

2.4 Berechne die folgenden Integrale:

$$(a) \int_{-1}^1 (3x^3 - 2x^2 + x - 1) dx$$

$$f(x) = 3x^3 - 2x^2 + x - 1 \Rightarrow F(x) = \frac{3}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2 - x + c$$

$$\Rightarrow \int_{-1}^1 (3x^3 - 2x^2 + x - 1) dx = F(x) \Big|_{-1}^1 = F(1) - F(-1) = -\frac{10}{3}$$

$$(b) \int_{-1}^1 \frac{1}{1+x^2} dx$$

$$f(x) = \frac{1}{1+x^2} \Rightarrow F(x) = \arctan(x)$$

$$\Rightarrow \int_{-1}^1 \frac{1}{1+x^2} dx = F(x) \Big|_{-1}^1 = F(1) - F(-1) = \frac{\pi}{2}$$

$$(c) \int_{-1}^2 \frac{e^x - 1}{e^x + 1} dx$$

$$\begin{aligned} & \int_{-1}^2 \frac{e^x - 1}{e^x + 1} dx \\ &= \int_{-1}^2 \frac{e^x}{e^x + 1} dx - \int_{-1}^2 \frac{1}{e^x + 1} dx \end{aligned}$$

$$\text{Mit } g(x) = e^x + 1 \Rightarrow g'(x) = e^x \text{ folgt aus } \int_a^b \frac{g'(x)}{g(x)} dx = \ln(|g(x)|) \Big|_a^b :$$

$$= \ln(e^x + 1) \Big|_{-1}^2 - \int_{-1}^2 \frac{1}{e^x + 1} dx$$

Mit $s := e^x$ folgt aus der Substitutionsregel:

$$\begin{aligned} &= \ln(e^x + 1) \Big|_{-1}^2 - \int_{e^{-1}}^{e^2} \frac{1}{s+1} \frac{ds}{s} \\ &= \ln(e^x + 1) \Big|_{-1}^2 - \int_{e^{-1}}^{e^2} \frac{1}{s(s+1)} ds \end{aligned}$$

Aus der Partialbruchzerlegung von $\frac{1}{s(s+1)}$ zu $\frac{1}{2} - \frac{1}{s+1}$ folgt:

$$= \ln(e^x + 1) \Big|_{-1}^2 - \int_{e^{-1}}^{e^2} \frac{1}{s} ds + \int_{e^{-1}}^{e^2} \frac{1}{s+1} ds$$

Mit $u := s + 1$ folgt aus der Substitutionsregel:

$$\begin{aligned} &= \ln(e^x + 1) \Big|_{-1}^2 - \int_{e^{-1}}^{e^2} \frac{1}{s} ds + \int_{e^{-1}+1}^{e^2+1} \frac{1}{u} du \\ &= \ln(e^x + 1) \Big|_{-1}^2 - \ln(s) \Big|_{e^{-1}}^{e^2} + \ln(u) \Big|_{e^{-1}+1}^{e^2+1} \\ &= -2 \ln(e^2 + 1) + 2 \ln(e + 1) - 1 \end{aligned}$$

$$\Rightarrow \int_{-1}^2 \frac{e^x - 1}{e^x + 1} dx = -2 \ln(e^2 + 1) + 2 \ln(e + 1) - 1$$

(d) $\int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}} dx$

partielle Integration mit $f_1(x) = x, g_2(x) = \frac{1}{\sqrt{1-x^2}}$:

$$\begin{aligned} &\Rightarrow \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}} dx \\ &\rightarrow x \arcsin(x) \Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} 1 \cdot \arcsin(x) dx \end{aligned}$$

Substitution mit $v := \arcsin(x) \Rightarrow dx = \cos(v) dv$

$$\rightarrow x \arcsin(x) \Big|_0^{\frac{1}{2}} - \int_0^{\arcsin(\frac{1}{2})} v \cos(v) dv$$

partielle Integration mit $f_2(v) = v, g_2(v) = \cos(v)$

$$\begin{aligned} &= x \arcsin(x) \Big|_0^{\frac{1}{2}} - v \sin(v) \Big|_0^{\arcsin(\frac{1}{2})} + \int_0^{\arcsin(\frac{1}{2})} 1 \cdot \sin(v) dv \\ &= x \arcsin(x) \Big|_0^{\frac{1}{2}} - v \sin(v) \Big|_0^{\arcsin(\frac{1}{2})} - \cos(v) \Big|_0^{\arcsin(\frac{1}{2})} \\ &= \frac{1}{2} \arcsin\left(\frac{1}{2}\right) - 0 - \arcsin\left(\frac{1}{2}\right) \frac{1}{2} + 0 - \cos\left(\arcsin\left(\frac{1}{2}\right)\right) + 1 \\ &= -\cos\left(\arcsin\left(\frac{1}{2}\right)\right) + 1 \\ &= \frac{2 - \sqrt{3}}{2} \approx 0.133975 \end{aligned}$$

