If I try to build the figure from ground up, it will take way too much time. But blackboxing reduces what I can learn fully from this task. This is a trade-off between time & understanding deeper. I will find the sweet spot for now, send the figure reproduced & ask questions later. Don't forget to ask questions!

→ Blackboxing helps with time (I want to appreciate home).

→ Understanding deeper is for me (I want to learn, so).

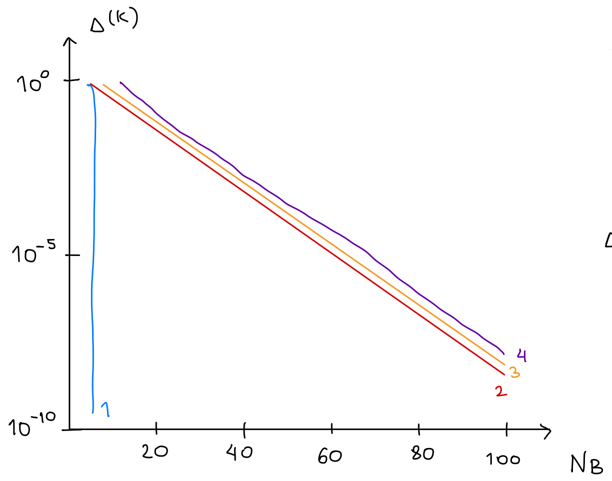
Things to track so I don't lose my mind:

- What I have understood so far
- Questions I have
- Terms to look up.
- What I might blackbox.

Depth vs Speed (Notion tracker)

THEE FIGURE.

3b.



When $k \geq 1$, almost linear ?

Δ - trace distance

$\Delta^{(K)}$ - k^{th} moment of projected ensemble to a Haar random ensemble

N_B - projected subsystem size N_B

· System size.

Here: $N_A = t = 3$ & $g = \pi/9$
what is this.?

$\Delta^{(K)} = f(k, N_A, t, g, N_B)$.

$k = 1$ is a special case.

Questions:

1. Wh: distance ?

2. What a Haar random ensemble?

Answers / Solution:

Q: What is trace distance ?

There.

A: A metric that shows how different two quantum states are from each other.

- Generalization of the classical total variation distance (how's it generalized?)

- Important in error analysis. $\longrightarrow$ cryptography

Definition:

For density matrices ρ & σ, the trace distance is defined as

$$T(\rho, \sigma) = \frac{1}{2} \|\rho - \sigma\|_1 = \frac{1}{2} \mathrm{Tr}\left(\sqrt{(\rho-\sigma)^\dagger(\rho-\sigma)}\right)$$

So. ρ & σ have to be same dimensional & square.

When $\rho = \sigma$, $T(\rho, \sigma) = 0$.

Physical meaning:

It equals the maximum probability advantage for distinguishing ρ & σ using optimal measurements.

$$P_{guess}^{max} = \frac{1}{2}(1 + T(\rho, \sigma))$$

observables are operators.
observable $\Rightarrow$ operator
operator $\not\Rightarrow$ observable.

When $T(\rho, \sigma) = 1$, P_{guess}^{max} is 1.

When $T(\rho, \sigma) = 0$, P_{guess}^{max} is 0.5.

This follows from maximizing over all possible POVMs. (positive operator valued measurements).

Key Properties:

1. Metric: nonnegative, symmetric, & satisfies the Δ inequality.

2. $0 \leq T(\rho, \sigma) \leq 1$. $T(\rho, \sigma) = 1$ if & only if ρ & σ have orthogonal supports.
(another way to say totally different),

3. Unitary invariance: $T(U\rho U^\dagger, U\sigma U^\dagger) = T(\rho, \sigma)$
time evolution I know is unitary. (Can it not be?) $\rightarrow$ Yes, it seems linguistically.

4. Contractivity: Non-increasing under trace-preserving quantum channels.

5. Pure states: $T(|\psi\rangle\langle\psi|, |\phi\rangle\langle\phi|) = \sqrt{1 - |\langle\psi|\phi\rangle|^2}$

6. Qubit case: Equals half the Euclidean distance in Bloch sphere representation.

Q: What is a Haar random ensemble? (In general, what is random?)

A: A Haar random ensemble refers to a collection of quantum states or unitary operators sampled via the Haar measure, which is the unique uniform (unitarily invariant) probability measure on the group of unitary operators or on the space of pure quantum states. (high dimensional sphere $\rightarrow$ pick points fairly).

Term	Description.
Haar measure	A mathematically rigorous way to define unique, uniform, unitarily invariant measure on unitary group or pure state space (No region is favored over another).
Haar random state	Pure quantum state sampled uniformly from the Hilbert space sphere.
Haar random unitary	Unitary operator sampled uniformly from the unitary group $U(d)$.

What can be concluded currently:

When the projected subsystem size increases & $k > 1$, the projected ensemble starts resembling a Haar random ensemble.

$\boxed{\text{Q: What does the subsystem size have to do with it?}}$

Projected Ensembles & Quantum State Designs

N qubits

$N = N_A + N_B$.

