1. Strength Analysis of the Robot Parts

All components were designed and assembled by using Solidworks 2017. Strength Analysis will be analyzed by using Solidworks. And also, Dynamic components will be analyzed by solving standard mechanical formulas. All components will be manufactured from PLA due to the characteristics of its strength, resistance to temperature, toughness, flexibility, resistance to impact.

2. Kinematic Analysis of the Robot Arm

The problem of kinematics is to describe the motion of the manipulator without consideration of the forces and torques causing the motion. Denavit-Hartenberg demonstrates that a general transformation between two joints requires four parameters. These parameters, known as the Denavit-Hartenberg (D-H) parameters which became a standard to describe robot kinematics (Funda et. Al - 1990).

2.1. Forward Kinematic Analysis

The transformation of coordinates of the end-effector point from the joint space to the world space is known as ***forward kinematic*** transformation. It is also called more clearly, forward kinematic which is to determine the position and orientation of the end effector given the values for the joint variables of the robots.

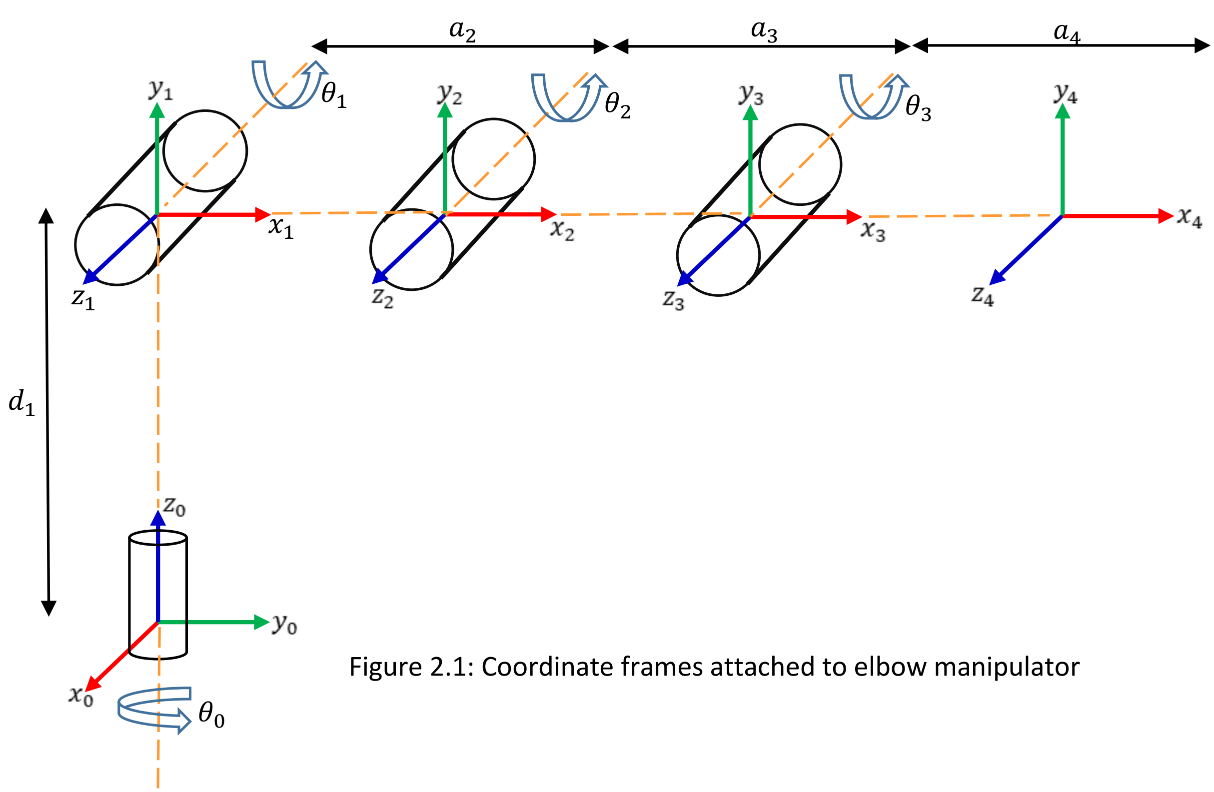

Figure 2.1: Coordinate frames attached to elbow manipulator

Axis Number i	Link Length a_i	Twist Angle α_i	Link offset d_i	Joint angle θ_i
1	0	$\pi/2$	d_1	θ_1^*
2	a_2	0	0	θ_2^*
3	a_3	0	0	θ_3^*
4	a_4	0	0	θ_4^*

Table 2.1: D-H Parameters of Robot Arm

Denavit – Hartenberg (D-H) method uses the four parameters including a_i,α_i, d_i and θ_i; which are the link length, twist angle, link offset and joint angle, respectively.

- θ_i , joint angle is angle from x_{i-1} to x_i measured around the z_{i-1}.
- d_i, link offset is distance from O_{i-1} to O_i measured along z_{i-1}.
