UNIT : II

Lattices and Boolean Algebra

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Partially Ordered Set :

A relation R on a set A is called partial ordering, if it is reflexive, antisymmetric and transitive.
The set A together with partial order relation R is called a partial order set or Poset.

Power Set :

Power set is a set of all possible subsets of a given set and it is denoted by P(A).
P(A) contains 2^n elements. Here n is the no. of elements of set A.

For eg: If $A = \{a, b\}$

$O(A) = 2$

$\therefore P(A) = 2^{O(A)} = 2^2 = 4$

$P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

Special elements in Poset :

Let $(P, \leq)$ be a Poset.

- An element $a \in P$ is the greatest element of P if $x \leq a$, $\forall x \in P$
- An element $b \in P$ is the least element if $b \leq x, \forall x \in$
- The greatest & least elements are unique if exists.
- An element 'a' in the Poset is called the maximal element of P if $a < x$ for no x in P i.e. no elements of P strictly succeds 'a'
- An element 'b' in P is called minimal element if $x < b$, for no x in P.
- Maximal & minimal elements are easy to spot using

Hasse diagram. They are top & bottom elements in the diagram.

- A Poset may have more than one maximal or minimal elements.

Hasse Diagram : A diagram that is used to describe partial order relation associated with a set is called Hasse Diagram.

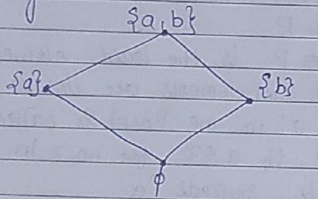
We represent the elements of A by dots and joining the points by straight line in such a way that higher value assign at higher level.

Que-1 For example : Let A be given finite set and P(A) be its power set. Let $\subseteq$ subset with the relation of P(A). Draw Hasse diagram of $(P(A), \subseteq)$ for $A = \{a, b\}$

Soln ⇒ Given that : $A = \{a, b\}$

$P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

The Hass digram is :→

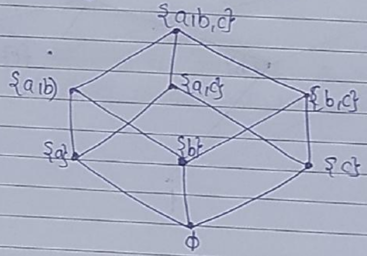


Q-2 Let $A = \{a, b, c\}$ ~~be~~ and P(A) be its power set. Let subset $\subseteq$ be the partial ordered relation on it. Draw a hasse diagram of $(P(A), \subseteq)$

Soln : Given : $A = \{a, b, c\}$

$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$

The Hasse diagram of $(P(A), \subseteq)$ is

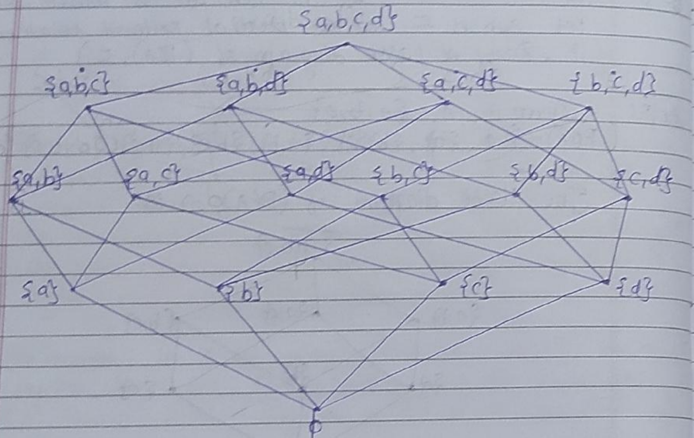


Q-3 Let $A = \{a, b, c, d\}$ and P(A) is its power set. Let subset, $\subseteq$ be the relation on the elements on P(A). Draw Hasse diagram of $(P(A), \subseteq)$.

