2. Analyses of Feeding Assistance Robot

2.1. Materials and Methods:

In this study, a 4 DOF artificial intelligence and image processing based feeding assistance robotic arm for disabled people was developed which is shown schematically in Figure 2.1. The presented robotic arm is composed of serial link which are affixed to each other with revolute joints from the base frame to the end-effector. All components were designed, assembled and analyzed by using Solidworks 2017 and MATLAB 2017b. Forward and inverse kinematic analysis were developed by using Denavit-Hartenberg Parameters. Dynamic components were analyzed by using standard mechanical formula and Solidworks 2018. The feeding assistance robot will be attached with people.

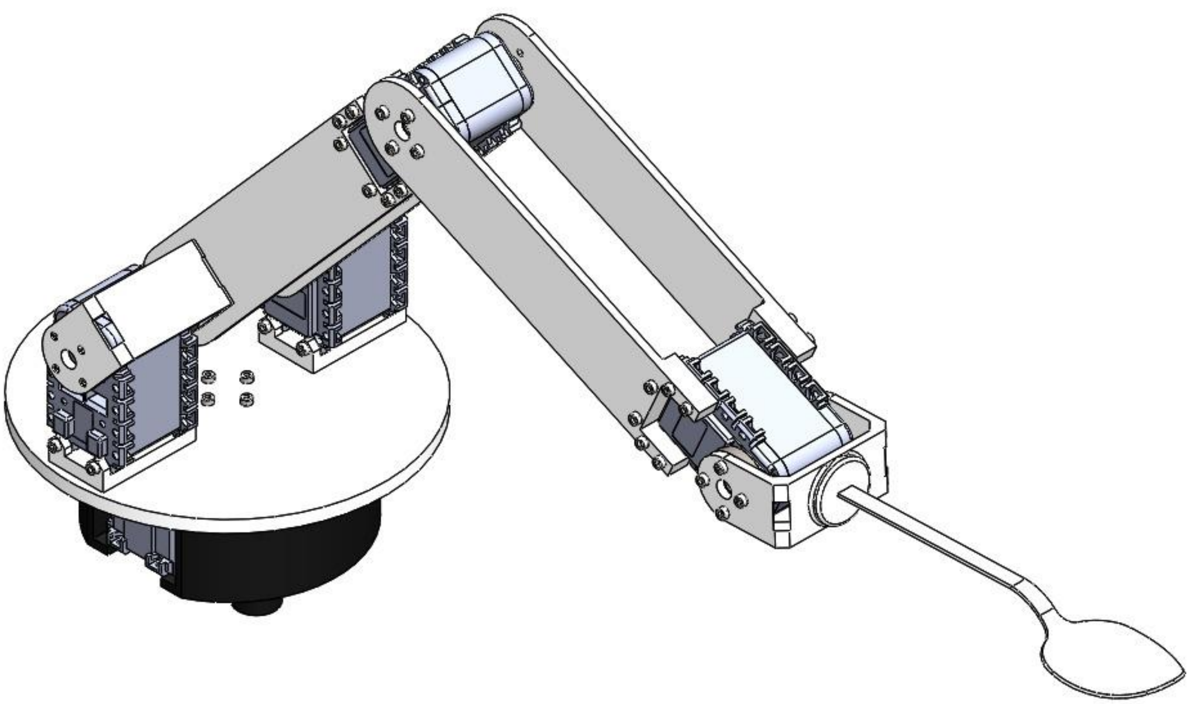


Figure 2.1: Assembled model of the feeding assistance robot

As a material selection, all components were manufactured from PLA (Polylactic Acid) due to the properties of this organic biopolymer and thermoplastic material. It is not harmful to people's health. When we compare with ABS, PLA has sleekier surface. 3D Printer using FDM technology can print PLA properly. The specifications of PLA are as follow:

- Hard structure. Therefore, PLA is durable and resistance to impacts.
- Slight flexibility but some brittle.
- No problems of plastic deformation during cooling process.
- Resistance to temperature.
- Difficult to dissolve with acetone.
- Very easy to print according to ABS.

2.2. Description of the Feeding Assistance Robot Workspace

The workspace of a manipulator is the total volume swept out by the end effector as the manipulator executes all possible motions. The workspace is constrained by the geometry of the manipulator as well as mechanical constraints on the joints. For example, a revolute joint may be limited to less than a full 360° of motion. The workspace is often broken down into a **reachable workspace** and a **dexterous workspace**. The reachable workspace is the entire set of points reachable by the manipulator, whereas the dexterous workspace consists of those points that the manipulator can reach with an arbitrary orientation of the end effector. Obviously the dexterous workspace is a subset of the reachable workspace. In shortly, the robot workspace or reachable spaces consist of all the points in the Cartesian space that the end effector of the robotic arm can access. The workspace and a quick access to a certain point in all robotic arms are strongly dependent on linkages properties, joint properties (length, angles, angular velocity and torque), degree of freedom, angle/translation limitations and robot configurations.

Consider the feeding assistance robot in the Figure 2.2. The left side shows the complete work envelope of the robot from the side view. The right side of Figure 2.2 shows the whole workspace from the top view. The maximum frontal distance covered by the arm is 45.826 cm. All dimensions have a tolerance in the range of 2–5 mm. The maximum height of access point is 50.176 cm. According to the accessible points in X and Z directions, the feeding assistance robot gas enough range of motion to be used in medical applications. In shortly, the workspace of feeding assistance robot has a suitable reach to user's mouth and bowls easily.

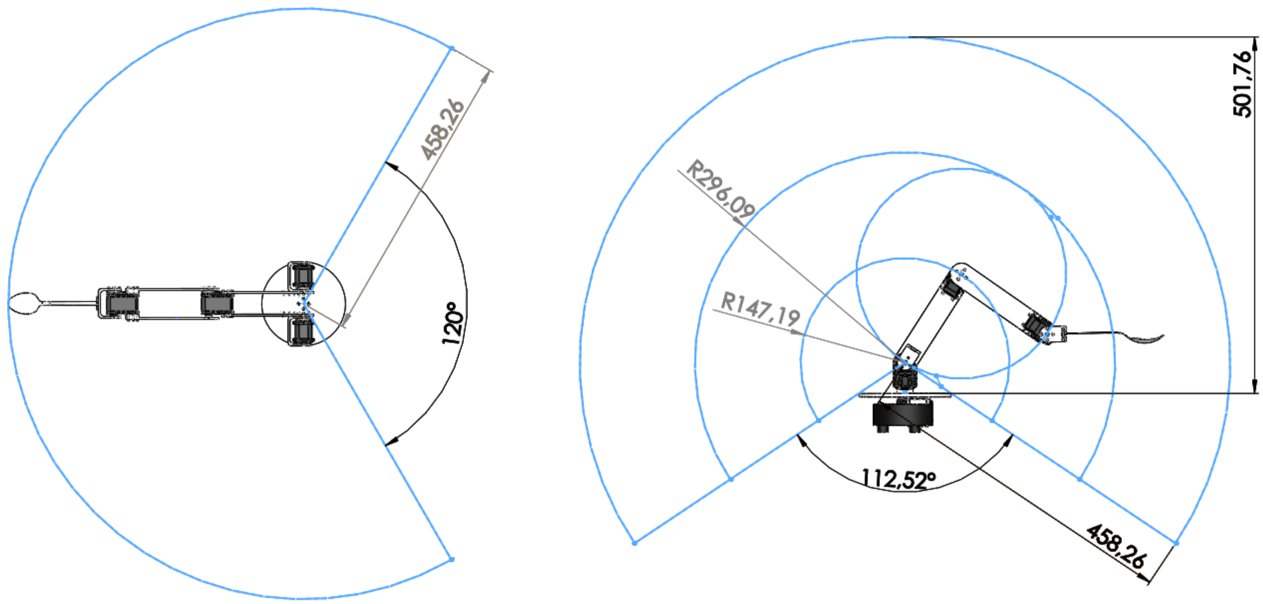
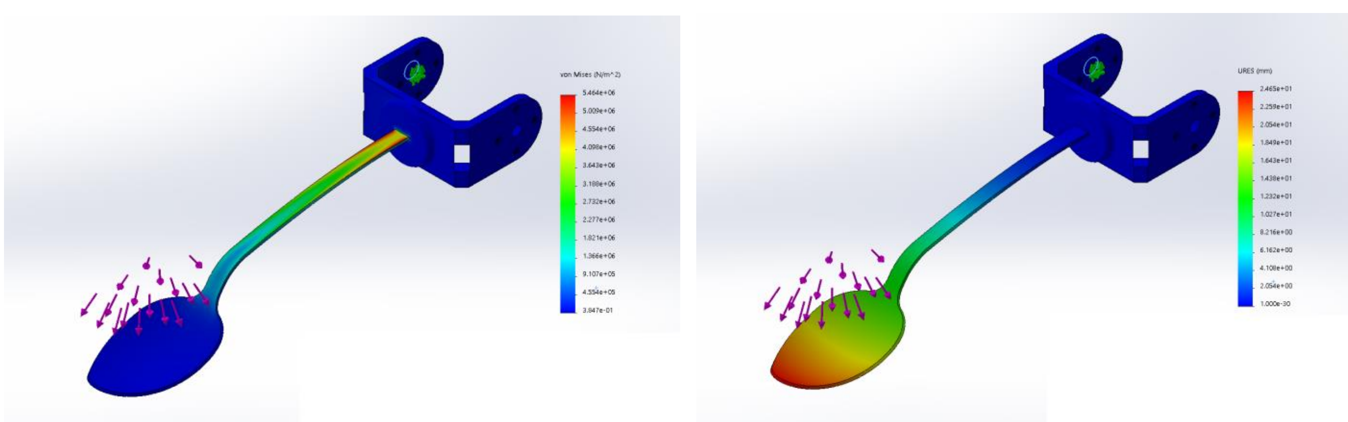


