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An Isobar-Surfaces Approach to Multidimensional Outlier-Proneness

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Abstract. The aim of this paper is to define and investigate outlier-proneness for multivariate distributions. This is done by using a concept of ordering multivariate data based on isobar-surfaces, which yields an analogy of the results to the univariate case.

Key words. isobars, order statistics, extreme values, weak stability, outliers, outlier-proneness

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1. Introduction

When modeling multivariate data, we often have some feeling about how outlier prone the underlying distribution should be. That is, there are phenomena where some outliers in the data will be a rule and are to be expected, rather than they are considered as very rare exceptions. In such situations, we would not want of course to choose e.g. a multivariate normal distribution with very light tails as a model of the data-generating mechanism.

There are other multivariate distributions, different from the normal where even such a vague knowledge about their tendency to produce outliers is not known. Therefore, a classification of multivariate distributions w.r.t. their outlier-resistance and proneness as available in the univariate case (Green, 1976; Gather and Rauhut, 1990; Schuster, 1984) would be useful.

This paper gives a formal framework and definitions of the terms outlier-proneness and outlier-resistance of multivariate distributions based on an isobar—surfaces approach to multivariate extremes (Delcroix and Jacob, 1991).

More exactly, the limit behavior of the difference of the two largest multivariate ''extremes'' is used—in the sense of Green (1976)—to define the outlier-proneness of the underlying distribution. Having to decide then, if some class of multivariate distribution functions is outlier-prone, we need a characterization, directly in terms of the distribution function, too. This paper gives such equivalence theorems which allow one to check for outlier-proneness or outlier-resistance of a distribution in many different ways.

As mentioned, we choose a concept of ordering multivariate data based on the isobar-surfaces of the underlying distribution. Though this is a natural way of ordering multivariate data, in contexts with just a given data set, it cannot be applied, when the data generating distribution is not completely known. This is usually a deficiency but in our situation, where we want to check if a given distribution at hand is suitable for modeling a data structure, we are able to use this natural notion of ordering in terms of isobar-extremes, since the distribution is known then.

Also, as in the univariate case, this new notion of outlier-proneness via isobar-extremes is strongly related to weak stability of the extremes. Our approach differs from the one by Mathar (1985) who defines outlier-resistance via the limit behavior of the distance of the upper extremes of the real valued norms of the sample points as ordering principle. His approach therefore yields a characterization of multivariate outlier-proneness via the minimum of the distribution function of the marginals, whereas our definition leads to characterizations depending on the behavior of the conditional distribution functions given the radial directions—i.e. the ‘‘angles’’—as it is defined in the next section. Hence, we take into account the complete shape of the multivariate distribution.

This paper is organized as follows. In Section 2, we start with defining weak stability of multivariate extremes by the isobar surface ordering. In Section 3, we define outlier-resistance and outlier-proneness of multivariate distributions, we relate these properties to weak stability of the extremes and characterize outlier-resistance by the tail behavior of the conditional radial distributions. Section 4 gives a generalization and examples.

2. Weak stability of multivariate extremes

We first recall the definition of the largest value of a multivariate sample, as given in Delcroix and Jacob (1991). The motivation was to describe the asymptotic position of a multivariate sample (Barme-Delcroix, 1993) without using classical convexity notions (Geffroy, 1961).

We consider random variables with values in the Euclidean space $\mathbf{R}^k$.

For every x in $\mathbf{R}^k \setminus \{0\}$ we define a pair $(\|x\|, x/\|x\|) = (r, \theta)$ in $\mathbf{R}^{+*} \times \mathbf{S}^{k-1}$, where $\|\cdot\|$ is the Euclidean norm and $\mathbf{R}^{+*}$ is the set of the strictly positive real numbers. The unit sphere $\mathbf{S}^{k-1}$ in $\mathbf{R}^k$ is endowed with the induced topology of $\mathbf{R}^k$.

For each random variable (r.v. for short) $X = (R, \Theta)$ in $\mathbf{R}^k$ with radius R and angle Θ, we assume that the distribution of Θ, and for all θ in $\mathbf{S}^{k-1}$, the distribution of R given $\Theta = \theta$ respectively, has a continuous density. F_θ denotes the continuous and one-to-one conditional distribution function of R given $\Theta = \theta$. This means in particular that we suppose $F_\theta(r) < 1$ for all $r > 0$ and for all θ, and that the support of the distribution is not finite.

