# Weighted Projection Quantiles Algorithm

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**Algorithm to calculate weighted projection quantile** along the vector $\mathbf{u} \in \mathbb{R}^p$, given a set of observations $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_n$:

1. ***Compute $Q_{proj}(\mathbf{u})$, the projection quantile along $\mathbf{u}$***
   - Project each $\mathbf{X}_i$ along $\mathbf{u}$ to obtain $X_{\mathbf{u}i} = \frac{\langle \mathbf{X}_i, \mathbf{u} \rangle}{\|\mathbf{u}\|}$, for $i = 1, 2, ..., n$.
   - Find $\alpha = \frac{1+\|\mathbf{u}\|}{2}$-th quantile of $X_{\mathbf{u}1}, ..., X_{\mathbf{u}n}$, say $q_{\mathbf{u}}$..
   - $Q_{proj}(\mathbf{u}) = q_{\mathbf{u}} \mathbf{e}_{\mathbf{u}}$, $\mathbf{e}_{\mathbf{u}} = \mathbf{u}/\|\mathbf{u}\|$ being the unit vector along $\mathbf{u}$.
2. ***Compute Weights corresponding to this projection quantile $Q_{proj}(\mathbf{u})$***
   - Compute global weights for the direction vector $\mathbf{u}$ by $k$-mean distance:
     - Compute $k$-mean distance corresponding to $Q_{proj}(\mathbf{u})$ using $d_k = \frac{1}{n} \sum_{i=1}^{n} d_i \mathbb{I}_{\{d_i < d_{(k)}\}}$, where $d_i$ is the euclidean distace of $\mathbf{X}_i$ from $Q_{proj}(\mathbf{u})$ given by $\|\mathbf{X}_i - Q_{proj}(\mathbf{u})\|$. $k$ is a tuning parameter.
     - Compute the weights corresponding to $\mathbf{u}$:
       $$w_{\mathbf{u}} = \exp(-a.d_k)$$
       where $a$ is a tuning parameter.
   - Compute weights for each sample point $\mathbf{X}_i; i = 1, 2, ..., n$:
     - Compute the orthogonal Norms by $\|\mathbf{X}_{\mathbf{u}\perp i}\| = \|\mathbf{X}_i - X_{\mathbf{u}i} \mathbf{e}_{\mathbf{u}}\|$.
     - Compute weight of $i^{th}$ sample:
       $$w_{2i} = \begin{cases} \exp\left[-b \frac{\|\mathbf{X}_{\mathbf{u}\perp i}\|}{\|\mathbf{X}_i\|}\right] & \text{if} \|\mathbf{X}_i\| \le \epsilon \\ 0 & \text{otherwise} \end{cases}$$
       $b, \epsilon$ being tuning parameters.
3. ***Compute the weighted projection quantile***
   - Define $\tilde{X}_{\mathbf{u}i} = w_{\mathbf{u}} w_{2i} X_{\mathbf{u}i}$ for $i = 1, ..., n$.
   - Find $\alpha = \frac{1+\|\mathbf{u}\|}{2}$ th quantile of $\tilde{X}_{\mathbf{u}1}, ..., \tilde{X}_{\mathbf{u}n}$. Let it be $\tilde{q}_{\mathbf{u}}$.
   - Find the weighted projection quantile as $\tilde{Q}_{proj}(\mathbf{u}) = \tilde{q}_{\mathbf{u}} \mathbf{e}_{\mathbf{u}}$.

***Algorithm for weighted projection quantile based data depth given data $\mathbf{X}_1, ..., \mathbf{X}_n$ and a new point $\mathbf{p}$***

1. Compute Projection Quantiles for all $n$ sample points along the direction of $\mathbf{p}$.
2. Find two closest points along the grid and their corresponding quantile values, say $\alpha_1$ and $\alpha_2$ corresponding to $u_{p1}$ and $u_{p2}$.
3. Approximate the quantile value for $p$ by interpolation $\alpha_p = \frac{(\alpha_1-\alpha_2)(\|p\|-\|u_1\|}{\|u_1\|-\|u_2\|} + \alpha_1$.
4. Define **projection quantile depth** as $d_p = \exp(-\alpha_p)$.

***Algorithm for weighted projection depth based classification scheme***

1. Given data with $k$ classes and a new observation $p$ find the data depth for $p$ for each class $k$.
2. Classify $p$ in class $j$ if $d_{pj} = min\{d_{pi}; i \in \{1, ..., k\}\}$