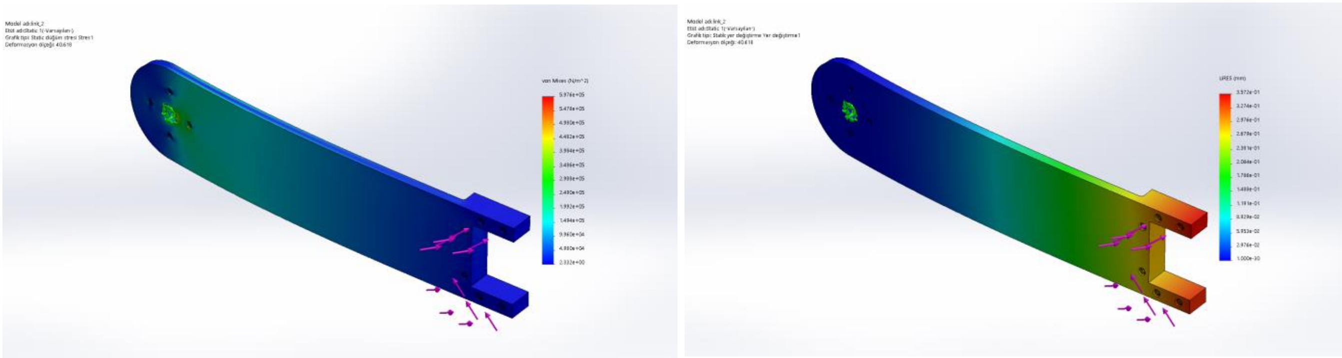
Figure 2.2: Workspace of the feeding assistance robot

2.3. Strength Analysis of the Robot Parts



(a) (b)

Figure 2.3.1: (a) Statically Stress Analysis of the end-effector of the robot. (b) Statically displacement Analysis of the end-effector of the robot



(a) (b)

Figure 2.3.2: (a) Statically Stress Analysis of the second link of the robot. (b) Statically displacement Analysis of the second link of the robot

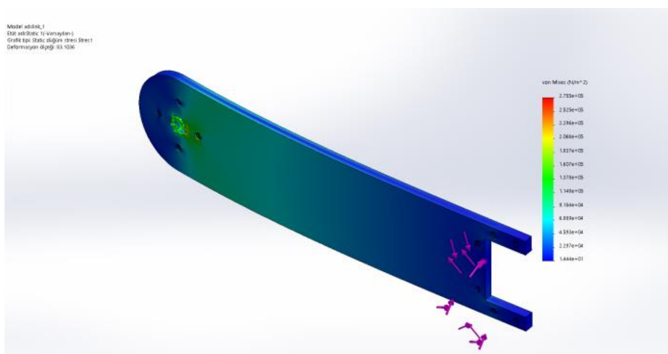
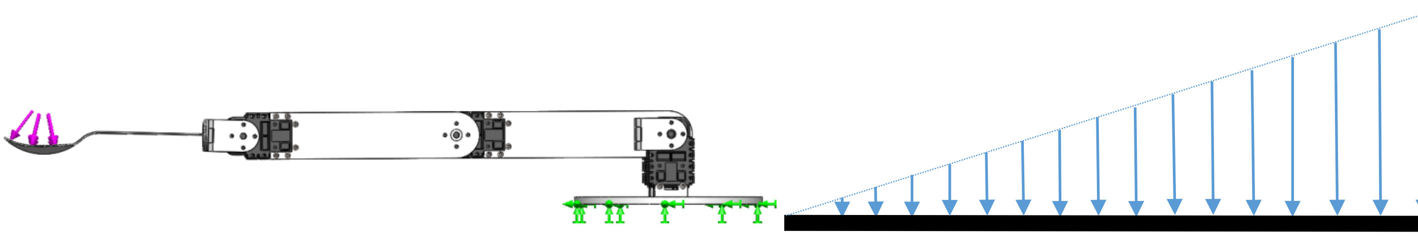


Figure 2.3.3: Statically Stress Analysis of the first link of the robot.



(a) (b)

Figure 2.3.4: (a) Orientation of Maximum Torque of the Robot. (b) Representation of Torque Distribution on the Robot

As shown in Figure 2.3.4, orientation of the robot must be like that to obtain maximum torque in each links and joints because of the minimize the cosine angle of joints. Maximum Torque will be in the farthest point because of the torque formula which is $\tau = Fr$.

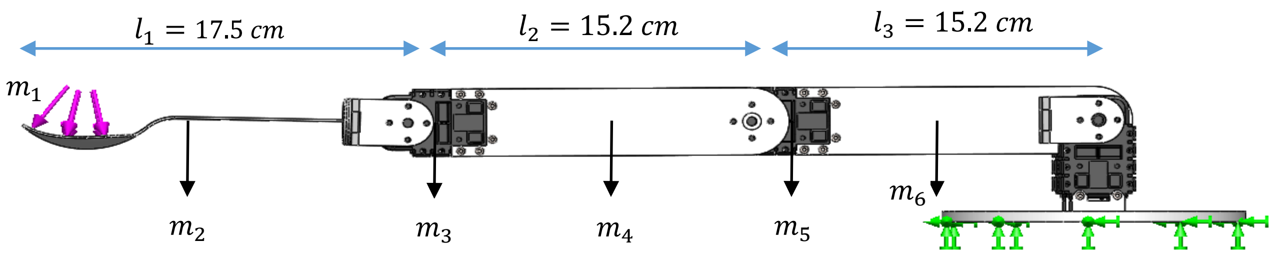


Figure 2.3.5: Orientation of Maximum Torque of the Robot

$m_1 = 0.06\ gram$ $\quad m_2 = 9.59\ gram$ $\quad m_3 = 58\ gram$

$m_4 = 14.24x2\ gram$ $\quad m_5 = 58\ gram$ $\quad m_6 = 12.59x2\ gram$

While we are finding the maximum torque value of the last joint, the load being lifted and mass of the end-effector are taken into account. Mass of the end-effector is equal to 9.59 gr and load being lifted is equal to 6 gr. Otherwise, the length of the end-effector is equal to 17.5 cm.

$$\tau_1 = m_1 l_1 + m_2 \frac{l_1}{2} = 6x10^{-3} x17.5 + 9.59x10^{-3} x \frac{17.5}{2} = 0.1889125\ kgcm$$

In the second joint, the mass of link 2 is equal to 14.24 gr and mass of the servo motor is equal to 58 gr. And, length of the second link is equal to 15.2 cm.

$$\tau_2 = m_1(l_1 + l_2) + m_2\left(\frac{l_1}{2} + l_2\right) + m_3 l_2 + m_4 \frac{l_2}{2}$$

$$6x10^{-3}x(17.5 + 15.2) + 9.59x10^{-3}x\left(\frac{17.5}{2} + 15.2\right) + 58x10^{-3}x15.2 + 14.24x10^{-3}x\frac{15.2}{2} = 1.524\ kg\ cm$$

In the first joint, the mass of link 3 is equal to 12.59 gr and mass of the servo motor is equal to 58 gr. And, length of the first link is equal to 15.2 cm.

$$\tau_3 = m_1(l_1 + l_2 + l_3) + m_2\left(\frac{l_1}{2} + l_2 + l_3\right) + m_3(l_2 + l_3) + m_4\left(\frac{l_2}{2} + l_3\right) + m_5 l_3 + m_6 \frac{l_3}{2} = 4.148\ kgcm$$

2.3.1. Equilibrium of the Feeding Assistance Robot Statically

Industrial Robots are exposed to many external forces during the motion. First one of the external forces is gravity. Gravity effect the all parts of robot continuously. In the Robot Control, gravity forces should be calculated in order to obtain requested performances. Second one of the external forces, the reaction forces resulting from the interaction of the robot with its environment. This type of the forces are external forces that is not affect the robot continuously in the end-effector. Especially, in the application of force control, external forces that affects to the end-effector should be calculated and feedback mechanism to obtain more smooth motion.

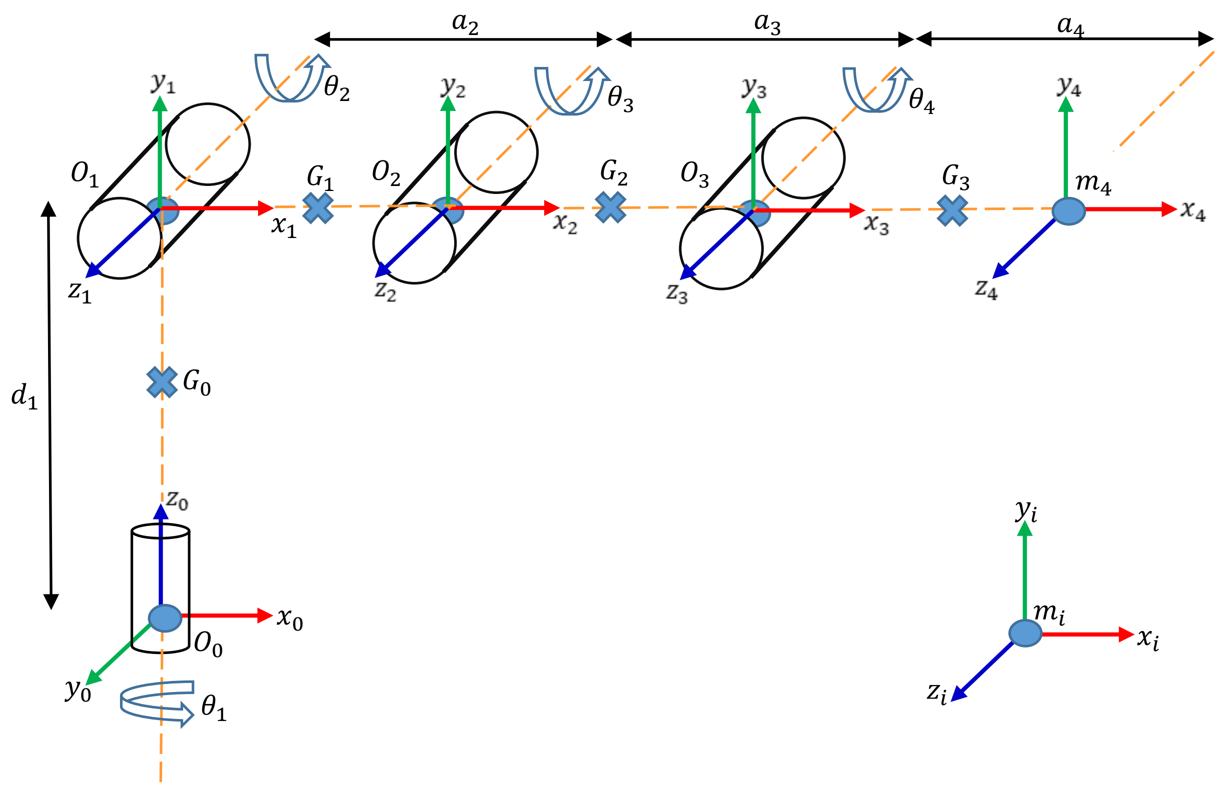


Figure 2.3.1: Representation of the center of gravity

m_1, m_2, m_3 *and* m_4 are the masses of the robot links and G_0, G_1, G_2 *and* G_3 are the center of gravities of the robot. z_i is the projection vector of the center of the gravities.

$$O_0G_0 = d_1 z_0 \quad , \quad O_1G_1 = a_{G_1} z_1 \quad , \quad O_2G_2 = a_{G_2} z_2 \quad , \quad O_3G_3 = a_{G_3} z_3$$

O_0G_3 should be found by using homogenous transformation matrices.