Soln : Given : $A = \{a, b, c, d\}$

$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{c, d\}, \{a, c\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{b, c, d\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}\}$

The Hasse diagram is :

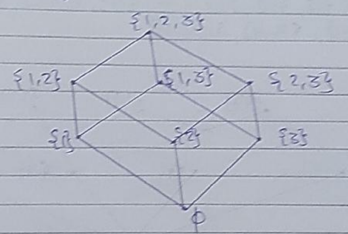


Q-4 Let $A = \{1, 2, 3\}$ & $P(A)$ be its power set. Let ($\subseteq$) subset be the partial order relation on it. Draw Hassey diagram of $(P(A), \subseteq)$.

Sol^n: Given : $A = \{1, 2, 3\}$

$P(A) = \{\{1\}\{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}, \phi\}$

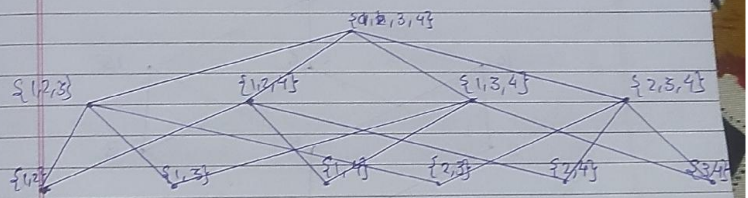
The Hasse diagram of $(P(A), \subseteq)$ is :



Q-5 Draw Hasse diagram of the set of all subsets of $\{1,2,3,4\}$ having atleast two no.'s partially ordered by $\subseteq$.

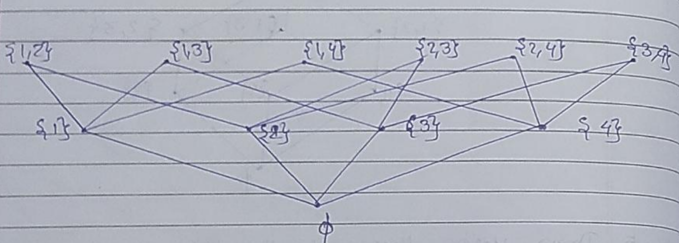
Sol^n → Given that : $A = \{1,2,3,4\}$

$P(A) = \{\{1\}\{2\}, \{3\}, \{4\}, \{1,2\}, \{2,3\}, \{3,4\}, \{1,4\}, \{1,3\}, \{2,4\}, \{1,2,3\}, \{2,3,4\}, \{3,4,1\}, \{1,2,4\}, \{1,2,3,4\}, \phi\}$



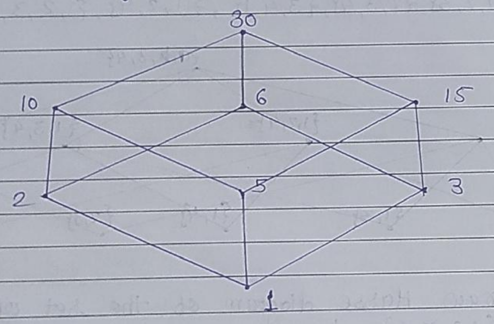
Q-6 Draw Hasse diagram of the set of all subsets of $\{1,2,3,4\}$ having atmost two no.'s partially ordered by $\subseteq$.

→ Given that $A = \{1,2,3,4\}$
$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\},$

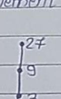


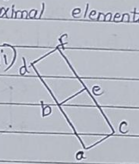
Q-7 Draw the Hasse diagram of D_{30}.

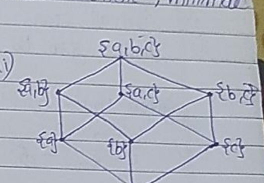
Sol^n: $D_{30} = \{1,2,3,5,6,10,15,30\}$
The Hasse diagram of D_{30} is :

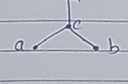


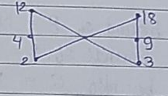
Q-8 Determine whether the posets represented by following Hasse diagram have a greatest element, least element, minimal element and maximal elements.

i) 

ii) 

iii) 

iv) 

v) 

Sol^n ⇒ i) Greatest element = 27
Maximum element = 27
Least element = 1
Minimal element = 1

ii) Greatest element = f ; Maximum element = f
Least element = a ; Minimal element = a

iii) Greatest element = $\{a,b,c\}$; Maximum element = $\{a,b,c\}$
Least element = $\{\phi\}$ Minimal element = ϕ

iv) Greatest element = d ; Maximum element = $\{d\}$
Least element = None ; Minimal ~~eleme~~ element = a, b

v) Greatest element : There is no greast element
Least element : There is no least element
maximum elements are 12 & 18 ; Minimal elements are 2 & 3.