- a_i, link length is distance from z_{i-1} to z_i measured along x_i.
- α_i, twist angle is angle from z_{i-1} to z_i measured along x_i

The matrix T_i^{i-1} is known as a D-H convention matrix given in this equation. In the matrix T_i^{i-1}, it means transformation matrix, the quantities of a_{i-1},α_{i-1}, d_i are constant for a given link while the parameter θ_i for a revolute joint is variable.

$$A_i = T_i^{i-1} = \begin{bmatrix} cos\theta_i & -cos\alpha_i sin\theta_i & sin\alpha_i sin\theta_i & a_i\, cos\theta_i \\ sin\theta_i & cos\alpha_i cos\theta_i & -sin\alpha_i cos\theta_i & a_i\, sin\theta_i \\ 0 & sin\alpha_i & cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We obtain the matrices that is shown below by using this formulation;

$$A_1 = T_1^0 = \begin{bmatrix} cos\theta_1 & 0 & sin\theta_1 & 0 \\ sin\theta_1 & 0 & -cos\theta_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = T_2^1 = \begin{bmatrix} cos\theta_2 & -sin\theta_2 & 0 & a_2\, cos\theta_2 \\ sin\theta_2 & cos\theta_2 & 0 & a_2 sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = T_3^2 = \begin{bmatrix} cos\theta_3 & -sin\theta_3 & 0 & a_3\, cos\theta_3 \\ sin\theta_3 & cos\theta_3 & 0 & a_3 sin\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = T_4^3 = \begin{bmatrix} cos\theta_4 & -sin\theta_4 & 0 & \mathrm{a}_4\, cos\theta_4 \\ sin\theta_4 & cos\theta_4 & 0 & \mathrm{a}_4 sin\theta_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then, the Transformation matrix of overall system is created by multiplying each matrix. T_4^0 matrix defined as follows,

$$T_4^0 = \prod_{i=1}^{4} T_i^{i-1} = T_1^0\, T_2^1\, T_3^2\, T_4^3 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ r_{41} & r_{42} & r_{43} & r_{44} \end{bmatrix} = A_1 A_2 A_3 A_4$$