$g_1(\theta) = 0$

$g_2(\theta) = (m_2 + m_3 + m_4)ga_2cos\theta_2 + (m_3 + m_4)ga_3cos(\theta_2 + \theta_3) + m_4ga_4cos(\theta_2 + \theta_3 + \theta_4)$

$g_3(\theta) = (m_3 + m_4)ga_3cos(\theta_2 + \theta_3) + m_4ga_4cos(\theta_2 + \theta_3 + \theta_4)$

$g_4(\theta) = m_4ga_4cos(\theta_2 + \theta_3 + \theta_4)$

We obtained the gravity matrices like that:

$$g(\theta) = \begin{bmatrix} 0 \\ (m_2 + m_3 + m_4)ga_2cos\theta_2 + (m_3 + m_4)ga_3cos(\theta_2 + \theta_3) + m_4ga_4cos(\theta_2 + \theta_3 + \theta_4) \\ (m_3 + m_4)ga_3cos(\theta_2 + \theta_3) + m_4ga_4cos(\theta_2 + \theta_3 + \theta_4) \\ m_4ga_4cos(\theta_2 + \theta_3 + \theta_4) \end{bmatrix}$$

2.4. Kinematic Analysis of the Robot Arm

The problem of kinematics is to describe the motion of the manipulator without consideration of the forces and torques causing the motion. Denavit-Hartenberg demonstrates that a general transformation between two joints requires four parameters. These parameters, known as the Denavit-Hartenberg (D-H) parameters which became a standard to describe robot kinematics (Funda et. Al - 1990).

2.4.1. Forward Kinematic Analysis

The transformation of coordinates of the end-effector point from the joint space to the world space is known as ***forward kinematic*** transformation. It is also called more clearly, forward kinematic which is to determine the position and orientation of the end effector given the values for the joint variables of the robots.

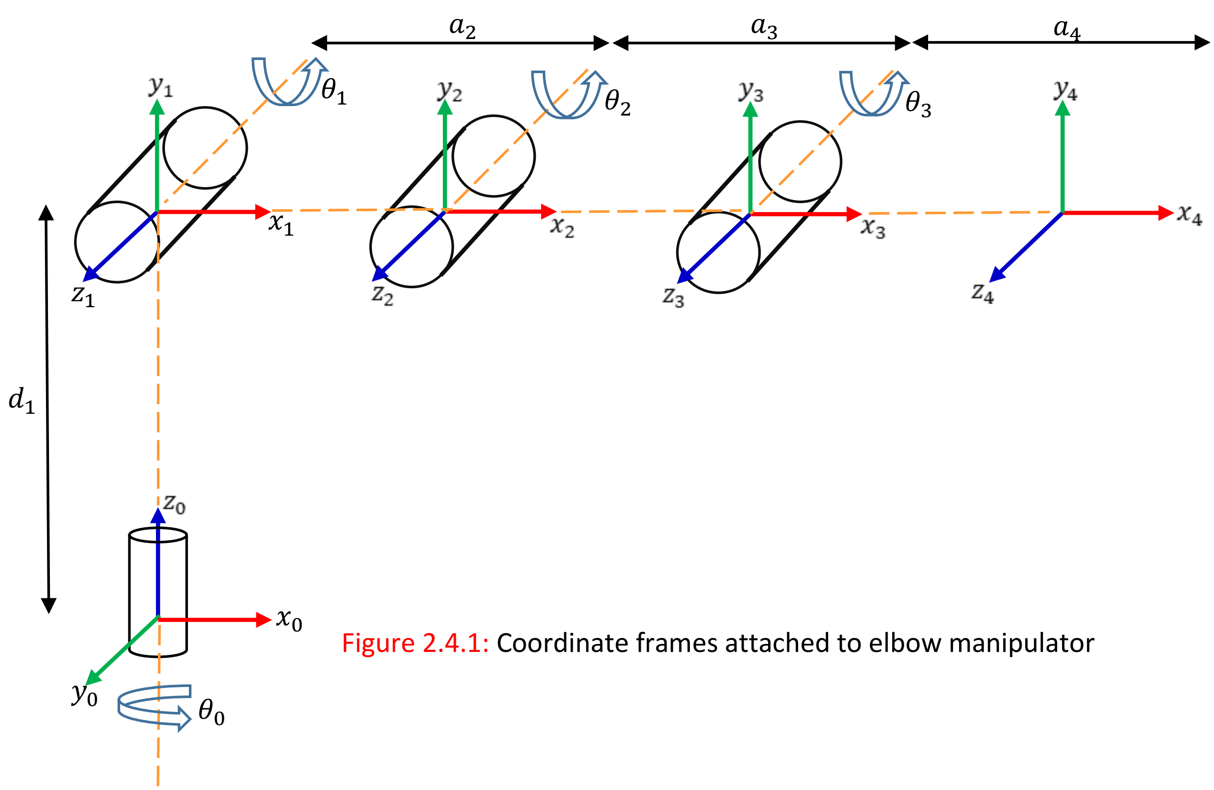


Figure 2.4.1: Coordinate frames attached to elbow manipulator

Axis Number i	Link Length a_i	Twist Angle α_i	Link offset d_i	Joint angle θ_i
1	0	$\pi/2$	d_1	θ_1^*
2	a_2	0	0	θ_2^*
3	a_3	0	0	θ_3^*
4	a_4	0	0	θ_4^*

Table 2.4.1: D-H Parameters of Robot Arm

Denavit – Hartenberg (D-H) method uses the four parameters including a_i,α_i, d_i and θ_i; which are the link length, twist angle, link offset and joint angle, respectively.

- θ_i , joint angle is angle from x_{i-1} to x_i measured around the z_{i-1}.
- d_i, link offset is distance from O_{i-1} to O_i measured along z_{i-1}.
- a_i, link length is distance from z_{i-1} to z_i measured along x_i.
- α_i, twist angle is angle from z_{i-1} to z_i measured along x_i

The matrix T_i^{i-1} is known as a D-H convention matrix given in this equation. In the matrix T_i^{i-1}, it means transformation matrix, the quantities of a_{i-1},α_{i-1}, d_i are constant for a given link while the parameter θ_i for a revolute joint is variable.

$$A_i = T_i^{i-1} = \begin{bmatrix} cos\theta_i & -cos\alpha_i sin\theta_i & sin\alpha_i sin\theta_i & a_i\, cos\theta_i \\ sin\theta_i & cos\alpha_i cos\theta_i & -sin\alpha_i cos\theta_i & a_i\, sin\theta_i \\ 0 & sin\alpha_i & cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We obtain the matrices that is shown below by using this formulation;

$$A_1 = T_1^0 = \begin{bmatrix} cos\theta_1 & 0 & sin\theta_1 & 0 \\ sin\theta_1 & 0 & -cos\theta_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = T_2^1 = \begin{bmatrix} cos\theta_2 & -sin\theta_2 & 0 & a_2\, cos\theta_2 \\ sin\theta_2 & cos\theta_2 & 0 & a_2 sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = T_3^2 = \begin{bmatrix} cos\theta_3 & -sin\theta_3 & 0 & a_3\, cos\theta_3 \\ sin\theta_3 & cos\theta_3 & 0 & a_3 sin\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = T_4^3 = \begin{bmatrix} cos\theta_4 & -sin\theta_4 & 0 & a_4\, cos\theta_4 \\ sin\theta_4 & cos\theta_4 & 0 & a_4 sin\theta_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then, the Transformation matrix of overall system is created by multiplying each matrix. T_4^0 matrix defined as follows,

$$T_4^0 = \prod_{i=1}^{4} T_i^{i-1} = T_1^0\, T_2^1\, T_3^2\, T_4^3 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ r_{41} & r_{42} & r_{43} & r_{44} \end{bmatrix} = A_1 A_2 A_3 A_4$$

$r_{11} = -c_1 c_4 (s_2 s_3 - c_2 c_3) - c_1 s_4 (c_2 s_3 + c_3 s_2)$
$r_{12} = c_1 s_4 (s_2 s_3 - c_2 c_3) - c_1 c_4 (c_2 s_3 + c_3 s_2)$
$r_{13} = sin\theta_1$
$r_{14} = a_2 c_1 c_2 - a_4 c_1 c_4 (s_2 s_3 - c_2 c_3) - a_4 c_1 s_4 (c_2 s_3 + c_3 s_2) + a_3 c_1 (c_2 c_3 - s_2 s_3)$
$r_{21} = -c_4 s_1 (s_2 s_3 - c_2 c_3) - s_1 s_4 (c_2 s_3 + c_3 s_2)$
$r_{22} = s_1 s_4 (s_2 s_3 - c_2 c_3) - c_4 s_1 (c_2 s_3 + c_3 s_2)$
$r_{23} = -cos\theta_1$
$r_{24} = a_2 c_2 s_1 - a_4 c_4 s_1 (s_2 s_3 - c_2 c_3) - a_4 s_1 s_4 (c_2 s_3 + c_3 s_2) + a_3 s_1 (c_2 c_3 - s_2 s_3)$
$r_{31} = c_4 (c_2 s_3 + c_3 s_2) + s_4 (c_2 c_3 - s_2 s_3)$
$r_{32} = c_4 (c_2 c_3 - s_2 s_3) - s_4 (c_2 s_3 + c_3 s_2)$
$r_{33} = 0$
$r_{34} = d_1 + a_2 s_2 + a_4 c_4 (c_2 s_3 + c_3 s_2) + a_4 s_4 (c_2 c_3 - s_2 s_3) + a_3 (c_2 s_3 + c_3 s_2)$
$r_{41} = 0$
$r_{42} = 0$
$r_{43} = 0$
$r_{44} = 1$

In the expressions of these elements of transformation matrix, the variables are defined as follow:

$$c_i = cos\theta_i \quad , s_i = sin\theta_i \quad , c_{ij} = cos(\theta_i + \theta_j) \quad , s_{ij} = sin(\theta_i + \theta_j)$$

