Some definitions

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In this note I will write down some of the definitions that are often used in the course.

Definition. Courant condition
The Courant condition (also called Courant-Friedrichs-Lewy (CFL) condition) is a condition that must (often) hold for the finite difference scheme to be stable.

For the two dimentional wave equation

$$\frac{\partial^2 u}{\partial^2 t} = q(x, y)(\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y})$$

it can be deduced that the Courant condition is

$$c^2(\frac{dt^2}{dx^2} + \frac{dt^2}{dy^2}) \leq 1,$$

where $c = \sqrt{\max q(x)}$. This gives the stability condition

$$dt \leq \frac{1}{c}\frac{1}{\sqrt{\frac{1}{dx^2} + \frac{1}{dy^2}}}$$

on dt. Usally we let dt be a little smaller than this.

Definition. Neumann/Dirichlet boundary conditions
A von Neumann boundary condition states that the solution to the PDE should satisfy

$$\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = f(\mathbf{x}) \quad \forall \mathbf{x} \in \partial\Omega,$$

where Ω is the domain where the function u is defined. Often the function f is equal to 0. The normal derivative is defined by

$$\frac{\partial y}{\partial \mathbf{n}}(\mathbf{x}) = \nabla y(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}),$$

where $\mathbf{n}(\mathbf{x})$ denotes the outward pointing normal vector.

While the von Neumann condition gives restrictions on the derivative of the solution at the boundary, a Dirichlet boundary condition says what the value of the solution should be at the boundary:

$$u(\mathbf{x}) = f(\mathbf{x}) \quad \forall \mathbf{x} \in \partial\Omega.$$

Definition. Convergence rate
For many numerical schemes we have the relation

$$E = Ch^r$$

where E is the error, h is some parameter proportional to the mesh size, and C and r are constants. The convergence rate of the scheme is r. To find r we consider the error with two different h values, say (E_i, h_i) and (E_{i+1}, h_{i+1}). Since both errors should satify the relation above, we have

$$E_i = Ch_i^r,$$
$$E_{i+1} = Ch_{i+1}^r.$$

Divide the two equations and take the logarithm,

$$\frac{E_i}{E_{i+1}} = \left(\frac{h_i}{h_{i+1}}\right)^r,$$

$$\log\left(E_i/E_{i+1}\right) = r\log\left(h_i/h_{i+1}\right),$$

$$r = \frac{\log\left(E_i/E_{i+1}\right)}{\log\left(h_i/h_{i+1}\right)}.$$

The r above is often denoted r_i, and you should compute a series of r_i's to see if the convergence rate is stabilizing when h is getting small.

Note that if h_i and h_{i+1} are very small you may get problems due to precision errors when the computer is dividing small numbers.