The state of B is projectively measured in the local computational basis.

The bit-string outcome:

$z_B = (z_{B,1},\ z_{B,2},\ z_{B,3},\ z_{B,4} \dots\ z_{B,N_B}) \in \{0, 1\}^{N_B}$.

B's associated pure quantum state on A:

$$|\psi(z_B)\rangle = \frac{(\mathbb{I}_A \otimes \langle z_B|)|\psi\rangle}{\sqrt{P(z_B)}} \quad (1)$$

$\otimes$ - tensor product (Kronecker product).

$$P(z_B) = \langle\psi| \mathbb{I}_A \otimes |z_B\rangle\langle z_B|\psi\rangle \quad (1a)$$

The set of projected states over all 2^{N_B} outcomes with respective probabilities forms the projected ensemble:

$\mathcal{E} := \{P(z_B),\ |\psi(z_B)\rangle\}$.

Understanding equation (1): $|\psi(z_B)\rangle = \dfrac{(\mathbb{I}_A \otimes \langle z_B|)|\psi\rangle}{\sqrt{P(z_B)}}$

- $|\psi\rangle$ lives on $A \otimes B$.
- $\langle z_B|$ is the computational - basis bra on B.
- $\mathbb{I}_A |\varphi\rangle = |\varphi\rangle$ if $\forall |\varphi\rangle \in \mathcal{H}_A$.

I still don't think I fully get this.

Thus,

- $\mathbb{I}_A \otimes \langle z_B|$ preserves the A part of $|\psi\rangle$ while projecting the B - part onto the basis state $\langle z_B|$.
- Dividing by $\sqrt{P(z_B)}$ normalizes $|\psi(z_B)\rangle$

The statistical properties of the projected ensemble is characterized by moments of its distribution. The k^{th} moment is captured by a density matrix:

$$\rho_{\mathcal{E}}^{(k)} = \sum_{z_B} P(z_B) \left(|\psi(z_B)\rangle\langle\psi(z_B)| \right)^{\otimes k}.$$

Q: What does tensor product mean here?

A: Tensor product accommodates composite systems.

Single system $\longrightarrow$ Composite system

Single system: The state is represented by a vector in Hilbert space $\mathcal{H}$. $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

Composite system: When two quantum systems A & B are combined, their joint state is not the Cartesian but the tensor product.

$H_{AB} = H_A \otimes H_B$.

Two qubits: $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$

Three qubits: $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^8$

This structure allows us to describe both product states & entangled states (How & why?)

Q: Why & how tensor product?

A: Entanglement is reflected only via tensor product. (How do you entangle stuff?)

OPERATORS ON COMPOSITE SYSTEMS

If $\hat{A}$ acts only on system A, then on the joint system it becomes:

$\hat{A} \otimes \mathbb{1}_B$.

Then the Hamiltonian may look like:

$$\hat{H} = \hat{H}_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes \hat{H}_B + \underbrace{\hat{H}_{int}}_{\text{interaction between A \& B}}$$

MEASUREMENT & THE TENSOR PRODUCT STRUCTURE.

When subsystem A is measured, we're effectively applying an operator $M_A \otimes \mathbb{1}_B$. This structure ensures that measuring one part of the system affects the total state, even without directly interacting with the other subsystem.

I will blackbox this for now.

Q. What is a density matrix?

A: $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ (It represents a statistical mixture of states).

$\sum_{i=1} p_i = 1$

$|\psi_i\rangle$ are pure states

TENSOR PRODUCTS IN COMPOSITE SYSTEMS:

For composite systems A & B, the joint state is represented by a density matrix on the tensor product space:

$\rho_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$.

- If ρ_A & ρ_B are <u>independent</u>, their joint state is:

$\rho_{AB} = \rho_A \otimes \rho_B$.

PARTIAL TRACE & REDUCED DENSITY MATRICES.

To describe a part of a system, we use the partial trace operation.

For a joint state ρ_{AB}, the state of subsystem A is:

$\rho_A = \mathrm{Tr}_B(\rho_{AB})$

Likewise:

$\rho_B = \mathrm{Tr}_A(\rho_{AB})$.

The partial trace "traces out" the degrees of freedom of the other subsystem.

Q: What is a moment in this context?

A: It is an extension of the classical counterpart.

In classical probability theory, the n^{th} moment of a random variable X is:

$\mu_n = \mathbb{E}[X^n]$

μ_1: mean

Higher moments describe shape: skewness, kurtosis etc.

$\mu_2 - \mu_1^2$: variance

In summary, these moments characterize the distribution.

Extension into quantum:

$\langle \hat{O}^n \rangle = \mathrm{Tr}(\rho \hat{O}^n)$

Thus: $\langle \hat{O} \rangle = \mathrm{Tr}(\rho \hat{O})$ is the expectation value.

$\mathrm{Var}(\hat{O}) = \langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2$

In general, I need to understand why it is formalized this way.

- Why Tr
- How tensor?

Q: What is purity & mixedness? How to express them? Why do we even bother?