When we regulate these elements, we obtain these notations

$r_{11} = c_1 c_4 c_{23} - c_1 s_4 s_{23} = \cos(\theta_2 + \theta_3 + \theta_4)\cos(\theta_1)$
$r_{12} = -c_1 s_4 c_{23} - c_1 c_4 s_{23} = -\sin(\theta_2 + \theta_3 + \theta_4)\cos(\theta_1)$
$r_{13} = s_1$
$r_{14} = a_2 c_1 c_2 + a_4 c_1 c_4 c_{23} - a_4 c_1 s_4 s_{23} + a_3 c_1 c_{23}$
$r_{21} = c_4 s_1 c_{23} - s_1 s_4 s_{23} = \cos(\theta_2 + \theta_3 + \theta_4)\sin(\theta_1)$
$r_{22} = -s_1 s_4 c_{23} - c_4 s_1 s_{23} = -\sin(\theta_2 + \theta_3 + \theta_4)\sin(\theta_1)$
$r_{23} = -c_1$
$r_{24} = a_2 c_2 s_1 + a_4 c_4 s_1 c_{23} - a_4 s_1 s_4 s_{23} + a_3 s_1 c_{23}$
$r_{31} = c_4 s_{23} + s_4 c_{23} = \sin(\theta_2 + \theta_3 + \theta_4)$
$r_{32} = c_4 c_{23} - s_4 s_{23} = \cos(\theta_2 + \theta_3 + \theta_4)$
$r_{33} = 0$
$r_{34} = d_1 + a_2 s_2 + a_4 c_4 s_{23} + a_4 s_4 c_{23} + a_3 s_{23}$
$r_{41} = 0$
$r_{42} = 0$
$r_{43} = 0$
$r_{44} = 1$

And transformation matrix (T_4^0) consist of as indicated:

$$\begin{bmatrix} c_{234}c_1 & -s_{234}c_1 & s_1 & a_2c_1c_2 + a_4c_1c_4c_{23} - a_4c_1s_4s_{23} + a_3c_1c_{23} \\ c_{234}s_1 & -s_{234}s_1 & -c_1 & a_2c_2s_1 + a_4c_4s_1c_{23} - a_4s_1s_4s_{23} + a_3s_1c_{23} \\ s_{234} & c_{234} & 0 & d_1 + a_2s_2 + a_4c_4s_{23} + a_4s_4c_{23} + a_3s_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It is possible to calculate the values of (P_x, P_y, P_z) with respect to the fixed coordinate system by using transformation matrix.

$$P_x = a_2cos\theta_1cos\theta_2 - a_4cos\theta_1cos\theta_4(sin\theta_2sin\theta_3 - cos\theta_2cos\theta_3)$$
$$-a_4cos\theta_1sin\theta_4(cos\theta_2sin\theta_3 + cos\theta_3sin\theta_2) + a_3cos\theta_1(cos\theta_2cos\theta_3 - sin\theta_2sin\theta_3)$$

$$P_y = a_2cos\theta_2sin\theta_1 - a_4cos\theta_4sin\theta_1(sin\theta_2sin\theta_3 - cos\theta_2cos\theta_3)$$
$$-a_4sin\theta_1sin\theta_4(cos\theta_2sin\theta_3 + cos\theta_3sin\theta_2) + a_3sin\theta_1(cos\theta_2cos\theta_3 - sin\theta_2sin\theta_3)$$

$$P_z = d_1 + a_2sin\theta_2 + a_4cos\theta_4(cos\theta_2sin\theta_3 + cos\theta_3sin\theta_2) + a_4sin\theta_4(cos\theta_2cos\theta_3 - sin\theta_2sin\theta_3) + a_3(cos\theta_2sin\theta_3 + cos\theta_3sin\theta_2)$$

Also, the transformation matrix can be demonstrated more clearly as follow:

$$\begin{bmatrix} c_{234}c_1 & -s_{234}c_1 & s_1 & c_1(a_3c_{23} + a_2c_2 + a_4c_{234}) \\ c_{234}s_1 & -s_{234}s_1 & -c_1 & s_1(a_3c_{23} + a_2c_2 + a_4c_{234}) \\ s_{234} & c_{234} & 0 & d_1 + a_3s_{23} + a_2s_2 + a_4s_{234} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.4.2. Inverse Kinematic Analysis

The transformation of coordinates from world space to joint space is known as *backward or* ***inverse kinematic*** transformation. The inverse kinematics solution uses the position and orientation (P_x, P_y, P_z) of the robot's end effector which has been known to solve the joint angles $(\theta_1, \theta_2, \theta_3, \theta_4)$. In this study, the geometrical method was used.

We can demonstrate the (P_x, P_y, P_z) in shorter notation:

$$P_x = a_2 cos\theta_1 cos\theta_2 + a_4 cos\theta_1 cos\theta_4\, cos(\theta_2+\theta_3) - a_4 cos\theta_1 sin\theta_4\, sin(\theta_2+\theta_3) + a_3 cos\theta_1 cos(\theta_2+\theta_3)$$

$$P_y = a_2 cos\theta_2 sin\theta_1 + a_4 cos\theta_4 sin\theta_1\, cos(\theta_2+\theta_3) - a_4 sin\theta_1 sin\theta_4\, sin(\theta_2+\theta_3) + a_3 sin\theta_1 cos(\theta_2+\theta_3)$$

$$P_z = d_1 + a_2 sin\theta_2 + a_4 cos\theta_4 sin(\theta_2+\theta_3) + a_4 sin\theta_4 cos(\theta_2+\theta_3) + a_3 sin(\theta_2+\theta_3)$$

Solutions of the arm joint angles $(\theta_1, \theta_2, \theta_3, \theta_4)$;

The position of first point can be determined from the homogeneous transformation matrix, which is derived from $T_1^0, T_2^1, T_2^3, T_3^4$ as shown:

$$T_4^0 = \prod_{i=1}^{4} T_i^{i-1} = T_1^0\, T_2^1\, T_3^2\, T_4^3 \Rightarrow$$

$$\Rightarrow \begin{bmatrix} c_{234}c_1 & -s_{234}c_1 & s_1 & c_1(a_3c_{23}+a_2c_2+a_4c_{234}) \\ c_{234}s_1 & -s_{234}s_1 & -c_1 & s_1(a_3c_{23}+a_2c_2+a_4c_{234}) \\ s_{234} & c_{234} & 0 & d_1+a_3s_{23}+a_2s_2+a_4s_{234} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Elements of P can be described like this:

$$P_x = a_2 cos\theta_1 cos\theta_2 + a_4 cos\theta_1 cos\theta_4\, cos(\theta_2+\theta_3) - a_4 cos\theta_1 sin\theta_4\, sin(\theta_2+\theta_3) + a_3 cos\theta_1 cos(\theta_2+\theta_3)$$

- $P_x \Rightarrow cos\theta_1(a_3\, cos(\theta_2+\theta_3) + a_2 cos\theta_2 + a_4 cos(\theta_2+\theta_3+\theta_4))$

$$P_y = a_2 cos\theta_2 sin\theta_1 + a_4 cos\theta_4 sin\theta_1\, cos(\theta_2+\theta_3) - a_4 sin\theta_1 sin\theta_4\, sin(\theta_2+\theta_3) + a_3 sin\theta_1 cos(\theta_2+\theta_3)$$

- $P_y \Rightarrow sin\theta_1(a_3\, cos(\theta_2+\theta_3) + a_2 cos\theta_2 + a_4 cos(\theta_2+\theta_3+\theta_4))$

$$P_z = d_1 + a_2 sin\theta_2 + a_4 cos\theta_4 sin(\theta_2+\theta_3) + a_4 sin\theta_4 cos(\theta_2+\theta_3) + a_3 sin(\theta_2+\theta_3)$$

- $P_z \Rightarrow d_1 + a_3 \sin(\theta_2+\theta_3) + a_2 sin\theta_2 + a_4 sin(\theta_2+\theta_3+\theta_4)$

$$\frac{P_y}{P_x} = \frac{sin\theta_1}{cos\theta_1} = tan\theta_1 \Rightarrow \theta_1 = tan^{-1}\left(\frac{P_y}{P_x}\right) \; or \; \theta_1 = Atan2(P_y, P_x)$$

We obtained more accurate result by using simplify(T) command on the command window in MATLAB. T is equal to the $\frac{P_y}{P_x}$.

If P_x is multiplied by c_1 and P_y is multiplied by s_1, A can be obtained explicitly by using MATLAB simplify command.

$$P_x.\,c_1 + P_y.\,s_1 = a_3\cos(\theta_2 + \theta_3) + a_2\cos\theta_2 + a_4\cos(\theta_2 + \theta_3 + \theta_4) = A$$

$$c_{23} = \frac{\left(P_x.\,c_1 + P_y.\,s_1\right) - a_2\cos\theta_2 + a_4\cos(\theta_2 + \theta_3 + \theta_4)}{a_3}$$

And, also we obtained s_{23} by solving P_z

$$s_{23} = \frac{P_z - d_1 - a_2 sin\theta_2 - a_2 sin(\theta_2 + \theta_3 + \theta_4)}{a_3}$$

Substituting last two equations into the $c_{23}^2 + s_{23}^2 = 1$

$$\left(\left(P_x.\,c_1 + P_y.\,s_1\right) - a_2\cos\theta_2 + a_4\cos(\theta_2 + \theta_3 + \theta_4)\right)^2 + \left(P_z - d_1 - a_2 sin\theta_2 - a_2 sin(\theta_2 + \theta_3 + \theta_4)\right)^2 = a_3^2$$

$$\left(\left(P_x.\,c_1 + P_y.\,s_1 + a_4\cos(\theta_2 + \theta_3 + \theta_4)\right) - a_2\cos\theta_2)\right)^2 + \left(\left(P_z - d_1 - a_2 sin(\theta_2 + \theta_3 + \theta_4)\right) - a_2 sin\theta_2\right)^2 = a_3^2$$

$$\begin{aligned}\left(P_x.\,c_1 + P_y.\,s_1 + a_4\cos(\theta_2 + \theta_3 + \theta_4)\right)^2 - 2\left(P_x.\,c_1 + P_y.\,s_1 + a_4\cos(\theta_2 + \theta_3 + \theta_4)\right)a_2\cos\theta_2) + a_2^2c_2^2 \\ + (P_z - d_1 - a_2 sin(\theta_2 + \theta_3 + \theta_4)^2 - 2(P_z - d_1 - a_2 sin(\theta_2 + \theta_3 + \theta_4)a_2 sin\theta_2 + a_2^2 s_2^2 \quad = a_3^2\end{aligned}$$

$$\left(P_x.c_1 + P_y.s_1 + a_4\cos(\theta_2+\theta_3+\theta_4)\right)\cos\theta_2 + \left(P_z - d_1 - a_2 sin(\theta_2+\theta_3+\theta_4)\right)sin\theta_2 =$$

$$\Rightarrow \frac{\left(P_x.c_1 + P_y.s_1 + a_4\cos(\theta_2+\theta_3+\theta_4)\right)^2 + a_2^2 + \left(P_z - d_1 - a_2 sin(\theta_2+\theta_3+\theta_4)\right)^2 - a_3^2}{2a_2} = A$$