$r_{11} = -c_1 c_4 (s_2 s_3 - c_2 c_3) - c_1 s_4 (c_2 s_3 + c_3 s_2)$
$r_{12} = c_1 s_4 (s_2 s_3 - c_2 c_3) - c_1 c_4 (c_2 s_3 + c_3 s_2)$
$r_{13} = sin\theta_1$
$r_{14} = a_2 c_1 c_2 - a_4 c_1 c_4 (s_2 s_3 - c_2 c_3) - a_4 c_1 s_4 (c_2 s_3 + c_3 s_2) + a_3 c_1 (c_2 c_3 - s_2 s_3)$
$r_{21} = -c_4 s_1 (s_2 s_3 - c_2 c_3) - s_1 s_4 (c_2 s_3 + c_3 s_2)$
$r_{22} = s_1 s_4 (s_2 s_3 - c_2 c_3) - c_4 s_1 (c_2 s_3 + c_3 s_2)$
$r_{23} = -cos\theta_1$
$r_{24} = a_2 c_2 s_1 - a_4 c_4 s_1 (s_2 s_3 - c_2 c_3) - a_4 s_1 s_4 (c_2 s_3 + c_3 s_2) + a_3 s_1 (c_2 c_3 - s_2 s_3)$
$r_{31} = c_4 (c_2 s_3 + c_3 s_2) + s_4 (c_2 c_3 - s_2 s_3)$
$r_{32} = c_4 (c_2 c_3 - s_2 s_3) - s_4 (c_2 s_3 + c_3 s_2)$
$r_{33} = 0$
$r_{34} = d_1 + a_2 s_2 + a_4 c_4 (c_2 s_3 + c_3 s_2) + a_4 s_4 (c_2 c_3 - s_2 s_3) + a_3 (c_2 s_3 + c_3 s_2)$
$r_{41} = 0$
$r_{42} = 0$
$r_{43} = 0$
$r_{44} = 1$

In the expressions of these elements of transformation matrix, the variables are defined as follow:

$$c_i = cos\theta_i \quad , s_i = sin\theta_i \quad , c_{ij} = cos(\theta_i + \theta_j) \quad , s_{ij} = sin(\theta_i + \theta_j)$$

When we regulate these elements, we obtain these notations

$r_{11} = c_1c_4c_{23} - c_1s_4s_{23}$
$r_{12} = -c_1s_4c_{23} - c_1c_4s_{23}$
$r_{13} = s_1$
$r_{14} = a_2c_1c_2 + a_4c_1c_4c_{23} - a_4c_1s_4s_{23} + a_3c_1c_{23}$
$r_{21} = c_4s_1c_{23} - s_1s_4s_{23}$
$r_{22} = -s_1s_4c_{23} - c_4s_1s_{23}$
$r_{23} = -c_1$
$r_{24} = a_2c_2s_1 + a_4c_4s_1c_{23} - a_4s_1s_4s_{23} + a_3s_1c_{23}$
$r_{31} = c_4s_{23} + s_4c_{23}$
$r_{32} = c_4c_{23} - s_4s_{23}$
$r_{33} = 0$
$r_{34} = d_1 + a_2s_2 + a_4c_4s_{23} + a_4s_4c_{23} + a_3s_{23}$
$r_{41} = 0$
$r_{42} = 0$
$r_{43} = 0$
$r_{44} = 1$

And transformation matrix (T_4^0) consist of as indicated:

$$\begin{bmatrix} c_1c_4c_{23} - c_1s_4s_{23} & -c_1s_4c_{23} - c_1c_4s_{23} & s_1 & a_2c_1c_2 + a_4c_1c_4c_{23} - a_4c_1s_4s_{23} + a_3c_1c_{23} \\ c_4s_1c_{23} - s_1s_4s_{23} & -s_1s_4c_{23} - c_4s_1s_{23} & -c_1 & a_2c_2s_1 + a_4c_4s_1c_{23} - a_4s_1s_4s_{23} + a_3s_1c_{23} \\ c_4s_{23} + s_4c_{23} & c_4c_{23} - s_4s_{23} & 0 & d_1 + a_2s_2 + a_4c_4s_{23} + a_4s_4c_{23} + a_3s_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It is possible to calculate the values of (P_x, P_y, P_z) with respect to the fixed coordinate system by using transformation matrix.