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Q-9 Consider a Poset $A = \{1,2,3,4,5,6,7,8\}$ under the partial order relation whose Hasse diagram is shown below. Let $B = \{1,2\}$, $C = \{3,4,5\}$

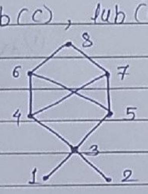
i) Find all Lower bounds & upper bounds of B & C.

ii) Find : glb(B), lub(B), glb(C), lub(C)

Given that: $A = \{1,2,3,4,5,6,7,8\}$

$B = \{1,2\}$

$C = \{3,4,5\}$



i) Lower bound of B = None

Upper bounds of B = 3, 4, 5, 6, 7, 8

Lower bound of C = 1, 2

Upper bound of C = 6, 7, 8

ii) Greatest lower bound, glb(B) = None

~~lub~~ lowest ~~lower~~ upper bound, lub(B) = 3

Greatest lower bound glb(C) = None

Lowest upper bound, lub(C) = 6, 7

Lattice :

A poset $(L, \leq)$ is called Lattice if every two elements of L has both least upper bound & greatest lower bound.

In this case, we write :

$x \vee y = lub(x,y) = lcm(x,y)$ (read as x join y)

$x \wedge y = glb(x,y) = gcd(x,y)$ (read as x meet y)

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NOTE : Let D_n be the set of all positive divisor of a positive integer n.

For eg : if $n = 8$

$D_8 = \{1,2,4,8\}$

NOTE : The total no. of divisors of a number is the product of 1 more than each exponent in it's prime factorization.

For eg : $30 = 15 \times 2$

$= 5^1 \times 3^1 \times 2^1$

Total number of divisors of $30 = (1+1) \times (1+1) \times (1+1)$

$= 2 \times 2 \times 2$

$= 8$

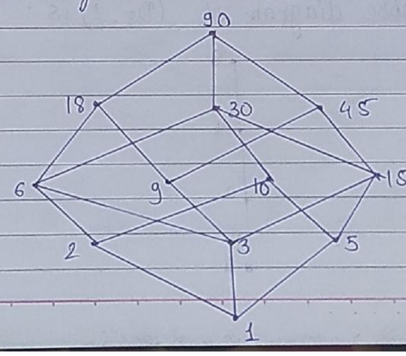
Q-10 Draw the Hasse diagram of $(D_{90}, /)$.

$90 = 10 \times 9$
$= 5^1 \times 2^1 \times$
$= 5^1 \times 2^1 \times$
$= (1+1) \times (1+$
$= 2 \times 2 \times$

Sol^n: $D_{90} = \{1,2,3,5,6,9,10,15,18,30,45,90\}$

The Hasse diagram of $(D_{90}, /)$:

18, 30, 45 divide 90 but 30 does not divide 45 (i.e 18 does not divide 30 & 45 but 15 divide 30 & 45 so, 15 will not come



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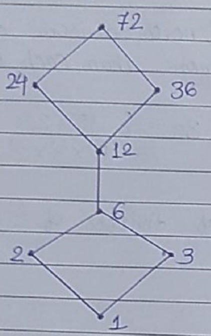
Q-9 Q-11 Let $A = \{1, 2, 3, 6, 12, 24, 36, 72\}$. Draw Hasse diagram of $(A, /)$

i) ii) Sol^n: Given : $A = \{1, 2, 3, 6, 12, 24, 36, 72\}$

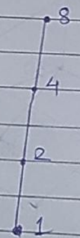
The Hasse diagram of $(A, /)$:



Q-12 Draw the hasse diagram of $(D_8, /)$.

Sol^n: $D_8 = \{1, 2, 4, 8\}$

The Hasse diagram of $(D_8, /)$ is :



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Types of Lattice

1. Complete Lattice : A lattice is said to be complete, if each of its non-empty subset has a least upper bound (lub) and greatest lower bound (glb).