The equation can be regulated more comprehensible:

$$f = \left(P_z - d_1 - a_2 sin(\theta_2+\theta_3+\theta_4)\right)$$

$$g = P_x.c_1 + P_y.s_1 + a_4\cos(\theta_2+\theta_3+\theta_4)$$

$$h = \frac{\left(P_x.c_1 + P_y.s_1 + a_4\cos(\theta_2+\theta_3+\theta_4)\right)^2 + a_2^2 + \left(P_z - d_1 - a_2 sin(\theta_2+\theta_3+\theta_4)\right)^2 - a_3^2}{2a_2}$$

$$f\ sin\theta_2 + g\cos\theta_2 = h$$

If the approximations are considered,

$$g + h \neq 0, \qquad f\sqrt{f^2+g^2-h^2} - f^2 - g^2 - gh \neq 0 \rightarrow \theta_2$$

$$\approx 2.\left(3.14159n + tan^{-1}\left(\frac{f - \sqrt{f^2+g^2-h^2}}{f+g}\right)\right), n \in Z$$

$$g + h \neq 0, \qquad f\sqrt{f^2+g^2-h^2} + f^2 + g^2 + gh \neq 0 \rightarrow \theta_2$$

$$\approx 2.\left(3.14159n + tan^{-1}\left(\frac{f + \sqrt{f^2+g^2-h^2}}{f+g}\right)\right), n \in Z$$

$$f \neq 0, \quad f^2 + g^2 \neq 0, \quad h \approx -f \rightarrow \theta_2 \approx 2.\left(3.14159n + tan^{-1}\left(\frac{g}{f}\right)\right), n \in Z$$

$g = -f \rightarrow \theta_2 = 2\pi n + \pi, n \ \in Z$

And if $g = -f$, $x = 2\pi n + \pi$ it is possible to obtain as follows:

$$\theta_2 = Atan2\left(\frac{gh - \sqrt{f^4 + f^2 g^2 - f^2 h^2}}{f^2 + g^2}, \frac{1}{f}\left(\frac{g\sqrt{-f^2(-f^2 - g^2 + h^2)} - g^2 h}{f^2 + g^2} + h\right)\right)$$

$$\theta_2 = Atan2\left(\frac{gh + \sqrt{f^4 + f^2 g^2 - f^2 h^2}}{f^2 + g^2}, \frac{1}{f}\left(\frac{-g\sqrt{-f^2(-f^2 - g^2 + h^2)} - g^2 h}{f^2 + g^2} + h\right)\right)$$

If we consider c_{23} and s_{23} from the previous equations to obtain $tan(\theta_2 + \theta_3)$

$$tan(\theta_2 + \theta_3) = \frac{P_z - d_1 - a_2 sin\theta_2 - a_2 sin(\theta_2 + \theta_3 + \theta_4)}{\left(P_x . c_1 + P_y . s_1\right) - a_2 \cos\theta_2 + a_4 \cos(\theta_2 + \theta_3 + \theta_4)}$$

Now, θ_3 can obtained easily.

$$\theta_3 = Atan2\left(P_z - d_1 - a_2 sin\theta_2 - a_2 sin(\theta_2 + \theta_3 + \theta_4), \left(P_x . c_1 + P_y . s_1\right) - a_2 \cos\theta_2 + a_4 \cos(\theta_2 + \theta_3 + \theta_4)\right) - \theta_2$$

$$\frac{P_{y_4^3}}{P_{x_4^3}} = \frac{s_4}{c_4} = \theta_4 = Atan2\left(P_{y_4^3}, P_{x_4^3}\right)$$

2.5. Dynamic Analysis of the Robot Arm

The kinematic equations describe the motion of the robot without consideration of the forces and torques producing the motion, the dynamics equations describe the relationship between force and motion. The equations of motion are important to consider in the design of robots, in simulation and animation of robot motion and in the design of control algorithms.

Euler Lagrange equations, it means Lagrangian Formulation, describes the evolution of a mechanical system subject to holonomic constraints. In order to determine the Euler-Lagrange equations in a specific situation, one has to form the Lagrangian of the system which is difference between the kinetic energy and potential energy.

Newton-Euler Formulation which is a recursive formulation of the dynamic equations that is often used for numerical calculation.

$$Forward\ Dynamics: Given\ \theta, \dot{\theta}, \tau\ and\ find\ \ddot{\theta}$$
$$Inverse\ Dynamics: Given\ \theta, \dot{\theta}, \ddot{\theta}\ and\ find\ \tau$$

The Forward dynamics problem is to calculate the joint accelerations $\ddot{\theta}$, given the current joint positions θ, the joint velocities $\dot{\theta}$, and the forces and torques τ applied at each joint. The forward dynamics is useful for simulation. The inverse dynamics problem is to find the joint forces and torques τ needed to create the acceleration $\ddot{\theta}$ for the given joint positions and velocities that is $\theta, \dot{\theta}$ respectively. The inverse dynamics is useful in control of robots. Robot dynamics is necessary not just for simulation and control but also for the analysis of robot motion planners and controller.

- Lagrangian Formulation, a variational approach based on the kinetic and potential energy of the robot.
- Newton-Euler Formulation, which relies on $F = m.a$ applied to each individual link of the robot.

2.5.1. Lagrangian Formulation

Lagrangian mechanics is based on the differentiation energy terms only with respect to the System's variables and time.

$$\mathcal{L}(\theta, \dot{\theta}) = K(\theta, \dot{\theta}) - P(\theta)$$

which is $K(\theta, \dot{\theta})$ is Kinetic Energy of the System and $P(\theta)$ is Potential Energy of the System.

The vector of joint forces and torques can be demonstrated as follows:

$$F = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} \quad , \quad F_i = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) - \frac{\partial \mathcal{L}}{\partial x_i}$$

$$\tau = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} \quad , \quad \tau_i = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_i}\right) - \frac{\partial \mathcal{L}}{\partial \theta_i}$$

which is the meaning of $\mathcal{L}$ is Lagrange function of joint variables in the Euler Lagrange equation. $\mathcal{L}$ is defined as the difference between the total kinetic energy of the system and the total internal energy. The internal energy of the rigid linked robot can be neglected.

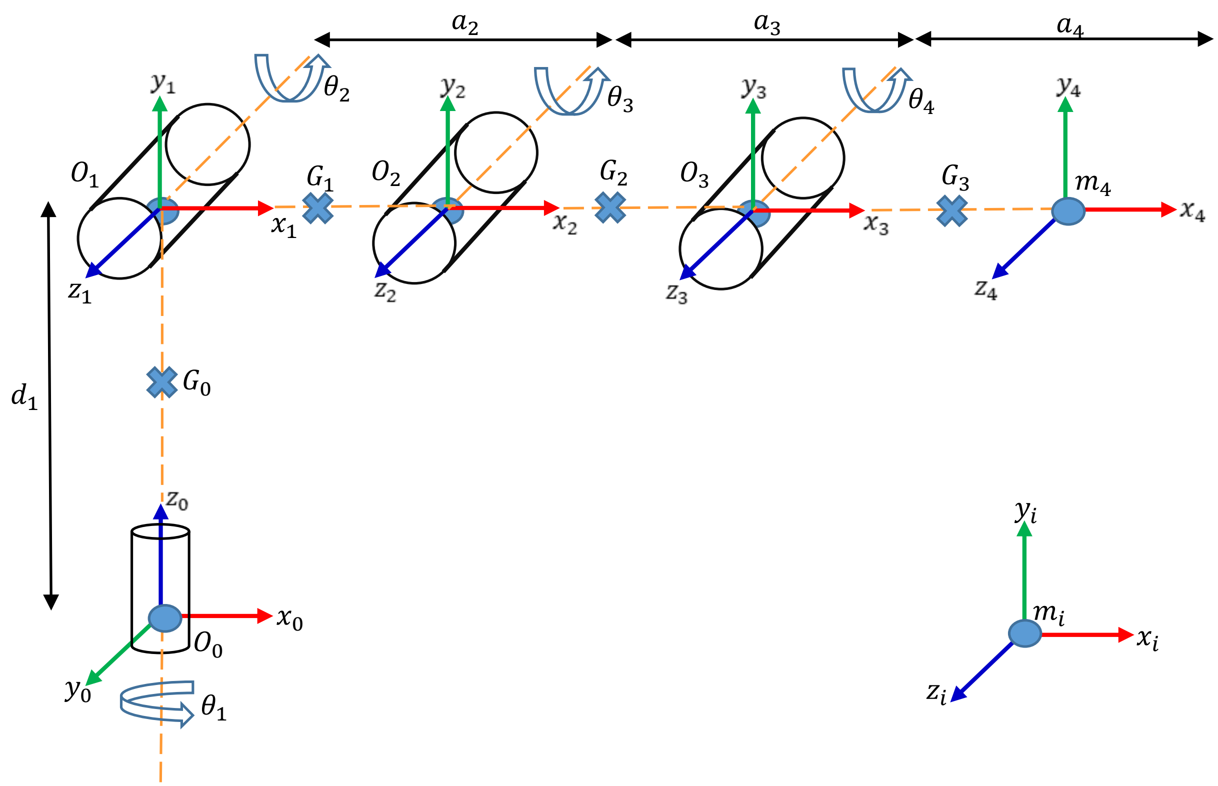


Figure 2.5.1: Coordinate frames attached to elbow manipulator

2.5.1.1. Defining and calculating the rotation matrices of the system:

$$D_0^1 = \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad D_1^2 = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D_2^3 = \begin{bmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad D_3^4 = \begin{bmatrix} c_4 & s_4 & 0 \\ -s_4 & c_4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D_0^2 = \begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad D_0^3 = \begin{bmatrix} c_{123} & s_{123} & 0 \\ -s_{123} & c_{123} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad D_0^4 = \begin{bmatrix} c_{1234} & s_{1234} & 0 \\ -s_{1234} & c_{1234} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.5.1.2. Defining the angular velocities of the sequential links:

$$\omega_0^0(R_0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \omega_1^0(R_0) = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \qquad \omega_2^0(R_1) = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\omega_3^0(R_2) = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{bmatrix} \qquad \omega_4^0(R_3) = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_4 \end{bmatrix}$$

2.5.1.3. Calculating the angular velocities of the overall links:

$$\omega_1^0(R_1) = \omega_1^0(R_0) + D_0^1\, \omega_0^0(R_0) = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$$

$$\omega_2^0(R_2) = \omega_2^1(R_1) + D_1^2\, \omega_1^0(R_1) = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}$$

$$\omega_3^0(R_3) = \omega_3^2(R_2) + D_2^3\, \omega_2^0(R_2) = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{bmatrix} + \begin{bmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}$$

$$\omega_4^0(R_4) = \omega_4^3(R_3) + D_3^4\, \omega_3^0(R_3) = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_4 \end{bmatrix} + \begin{bmatrix} c_4 & s_4 & 0 \\ -s_4 & c_4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 + \dot{\theta}_4 \end{bmatrix}$$

There are 2 types of kinetic energy of the system and one of them is translational kinetic energy and other is rotational kinetic energy.