$$P_x = a_2cos\theta_1cos\theta_2 - a_4cos\theta_1cos\theta_4(sin\theta_2sin\theta_3 - cos\theta_2cos\theta_3)$$
$$-a_4cos\theta_1sin\theta_4(cos\theta_2sin\theta_3 + cos\theta_3sin\theta_2) + a_3cos\theta_1(cos\theta_2cos\theta_3 - sin\theta_2sin\theta_3)$$

$$P_y = a_2cos\theta_2sin\theta_1 - a_4cos\theta_4sin\theta_1(sin\theta_2sin\theta_3 - cos\theta_2cos\theta_3)$$
$$-a_4sin\theta_1sin\theta_4(cos\theta_2sin\theta_3 + cos\theta_3sin\theta_2) + a_3sin\theta_1(cos\theta_2cos\theta_3 - sin\theta_2sin\theta_3)$$

$$P_z = d_1 + a_2sin\theta_2 + a_4cos\theta_4(cos\theta_2sin\theta_3 + cos\theta_3sin\theta_2) + a_4sin\theta_4(cos\theta_2cos\theta_3 - sin\theta_2sin\theta_3) + a_3(cos\theta_2sin\theta_3 + cos\theta_3sin\theta_2)$$

2.2. Inverse Kinematic Analysis

The transformation of coordinates from world space to joint space is known as *backward or* ***inverse kinematic*** transformation. The inverse kinematics solution uses the position and orientation (P_x, P_y, P_z) of the robot's end effector which has been known to solve the joint angles $(\theta_1, \theta_2, \theta_3, \theta_4)$. In this study, the geometrical method was used.

We can demonstrate the (P_x, P_y, P_z) in shorter notation:

$$P_x = a_2 cos\theta_1 cos\theta_2 + a_4 cos\theta_1 cos\theta_4 \, cos(\theta_2 + \theta_3) - a_4 cos\theta_1 sin\theta_4 \, sin(\theta_2 + \theta_3) + a_3 cos\theta_1 cos(\theta_2 + \theta_3)$$

$$P_y = a_2 cos\theta_2 sin\theta_1 + a_4 cos\theta_4 sin\theta_1 \, cos(\theta_2 + \theta_3) - a_4 sin\theta_1 sin\theta_4 \, sin(\theta_2 + \theta_3) + a_3 sin\theta_1 cos(\theta_2 + \theta_3)$$

$$P_z = d_1 + a_2 sin\theta_2 + a_4 cos\theta_4 sin(\theta_2 + \theta_3) + a_4 sin\theta_4 cos(\theta_2 + \theta_3) + a_3 sin(\theta_2 + \theta_3)$$

2.2.1. Solutions of the arm joint angles $(\boldsymbol{\theta_1}, \boldsymbol{\theta_2}, \boldsymbol{\theta_3}, \boldsymbol{\theta_4})$

The position of first point can be determined from the homogeneous transformation matrix, which is derived from $T_1^0, T_2^1, T_2^3, T_3^4$ as shown:

$$T_4^0 = \prod_{i=1}^{4} T_i^{i-1} = T_1^0 \, T_2^1 \, T_3^2 \, T_4^3 \Rightarrow$$

$$\Rightarrow \begin{bmatrix} c_1c_4c_{23} - c_1s_4s_{23} & -c_1s_4c_{23} - c_1c_4s_{23} & s_1 & a_2c_1c_2 + a_4c_1c_4c_{23} - a_4c_1s_4s_{23} + a_3c_1c_{23} \\ c_4s_1c_{23} - s_1s_4s_{23} & -s_1s_4c_{23} - c_4s_1s_{23} & -c_1 & a_2c_2s_1 + a_4c_4s_1c_{23} - a_4s_1s_4s_{23} + a_3s_1c_{23} \\ c_4s_{23} + s_4c_{23} & c_4c_{23} - s_4s_{23} & 0 & d_1 + a_2s_2 + a_4c_4s_{23} + a_4s_4c_{23} + a_3s_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Elements of P can be described like this:

$$P_x = a_2 cos\theta_1 cos\theta_2 + a_4 cos\theta_1 cos\theta_4 \, cos(\theta_2 + \theta_3) - a_4 cos\theta_1 sin\theta_4 \, sin(\theta_2 + \theta_3) + a_3 cos\theta_1 cos(\theta_2 + \theta_3)$$

