2.4 Berechne die folgenden Integrale:

(a) $\int\limits_{-1}^{1}(3x^3-2x^2+x-1)dx$

$$f(x)=3x^3-2x^2+x-1 \Rightarrow F(x)=\frac{3}{4}x^4-\frac{2}{3}x^3+\frac{1}{2}x^2-x+c$$

$$\Rightarrow \int\limits_{-1}^{1}(3x^3-2x^2+x-1)dx = F(x)\Big|_{-1}^{1} = F(1)-F(-1) = -\frac{10}{3}$$

(b) $\int_{-1}^{1}\frac{1}{1+x^2}dx$

$$f(x)=\frac{1}{1+x^2} \Rightarrow F(x)=\arctan(x)$$

$$\Rightarrow \int_{-1}^{1}\frac{1}{1+x^2}dx = F(x)\Big|_{-1}^{1} = F(1)-F(-1) = \frac{\pi}{2}$$

(c) $\int_{-1}^{2}\frac{e^x-1}{e^x+1}dx$

$$\int_{-1}^{2}\frac{e^x-1}{e^x+1}dx$$
$$=\int_{-1}^{2}\frac{e^x}{e^x+1}dx-\int_{-1}^{2}\frac{1}{e^x+1}dx$$

Mit $g(x)=e^x+1 \Rightarrow g'(x)=e^x$ folgt aus $\int_a^b\frac{g'(x)}{g(x)}dx=\ln(|g(x)|)\Big|_a^b$:

$$=\ln(e^x+1)\Big|_{-1}^{2}-\int_{-1}^{2}\frac{1}{e^x+1}dx$$

Mit $s:=e^x$ folgt aus der Substitutionsregel:

$$=\ln(e^x+1)\Big|_{-1}^{2}-\int_{e^{-1}}^{e^2}\frac{1}{s+1}\frac{ds}{s}$$
$$=\ln(e^x+1)\Big|_{-1}^{2}-\int_{e^{-1}}^{e^2}\frac{1}{s(s+1)}ds$$

Aus der Partialbruchzerlegung von $\frac{1}{s(s+1)}$ zu $\frac{1}{2}-\frac{1}{s+1}$ folgt:

$$=\ln(e^x+1)\Big|_{-1}^{2}-\int_{e^{-1}}^{e^2}\frac{1}{s}ds+\int_{e^{-1}}^{e^2}\frac{1}{s+1}ds$$

Mit $u \coloneqq s+1$ folgt aus der Substitutionsregel:

$$\begin{aligned}
&= \ln(e^x+1)\Big|_{-1}^{2} - \int_{e^{-1}}^{e^2} \frac{1}{s} ds + \int_{e^{-1}+1}^{e^2+1} \frac{1}{u} du \\
&= \ln(e^x+1)\Big|_{-1}^{2} - \ln(s)\Big|_{e^{-1}}^{e^2} + \ln(u)\Big|_{e^{-1}+1}^{e^2+1} \\
&= -2\ln(e^2+1) + 2\ln(e+1) - 1
\end{aligned}$$

$\Rightarrow \int_{-1}^{2} \frac{e^x-1}{e^x+1} dx = -2\ln(e^2+1) + 2\ln(e+1) - 1$

(d) $\int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}} dx$

partielle Integration mit $f_1(x) = x, g_2(x) = \frac{1}{\sqrt{1-x^2}}$:

$$\begin{aligned}
&\Rightarrow \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}} dx \\
&\rightarrow x\arcsin(x)\Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} 1 \cdot \arcsin(x) dx
\end{aligned}$$

Substitution mit $v \coloneqq \arcsin(x)$

$$\rightarrow x\arcsin(x)\Big|_0^{\frac{1}{2}} - \int_0^{\arcsin(\frac{1}{2})} v\cos(v) dv$$

partielle Integration mit $f_2(v) = v, g_2(v) = \cos(v)$

$$\begin{aligned}
&= x\arcsin(x)\Big|_0^{\frac{1}{2}} - v\sin(v)\Big|_0^{\arcsin(\frac{1}{2})} + \int_0^{\arcsin(\frac{1}{2})} 1 \cdot \sin(v) dv \\
&= x\arcsin(x)\Big|_0^{\frac{1}{2}} - v\sin(v)\Big|_0^{\arcsin(\frac{1}{2})} - \cos(v)\Big|_0^{\arcsin(\frac{1}{2})} \\
&= \tfrac{1}{2}\arcsin(\tfrac{1}{2}) - 0 - \arcsin(\tfrac{1}{2})\tfrac{1}{2} + 0 - \cos(\arcsin(\tfrac{1}{2})) + 1 \\
&= -\cos(\arcsin(\tfrac{1}{2})) + 1 \\
&= \frac{2-\sqrt{3}}{2} \approx 0.133975
\end{aligned}$$