2.5.1.4. Calculating of the Rotational Kinetic Energy:

$$K_{E_1}^{(R)} = \frac{1}{2}[\omega_1^0(R_1)]^T\, I^1\, [\omega_1^0(R_1)] = \frac{1}{2}[0 \quad 0 \quad \dot{\theta}_1].\begin{bmatrix} I_{xx}^1 & I_{xy}^1 & I_{xz}^1 \\ I_{yx}^1 & I_{yy}^1 & I_{yz}^1 \\ I_{zx}^1 & I_{zy}^1 & I_{zz}^1 \end{bmatrix}.\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} = \frac{1}{2} I_{zz}^1 \left(\dot{\theta}_1\right)^2$$

$$K_{E_2}^{(R)} = \frac{1}{2}[\omega_2^0(R_2)]^T\, I^2\, [\omega_2^0(R_2)] = \frac{1}{2}[0 \quad 0 \quad \dot{\theta}_1 + \dot{\theta}_2].\begin{bmatrix} I_{xx}^2 & I_{xy}^2 & I_{xz}^2 \\ I_{yx}^2 & I_{yy}^2 & I_{yz}^2 \\ I_{zx}^2 & I_{zy}^2 & I_{zz}^2 \end{bmatrix}.\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}$$

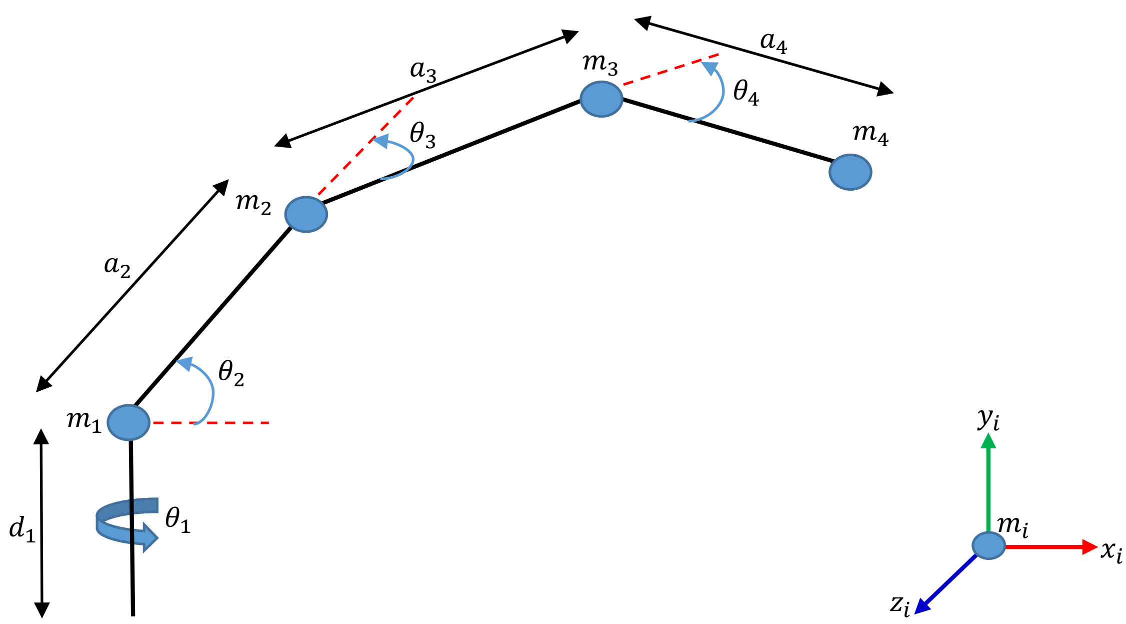
$$= \frac{1}{2} I_{zz}^2 \left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 = \frac{1}{2} I_{zz}^2 \left(\dot{\theta}_1\right)^2 + \frac{1}{2} I_{zz}^2 \left(\dot{\theta}_2\right)^2 + \frac{1}{2} I_{zz}^2 \left(\dot{\theta}_1 \dot{\theta}_2\right)$$

$$K_{E_3}^{(R)} = \frac{1}{2}[\omega_3^0(R_3)]^T\, I^3\, [\omega_3^0(R_3)] = \frac{1}{2}[0 \quad 0 \quad \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3].\begin{bmatrix} I_{xx}^3 & I_{xy}^3 & I_{xz}^3 \\ I_{yx}^3 & I_{yy}^3 & I_{yz}^3 \\ I_{zx}^3 & I_{zy}^3 & I_{zz}^3 \end{bmatrix}.\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}$$

$$= \frac{1}{2} I_{zz}^3 \left(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3\right)^2 = \frac{1}{2} I_{zz}^3 \left(\dot{\theta}_1\right)^2 + \frac{1}{2} I_{zz}^3 \left(\dot{\theta}_2\right)^2 + \frac{1}{2} I_{zz}^3 \left(\dot{\theta}_3\right)^2 + \frac{1}{2} I_{zz}^3 \left(\dot{\theta}_1 \dot{\theta}_2\right) + \frac{1}{2} I_{zz}^3 \left(\dot{\theta}_1 \dot{\theta}_3\right) + \frac{1}{2} I_{zz}^3 \left(\dot{\theta}_2 \dot{\theta}_3\right)$$

$$K_{E_4}^{(R)} = \frac{1}{2}[\omega_4^0(R_4)]^T\, I^4\, [\omega_4^0(R_4)] = \frac{1}{2}[0 \quad 0 \quad \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 + \dot{\theta}_4].\begin{bmatrix} I_{xx}^4 & I_{xy}^4 & I_{xz}^4 \\ I_{yx}^4 & I_{yy}^4 & I_{yz}^4 \\ I_{zx}^4 & I_{zy}^4 & I_{zz}^4 \end{bmatrix}.\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 + \dot{\theta}_4 \end{bmatrix}$$

$$= \frac{1}{2} I_{zz}^4 \left(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 + \dot{\theta}_4\right)^2$$



2.5.1.5. Calculating of the linear velocity vectors of the center of masses:

$$v^0_{G_0}(R_0) = 0$$

$$v^0_{G_1}(R_1) = v^0_1(R_1) + v^1_{G_1}(R_1) + \omega^0_1(R_1) x O_1G_1(R_1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} x \begin{bmatrix} a_{G_2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ a_{G_1}\dot{\theta}_1 \\ 0 \end{bmatrix}$$

which is the meaning of a_{G_1} is the length from G_1 to O_1 on the second link that its length is a_2.

$$v^0_2(R_1) = v^0_1(R_1) + v^1_2(R_1) + \omega^0_1(R_1) x O_1O_2(R_1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} x \begin{bmatrix} a_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ a_2\dot{\theta}_1 \\ 0 \end{bmatrix}$$

which is the definition of a_2 is the length of the second link.

$$v^0_2(R_2) = D^2_1\, v^0_2(R_1) = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ a_2\dot{\theta}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_2\dot{\theta}_1 s_2 \\ a_2\dot{\theta}_1 c_2 \\ 0 \end{bmatrix}$$

$$v_{G_2}^0(R_2) = v_2^0(R_2) + v_{G_2}^2(R_2) + \omega_2^0(R_2) x O_2G_2(R_2) = \begin{bmatrix} a_2\dot{\theta}_1 s_2 \\ a_2\dot{\theta}_1 c_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} x \begin{bmatrix} a_{G_2} \\ 0 \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} a_2\dot{\theta}_1 s_2 \\ a_{G_2}(\dot{\theta}_1 + \dot{\theta}_2) + a_2\dot{\theta}_1 c_2 \\ 0 \end{bmatrix}$$

which is the meaning of a_{G_2} is the length from G_2 to O_2 on the third link that its length is a_3.

$$v_3^0(R_2) = v_2^0(R_2) + v_3^2(R_2) + \omega_2^0(R_2) x O_2O_3(R_2) = \begin{bmatrix} a_2\dot{\theta}_1 s_2 \\ a_2\dot{\theta}_1 c_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} x \begin{bmatrix} a_3 \\ 0 \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} a_2\dot{\theta}_1 s_2 \\ a_3(\dot{\theta}_1 + \dot{\theta}_2) + a_2\dot{\theta}_1 c_2 \\ 0 \end{bmatrix}$$

which is the definition of a_3 is the length of the third link.

