Übung 2

2.4 Berechne die folgenden Integrale:

(a) $\int\limits_{-1}^{1} (3x^3 - 2x^2 + x - 1)dx$

$$f(x) = 3x^3 - 2x^2 + x - 1 \Rightarrow F(x) = \frac{3}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2 - x + c$$

$$\Rightarrow \int\limits_{-1}^{1} (3x^3 - 2x^2 + x - 1)dx = F(x)\Big|_{-1}^{1} = F(1) - F(-1) = -\frac{10}{3}$$

(b) $\int_{-1}^{1} \frac{1}{1+x^2} dx$

$$f(x) = \frac{1}{1+x^2} \Rightarrow F(x) = \arctan(x)$$

$$\Rightarrow \int_{-1}^{1} \frac{1}{1+x^2} dx = F(x)\Big|_{-1}^{1} = F(1) - F(-1) = \frac{\pi}{2}$$

(c) $\int_{-1}^{2} \frac{e^x - 1}{e^x + 1} dx$

$$\int_{-1}^{2} \frac{e^x - 1}{e^x + 1} dx$$
$$= \int_{-1}^{2} \frac{e^x}{e^x + 1} dx - \int_{-1}^{2} \frac{1}{e^x + 1} dx$$

Mit $g(x) = e^x + 1 \Rightarrow g'(x) = e^x$ folgt aus $\int_a^b \frac{g'(x)}{g(x)} dx = \ln(|g(x)|)\Big|_a^b$:

$$= \ln(e^x + 1)\Big|_{-1}^{2} - \int_{-1}^{2} \frac{1}{e^x + 1} dx$$

Mit $s := e^x$ folgt aus der Substitutionsregel:

$$= \ln(e^x + 1)\Big|_{-1}^{2} - \int_{e^{-1}}^{e^2} \frac{1}{s+1} \frac{ds}{s}$$

$$= \ln(e^x + 1)\Big|_{-1}^{2} - \int_{e^{-1}}^{e^2} \frac{1}{s(s+1)} ds$$

Aus der Partialbruchzerlegung von $\frac{1}{s(s+1)}$ zu $\frac{1}{2} - \frac{1}{s+1}$ folgt:

$$= \ln(e^x + 1)\Big|_{-1}^{2} - \int_{e^{-1}}^{e^2} \frac{1}{s} ds + \int_{e^{-1}}^{e^2} \frac{1}{s+1} ds$$

Mit $u \coloneqq s+1$ folgt aus der Substitutionsregel:

$$\begin{aligned}
&= \ln(e^x+1)\Big|_{-1}^{2} - \int_{e^{-1}}^{e^2} \frac{1}{s} ds + \int_{e^{-1}+1}^{e^2+1} \frac{1}{u} du \\
&= \ln(e^x+1)\Big|_{-1}^{2} - \ln(s)\Big|_{e^{-1}}^{e^2} + \ln(u)\Big|_{e^{-1}+1}^{e^2+1} \\
&= -2\ln(e^2+1) + 2\ln(e+1) - 1
\end{aligned}$$

$\Rightarrow \int_{-1}^{2} \frac{e^x-1}{e^x+1} dx = -2\ln(e^2+1) + 2\ln(e+1) - 1$

(d) $\int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}} dx$

partielle Integration mit $f_1(x) = x, g_2(x) = \frac{1}{\sqrt{1-x^2}}$:

$$\begin{aligned}
&\Rightarrow \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}} dx \\
&\rightarrow x\arcsin(x)\Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} 1 \cdot \arcsin(x) dx
\end{aligned}$$

Substitution mit $v \coloneqq \arcsin(x)$

$$\rightarrow x\arcsin(x)\Big|_0^{\frac{1}{2}} - \int_0^{\arcsin(\frac{1}{2})} v\cos(v) dv$$

partielle Integration mit $f_2(v) = v, g_2(v) = \cos(v)$

$$\begin{aligned}
&= x\arcsin(x)\Big|_0^{\frac{1}{2}} - v\sin(v)\Big|_0^{\arcsin(\frac{1}{2})} + \int_0^{\arcsin(\frac{1}{2})} 1 \cdot \sin(v) dv \\
&= x\arcsin(x)\Big|_0^{\frac{1}{2}} - v\sin(v)\Big|_0^{\arcsin(\frac{1}{2})} - \cos(v)\Big|_0^{\arcsin(\frac{1}{2})} \\
&= \tfrac{1}{2}\arcsin(\tfrac{1}{2}) - 0 - \arcsin(\tfrac{1}{2})\tfrac{1}{2} + 0 - \cos(\arcsin(\tfrac{1}{2})) + 1 \\
&= -\cos(\arcsin(\tfrac{1}{2})) + 1 \\
&= \frac{2-\sqrt{3}}{2} \approx 0.133975
\end{aligned}$$