For each $0 < u < 1$, we call the mapping $\theta \rightsquigarrow F_\theta^{-1}(u)$ a u-level isobar of the distribution of R given $\Theta = \theta$. We suppose that this mapping is continuous and strictly positive. The surface given by $\rho_u(\theta) = F_\theta^{-1}(u)$, considered as a function of θ, is also called a u-isobar for

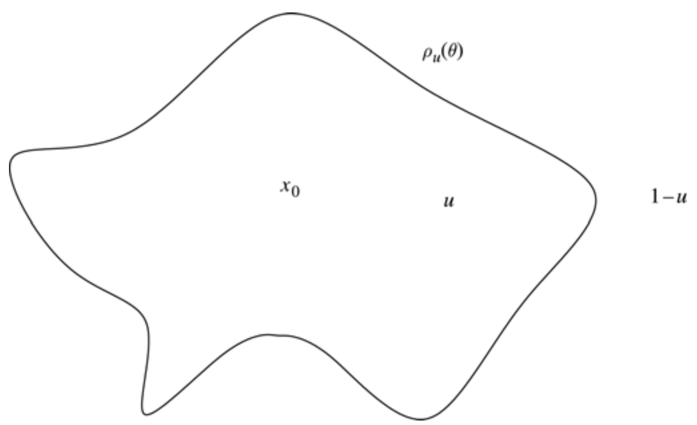

Figure 1. u-level isobar.

all $0 < u < 1$; see Figure 1. So, isobars are closed surfaces included in each other for increasing levels. For bivariate distributions isobars are classical polar curves. Very different shapes of isobars can be considered according to the choice of the distribution, for instance, in Remark 2 the equation of isobars for a bivariate Gaussian distribution is given.

Let $x_1 = (r_1, \theta_1), \ldots, x_n = (r_n, \theta_n)$ be observations of an i.i.d. sample $X_1, \ldots, X_n$ from the distribution of $X = (R, \Theta)$.

Let $u_j = F_{\theta_j}(r_j)$ for $1 \leq j \leq n$, $u_n^* = \max_{1 \leq j \leq n} u_j$, and define $x_n^* = (r_n^*, \theta_n^*)$ by

$$F_{\theta_n^*}(r_n^*) = u_n^*.$$

Since F_θ is continuous and strictly increasing for all θ, like this we have defined almost surely unique r.v.'s $U_1, \ldots, U_n$ as well as an almost surely unique r.v. $X_n^* = (R_n^*, \Theta_n^*)$ which is an element of $\{X_1, \ldots, X_n\}$ for which

$$F_{\Theta_n^*}(R_n^*) = \max_{1 \leq j \leq n} U_j.$$

We call X_n^* the isobar-maximum of $X_1, \ldots, X_n$; see Figure 2.

Obviously, to find this isobar-maximum of a multivariate sample, the underlying distribution has to be known. However, this kind of extreme value, and more generally, the ordering of the sample according to the isobars, does not give up any information the sample carries, like the ordering by norms, e.g. it is possible to give an estimation of the isobars by regression methods for particular cases (Jacob and Suquet, 1997). One can also estimate the origin by using the barycenter of the sample points. However for many situations this origin is given in a natural way.

It has been shown in Delcroix and Jacob (1991) that the conditional distribution of R_n^* given Θ_n^* is F_θ^n, hence the distributions of (R_n^*, Θ_n^*) and (R, Θ) have the same set of isobars which led to the following definition.

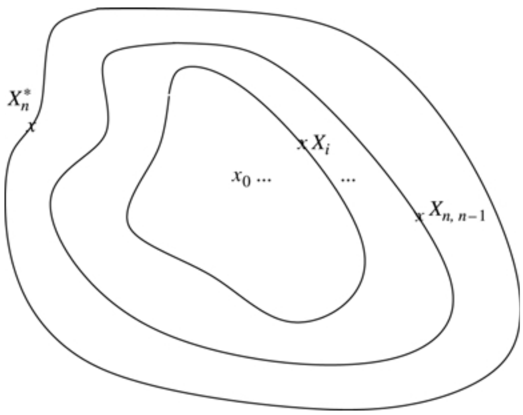

Figure 2. Isobar-maximum.

Definition 1: For a sequence $(E_n)_n$ of multivariate r.v.s, the sequence $(X_n^*)_n = ((R_n^*, \Theta_n^*))_n$ of the isobar-maxima is called stable in probability if and only if there is a sequence $(g_n)_n$ of isobars satisfying

$$R_n^* - g_n(\Theta_n^*) \xrightarrow{P} 0. \tag{1}$$

Following Geffroy (1958), it is possible to choose $g_n(\theta) = F_\theta^{-1}(1 - \frac{1}{n})$.

It is convenient to fix a point $x_1 = (1, \theta_1)$, θ_1 in $\mathbf{S}^{k-1}$. For every point $x = (r, \theta_1)$, there is a unique surface $g(\theta, r)$, θ in $\mathbf{S}^{k-1}$, containing x, which has a level denoted by $u(r)$ and which is given by

$$g(\theta, r) = \rho_{u(r)}(\theta) = F_\theta^{-1}(F_{\theta_1}(r)). \tag{2}$$

Note that $g(\theta_1, r) = r$; see Figure 3. Moreover the mapping $r \rightsquigarrow u(r)$ from $\mathbf{R}_+^*$ into $\mathbf{R}_+^*$ is increasing and one-to-one.