$$v_3^0(R_3) = D_2^3\, v_3^0(R_2) = \begin{bmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} a_2\dot{\theta}_1 s_2 \\ a_3(\dot{\theta}_1 + \dot{\theta}_2) + a_2\dot{\theta}_1 c_2 \\ 0 \end{bmatrix} = \begin{bmatrix} (a_2 s_{23} + a_3 s_3)\dot{\theta}_1 + a_3 s_3 \dot{\theta}_2 \\ (a_2 c_{23} + a_3 c_3)\dot{\theta}_1 + a_3 c_3 \dot{\theta}_2 \\ 0 \end{bmatrix}$$

$$v_{G_3}^0(R_3) = v_3^0(R_3) + v_{G_3}^3(R_3) + \omega_3^0(R_3) x O_3G_3(R_3) =$$

$$= \begin{bmatrix} (a_2 s_{23} + a_3 s_3)\dot{\theta}_1 + a_3 s_3 \dot{\theta}_2 \\ (a_2 c_{23} + a_3 c_3)\dot{\theta}_1 + a_3 c_3 \dot{\theta}_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix} x \begin{bmatrix} a_{G_3} \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} (a_3 s_3 + a_2 s_{23})\dot{\theta}_1 + a_3 s_3 \dot{\theta}_2 \\ (a_{G_3} + a_3 c_3 + a_2 c_{23})\dot{\theta}_1 + (a_{G_3} + a_3 c_3)\dot{\theta}_2 + a_{G_3}\dot{\theta}_3 \\ 0 \end{bmatrix}$$

$$v_4^0(R_3) = v_3^0(R_3) + v_4^3(R_3) + \omega_3^0(R_3) x O_3 O_4(R_3) =$$

$$= \begin{bmatrix} (a_3 s_3 + a_2 s_{23})\dot{\theta}_1 + a_3 s_3 \dot{\theta}_2 \\ \left(a_{G_3} + a_3 c_3 + a_2 c_{23}\right)\dot{\theta}_1 + \left(a_{G_3} + a_3 c_3\right)\dot{\theta}_2 + a_{G_3}\dot{\theta}_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix} x \begin{bmatrix} a_4 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} (a_3 s_3 + a_2 s_{23})\dot{\theta}_1 + a_3 s_3 \dot{\theta}_2 \\ \left(a_{G_3} + a_4 + a_3 c_3 + a_2 c_{23}\right)\dot{\theta}_1 + \left(a_{G_3} + a_4 + a_3 c_3\right)\dot{\theta}_2 + \left(a_{G_3} + a_4\right)\dot{\theta}_3 \\ 0 \end{bmatrix}$$

$$v_4^0(R_4) = D_3^4 \, v_4^0(R_3) =$$

$$= \begin{bmatrix} c_4 & s_4 & 0 \\ -s_4 & c_4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (a_3 s_3 + a_2 s_{23})\dot{\theta}_1 + a_3 s_3 \dot{\theta}_2 \\ \left(a_{G_3} + a_4 + a_3 c_3 + a_2 c_{23}\right)\dot{\theta}_1 + \left(a_{G_3} + a_4 + a_3 c_3\right)\dot{\theta}_2 + \left(a_{G_3} + a_4\right)\dot{\theta}_3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\left(a_{G_3} + a_4 + a_3 c_3 + a_2 c_{23}\right)s_4 + (a_3 s_3 + a_2 s_{23})c_4\right)\dot{\theta}_1 + \left(s_4\left(a_{G_3} + a_4 + a_3 c_3\right) + a_3 s_3 c_4\right)\dot{\theta}_2 + \left(a_{G_3} + a_4\right)s_4 \dot{\theta}_3 \\ \left(\left(a_{G_3} + a_4 + a_3 c_3 + a_2 c_{23}\right)c_4 - (a_3 s_3 + a_2 s_{23})s_4\right)\dot{\theta}_1 + \left(c_4\left(a_{G_3} + a_4 + a_3 c_3\right) - a_3 s_3 s_4\right)\dot{\theta}_2 + \left(a_{G_3} + a_4\right)c_4 \dot{\theta}_3 \\ 0 \end{bmatrix}$$

$$v_{G_4}^0(R_4) = v_4^0(R_4) + v_{G_4}^4(R_4) + \omega_4^0(R_4) x O_4 G_4(R_4) =$$

$$= \begin{bmatrix} \left(\left(a_{G_3} + a_4 + a_3 c_3 + a_2 c_{23}\right)s_4 + (a_3 s_3 + a_2 s_{23})c_4\right)\dot{\theta}_1 + \left(s_4\left(a_{G_3} + a_4 + a_3 c_3\right) + a_3 s_3 c_4\right)\dot{\theta}_2 + \left(a_{G_3} + a_4\right)s_4 \dot{\theta}_3 \\ \left(\left(a_{G_3} + a_4 + a_3 c_3 + a_2 c_{23}\right)c_4 - (a_3 s_3 + a_2 s_{23})s_4\right)\dot{\theta}_1 + \left(c_4\left(a_{G_3} + a_4 + a_3 c_3\right) - a_3 s_3 s_4\right)\dot{\theta}_2 + \left(a_{G_3} + a_4\right)c_4 \dot{\theta}_3 \\ 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 + \dot{\theta}_4 \end{bmatrix} x \begin{bmatrix} a_4 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\left(a_{G_3} + a_4 + a_3 c_3 + a_2 c_{23}\right)s_4 + (a_3 s_3 + a_2 s_{23})c_4\right)\dot{\theta}_1 + \left(s_4\left(a_{G_3} + a_4 + a_3 c_3\right) + a_3 s_3 c_4\right)\dot{\theta}_2 + \left(a_{G_3} + a_4\right)s_4 \dot{\theta}_3 \\ \left(\left(a_{G_3} + a_4 + a_3 c_3 + a_2 c_{23}\right)c_4 - (a_3 s_3 + a_2 s_{23})s_4 + a_4\right)\dot{\theta}_1 + \left(c_4\left(a_{G_3} + a_4 + a_3 c_3\right) - a_3 s_3 s_4 + a_4\right)\dot{\theta}_2 + \left(c_4\left(a_{G_3} + a_4\right) + a_4\right)\dot{\theta}_3 + a_4 \dot{\theta}_4 \\ 0 \end{bmatrix}$$

2.5.1.6. Calculating of the translational kinetic energies of the system:

$$K_{E_1}^{(T)} = \frac{1}{2}m_1\left[V_{G_1}^0(R_1)\right]^T\left[V_{G_1}^0(R_1)\right] = \frac{1}{2}m_1\begin{bmatrix}0 & a_{G_1}\dot{\theta}_1 & 0\end{bmatrix}\begin{bmatrix}0\\ a_{G_1}\dot{\theta}_1\\ 0\end{bmatrix} = \frac{1}{2}m_1\left(a_{G_1}\dot{\theta}_1\right)^2$$

$$K_{E_2}^{(T)} = \frac{1}{2}m_2\left[V_{G_2}^0(R_2)\right]^T\left[V_{G_2}^0(R_2)\right] = \frac{1}{2}m_2\begin{bmatrix}a_2s_2\dot{\theta}_1 & (a_{G_2}+a_2c_2)\dot{\theta}_1 + a_{G_2}\dot{\theta}_2 & 0\end{bmatrix}\begin{bmatrix}a_2s_2\dot{\theta}_1\\ (a_{G_2}+a_2c_2)\dot{\theta}_1 + a_{G_2}\dot{\theta}_2\\ 0\end{bmatrix}$$

$$\Rightarrow \left(\frac{1}{2}m_2\left(a_2^2 + a_{G_2}^2\right) + m_2a_{G_2}a_2c_2\right)\left(\dot{\theta}_1\right)^2 + \frac{1}{2}m_2a_{G_2}^2\left(\dot{\theta}_2\right)^2 + m_2\left(a_{G_2} + a_2c_2\right)a_{G_2}\dot{\theta}_1\dot{\theta}_2$$

$$K_{E_3}^{(T)} = \frac{1}{2}m_3\left[V_{G_3}^0(R_3)\right]^T\left[V_{G_3}^0(R_3)\right] \Rightarrow$$

$$= \frac{1}{2}m_3\begin{bmatrix}(a_3s_3 + a_2s_{23})\dot{\theta}_1 + a_3s_3\dot{\theta}_2 & (a_{G_3} + a_3c_3 + a_2c_{23})\dot{\theta}_1 + (a_{G_3} + a_3c_3)\dot{\theta}_2 + a_{G_3}\dot{\theta}_3 & 0\end{bmatrix}\begin{bmatrix}(a_3s_3 + a_2s_{23})\dot{\theta}_1 + a_3s_3\dot{\theta}_2\\ (a_{G_3} + a_3c_3 + a_2c_{23})\dot{\theta}_1 + (a_{G_3} + a_3c_3)\dot{\theta}_2 + a_{G_3}\dot{\theta}_3\\ 0\end{bmatrix}$$

$$= \frac{1}{2}m_3\left(a_2^2 + a_3^2 + 2a_3a_2c_2 + 2a_{G_3}(a_3c_3 + a_2c_{23})\right)\left(\dot{\theta}_1\right)^2 + \frac{1}{2}m_3\left(a_3^2 + a_{G_3}^2 + 2a_{G_3}a_3c_3\right)\left(\dot{\theta}_2\right)^2$$
$$+ m_3a_{G_3}^2\left(\dot{\theta}_3\right)^2$$
$$+\frac{1}{2}m_3\left(a_3^2s_3^2 + a_2a_3s_3s_{23} + 2\left(a_{G_3}^2 + 2a_{G_3}a_3c_3 + a_3^2c_3^2 + a_{G_3}a_2c_{23} + a_2a_3c_3c_{23}\right)\right)\dot{\theta}_1\dot{\theta}_2$$
$$+m_3\left(a_{G_3}^2 + a_{G_3}a_3c_3 + a_{G_3}a_2c_{23}\right)\dot{\theta}_1\dot{\theta}_3 + m_3\left(a_{G_3} + a_3c_3\right)\dot{\theta}_2\dot{\theta}_3$$

$$K_{E_4}^{(T)} = \frac{1}{2}m_4\left[V_{G_4}^0(R_4)\right]^T\left[V_{G_4}^0(R_4)\right] \Rightarrow$$

$$= \frac{1}{2}m_4\left(2a_4 + a_{G_3} + a_2c_{23} + a_3c_3 + a_4c_4 + a_{G_3}c_4 + a_2c_{23}c_4 + a_3c_3c_4 - a_2s_{23}s_4 - a_3s_3s_4 - a_2s_{23}c_4 - a_3s_3c_4\right)\left(\dot{\theta}_1\right)^2$$

$$+\frac{1}{2}m_4\left(a_4c_4 + a_{G_3}c_4 + a_3c_3c_4 - a_3s_3s_4 - a_3s_3c_4 + a_4s_4 + a_{G_3}s_4 + a_3c_3s_4 + a_4\right)\left(\dot{\theta}_2\right)^2 +$$

$$+\frac{1}{2}m_4\left(a_4c_4 + a_{G_3}c_4 + a_4 + a_{G_3} + a_4\right)\left(\dot{\theta}_3\right)^2 + \frac{1}{2}m_4a_4\left(\dot{\theta}_4\right)^2$$

$$2\dot{\theta}_1\dot{\theta}_2 + 2\dot{\theta}_1\dot{\theta}_3 + 2\dot{\theta}_1\dot{\theta}_4 + 2\dot{\theta}_2\dot{\theta}_3 + 2\dot{\theta}_2\dot{\theta}_4 + 2\dot{\theta}_3\dot{\theta}_4$$