- $P_x \Rightarrow cos\theta_1(a_3 \, cos(\theta_2 + \theta_3) + a_2 cos\theta_2 + a_4 cos(\theta_2 + \theta_3 + \theta_4))$

$$P_y = a_2 cos\theta_2 sin\theta_1 + a_4 cos\theta_4 sin\theta_1 \, cos(\theta_2 + \theta_3) - a_4 sin\theta_1 sin\theta_4 \, sin(\theta_2 + \theta_3) + a_3 sin\theta_1 cos(\theta_2 + \theta_3)$$

- $P_y \Rightarrow sin\theta_1(a_3 \, cos(\theta_2 + \theta_3) + a_2 cos\theta_2 + a_4 cos(\theta_2 + \theta_3 + \theta_4))$

$$P_z = d_1 + a_2 sin\theta_2 + a_4 cos\theta_4 sin(\theta_2+\theta_3) + a_4 sin\theta_4 cos(\theta_2+\theta_3) + a_3 sin(\theta_2+\theta_3)$$

- $P_z \Rightarrow d_1 + a_3 \sin(\theta_2+\theta_3) + a_2 sin\theta_2 + a_4 sin(\theta_2+\theta_3+\theta_4)$

$$\frac{P_y}{P_x} = \frac{sin\theta_1}{cos\theta_1} = \theta_1 = A\ tan2(P_y, P_x)$$

We obtained more accurate result by using simplify(T) command on the command window in MATLAB. T is equal to the $\frac{P_y}{P_x}$.

If P_x is multiplied by c_1 and P_y is multiplied by s_1, A can be obtained explicitly by using MATLAB simplify command.

$$P_x.c_1 + P_y.s_1 = a_3\cos(\theta_2+\theta_3) + a_2\cos\theta_2 + a_4\cos(\theta_2+\theta_3+\theta_4) = A$$

$$c_{23} = \frac{(P_x.c_1 + P_y.s_1) - a_2\cos\theta_2 + a_4\cos(\theta_2+\theta_3+\theta_4)}{a_3}$$

And, also we obtained s_{23} by solving P_z

$$s_{23} = \frac{P_z - d_1 - a_2 sin\theta_2 - a_2 sin(\theta_2+\theta_3+\theta_4)}{a_3}$$

Substituting last two equations into the $c_{23}^2 + s_{23}^2 = 1$

$$\left((P_x.c_1 + P_y.s_1) - a_2\cos\theta_2 + a_4\cos(\theta_2+\theta_3+\theta_4)\right)^2 + \left(P_z - d_1 - a_2 sin\theta_2 - a_2 sin(\theta_2+\theta_3+\theta_4)\right)^2 = a_3^2$$

$$\left((P_x.c_1 + P_y.s_1 + a_4\cos(\theta_2+\theta_3+\theta_4)) - a_2\cos\theta_2\right)^2 + \left((P_z - d_1 - a_2 sin(\theta_2+\theta_3+\theta_4)) - a_2 sin\theta_2\right)^2 = a_3^2$$

$$\begin{aligned}(P_x.c_1 + P_y.s_1 + a_4\cos(\theta_2+\theta_3+\theta_4))^2 - 2(P_x.c_1 + P_y.s_1 + a_4\cos(\theta_2+\theta_3+\theta_4))a_2\cos\theta_2) + a_2^2c_2^2 \\ + (P_z - d_1 - a_2 sin(\theta_2+\theta_3+\theta_4)^2 - 2(P_z - d_1 - a_2 sin(\theta_2+\theta_3+\theta_4)a_2 sin\theta_2 + a_2^2s_2^2 = a_3^2\end{aligned}$$