The following conditions (H) and (K) will be needed.

(H) There exist $0 < \alpha \le \beta < \infty$ such that for all θ in $\mathbf{S}^{k-1}$ and for all $r > 0$:

$$\alpha \le \frac{\partial g}{\partial r}(\theta, r) \le \beta.$$

(K) For all $\varepsilon > 0$, there exists $\eta > 0$ such that for all $r > 0$:

$$\sup_\theta \{g(\theta, r + \eta) - g(\theta, r - \eta)\} < \varepsilon.$$

Clearly, (H) implies (K).

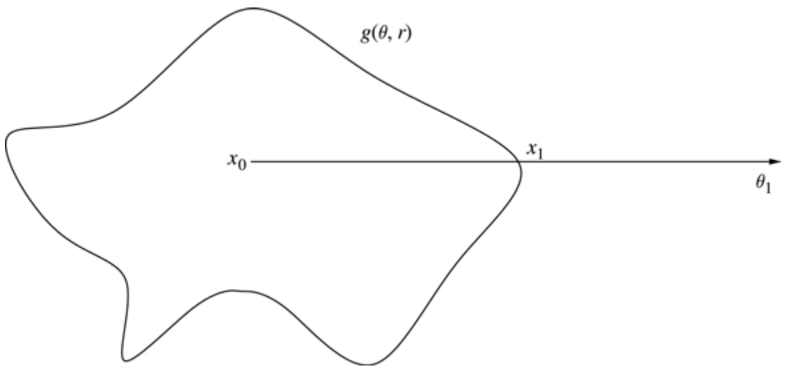

Figure 3. Isobar containing $x_1 = (1, \theta_1)$.

Remark 1: Condition (H) entails a regularity property of the isobars following from the mean value theorem:

For all $\beta_0 > 0$ there exists $\eta = \beta_0 \alpha / \beta > 0$ and for all $r > 0$, there exist two isobars $h_{\beta_0}(\theta, r) = g(\theta, r + \beta_0/\beta)$ and $\tilde{h}_{\beta_0}(\theta, r) = g(\theta, r - \beta_0/\beta)$ such that for all θ, $g(\theta, r) - \beta_0 < \tilde{h}_{\beta_0}(\theta, r) < g(\theta, r) - \eta < g(\theta, r) + \eta < h_{\beta_0}(\theta, r) < g(\theta, r) + \beta_0$.

Note that η does not depend on r.

Remark 2: For a bivariate Gaussian sample with covariance matrix $\left(\begin{smallmatrix} \sigma^2 & 0 \\ 0 & \tau^2 \end{smallmatrix}\right)$, we have $g(\theta, r) = r\phi(\theta)$ with $\phi(\theta) = 1/\sqrt{2}\sigma(\cos^2\theta/2\sigma^2 + \sin^2\theta/2\tau^2)^{\frac{-1}{2}}$ and the isobars are the density contours. Note that condition (H) is satisfied. For $\sigma = \tau = 1$, the distribution is spherically symmetric and the isobars are circles. Hence in this particular case the ordering of the sample is the ordering of the norms of the sample points.

The next theorem gives conditions for stability similar to those of Geffroy (1958) in the univariate case (see Delcroix and Jacob, 1991, for a proof). For this purpose, we define W_n^* by $F_{\theta_1}(W_n^*) = F_{\Theta_n^*}(R_n^*)$ for $X_n^* = (R_n^*, \Theta_n^*)$, i.e. $W_n^* = F_{\theta_1}^{-1}(F_{\Theta_n^*}(R_n^*))$ is the intersection of the half axis containing $x_1 = (1, \theta_1)$ and the isobar containing X_n^* (see Figure 4).

Theorem 1:

a. *Under condition (K) the sequence* $(X_n^*)_n$ *is stable in probability if* $(W_n^*)_n$ *is stable in probability.*
b. *Under condition (H) the sequence* $(W_n^*)_n$ *is stable in probability if and only if* $(X_n^*)_n$ *is stable in probability.*
c. *Consider for some fixed integer* α *the sequence* $(X_{n,n-\alpha+1})_n$, *this being defined by ordering the sample according to increasing levels by*

$$X_{n,1}, \ldots, X_{n,n-\alpha+1}, \ldots, X_{n,n} = X_n^*.$$

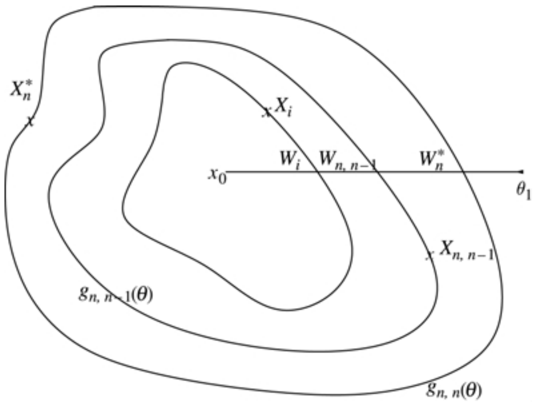

Figure 4. The order statistics of the real sample $W_1, \ldots, W_n$.

