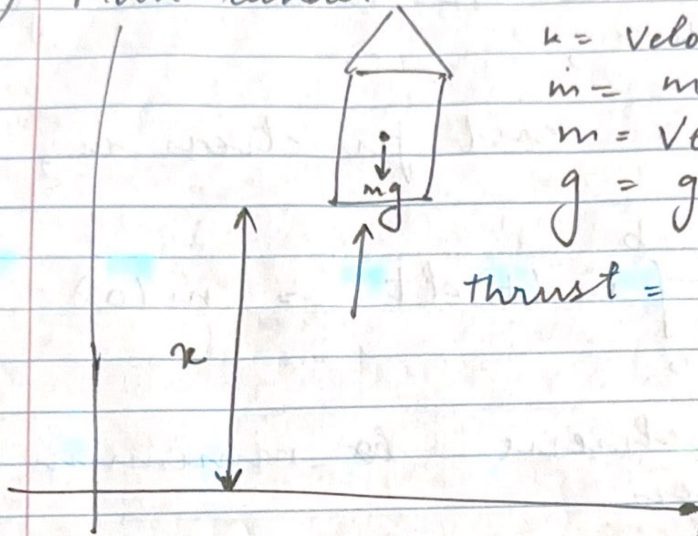
Q.2) Moon lander



k = velocity of exhaust gases

$\dot{m}$ = mass flow rate

m = vehicle mass

g = gravitational acc. of moon.

thrust $= -k\dot{m}$

$h(t)$ – height of space craft at time 't'.

$v(t) = \dot{h}(t)$ – velocity of spacecraft

$m(t)$ = total mass of spacecraft.

$\alpha(t)$ = thrust at time t

$\alpha(t) \in [0, 1]$

$\alpha(t) = 0 \rightarrow$ freefall

$\alpha(t) = 1 \rightarrow$ max. thrust

the whole operation takes place for time $0 < t < t^*$.

where, $h(t^*) = v(t^*) = 0 \rightarrow$ End cond^n to achieve soft landing

The goal is to min. the fuel consumption,

$\therefore$ Our cost function is,

$$f = -\int_0^b \dot{m}(t)\,dt = m(0) - m(b)$$

Now to achieve ~~the~~ minimum fuel consumption,

we know, motion of vehicle is governed by,

$$\alpha = \ddot{x} = -k\frac{\dot{m}}{m} - g \quad -(1)$$

$$\text{Now, } \frac{\dot{m}}{m} = \cancel{\ln\left(\frac{m}{m}\right)}\ \frac{d}{dt}(\ln(m))$$

$$\cancel{\ddot{x} = -k \ln \frac{m(t)}{m(0)}}$$

$$S = \ddot{x} = -k\frac{d}{dt}(\ln m) - g$$

integrating between limits 0 & t.

$$\dot{x}(t) = -k \ln \frac{m(t)}{m(0)} - gt + x(0)$$

Now, $v(t^*) = \dot{x}(t^*) = 0$ if

$$k \ln\left(\frac{m(t^*)}{m(0)}\right) = \dot{x}(0) - gt^*$$

$$\Rightarrow \quad m(t^*) = m(0) \exp\left[\frac{\dot{x}(0) - gt^*}{k}\right]$$

$$\Rightarrow \quad S = m(0)\left\{1 - \exp\left[\frac{\dot{x}(0) - gt^*}{k}\right]\right\}$$

hence for a given, $m(0)$, $\dot{x}(0)$, g and k, the optimal thrust is dependent on terminal time t^*.

$\therefore$ min. t^* is equivalent to min. fuel consumption.

