Weighted Projection Quantiles Algorithm

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Algorithm to calculate weighted projection quantile along the vector $\mathbf{u} \in \mathcal{B}_p$, given a set of observations $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_n$:

1. ***Compute $Q_{proj}(\mathbf{u})$, the projection quantile along $\mathbf{u}$***
   - Project each $\mathbf{X}_i$ along $\mathbf{u}$ to obtain $X_{\mathbf{u}i} = \frac{\langle \mathbf{X}_i, \mathbf{u} \rangle}{\|\mathbf{u}\|}$, for $i = 1, 2, ..., n$.
   - Find $\alpha = \frac{1+\|\mathbf{u}\|}{2}$-th quantile of $X_{\mathbf{u}1}, ..., X_{\mathbf{u}n}$, say $q_{\mathbf{u}}$..
   - $Q_{proj}(\mathbf{u}) = q_{\mathbf{u}} \mathbf{e}_{\mathbf{u}}$, $\mathbf{e}_{\mathbf{u}} = \mathbf{u}/\|\mathbf{u}\|$ being the unit vector along $\mathbf{u}$.
2. ***Compute Weights corresponding to this projection quantile $Q_{proj}(\mathbf{u})$***
   - Compute global weights for the direction vector $\mathbf{u}$ by k-mean distance:
     - Compute k-mean distance corresponding to $Q_{proj}(\mathbf{u})$ using $\bar{d}_k = \frac{1}{n} \sum_{i=1}^{n} d_i \mathbb{I}_{\{d_i < d_{(k)}\}}$, where d_i is the euclidean distace of $\mathbf{X}_i$ from $Q_{proj}(\mathbf{u})$ given by $\|\mathbf{X}_i - Q_{proj}(\mathbf{u})\|$. k is a tuning parameter.
     - Compute the weights corresponding to $\mathbf{u}$:
       $$w_{\mathbf{u}} = \exp(-a.d_k)$$
       where a is a tuning parameter.
   - Compute weights for each sample point $\mathbf{X}_i; i = 1, 2, ..., n$:
     - Compute the orthogonal Norms by $\|\mathbf{X}_{\mathbf{u}\perp i}\| = \|\mathbf{X}_i - X_{\mathbf{u}i} \mathbf{e}_{\mathbf{u}}\|$.
     - Compute weight of i^{th} sample:
       $$w_{2i} = \exp\left[-b \frac{\|\mathbf{X}_{\mathbf{u}\perp i}\|}{\|\mathbf{X}_i\|}\right] \mathbb{I}_{\{\|\mathbf{X}_{\mathbf{u}\perp i}\| \le \epsilon\}}$$
       b, ϵ being tuning parameters.
3. ***Compute the weighted projection quantile***
   - Suppose there are m observations with non-zero weights w_{2i}, with indices $i_1, i_2, ..., i_m$. Define $\tilde{X}_{\mathbf{u}i_j} = w_{\mathbf{u}} w_{2i_j} X_{\mathbf{u}i_j}$.
   - Find $\alpha = \frac{1+\|\mathbf{u}\|}{2}$-th quantile of $\tilde{X}_{\mathbf{u}i_1}, ..., \tilde{X}_{\mathbf{u}i_m}$. Let it be $\tilde{q}_{\mathbf{u}}$.
   - Find the weighted projection quantile as $\tilde{Q}_{proj}(\mathbf{u}) = \tilde{q}_{\mathbf{u}} \mathbf{e}_{\mathbf{u}}$.

Definition Given a random vector $\mathbf{X} \in \mathbb{R}^p$ that follows a multivariate distribution F, and a point $\mathbf{p} \in \mathbb{R}^p$, find $\alpha_{\mathbf{p}}$ such that $\|\mathbf{p}\|$ is the $\alpha_{\mathbf{p}}$-th quantile for the projection of $\mathbf{X}$ on $\mathbf{p}$, say $X_{\mathbf{p}}$. Then the **Projection Quantile Depth** (PQD) at $\mathbf{p}$ with respect to F is defined as

$$D(\mathbf{p}, F) = \exp(-\alpha_{\mathbf{p}})$$

Given data $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_n$, the PQD at a given $\mathbf{p}$ can be estimated by finding the two nearest points on either side of $\|\mathbf{p}\|$ along $\mathbf{p}$, say $\mathbf{p}_1, \mathbf{p}_2$, obtain their corresponding quantiles, say α_1, α_2 respectively, then estimate $\alpha_{\mathbf{p}}$ by a linear approximation:

$$\hat{\alpha}_{\mathbf{p}} = \frac{(\alpha_1 - \alpha_2)(\|\mathbf{p}\| - \|\mathbf{p}_1\|)}{\|\mathbf{p}_1\| - \|\mathbf{p}_2\|} + \alpha_1$$

and plugging it in the above definition.

Algorithm 1 Algorithm for PQD-based classification

```
procedure PQDClassifier(training data X_i ∈ R^{n_i×p} with class labels i;
i = 1, 2, ..., k, new data x_new ∈ R^p)
    Set i = 1.
    top:
    Estimate from the sample the PQD of p with respect to the i^th population, say D(x_new, X_i).
    if i = k then Stop
    else
        Set i ← i + 1, goto top

    Find c that maximizes the PQD of x_new w.r.t. all possible classes:
        D(x_new, X_c) = max{D(x_new, X_i) : i = 1, 2, ..., k}
    Assign class c to new data x_new.
```

Note One can define a weighted version of PQD by replacing $X_{\mathbf{p}}$ by their weighted version $\tilde{X}_{\mathbf{p}}$. A weighted classification scheme follows similarly.

Modifications

1. $w_{2i} = \mathbb{I}_{\{\|X_{\mathbf{u}\perp i}\| \leq \epsilon\}}$ Wouldn't work. The objective function here is

$$\begin{aligned}
\tilde{\Psi}_{\mathbf{u}}(q) &= \mathbb{E}\left[\{|X_{\mathbf{u}} - q| + \alpha(X_{\mathbf{u}} - q)\}\mathbb{I}_{\{\|\mathbf{X}_{\mathbf{u}\perp}\| \leq \epsilon\}}\right] \\
&= \mathbb{E}\left[|X_{\mathbf{u}} - q| + \alpha(X_{\mathbf{u}} - q)\right] P\left[\|\mathbf{X}_{\mathbf{u}\perp}\| \leq \epsilon\right] \\
&= \Psi_{\mathbf{u}}(q) P\left[\|\mathbf{X}_{\mathbf{u}\perp}\| \leq \epsilon\right]
\end{aligned}$$

because $Cov(X_{\mathbf{u}}\mathbf{e}_{\mathbf{u}}, \mathbf{X}_{\mathbf{u}\perp}) = \mathbf{0}$. Hence it gives out PQ as the minimizer.