Let (H) be satisfied. Then $(X_n^*)_n$ *is stable in probability if and only if* $(X_{n,n-\alpha+1})_n$ *is stable in probability.*

For the proof of a and b see Delcroix and Jacob (1991) c follows immediately from a and b and the univariate result of Geffroy (1958).

Possibilities to check for the weak stability of $(X_n^*)_n$ on the basis of the distribution function $F_\theta(r)$, $r > 0$, θ in $\mathbf{S}^{k-1}$, will be provided by the characterization results in Theorem 2 below.

3. Multidimensional outlier-prone and outlier-resistant distributions

We give an application of stability in probability of multivariate samples as defined in the previous section to the notion of outlier-resistant and outlier-prone distributions. In Neyman and Scott (1971) we find a definition which has been improved by Green (1976). The goal is to distinguish between two classes of distributions: distributions for which as a rule there exist observations far apart from the main group of the data, and distributions for which this phenomenon occurs with very small probability. So, Green called a univariate distribution F absolutely outlier-resistant if for all $\varepsilon > 0$:

$$\lim_{n \to +\infty} P(X_{n,n} - X_{n,n-1} > \varepsilon) = 0 \tag{3}$$

where $X_{n,1} \leq X_{n,2} \leq \cdots \leq X_{n,n}$ are the usual univariate order statistics of $X_1, \ldots, X_n$, distributed identically according to F.

On the other hand, a distribution F is called absolutely outlier-prone if there exist $\varepsilon > 0$, $\delta > 0$ and an integer n_0 such that for $n \geq n_0$:

$$P(X_{n,n} - X_{n,n-1} > \varepsilon) \geq \delta. \tag{4}$$

Remark 3: The AOR- and AOP-property depend only on the tail-behavior of F, e.g. AOR is equivalent to $\lim_{x\to+\infty} \bar{F}(x)/\bar{F}(x-h) = 0$ for all $h > 0$, with $\bar{F} = 1 - F$. Moreover, the behavior of many types of univariate distributions is investigated in Gather (1979) yielding the following special results:

- The Gumbel distribution with $F(x) = \exp(-e^{-x})$ for $x \in \mathbf{R}$ is not AOR but is AOP.
- Also the Frechet distribution with $F(x) = \exp(-x^{-\alpha})\mathbf{1}_{\{x>0\}}$ is not AOR but is AOP for all $\alpha > 0$.
- Distribution functions of the type $F(x) = (1 - c\exp(-bx^{\alpha}))\mathbf{1}_{\{x>0\}}$ with constants c, b, $\alpha > 0$, are AOR as long as $\alpha > 1$.

Other definitions of outlier-proneness of univariate distributions have been given for example by O'Hagan (1979) and Goldstein (1982) in a Bayesian framework. Gather and Rauhut (1990) discuss these different notions for univariate data. Here, we will extend the notion of outlier-proneness given by Green to multidimensional samples by using isobars.

Recall that for all θ, F_θ denotes the distribution function of R given $\Theta = \theta$ and that $\bar{F}_\theta = 1 - F_\theta$. Note also that for each sample point $X_i = (R_i, \Theta_i)$, $i = 1, \ldots, n$, there exists almost surely a unique isobar containing X_i. Let $g_{n,n}$ denote the isobar containing $X_n^* = X_{n,n}$ and $g_{n,n-1}$ the isobar containing $X_{n,n-1}$. Thus for all θ in $\mathbf{S}^{k-1}$, $g_{n,n}(\theta)$ and $g_{n,n-1}(\theta)$ are real valued r.v.s. Since W_n^* was defined as the intersection of the half axis containing $(1, \theta_1)$ and the isobar containing X_n^*, we now have $W_n^* = g_{n,n}(\theta_1)$. If we define analogously for all $1 \le i \le n$, $W_i = {F_{\theta_1}}^{-1}(F_{\Theta_i}(R_i))$, we get $W_{n,1}, \ldots, W_{n,n} = W_n^*$ as the usual order statistics of the real valued sample $W_1, \ldots, W_n$ distributed identically according to F_{θ_1}; see Figure 4.