$$\left(P_x.c_1 + P_y.s_1 + a_4\cos(\theta_2+\theta_3+\theta_4)\right)\cos\theta_2 + (P_z - d_1 - a_2 sin(\theta_2+\theta_3+\theta_4) sin\theta_2 =$$

$$\Rightarrow \frac{\left(P_x.c_1 + P_y.s_1 + a_4\cos(\theta_2+\theta_3+\theta_4)\right)^2 + a_2^2 + (P_z - d_1 - a_2 sin(\theta_2+\theta_3+\theta_4)^2 - a_3^2}{2a_2} = A$$

The equation can be regulated more comprehensible:

$$f = (P_z - d_1 - a_2 sin(\theta_2+\theta_3+\theta_4)$$

$$g = P_x.c_1 + P_y.s_1 + a_4\cos(\theta_2+\theta_3+\theta_4)$$

$$h = \frac{\left(P_x.c_1 + P_y.s_1 + a_4\cos(\theta_2+\theta_3+\theta_4)\right)^2 + a_2^2 + (P_z - d_1 - a_2 sin(\theta_2+\theta_3+\theta_4)^2 - a_3^2}{2a_2}$$

$$f\ sin\theta_2 + g\cos\theta_2 = h$$

If the approximations are considered,

$$g + h \neq 0, \qquad f\sqrt{f^2+g^2-h^2} - f^2 - g^2 - gh \neq 0 \to \theta_2$$

$$\approx 2.\left(3.14159n + tan^{-1}\left(\frac{f - \sqrt{f^2+g^2-h^2}}{f+g}\right)\right), n \in Z$$

$$g + h \neq 0, \qquad f\sqrt{f^2+g^2-h^2} + f^2 + g^2 + gh \neq 0 \to \theta_2$$

$$\approx 2.\left(3.14159n + tan^{-1}\left(\frac{f + \sqrt{f^2+g^2-h^2}}{f+g}\right)\right), n \in Z$$

$$f \neq 0, \quad f^2 + g^2 \neq 0, \quad h \approx -f \to \theta_2 \approx 2.\left(3.14159n + tan^{-1}\left(\frac{g}{f}\right)\right), n \in Z$$

$g = -f \rightarrow \theta_2 = 2\pi n + \pi, n \in Z$

And if $g = -f$, $x = 2\pi n + \pi$ it is possible to obtain as follows:

$$\theta_2 = Atan2\left(\frac{gh - \sqrt{f^4 + f^2g^2 - f^2h^2}}{f^2 + g^2}, \frac{1}{f}\left(\frac{g\sqrt{-f^2(-f^2 - g^2 + h^2)} - g^2h}{f^2 + g^2} + h\right)\right)$$

$$\theta_2 = Atan2\left(\frac{gh + \sqrt{f^4 + f^2g^2 - f^2h^2}}{f^2 + g^2}, \frac{1}{f}\left(\frac{-g\sqrt{-f^2(-f^2 - g^2 + h^2)} - g^2h}{f^2 + g^2} + h\right)\right)$$

If we consider c_{23} and s_{23} from the previous equations to obtain $tan(\theta_2 + \theta_3)$

$$tan(\theta_2 + \theta_3) = \frac{P_z - d_1 - a_2 sin\theta_2 - a_2 sin(\theta_2 + \theta_3 + \theta_4)}{\left(P_x.c_1 + P_y.s_1\right) - a_2 \cos\theta_2 + a_4 \cos(\theta_2 + \theta_3 + \theta_4)}$$

Now, θ_3 can obtained easily.

$$\theta_3 = Atan2\left(P_z - d_1 - a_2 sin\theta_2 - a_2 sin(\theta_2 + \theta_3 + \theta_4), \left(P_x.c_1 + P_y.s_1\right) - a_2 \cos\theta_2 + a_4 \cos(\theta_2 + \theta_3 + \theta_4)\right) - \theta_2$$