2.5 Bestimme mittels geeigneter Integrationstechniken Stammfunktionen zu folgenden Funktionen:

(a) $f(x) = 3e^x\sqrt{e^x+1}$

$$\int 3e^x\sqrt{e^x+1}dx$$

Substituition mit $s \coloneqq e^x \Rightarrow dx = \frac{1}{s}$

$$= \int 3\sqrt{s+1}ds$$

partielle Integration mit $f(s) = 3, g(s) = \sqrt{s+1}$

$$=3G(s) - 0$$
$$=3\int \sqrt{s+1}ds$$

Substitution mit $u \coloneqq \sqrt{s+1} \Rightarrow ds = 2u$

$$=3\int \sqrt{2u^2}du$$
$$=3\frac{2}{3}u^3$$

$u = \sqrt{s+1} = \sqrt{e^x+1}$

$$=2\sqrt{e^x+1}^3$$

(b) $f(x) = x\ln(x) \quad (x > 0)$

$$\int x\ln(x)dx$$

partielle Integration mit $f(x) = x, g(x) = \ln(x)$

$$=\tfrac{1}{2}\ln(x)x^2 - \int \tfrac{1}{2}xdx$$
$$=\tfrac{1}{2}\ln(x)x^2 - \tfrac{1}{4}x^2$$
$$=\tfrac{1}{2}x^2(\ln(x) - \tfrac{1}{2})$$

(c) $f(x) = \frac{1}{\sqrt{x}(1+\sqrt[3]{x})} \quad (x > 0)$

$$\int \frac{1}{\sqrt{x}(1+\sqrt[3]{x})}dx$$
$$= \int \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{2}+\frac{1}{3}}}dx$$
$$= \int \frac{1}{x^{\frac{1}{6}^3} + x^{\frac{1}{6}^5}}dx$$

Substitution mit $s \coloneqq x^{\frac{1}{6}} \Rightarrow dx = 6s^5$

$$\begin{aligned}
&=6\int \frac{s^5}{s^3+s^5}ds \\
&=6\int \frac{s^2}{1+s^2}ds \\
&=6\left(\int ds - \int \frac{1}{1+s^2}ds\right) \\
&=6(s-\arctan(s))
\end{aligned}$$

$s = x^{\frac{1}{6}}$

$$=6(\sqrt[6]{x} - \arctan(\sqrt[6]{x}))$$

(d) $f(x) = \frac{x^2+9x+17}{x^3-3x^2-4}$

$$\int \frac{x^2+9x+17}{x^3-3x^2-4}dx$$

Partialbruchzerlegung zu $A = 3, B = -2, C = -1$

$$\begin{aligned}
&=\int \frac{3}{x-1} - \frac{2}{x+2} - \frac{1}{(x+2)^2}dx \\
&=3\int \frac{1}{x-1}dx - 2\int \frac{1}{x+2}dx - \int \frac{1}{(x+2)^2}dx
\end{aligned}$$

Substitution mit $s_1 \coloneqq x-1, s_2 \coloneqq x+2, s_3 \coloneqq (x+2)^2 \Rightarrow dx = \frac{1}{2\sqrt{s_3}}$

$$=3\ln(s_1) - 2\ln(s_2) - \int \frac{1}{2}s_3^{-\frac{3}{2}}ds_3$$

$s_1 = x-1, s_2 = x+2$

$$=3\ln(x-1) - 2\ln(x+2) + s_3^{-\frac{1}{2}}$$

$s_3 = (x+2)^2$

$$=3\ln(x-1) - 2\ln(x+2) + \frac{1}{x+2}$$

2.6 Die Graphen der Funktionen $f_1, f_2, g_1, g_2 : (0,\infty) \to (0,\infty)$

$$f_1(x) \coloneqq x^2, f_2(x) \coloneqq 2x^2, g_1(x) \coloneqq \frac{1}{x}, g_2(x) \coloneqq \frac{4}{x}$$

begrenzen eine Fläche im $\mathbb{R}^2$. Berechne den Flächeninhalt.