Now, lets replace,

$\dot{x}_1 = x_2$; $x = x_1$, $x_3 = m$, & $u = \dot{m}$

Our (1) can be represented as

$$\dot{x} = x_2, \quad \dot{x}_2 = -\frac{k}{x_3} u - g, \quad \dot{x}_3 = u \quad -(6)$$

in (6) u is control variable & is constrained by,

$-\alpha \le u(t) \le 0$ for $0 \le t \le t^*$.

boundary condⁿ. of (6) are

$$x_1(0) = x_2(0), \; x_2(0) = \dot{x}(0),$$

$$x_3(0) = m(0), \; x_1(t^*) = 0, \; x_2(t^*) = 0$$

and that $x_3(t^*)$, is the $m(t^*)$ (terminal mass) is free.

since, not all of the initial mass is propellant,

$x_3(t) > 0$, $0 \le t \le t^*$ along an optimal trajectory

The ~~pony~~ ~~Ponty~~ Pontryagin max. principle is now applied to ~~of~~ obtain the form of the optimal thrust program.

The Hamiltonian for the minimal time is formulated as below,

$$H = \psi_1 x_2 - \psi_2 \frac{k}{x_3} u - \psi_2 g + \psi_3 u$$

Where,

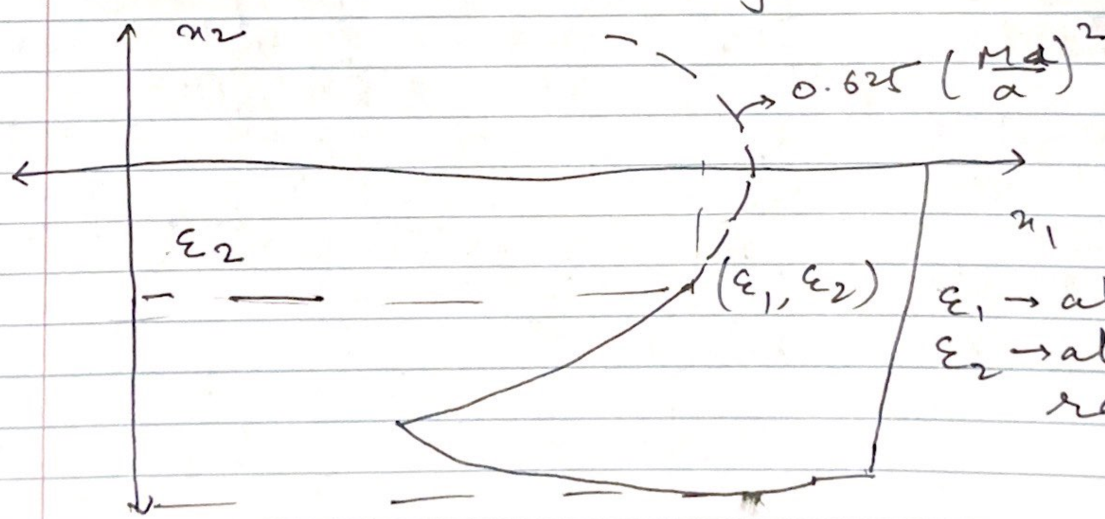
$$\psi_1 = 0, \quad \psi_2 = -\psi_1, \quad \psi_3 = -\psi_2 \frac{k}{(x_3)^2} u$$

the optimal thrust (mass flow rate) is achieved when, $u(t)$ maximizes the Hamiltonian.

$$u(t) = \begin{cases} -\alpha & \text{, when, } \psi_3(t) - \frac{k}{x_3(t)}\psi_2(t) < 0 \\ 0 & \text{, when } \psi_3(t) - \frac{k}{x_3(t)}\psi_2(t) > 0 \end{cases}$$

it is shown that there exists at most one switching in $[0, t^*]$ from initiation of mission until touchdown, as a free fall followed by full thrust.

the switching fn. is obtained by integrating the equations of motions under the assumption $u(t) = -\alpha$, existing in time $[0, t_1]$



$\varepsilon_1 \to$ altitude
$\varepsilon_2 \to$ altitude rate

the lander is allowed to free fall till $f(\varepsilon_1, \varepsilon_2)$ and then thruster is switched to achieve thrust till touchdown.

as ~~$f(\varepsilon_1$~~ $f(x_1, x_2) = 0$, thrust is switched on.

$$f(x_1, x_2) = \frac{b}{a} x_1 + 2a \sqrt{\frac{x_1}{a}} + x_2$$