3.1. Multivariate AOR distributions

Definition 2: The distribution of the multivariate r.v. (R, Θ) is absolutely outlier-resistant (AOR), if and only if for all θ:

$$g_{n,n}(\theta) - g_{n,n-1}(\theta) \xrightarrow{P} 0. \tag{5}$$

For a real sample, it has been shown in Geffroy (1958) and Gnedenko (1943), that $(X_{n,n})_n$ is stable in probability if and only if $X_{n,n} - X_{n,n-1} \xrightarrow{P} 0$. The following theorem gives an analogous result and as mentioned in Remark 3 a characterization of weak stability by the tail behavior of the underlying distribution.

Let condition (H) always be satisfied in the following.

Theorem 2: *All the following statements are equivalent:*

i. The distribution of (R, Θ) *is AOR.*
ii. $(X_n^*)_n$ *is stable in probability.*

iii. *For all* $1 \le \alpha \le n$, $(X_{n,n-\alpha+1})_n$ *is stable in probability.*
iv. *There exists* θ_1 *such that* $\lim_{x\to+\infty} \bar{F}_{\theta_1}(x)/\bar{F}_{\theta_1}(x-h) = 0$, *for all* $h > 0$.
v. *For all* θ, $\lim_{x\to+\infty} \bar{F}_{\theta_1}(x)/\bar{F}_{\theta_1}(x-h) = 0$, *for all* $h > 0$.
vi. $W_n^* - W_{n,n-1} \xrightarrow{P} 0$.
vii. $(W_n^*)_n$ *is stable in probability.*
viii. *For all* θ, *the distribution* F_θ *is AOR.*
ix. *There exists* θ_1 *such that the distribution* F_{θ_1} *is AOR.*

Proof: Theorem 1c shows that ii and iii are equivalent. Gnedenko's Theorem and Theorem 1b show that ii, iv, v and vii are equivalent. Moreover, from Geffroy (1958), vii and vi are equivalent. Now, i involves ii: if for all θ, $g_{n,n}(\theta) - g_{n,n-1}(\theta) \xrightarrow{P} 0$ we get $g_{n,n}(\theta_1) - g_{n,n-1}(\theta_1) \xrightarrow{P} 0$, that is $W_n^* - W_{n,n-1} \xrightarrow{P} 0$; and since vi and vii are equivalent, $(W_n^*)_n$ is stable in probability and from Theorem 1b $(X_n^*)_n$ is stable in probability.

Conversely, if $(X_n^*)_n$ is stable in probability, $(W_n^*)_n$ is also stable and $W_n^* - W_{n,n-1} \xrightarrow{P} 0$. Then $g_{n,n}(\theta_1) - g_{n,n-1}(\theta_1) \xrightarrow{P} 0$; but θ_1 being arbitrary, we obtain (i).

Clearly, these properties are equivalent to viii and ix. □

For univariate samples, it is possible, following Gather and Rauhut, (1990) to give other characterizations of AOR distributions based on the mean residual life function (mrlf), which is defined for a real r.v. X by

$$\mathrm{e}(x) = E(X - x \mid X > x).$$

For $i = 1, \ldots, n-1$ and for $x > 0$, $y > 0$ and $n > 2$, let

$$M_{i,n}(x, y, \theta) = P\{g_{n,i+1}(\theta) - g_{n,i}(\theta) > y \mid g_{n,i}(\theta) = x)\}. \tag{6}$$

For fixed, $\theta = \theta_1$, $M_{i,n}(x, y, \theta)$ can be written as

$$M_{i,n}(x, y, \theta_1) = P\{W_{n,i+1} - W_{n,i} > y \mid W_{n,i} = x\}. \tag{7}$$

From Gather and Rauhut (1990) and Theorem 2 we obtain the following result.

Theorem 3: *The distribution of* (R, Θ) *is AOR if and only if there exists* θ_1 *such that for all* $y > 0$:

$$\lim_{x\to+\infty} M_{i,n}(x, y, \theta_1) = 0, \tag{8}$$

for some $1 \le i \le n$.

Proof: From Theorem 2, the distribution of (R, Θ) is AOR if and only if there exists θ_1 such that F_{θ_1} is AOR. But, from Gather and Rauhut (1990), F_{θ_1} is AOR if and only if (8) is valid. To show this we observe that the order statistics $W_{n,1}, \ldots, W_{n,n}$ form a Markov chain (Arnold et al., 1984; David, 1981) and that

$$M_{i,n}(x, y, \theta_1) = P\{W_{n,i+1} > x + y \mid W_{n,i} = x\} = \left(\frac{\bar{F}_{\theta_1}(x+y)}{\bar{F}_{\theta_1}(x)}\right)^{n-i}.$$

Assertion (iv) of the previous theorem completes the proof. □

In Definition 2, the sample size increases; but in Theorem 3, the sample size is fixed which makes it intuitively easier to relate the definition of outlier-resistance of the distribution to the non-occurence of outliers in the sample: the larger $X_{n,i}$ gets, the smaller the probability for the difference $X_{n,i+1} - X_{n,i}$ to be larger than an arbitrary positive number. The next theorem describes this fact in average.