2.5 Bestimme mittels geeigneter Integrationstechniken Stammfunktionen zu folgenden Funktionen:

(a) $f(x) = 3e^x \sqrt{e^x + 1}$

$$\int 3e^x \sqrt{e^x + 1} dx$$

Substitution mit $s := e^x \Rightarrow dx = \frac{1}{s}$

$$= \int 3\sqrt{s+1} ds$$

partielle Integration mit $f(s) = 3, g(s) = \sqrt{s+1}$

$$= 3G(s) - 0$$

$$= 3 \int \sqrt{s+1} ds$$

Substitution mit $u := \sqrt{s+1} \Rightarrow ds = 2u du$

$$= 3 \int \sqrt{2} u^2 du$$

$$= 3 \frac{2}{3} u^3$$

$$u = \sqrt{s+1} = \sqrt{e^x + 1}$$

$$= 2\sqrt{e^x + 1}^3$$

(b) $f(x) = x \ln(x) \quad (x > 0)$

$$\int x \ln(x) dx$$

partielle Integration mit $f(x) = x, g(x) = \ln(x)$

$$= \frac{1}{2} \ln(x) x^2 - \int \frac{1}{2} x dx$$

$$= \frac{1}{2} \ln(x) x^2 - \frac{1}{4} x^2$$

$$= \frac{1}{2} x^2 (\ln(x) - \frac{1}{2})$$

(c) $f(x) = \frac{1}{\sqrt{x}(1+\sqrt[3]{x})} \quad (x > 0)$

$$\begin{aligned} & \int \frac{1}{\sqrt{x}(1+\sqrt[3]{x})} dx \\ &= \int \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{2}+\frac{1}{3}}} dx \\ &= \int \frac{1}{x^{\frac{1}{6}} + x^{\frac{1}{5}}} dx \end{aligned}$$

Substitution mit $s := x^{\frac{1}{6}} \Rightarrow dx = 6s^5 ds$

$$\begin{aligned} &= 6 \int \frac{s^5}{s^3 + s^5} ds \\ &= 6 \int \frac{s^2}{1 + s^2} ds \\ &= 6 \left(\int ds - \int \frac{1}{1 + s^2} ds \right) \\ &= 6(s - \arctan(s)) \end{aligned}$$

$$s = x^{\frac{1}{6}}$$

$$= 6(\sqrt[6]{x} - \arctan(\sqrt[6]{x}))$$

(d) $f(x) = \frac{x^2 + 9x + 17}{x^3 - 3x^2 - 4}$

$$\int \frac{x^2 + 9x + 17}{x^3 - 3x^2 - 4} dx$$

Partialbruchzerlegung zu $A = 3, B = -2, C = -1$

$$\begin{aligned} &= \int \frac{3}{x-1} - \frac{2}{x+2} - \frac{1}{(x+2)^2} dx \\ &= 3 \int \frac{1}{x-1} dx - 2 \int \frac{1}{x+2} dx - \int \frac{1}{(x+2)^2} dx \end{aligned}$$

Substitution mit $s_1 := x - 1, s_2 := x + 2, s_3 := (x + 2)^2 \Rightarrow dx = \frac{1}{2\sqrt{s_3}} ds_3$

$$= 3 \ln(s_1) - 2 \ln(s_2) - \int \frac{1}{2} s_3^{-\frac{3}{2}} ds_3$$

$$s_1 = x - 1, s_2 = x + 2$$

$$= 3 \ln(x - 1) - 2 \ln(x + 2) + s_3^{-\frac{1}{2}}$$

$$s_3 = (x + 2)^2$$

$$= 3 \ln(x - 1) - 2 \ln(x + 2) + \frac{1}{x + 2}$$

2.6 Die Graphen der Funktionen $f_1, f_2, g_1, g_2 : (0, \infty) \rightarrow (0, \infty)$

$$f_1(x) := x^2, f_2(x) := 2x^2, g_1(x) := \frac{1}{x}, g_2(x) := \frac{4}{x}$$

begrenzen eine Fläche im \mathbb{R}^2 . Berechne den Flächeninhalt.