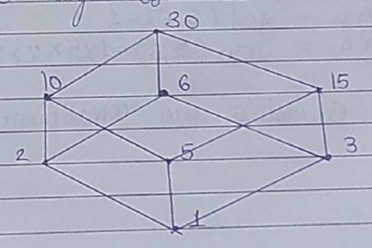
2. Bounded Lattice : A lattice is said to be bounded, if it contains zero and one as least and greatest element.

3. Complimented Lattice : In a bounded lattice $(L, \wedge, \vee, 0, 1)$ is called a compliment if $a \wedge b = 0$ and and $a \vee b = 1$, for $a, b \in L$.
where 0 and 1 are lower and upper bound, respectively.
A Lattice L is called complimented lattice, if it is bounded and every element in L has a compliment.

Q-13 Show that the lattice D_{30} is a complimented lattice.

Sol^n: $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

The Hasse diagram of D_{30} is :



Here 1 is lower bound and 30 is an upper bound.

For 1 & 30:-

$1 \wedge 30 = \gcd(1,30) = 1$

$1 \vee 30 = \text{lcm}(1,30) = 1 \times 30 = 30$

$\therefore$ 1 and 30 are compliment to each other.

For 2 & 15:

$2 \wedge 15 = \gcd(2,15) = 1$

$2 \vee 15 = \text{lcm}(2,15) = 1 \times 2 \times 3 \times 5 = 30$

$\therefore$ 2 and 15 are compliment to each other.

For 3 & 10:

$3 \wedge 10 = \gcd(3,10) = 1$

$3 \vee 10 = \text{lcm}(3,10) = 1 \times 3 \times 2 \times 5 = 30$

$\therefore$ 3 and 10 are compliment to each other

For 5 & 6:

$5 \wedge 6 = \gcd(5,6) = 1$

$5 \vee 6 = \text{lcm}(5,6) = 1 \times 5 \times 2 \times 3 = 30$

$\therefore$ 5 and 6 are compliment to each other

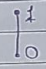
Product of Lattice

Let 'L' be a lattice then $L^2 = L \times L$, $L^3 = L \times L \times L$.

Q-14 Draw the lattice $(L, \leq_1)$, $(L^2, \leq_2)$, $(L^3, \leq_3)$ if $L = \{0,1\}$

Sol^n: We know that $L = \{0,1\}$

$(L, \leq_1)$:

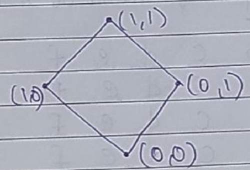


$L^2 = L \times L$

$= \{0,1\} \times \{0,1\}$

$= \{(0,0), (0,1), (1,0), (1,1)\}$

$(L^2, \leq_2)$:



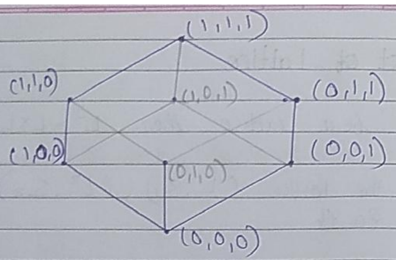
$L^3 = L \times L \times L$

$= \cancel{L^2} \times L \times L^2$

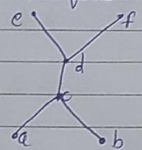
$= \{(0,0), (0,1), (1,0), (1,1)\} \times \{0,1\}$

$= \{0,1\} \times \{(0,0), (0,1), (1,0), (1,1)\}$

$= \{(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)\}$



Q-15 Check the following Hasse diagram is lattice or not.



Sol^n: Let, $A = \{a, b, c, d, e, f\}$

For every pair of element of A has lub & glb.

lub Table:-

(lub) ∨	a	b	c	d	e	f
a	a	c	c	d	e	f
b	c	b	c	d	e	f
c	c	c	c	d	e	f
d	d	d	d	d	e	f
e	e	e	e	e	e	–
f	f	f	f	f	–	f

From above table ~~go~~ lub of the elements e,f and f,e does not exists.

glb Table:-

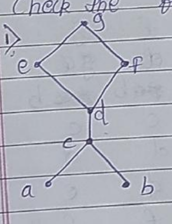
(glb) ∧	a	b	c	d	e	f
a	a	–	a	a	a	a
b	–	b	b	b	b	b
c	a	b	c	c	c	c
d	a	b	c	d	d	d
e	a	b	c	d	e	d
f	a	b	c	d	d	f

From above table glb of the elements a,b & b,a does not exists.