For all θ in $\mathbf{S}^{k-1}$ and for all $i = 1, \ldots, n$, consider

$$\tilde{M}_{i,n}(x, \theta) = E(g_{n,i+1}(\theta) - g_{n,i}(\theta) \mid g_{n,i}(\theta) = x). \tag{9}$$

For fixed $\theta = \theta_1, \tilde{M}_{i,n}(x, \theta)$ can be written as

$$\tilde{M}_{i,n}(x, \theta_1) = E(W_{n,i+1} - W_{n,i} \mid W_{n,i} = x).$$

Theorem 4: *Suppose that for all θ, $\int x\, \mathrm{d}F_\theta$ exists. Then, the distribution of (R, Θ) is AOR if and only if there exists θ_1 such that for all $n \geq 3$:*

$$\lim_{x \to +\infty} \tilde{M}_{n-1,n}(x, \theta_1) = 0.$$

The proof is again only an application of Theorem 1b and of Gather and Rauhut (1990).

3.2. AOP distributions

Definition 3: The distribution of (R, Θ) is called absolutely outlier-prone (AOP) if and only if for all θ there exist $\varepsilon > 0, \delta > 0$ and an integer n_θ, such that for all θ and for all $n \geq n_\theta$:

$$P(g_{n,n}(\theta) - g_{n,n-1}(\theta) > \varepsilon) > \delta. \tag{10}$$

That is, for all θ, the distribution F_θ is AOP.

Theorem 5: *All the following statements are equivalent*:

i. The distribution of (R, Θ) *is AOP.*
ii. For all θ, *there exist* $\alpha > 0, \beta > 0$ *such that for all* x

$$\frac{\bar{F}_\theta(x+\beta)}{\bar{F}_\theta(x)} \geq \alpha. \tag{11}$$

iii. There exist $\theta_0, \alpha_0 > 0$ *and* $\beta_0 > 0$ *such that for all* x

$$\frac{\bar{F}_{\theta_0}(x+\beta_0)}{\bar{F}_{\theta_0}(x)} \geq \alpha_0. \tag{12}$$

iv. There exists θ_0 *such that* F_{θ_0} *is AOP.*

Proof: From Green (1976, Theorem 3.3) we have that for fixed θ, the univariate distribution F_θ is AOP if and only if (11) is fulfilled. This proves that i and ii are equivalent.

Clearly, ii implies iii.

To show that iii implies ii we consider $\theta_1 \neq \theta_0$; for all $r > 0$. There exists an isobar $g(\theta, r), \theta$ in $\mathbf{S}^{k-1}$, containing the point (r, θ_1). Let $u(r)$ denote the level of this isobar. Since (H) is satisfied (see Remark 1) there exist $\eta > 0$ and an isobar $h_{\beta_0}(\theta, r)$ such that for all r and for all θ

$$g(\theta, r) + \eta < h_{\beta_0}(\theta, r) < g(\theta, r) + \beta_0.$$

Let $u_{\beta_0}(r)$ denote the level of $h_{\beta_0}(\theta, r)$. Since $\bar{F}_{\theta_1} = 1 - F_{\theta_1}$ is decreasing,

$$\bar{F}_{\theta_1}(r) = \bar{F}_{\theta_1}(g(\theta_1, r)) = 1 - u(r) = \bar{F}_{\theta_0}(g(\theta_0, r)),$$

and

$$\bar{F}_{\theta_1}(r+\eta) = \bar{F}_{\theta_1}(g(\theta_1, r) + \eta) > \bar{F}_{\theta_1}(h_{\beta_0}(\theta_1, r)) = 1 - u_{\beta_0}(r).$$

Moreover,

$$1 - u_{\beta_0}(r) = \bar{F}_{\theta_0}(h_{\beta_0}(\theta_0, r)) > \bar{F}_{\theta_0}(g(\theta_0, r) + \beta_0),$$

and

$$\frac{\bar{F}_{\theta_0}(g(\theta_0, r) + \beta_0)}{\bar{F}_{\theta_0}(g(\theta_0, r))} < \frac{\bar{F}_{\theta_1}(r+\eta)}{\bar{F}_{\theta_1}(r)}.$$

Thus, if $\bar{F}_{\theta_0}(r+\beta_0)/\bar{F}_{\theta_0}(r) \geq \alpha_0$ for all real x, then for all $\theta_1 \neq \theta_0$, there exist $\beta_1 = \eta > 0$ and $\alpha_1 = \alpha_0 > 0$ such that for all r

$$\frac{\bar{F}_{\theta_1(r+\beta_1)}}{\bar{F}_{\theta_1}(r)} \geq \alpha_1,$$

and we obtain ii.

