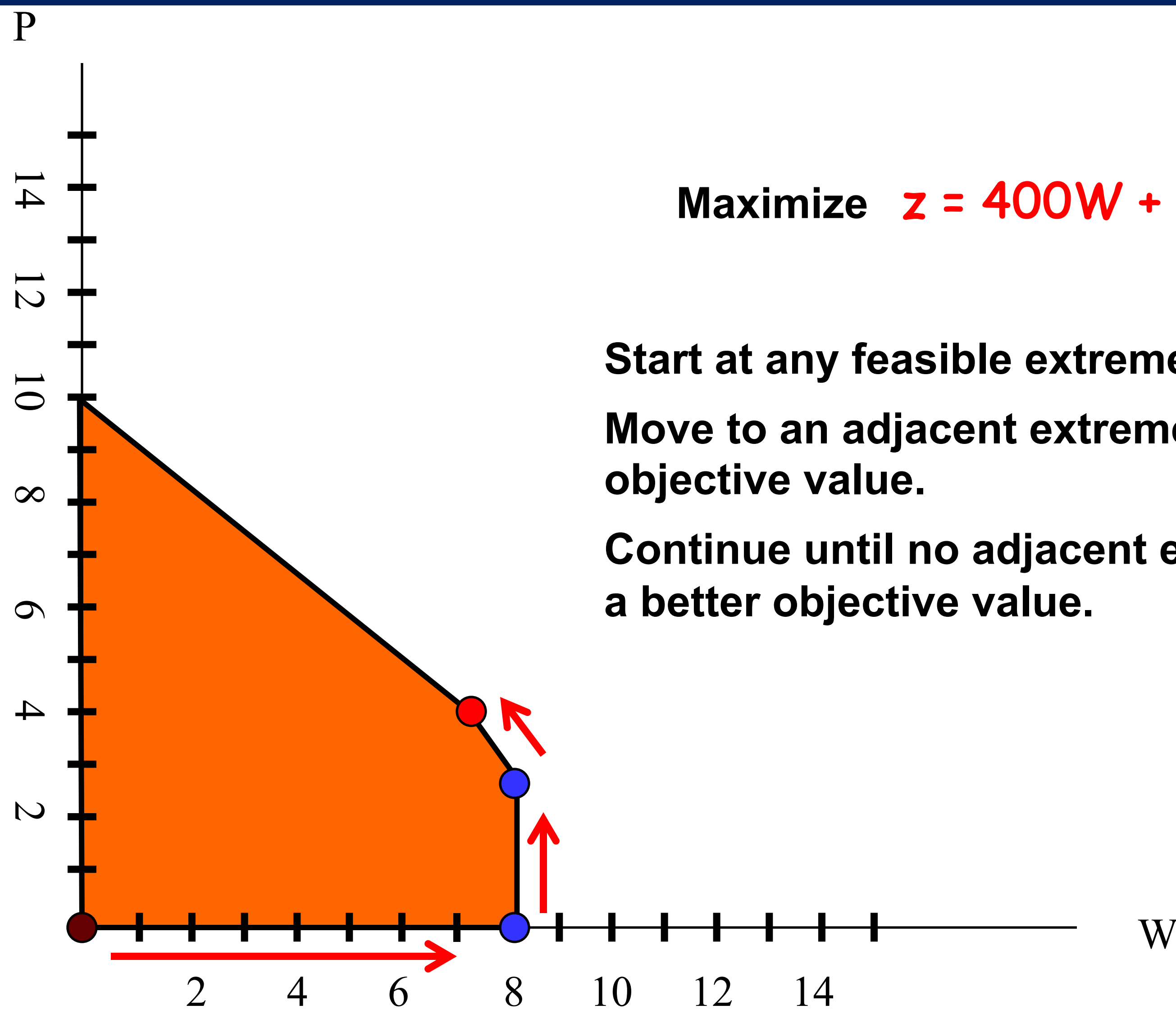
Data Required for an LP

Resource	Resource Usage per Unit of Activity 1	2	$\cdots$	n	Amount of Resource Avail.
1	a_{11}	a_{12}	$\cdots$	a_{1n}	b_1
2	a_{21}	a_{22}	$\cdots$	a_{2n}	b_2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
m	a_{m1}	a_{m2}	$\cdots$	a_{mn}	b_m
Contribution to Z per unit of activity	c_1	c_2	$\cdots$	c_n	

Standard Form of the LP MOdel

$$
\begin{array}{llccccccccc}
\text{maximize} & Z = & c_1x_1 & + & c_2x_2 & + & \dots & + & c_nx_n & & \\
\text{subject to} & & a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & \leq & b_1 \\
 & & a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & \leq & b_2 \\
 & & & & & & & & & \vdots & \\
 & & a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & \leq & b_m \\
 & & x_1 & & & & & & & \geq & 0 \\
 & & & & x_2 & & & & & \geq & 0 \\
 & & & & & & & & & \vdots & \\
 & & & & & & & & x_n & \geq & 0
\end{array}
$$

Preview of the Simplex Method



Maximize $z = 400W + 300P$

Start at any feasible extreme point.

Move to an adjacent extreme point with better objective value.

Continue until no adjacent extreme point has a better objective value.