From the calculated kinetic energies, the total kinetic will be as the following;

$$K_E = K_E{}^{(R)} + K_E^{(T)}$$

$$\begin{aligned}
\Rightarrow \frac{1}{2}\Big\{ & I_{zz}^1 + I_{zz}^2 + I_{zz}^3 + I_{zz}^4 + m_1 a_{G_1}^2 + m_2\left(a_2^2 + a_{G_2}^2 + 2a_{G_2}a_2c_2\right) \\
& + m_3\left(a_2^2 + a_3^2 + 2a_3a_2c_2 + 2a_{G_3}(a_3c_3 + a_2c_{23})\right) \\
& + m_4\left(a_4 + a_{G_3} + a_2c_{23} + a_3c_3 + a_4c_4 + a_{G_3}c_4 + a_2c_{23}c_4 + a_3c_3c_4 - a_2s_{23}s_4\right. \\
& \left. - a_3s_3s_4 - a_2s_{23}c_4 - a_3s_3c_4 + a_4\right)\Big\}(\dot{\theta}_1)^2 \\
& + \frac{1}{2}\left\{I_{zz}^2 + I_{zz}^3 + I_{zz}^4 + m_2a_{G_2}^2 + m_3\left(a_3^2 + a_{G_3}^2 + 2a_{G_3}a_3c_3\right)\right. \\
& + m_4\left(a_4c_4 + a_{G_3}c_4 + a_3c_3c_4 - a_3s_3s_4 - a_3s_3c_4 + a_4s_4 + a_{G_3}s_4 + a_3c_3s_4\right. \\
& \left.\left. + a_4\right)\right\}(\dot{\theta}_2)^2 + \frac{1}{2}\left\{I_{zz}^3 + I_{zz}^4 + 2m_3a_{G_3}^2 + m_4\left(a_4c_4 + a_{G_3}c_4 + a_4 + a_{G_3} + a_4\right)\right\}(\dot{\theta}_3)^2 \\
& + \frac{1}{2}\{I_{zz}^4 + m_4a_4\}(\dot{\theta}_4)^2 \\
& + \Big\{\frac{1}{2}I_{zz}^2 + \frac{1}{2}I_{zz}^3 + \frac{1}{2}I_{zz}^4 + m_2\left(a_{G_2}^2 + a_2c_2a_{G_2}\right) \\
& + \frac{1}{2}m_3\left(a_3^2s_3^2 + a_2a_3s_3s_{23} + 2\left(a_{G_3}^2 + 2a_{G_3}a_3c_3 + a_3^2c_3^2 + a_{G_3}a_2c_{23} + a_2a_3c_3c_{23}\right)\right) \\
& + 2\Big\}\dot{\theta}_1\dot{\theta}_2 + \left\{\frac{1}{2}I_{zz}^3 + \frac{1}{2}I_{zz}^4 + m_3\left(a_{G_3}^2 + a_{G_3}a_3c_3 + a_{G_3}a_2c_{23}\right) + 2\right\}\dot{\theta}_1\dot{\theta}_3 \\
& + \left\{\frac{1}{2}I_{zz}^3 + \frac{1}{2}I_{zz}^4 + m_3\left(a_{G_3} + a_3c_3\right) + 2\right\}\dot{\theta}_2\dot{\theta}_3 + \{I_{zz}^4 + 2\}\dot{\theta}_1\dot{\theta}_4 + \{I_{zz}^4 + 2\}\dot{\theta}_2\dot{\theta}_4 \\
& + \{I_{zz}^4 + 2\}\dot{\theta}_3\dot{\theta}_4
\end{aligned}$$

Now, some parameters must be defined to understand easily.

$$k_1 = I_{zz}^1 + I_{zz}^2 + I_{zz}^3 + I_{zz}^4 + m_1a_{G_1}^2$$

$$k_2 = m_2a_{G_2}^2$$

$$k_3 = m_2\left(a_2^2 + 2a_{G_2}a_2c_2\right) + m_3\left(a_2^2 + 2a_3a_2c_2 + 2a_{G_3}a_2c_{23}\right)$$

$$k_4 = m_3\left(a_3^2 + 2a_{G_3}a_3c_3\right)$$

$$k_5 = m_4\left(a_4 + a_4c_4 + a_{G_3}c_4 + a_3c_3c_4 - a_3s_3s_4 - a_3s_3c_4\right)$$

$$k_6 = m_4\left(a_{G_3} + a_2c_{23} + a_3c_3 + a_2c_{23}c_4 - a_2s_{23}s_4 - a_2s_{23}c_4 + a_4\right)$$

$$k_7 = I_{zz}^2 + I_{zz}^3 + I_{zz}^4$$

$k_8 = m_3 a_{G_3}^2$

$k_9 = m_4\left(a_4 s_4 + a_{G_3} s_4 + a_3 c_3 s_4\right)$

$k_{10} = I_{zz}^3 + I_{zz}^4$

$k_{11} = m_4\left(a_4 c_4 + a_{G_3} c_4 + a_{G_3} + 2a_4\right)$

$k_{12} = I_{zz}^4 + m_4 a_4$

$k_{13} = m_3\left(a_3^2 s_3^2 + a_2 a_3 s_3 s_{23} + 2\left(a_{G_3}^2 + 2a_{G_3} a_3 c_3 + a_3^2 c_3^2 + a_{G_3} a_2 c_{23} + a_2 a_3 c_3 c_{23}\right)\right)$

$k_{14} = m_3\left(a_{G_3} a_3 c_3 + a_{G_3} a_2 c_{23}\right) + 2$

$k_{15} = m_3\left(a_{G_3} + a_3 c_3\right) + 2$

$k_{16} = I_{zz}^4 + 2$

The total kinetic energy can be defined as the following:

$$K_E = \frac{1}{2}(k_1 + k_2 + k_3 + k_5 + k_6)\left(\dot{\theta}_1\right)^2 + \frac{1}{2}(k_7 + k_2 + k_4 + k_8 + k_5 + k_9)\left(\dot{\theta}_2\right)^2$$
$$+ \tfrac{1}{2}(k_{10} + 2k_8 + k_{11})\left(\dot{\theta}_3\right)^2 + \tfrac{1}{2}k_{12}\left(\dot{\theta}_4\right)^2 + \left(\tfrac{1}{2}k_7 + k_2 + \tfrac{1}{2}k_{13} + 2\right)\dot{\theta}_1\dot{\theta}_2$$
$$+ \left(\tfrac{1}{2}k_{10} + k_8 + k_{14}\right)\dot{\theta}_1\dot{\theta}_3 + \left(\tfrac{1}{2}k_{10} + k_{15}\right)\dot{\theta}_2\dot{\theta}_3 + k_{16}\dot{\theta}_1\dot{\theta}_4 + k_{16}\dot{\theta}_2\dot{\theta}_4 + k_{16}\dot{\theta}_3\dot{\theta}_4$$

2.5.1.7. Calculating of the potential energies of the system

$$P_1 = m_1.g.y_1 = m_1.g.d_1$$
$$P_2 = m_2.g.y_2 = m_2.g.(d_1 + a_2 sin\theta_2)$$
$$P_3 = m_3.g.y_3 = m_3.g.(d_1 + a_2 sin\theta_2 + a_3 sin\theta_3)$$
$$P_4 = m_4.g.y_4 = m_4.g.(d_1 + a_2 sin\theta_2 + a_3 sin\theta_3 + a_4 sin\theta_4)$$

$P_E = P_1 + P_2 + P_3 + P_4$

$\Rightarrow (m_1 + m_2 + m_3 + m_4)g.d_1 + (m_2 + m_3 + m_4)ga_2 sin\theta_2 + (m_3 + m_4)ga_3 sin\theta_3 + m_4.ga_4 sin\theta_4$

which is P_E is equal to the total potential energy of the system.

2.5.1.8. Calculating of Lagrangian Functions

$$\mathcal{L}(\theta, \dot{\theta}) = \sum_{i=1}^{4}(K_i - P_i) \qquad and \qquad \tau_i = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_i}\right) - \frac{\partial \mathcal{L}}{\partial \theta_i} \qquad i = 1,2,3,4$$

In order to determine the Euler-Lagrange equations in a specific situation, one has to form the Lagrangian of the system which is difference between the kinetic energy and potential energy.

$$\mathcal{L}(\theta, \dot{\theta}) = \sum_{i=1}^{4}(K_i - P_i) = (K_1 + K_2 + K_3 + K_4) - (P_1 - P_2 - P_3 - P_4) = K_E - P_E$$

$$\Rightarrow \frac{1}{2}(k_1 + k_2 + k_3 + k_5 + k_6)(\dot{\theta}_1)^2 + \frac{1}{2}(k_7 + k_2 + k_4 + k_8 + k_5 + k_9)(\dot{\theta}_2)^2$$

$$+\frac{1}{2}(k_{10} + 2k_8 + k_{11})(\dot{\theta}_3)^2 + \frac{1}{2}k_{12}(\dot{\theta}_4)^2 + \left(\frac{1}{2}k_7 + k_2 + \frac{1}{2}k_{13} + 2\right)\dot{\theta}_1\dot{\theta}_2$$

$$+\left(\frac{1}{2}k_{10} + k_8 + k_{14}\right)\dot{\theta}_1\dot{\theta}_3 + \left(\frac{1}{2}k_{10} + k_{15}\right)\dot{\theta}_2\dot{\theta}_3 + k_{16}\dot{\theta}_1\dot{\theta}_4 + k_{16}\dot{\theta}_2\dot{\theta}_4 + k_{16}\dot{\theta}_3\dot{\theta}_4$$

$$-(m_1 + m_2 + m_3 + m_4)g.d_1 + (m_2 + m_3 + m_4)ga_2 sin\theta_2 + (m_3 + m_4)ga_3 sin\theta_3 + m_4.ga_4 sin\theta_4$$

2.5.1.9. Calculating of the Partial Derivatives of the Lagrangian Function

$$\frac{\partial \mathcal{L}}{\partial \theta_1} =$$

$$\frac{\partial \mathcal{L}}{\partial \theta_2} =$$

$$\frac{\partial \mathcal{L}}{\partial \theta_3} =$$

$$\frac{\partial \mathcal{L}}{\partial \theta_4} =$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_3}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_4}$$