Clearly, iv is equivalent to the other statements. □

Examples:

a. For a bivariate Gaussian sample such as in Remark 2, we have $F_\theta(r) = 1 - \exp(-r^2\phi(\theta))$ and following Theorem 2 iv we can conclude that this distribution is AOR.

b. Suppose that $F_\theta(r) = 1 - c\exp(-br^{\alpha(\theta)})\mathbf{1}_{\{r>0\}}$ with α a strictly positive continuous function and $b, c > 0$ (Gumbel type distribution). It has been shown in Delcroix and Jacob (1991) that neither (H) nor the regularity property of isobars from Remark 1 is satisfied for this distribution. But if $\inf_\theta(\alpha(\theta)) > 1$, condition (K) is fulfilled for r large. Moreover, as in the univariate case, from Theorem 1a, $(X_n^*)_n$ is stable and the distribution of (R, Θ) is AOR. If α is constant and equal to 1, the distribution is AOP. And if there exists θ_0 such that $\alpha(\theta_0) < 1$ then the distribution is neither AOP nor AOR.

c. For the bivariate Morgenstern distribution with density $f(x,y) = e^{-(x+y)}(1+\alpha(2e^{-x}-1)(2e^{-y}-1))$ with $-1 \leq \alpha \leq 1$ it is possible to write down the distribution function F_θ explicitly:

$$\begin{aligned} F_\theta(r) = \frac{1}{d(\theta)}\Bigg\{&(1+\alpha)\left[\frac{(1-e^{-(\cos\theta+\sin\theta)r})}{(\cos\theta+\sin\theta)^2} - \frac{re^{-(\cos\theta+\sin\theta)r}}{(\cos\theta+\sin\theta)}\right] \\ &+ 4\alpha\left[\frac{1-e^{-2(\cos\theta+\sin\theta)r}}{4(\cos\theta+\sin\theta)^2} - \frac{re^{-2(\cos\theta+\sin\theta)r}}{2(\cos\theta+\sin\theta)}\right] \\ &+ 2\alpha\left[\frac{1-e^{-(2\cos\theta+\sin\theta)r}}{(2\cos\theta+\sin\theta)^2} - \frac{re^{-(2\cos\theta+\sin\theta)r}}{(2\cos\theta+\sin\theta)}\right] \\ &+ 2\alpha\left[\frac{1-e^{-(\cos\theta+2\sin\theta)r}}{(\cos\theta+2\sin\theta)^2} - \frac{re^{-(\cos\theta+2\sin\theta)r}}{(\cos\theta+2\sin\theta)}\right]\Bigg\} \end{aligned}$$

where $d(\theta)$ is a function of θ. Hence $F_\theta(r)$ is of the type

$$\begin{aligned} &1 - A\exp(-ar) - B\exp(-br) - C\exp(-cr) - D\exp(-dr) \\ &\quad - A'r\exp(-ar) - B'r\exp(-br) - C'r\exp(-cr) - D'r\exp(-dr) \end{aligned}$$

with $a = \cos(\theta) + \sin(\theta), b = 2a, c = \cos(\theta) + 2\sin(\theta), d = 2\cos(\theta) + \sin(\theta)$ and A, B, C, D, A′, B′, C′, D′ all depending only on θ and α. We can then apply Theorem 5 (iii) which yields after some manipulations that the bivariate Morgenstern distribution is AOP.

The following corollary is also obvious from Theorem 5 as well as from using (6), (7) and (9).

Corollary 1

a. *The distribution of* (R, Θ) *is AOP if and only if there exists* θ_0 *such that for all* $y > 0$, *there exist* α_0 *and* x_0 *such that*

$$M_i, n(x, y, \theta_0) \geq \alpha_0,$$

for all $x \geq x_0$, *for some* $1 \leq i \leq n-1$.

b. *Suppose that* $\int x dF_\theta$ *exists for all* θ *and that the distribution of* (R, Θ) *is AOP, then there exist* θ_0, δ_0 *and* x_0, *such that for* $x \geq x_0$ *and for all* $n \geq 3$

$$\tilde{M}_{n-1,n}(x, \theta_0) \geq \delta_0.$$

4. Generalization and examples

Of course, a lot of multidimensional distributions do not have stability properties. However, we can generalize the notion of weak stability, to φ-stability (see, for example, Delcroix and Jacob, 1991; Gather and Rauhut, 1990; Geffroy, 1958; Gnedenko, 1943; Green, 1976; Resnick, 1987; Tomkins and Wang, 1992). For a positive, increasing, concave, one-to-one C^1-function defined on $\mathbf{R}^+$, we consider the set of points $((\varphi(R_1), \Theta_1) \cdots (\varphi(R_n), \Theta_n))$ instead of the initial sample. Then, for a suitable function φ, we obtain stability properties for many usual multivariate distributions (exponential distributions, Cauchy distributions ...). Having defined φ-stability (Delcroix and Jacob, 1991), we can also define multidimensional φ-outlier-resistant or φ-outlier-prone distributions. It suffices to consider the distribution of $(\varphi(R), \Theta)$ instead of the distribution of (R, Θ). For example, if the distribution of (R, Θ) is AOR and if φ is a positive, increasing, concave, one-to-one C^1-function defined over $\mathbf{R}^+$, then the distribution of $(\varphi(R), \Theta)$ is also AOR. When $\varphi(x) = \max(0, \log x)$, we come to the notions of relatively outlier-resistant or relatively outlier-prone distributions. In this case, φ-outlier-resistant and φ-outlier-prone are denoted by ROR and ROP as they are given in Green (1976) for univariate distributions.