$$\begin{aligned}
w_1 &\Leftarrow f_2 = g_1 \Leftrightarrow 2x^2 = \frac{1}{x} && \Leftrightarrow x = \sqrt[3]{\frac{1}{2}} \\
w_2 &\Leftarrow f_1 = g_1 \Leftrightarrow x^2 = \frac{1}{x} && \Leftrightarrow x = 1 \\
w_3 &\Leftarrow f_2 = g_2 \Leftrightarrow 2x^2 = \frac{4}{x} && \Leftrightarrow x = \sqrt[3]{2} \\
w_4 &\Leftarrow f_1 = g_2 \Leftrightarrow x^2 = \frac{4}{x} && \Leftrightarrow x = \sqrt[3]{4}
\end{aligned}$$

$$\begin{aligned}
A &= \int_{w_1}^{w_2} f_2(x) - g_1(x)dx + \int_{w_2}^{w_3} f_2(x) - f_1(x)dx + \int_{w_3}^{w_4} g_2(x) - f_1(x)dx \\
&= \int_{\sqrt[3]{\frac{1}{2}}}^{1} 2x^2 - \frac{1}{x}dx + \int_{1}^{\sqrt[3]{2}} 2x^2 - x^2dx + \int_{\sqrt[3]{2}}^{\sqrt[3]{4}} \frac{4}{x} - x^2dx \\
&= \left(\frac{2}{3}x^3 - \ln(x)\right)\Bigg|_{\sqrt[3]{\frac{1}{2}}}^{1} + \left(\frac{1}{3}x^3\right)\Bigg|_{1}^{\sqrt[3]{2}} + \left(4\ln(x) - \frac{1}{3}x^3\right)\Bigg|_{\sqrt[3]{2}}^{\sqrt[3]{4}} \\
&= \frac{1}{3} - \frac{1}{3}\ln(2) + \frac{1}{3} - \frac{2}{3} + \frac{4}{3}\ln(2) \\
&= \ln(2)
\end{aligned}$$

2.7 (a) Seien $f, \varphi, \psi : \mathbb{R} \to \mathbb{R}$ differenzierbar. Berechne $\frac{d}{dr}\int_{\varphi(r)}^{\psi(r)} f(x)dx$

f differenzierbar
$\Rightarrow f$ stetig auf dem gesamten Definitionsbereich
$\Rightarrow f$ weist keine Sprünge auf.
$\Rightarrow f$ besitzt eine kontinuierliche Veränderung der Fläche die sie begrenzt.
$\Rightarrow f$ ist integrierbar.
$\Rightarrow f$ besitzt eine Stammfunktion

$$\begin{aligned}
&\frac{d}{dr}\int_{\varphi(r)}^{\psi(r)} f(x)dx \\
=&\frac{d}{dr}(F(\psi(r))) - F(\varphi(r)))
\end{aligned}$$

Kettenregel:

$$=f(\psi(r))\psi'(r) - f(\varphi(r))\varphi'(r)$$

(b) Berechne $\frac{d}{dr}\int_{\sqrt{\ln(r)}}^{2\sqrt{\ln(r)}} e^{x^2}dx$

$$\begin{aligned}
&\frac{d}{dr}\int_{\sqrt{\ln(r)}}^{2\sqrt{\ln(r)}} e^{x^2}dx \\
=&e^{(2\sqrt{\ln(r)})^2}\frac{1}{r\sqrt{\ln(r)}} - e^{\sqrt{\ln(r)}^2}\frac{1}{2r\sqrt{r}} \\
=&e^{4\ln(r)}\frac{1}{r\sqrt{\ln(r)}} - e^{\ln(r)}\frac{1}{2r}\frac{1}{\sqrt{\ln(r)}} \\
=&\frac{1}{\sqrt{\ln(r)}}(r^3 - \frac{1}{2})
\end{aligned}$$

2.8 Ermittle den Grenzwert

$$\begin{aligned}
&\lim_{n\to\infty} n \cdot \left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \ldots + \frac{1}{(2n-1)^2}\right) \\
=&\lim_{n\to\infty} \frac{1}{n}\left(\frac{n^2}{n^2} + \frac{n^2}{(n+1)^2} + \ldots + \frac{n^2}{(2n-1)^2}\right) \\
=&\lim_{n\to\infty} \frac{1}{n}\left(\frac{1}{(1+\frac{0}{n})^2} + \frac{1}{(1+\frac{1}{n})^2} + \ldots + \frac{1}{(1+\frac{n-1}{n})^2}\right) \\
=&\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{(1+\frac{k}{n})^2}
\end{aligned}$$

$x = \frac{k}{n}; 0 \leq x < 1$:

$$=\int_0^1 \frac{1}{(1+x)^2}dx$$

$a = x+1; 1 \leq a < 2$:

$$=\int_1^2 \frac{1}{a^2}da$$

$f(x) = \frac{1}{a^2} \Rightarrow F(x) = -\frac{1}{a}$:

$$\begin{aligned}
=&F(2) - F(1) \\
=&-\frac{1}{2} + 1 \\
=&\frac{1}{2}
\end{aligned}$$