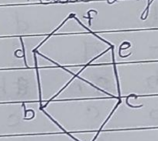
Therefore, the given Hasse diagram is not a lattice.

Q-16 Check the following Hasse diagram is lattice or not.

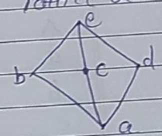
1)



2)



3)



Sol^n: 1) let $A = \{a, b, c, d, e, f, g\}$

For every pair of element of A has lub & glb

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Boolean algebra

A non-empty set B with two binary populations :- '+' and '.', Unary operation ' ' ' and to distinct elements 0 and 1 is called boolean algebra denoted by : (B, +, ., ', 0, 1). if and only if the following properties are satisfied :-

If $a, b, c \in B$ then -

1. Commutative Law
   a) $a+b = b+a$
   b) $a.b = b.a$

2. Distributive Law
   a) $a+(b.c) = (a+b).(a+c)$
   b) $a.(b+c) = (a.b)+(a.c)$

3. Identity Law
   a) $a+0 = a$
   b) $a.1 = a$

4. Complement Law
   a) $a+a' = 1$ b) $a.a' = 0$

Basic Theorems

Let $a, b, c \in B$ then

1. Idempodent Law : a) $a+a = a$ b) $a.a = a$
2. Boundedness Law : a) $a+1 = 1$ b) $a.0 = 0$
3. Absorption Law : a) $a+(a.b) = a$
   b) $a.(a+b) = a$

4. Associative Law

a) $(a+b)+c = a+(b+c)$

b) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

5. Uniqueness of Complement

$a+x = 1$ & $a \cdot x = 0$, then $x = a$

6. Involution Law

$$(a')' = a$$

7. a) $0' = 1$ b) $1' = 0$

8. De-Morgan's Law

a) $(a+b)' = a'b'$ b) $(a \cdot b)' = a'+b'$

De-Morgan's law:

a) $(a+b)' = a' \cdot b'$ b) $(a \cdot b)' = a'+b'$

<u>Proof</u>: a) $(a+b)' = a' \cdot b'$

We prove that: $(a+b) + a'b' = 1$

& $(a+b) \cdot a'b' = 0$

This shows that $(a+b)$ & $a'b'$ are compliments

1. Consider: $LHS = (a+b) + a' \cdot b'$

$= a + (b + a' \cdot b')$

$= a + (b+a') \cdot (b+b')$

$= a + (b+a') \cdot 1$

$$= a + (b + a')$$
$$= a + (a' + b)$$
$$= (a + a') + b$$
$$= \cancel{1}$$
$$= 1 + b$$

$$\therefore \ (a+b) + a' \cdot b' = 1$$

~~③ Consider LHS = $(a+b).(a'b')$~~
~~$= a.(a'.b') + b.(a'.b')$~~
~~$= (a.a').b' + b.(b'.a')$~~
$=$

② Consider LHS $= (a+b) \cdot (a' \cdot b')$

$$= a \cdot (a' \cdot b') + b \cdot (a' \cdot b')$$
$$= (a \cdot a') \cdot b' + b \cdot (b' \cdot a')$$
$$= 0 \cdot b' + (b \cdot b') \cdot a' \quad (\because a \cdot a' = 0)$$
$$= 0 \cdot b' + 0 \cdot a'$$
$$= 0 + 0$$
$$= 0$$

$$\therefore \ (a+b) \cdot (a' \cdot b') = 0$$

Hence, $(a+b)' = a' \cdot b'$

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Q-17 Find the Complement of the following Boolean expression.