Examples:

a. Exponentional-type distributions with

$$F_\theta(r) = (1 - c\exp(-b(\theta)r))1_{\{r>0\}},$$

$c > 0$ and b being a strictly positive and continuous function, are ROR and AOP (see Example 1b in Section 3).

b. Cauchy distributions with conditional density

$$f_\theta(r) = \frac{2}{\pi}\frac{\lambda(\theta)}{r^2 + \lambda(\theta)^2} 1_{\{r>0\}},$$

λ being a strictly positive, and continuous function, are ROP, but are φ-OR for $\varphi(x) = \log \log x$.

c. If $0 < m \leq 1$ and

$$F_\theta(r) = (1 - \exp(-\alpha(\theta)r^m))1_{\{r>0\}},$$

the distribution of (R, Θ) is φ-OR, with $\varphi(x) = x^{\frac{1}{2m}}$.

For each example the general form of φ is given in Delcroix and Jacob (1991).

References

Arnold, B.C., Becker, A., Gather, U., and Zahedi, H., ''On the Markov property of order statistics,'' *Journal of Statistical Planning and Inference* 147–154, (1984).

Barme-Delcroix, M.F., ''Localisation asymptotique des échantillons multidimensionnels,'' *Bulletin de la société mathématique de Belgique* 45, fasc 2, série B, (1993).

David, H.A., *Orders Statistics* (second edition), Wiley, New York, 1981.

De Haan, L., ''On regular variation and its application to the weak convergence of sample extremes,'' *Mathematical Centre Tracts, no. 32*, Amsterdam, 1970.

Delcroix, M.F. and Jacob, P., ''Stability of extreme value for a multidimensional sample,'' *Statistique et analyse des Données* 16, 1–21, (1991).

Gather, U., ''Über Ausreissertests und Ausreisseranfälligkeit von Wahrschein-lichkeitsverteilungen,'' *Dissertation*, Aachen Technical University, 1979.

Gather, U. and Rauhut, B., ''Outlier behavior of probability distributions,'' *Journal of Statistical Planning and Inference*, North Holland, 26, 237–252, (1990).

Geffroy, J., ''Contribution à la théorie des valeurs extrêmes,'' *Publ. Inst. Statist. Univ. Paris*, 7(8), 37–185, (1958).

Geffroy, J., ''Localisation asymptotique du polyèdre d'appui d'un échantillon laplacien à k dimension,'' *Publ. Inst. Stat. Univ.* Paris, no. 59, 1961.

Gnedenko, B.V., ''Sur la distribution limite du terme maximum d'une série aléatoire,'' *Ann. Math.* 44, 423–453, (1943).

Goldstein, M., ''Contamination distribution,'' *Ann. Statist.* 10, 174–183, (1982).

Green, R.F., ''Outlier-prone and outlier-resistant distributions,'' *J. Amer. Statist. Assoc.* 71, 502–505, (1976).

Jacob, P. and Suquet, C., ''Regression and asymptotical location of a multivariate sample,'' *Statistics and Probability Letters* 35, 173–179, (1997).

Mathar, R., ''Ausreisser bei ein und mehrdimensionalen Wahrscheinlichkeitsverteilungen,'' *Dissertation*, Aachen Technical University, 1981.

Mathar, R., ''Outlier-prone and outlier resistant multidimensional distributions,'' *Statistics* 16, 451–456, (1985).

Neyman, J. and Scott, E.L., ''Outlier proneness of phenomena and of related distribution,'' In: *Optimizing Methods in Statistics*. (J. Rustagi, ed.) Academic Press, New York, 413–430, (1971).

O'Hagan, A., ''On outlier rejection phenomena in Bayes inference,'' *J. Roy. Statist. Soc. Ser.* B 41, 358–367, (1979).

Resnick, S.I., ''Extreme values, Regular Variation, and Point Processes,'' Springer-Verlag, 1987.

Schuster, E.F., ''Classification of probability laws by tail behavior,'' *J. Amer. Statist. Assoc.* 79, 936–939, (1984).

Tomkins, R.J. and Wang, H., ''Stability theorems for large order statistics with varying ranks,'' *Statistics and Probability letters* 14, 91–95, (1992).