A: We want to know how entangled a system is.

- How rapid does it have to be a quench dynamics?

Purity: $\gamma = \mathrm{Tr}(\rho^2)$

Pure state: $\gamma = 1$

Maximally mixed state in d-dimensional space: $\gamma = \frac{1}{d}$.

Q: What is quench dynamics?

A: A quantum quench is a sudden or rapid change in the parameters of a Hamiltonian governing a quantum system.

Q: What is Floquet unitary?

A: The Floquet unitary is the time evolution over one period.

$U_F \equiv U(T) = \mathcal{T} \exp\left(-i \int_0^T H(t)\,dt\right)$

$\mathcal{T}$ - time-ordering operator.

- Why is Floquet unitary chosen?

(Apparently, at high frequencies, the system resembles a thermal state for long times before heating. Eventually, it may heat to ∞ temperature). → Okay? (This is surface level explanation though).

Q: What is an Ising model?

A: A lattice model to describe a system of spins, each of which can point up or down, interacting with their neighbors.

Classical Ising Model: A model of magnetism describing how microscopic interactions lead to macroscopic order.

$H = -J \sum_{\langle i,j \rangle} s_i s_j - h \sum_i s_i$

- J coupling constant: $J > 0$: favors alignment (ferromagnetism)

  $J < 0$: favors anti-alignment (antiferromagnetism)

  $J = 0 \Rightarrow$ nonmagnetic?

- h: external magnetic field
- $\langle i,j \rangle$: sum over neighboring spins (e.g nearest neighbors in 1D or 2D

Quantum Ising Model:

The quantum transverse-field Ising model (TFIM) is:

$H = -J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z - h \sum_i \sigma_i^x$

- The second term, transverse magnetic field, induces quantum spin flips.

Key point: The second term doesn't commute with the interaction term, so the system becomes quantum mechanical. (Why how?)

The Ising Model in the paper:

$$H_{Ising} = J \sum_{i=1}^{N-1} \sigma_i^z \sigma_{i+1}^z + g \sum_{i=1}^{N} \sigma_i^z + \underbrace{(b_1 \sigma_1^z + b_N \sigma_N^z)}_{\text{boundary terms}}$$

J - nearest neighbor interaction strength

boundary terms (what is their function?)

g - longitudinal field

Okay, moving on:

$$U_F = U_h e^{-iH_{Ising}\tau} \quad (4)$$

I bet the answers are in the paper.

$\tau = 1$

Non-integrable for general values of (J, h, g) & possesses no global conservation laws.
(can't normalize?)

$J = h = \frac{\pi}{4}$ & arbitrary g except $g \in \mathbb{Z}\frac{\pi}{8}$ (why?)

So far, the current skeleton I have is:

1. There are N qubits, & we divide it into 2 parts: A the subsystem & B the bath. Bath just means the environment.
2. And instead of observing A which is what a lot of people do, they focused on B. However, to connect A & B, they proposed the projected ensemble. (The reason for their intuition is because they're entangled)
3. The projected ensemble is $\mathcal{E} := \{p(z_B), |\psi(z_B)\rangle\}$. • Measure B onto local z. $|\uparrow\rangle$ & $|\downarrow\rangle$.

   $|\psi(z_B)\rangle = (\mathbb{1}_A \otimes \langle z_B|)|\Psi\rangle / \sqrt{P(z_B)}$ $\qquad |\Psi\rangle = |\Psi(t)\rangle = U_F^t |+\rangle^{\otimes N}$
4. The statistical properties of the ensemble is captured by the below equation: $\rho_{\mathcal{E}}^{(k)} = \sum_{z_B} P(z_B) (|\psi(z_B)\rangle\langle\psi(z_B)|)^{\otimes k}$
5. How the ensemble connects A & B is that it is from the measurement outcomes of the bath.
6. What drives the system: $U_F = U_h e^{-iH_{Ising}\tau}$
7. Measure the bath in z from the generator state $|\Psi(t)\rangle = U_F^t |+\rangle^{\otimes N}$
8. So since there is entanglement, manipulating B affects A, thus creating the projected states to be a Haar random state distribution. Entanglement + manipulate B => randomness on A.
9. How the 8 is measured is via trace distance & that result is figure 3b.

Questions.

1. How is it ecoming random? Projecting onto a single computational basis state on B & getting a normalized pure state on A. How is this repeated led to the result? @ ∞ temperature, max entanglement? Apparently, this is how my brain is putting the 2 together?

   → Floquet dynamics will lead to ∞ temperature, where it is going to be very chaotic → random.

   How will it lead to ∞ temperature? Cause it is very rapid?

   How/why come very rapid?

   It doesn't allow the system to settle (so it keeps the chaos?)

   Probably the Ising model.

   $U_F = U_h e^{-iH_{Ising}\tau}$ ⟶ Repeating this ($t\uparrow$) ensures the chaos.

   If this was $\mathbb{1}$, $\Delta^k \to 0$ is not gonna happen. This is a mixer.

   This neighbor interaction probably is responsible for the entanglement keeping.