1) $xy' + x'z$ 3) $ab' + ac + b'c$

2) $x(yz' + yz)$ 4) $w + (ab + c')*(d'e + 1) + g'(h' + 0)$

→ 1) $xy' + x'z$

Let $f = xy' + x'z$

Taking complements on both sides

$$f' = (xy' + x'z)'$$
$$= (xy')' \cdot (x'z)' \qquad [\because (a+b)' = a' \cdot b']$$
$$= [x' + (y')'] \cdot [(x')' + z'] \qquad [\because (a \cdot b)' = a' + b']$$
$$= (x' + y) \cdot (x + z')$$

2) Let $f = x(y'z' + yz)$

Taking complements on both sides

$$f' = [x(y'z' + yz)]'$$

$$= x' + (y'z' + yz)'$$
$$= x' + (y'z')' \cdot (yz)'$$
$$= x' + [(y')' + (z')'] (y' + z')$$
$$= x' + (y + z)(y' + z')$$

3) Let $f = ab' + ac + b'c$

Taking Complements on both sides

$$f' = [ab' + ac + b'c]'$$

let, $a = ab' + ac$, $b = b'c$

$f' = [ab' + ac]' \cdot [b'c]'$

$f' = [(ab')' \cdot (ac)'] \cdot [(b')' + c']$

$f' = \{a' + (b')' \cdot a' + c' \cdot b + c'$

$f' = (a' + b)(a' + c')(b + c')$

4) Let, $f = w + (ab + c')(d'e + 1) + g'(h' + 0)$

Taking complement on both sides

$f' = [w + (ab + c')(d'e + 1) + g'(h' + 0)]'$

Let $a = w + (ab + c')(d'e + 1)$, $b = g'(h' + 0)$

$[\because (a+b)' = a' \cdot b']$

$\therefore f' = [\underbrace{w}_{a} + \underbrace{(ab + c')(d'e + 1)}_{b}]' \cdot [g'(h' + 0)]'$

$= w' \cdot [(ab + c')(d'e + 1)]' \cdot [(g')' + (\underbrace{h'}_{a} + \underbrace{0}_{b})']$

$= w' \cdot [(ab + c')' + (d'e + 1)'] \cdot [g + h \cdot 1]$

$= w' \cdot [(ab)' \cdot c + (d'e)' \cdot 0] \cdot [g + h]$

$= w' \cdot [\cancel{ab}(a' + b') \cdot c] \cdot (g + h)$

Q-18 Simplify the boolean expression :

i) $C(B + C) \cdot (A + B + C)$ ii) $A + B(A + B) + A(A' + B)$

iii) $AB'C' + AB'C + ABC + AB'C) \cdot (A + B)$

iv) $P + P'QR' + (Q + R)'$

$\rightarrow$ i) $C \cdot (B + C) \cdot (A + B + C)$

$= (C \cdot B + C \cdot C) \cdot (A + B + C)$

$= (C \cdot B + C) \cdot (A + B + C)$ $(\because C \cdot C = C)$

$= C(B + 1) \cdot (A + B + C)$

$= C \cdot 1 \cdot (A + B + C)$ $(\because B + 1 = 1)$

$= C \cdot A + C \cdot B + C \cdot C$

$= C \cdot A + C \cdot B + C$

$= C \cdot A + C(B + 1)$

$= C \cdot A + C \cdot 1$

$= C \cdot A + C$

$= C(A + 1)$

$= C \cdot 1$

$= C$

ii) $A + B(A + B) + A(A' + B)$

$\rightarrow A + (B \cdot A + B \cdot B) + (A \cdot A' + A \cdot B)$

$A + A \cdot B + B + 0 + A \cdot B$

$A + (A \cdot B + B) + (A \cdot B)$

$\cancel{A \cdot B(}$ $A + B(A + 1) + A \cdot B$

$A + B \cdot 1 + A \cdot B$

$A + B + A \cdot B$

$A + B(A + 1)$

$A + B \cdot 1$

iii iv) $P + P'QR' + (Q+R)'$

$\rightarrow p + P'QR' + Q'.R'$ $\quad [\because (a+b)' = a'.b']$

$= P + R'(P'Q + Q')$

$= (P + P'Q + Q')(P+R')$

$= [(P+P')(P+Q) + Q'](P+R')$ $\quad [\because a+bc = (a+b).(a+c)]$

$= [1.(P+Q) + Q'](P+R')$

$= [P + (Q+Q')](P+R')$

$= (P+1)(P+R')$ $\quad [\because Q+Q' = 1]$

$= 1.(P+R')$

$= P+R'$

iii) $(AB'C' + AB'C + ABC + AB'C).(A+B)$