$$\begin{aligned} w_1 \Leftarrow f_2 = g_1 &\Leftrightarrow 2x^2 = \frac{1}{x} && \Leftrightarrow x = \sqrt[3]{\frac{1}{2}} \\ w_2 \Leftarrow f_1 = g_1 &\Leftrightarrow x^2 = \frac{1}{x} && \Leftrightarrow x = 1 \\ w_3 \Leftarrow f_2 = g_2 &\Leftrightarrow 2x^2 = \frac{4}{x} && \Leftrightarrow x = \sqrt[3]{2} \\ w_4 \Leftarrow f_1 = g_2 &\Leftrightarrow x^2 = \frac{4}{x} && \Leftrightarrow x = \sqrt[3]{4} \end{aligned}$$

$$\begin{aligned} A &= \int_{w_1}^{w_2} f_2(x) - g_1(x) dx + \int_{w_2}^{w_3} f_2(x) - f_1(x) dx + \int_{w_3}^{w_4} g_2(x) - f_1(x) dx \\ &= \int_{\sqrt[3]{\frac{1}{2}}}^1 2x^2 - \frac{1}{x} dx + \int_1^{\sqrt[3]{2}} 2x^2 - x^2 dx + \int_{\sqrt[3]{2}}^{\sqrt[3]{4}} \frac{4}{x} - x^2 dx \\ &= \left(\frac{2}{3}x^3 - \ln(x) \right) \Big|_{\sqrt[3]{\frac{1}{2}}}^1 + \left(\frac{1}{3}x^3 \right) \Big|_1^{\sqrt[3]{2}} + \left(4\ln(x) - \frac{1}{3}x^3 \right) \Big|_{\sqrt[3]{2}}^{\sqrt[3]{4}} \\ &= \frac{1}{3} - \frac{1}{3}\ln(2) + \frac{1}{3} - \frac{2}{3} + \frac{4}{3}\ln(2) \\ &= \ln(2) \end{aligned}$$

2.7 (a) Seien $f, \varphi, \psi : \mathbb{R} \rightarrow \mathbb{R}$ differenzierbar. Berechne $\frac{d}{dr} \int_{\varphi(r)}^{\psi(r)} f(x) dx$

f differenzierbar

$\Rightarrow f$ stetig auf dem gesamten Definitionsbereich

$\Rightarrow f$ weist keine Sprünge auf.

$\Rightarrow f$ besitzt eine kontinuierliche Veränderung der Fläche die sie begrenzt.

$\Rightarrow f$ ist integrierbar.

$\Rightarrow f$ besitzt eine Stammfunktion

$$\begin{aligned} &\frac{d}{dr} \int_{\varphi(r)}^{\psi(r)} f(x) dx \\ &= \frac{d}{dr} (F(\psi(r)) - F(\varphi(r))) \end{aligned}$$

Kettenregel:

$$= f(\psi(r))\psi'(r) - f(\varphi(r))\varphi'(r)$$

(b) Berechne $\frac{d}{dr} \int_{\sqrt{\ln(r)}}^{2\sqrt{\ln(r)}} e^{x^2} dx$

$$\begin{aligned} & \frac{d}{dr} \int_{\sqrt{\ln(r)}}^{2\sqrt{\ln(r)}} e^{x^2} dx \\ &= e^{(2\sqrt{\ln(r)})^2} \frac{1}{r\sqrt{\ln(r)}} - e^{\sqrt{\ln(r)}^2} \frac{1}{2r\sqrt{r}} \\ &= e^{4\ln(r)} \frac{1}{r\sqrt{\ln(r)}} - e^{\ln(r)} \frac{1}{2r} \frac{1}{\sqrt{\ln(r)}} \\ &= \frac{1}{\sqrt{\ln(r)}} \left(r^3 - \frac{1}{2} \right) \end{aligned}$$

2.8 Ermittle den Grenzwert

$$\begin{aligned} & \lim_{n \rightarrow \infty} n \cdot \left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \dots + \frac{1}{(2n-1)^2} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left(\frac{n^2}{n^2} + \frac{n^2}{(n+1)^2} + \dots + \frac{n^2}{(2n-1)^2} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left(\frac{1}{(1+\frac{0}{n})^2} + \frac{1}{(1+\frac{1}{n})^2} + \dots + \frac{1}{(1+\frac{n-1}{n})^2} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{(1+\frac{k}{n})^2} \end{aligned}$$

$$x = \frac{k}{n}; 0 \leq x < 1:$$

$$= \int_0^1 \frac{1}{(1+x)^2} dx$$

$$a = x + 1; 1 \leq a < 2:$$

$$= \int_1^2 \frac{1}{a^2} da$$

$$f(x) = \frac{1}{a^2} \Rightarrow F(x) = -\frac{1}{a}:$$

$$= F(2) - F(1)$$

$$= -\frac{1}{2} + 1$$

$$= \frac{1}{2}$$