Logic

* Propositional logic -

Proposition :- A declarative sentence to which we can assign one and only one truth value i.e. true or false.
(T) (F)

ex. 1) London is a city. → T
2) 2 + 3 = 5 → F
}Propositions

3) x + 2 = 3
4) 15th August is independence day.
}Not propositions

* Law of excluded middle :-
If a proposition is not true, then it is false. Similarly if a proposition is not false, then it is true.

* Law of contradiction :-
A proposition cannot be simultaneously true and false.

* Types of propositions :-

1) Atomic propositions (simple statements) :-
A proposition which cannot be divided further into two or more propositions is said to be "Atomic".
Usually atomic propositions are denoted by 'p', 'q', 'r' and so on.

8) Compound proposition (Statement formula / propositional formula) →

Two or more atomic propositions can be combined using connectives to form a compound proposition.

not	$\sim$ $\neg$
and	$\wedge$
or	$\vee$
implies	$\rightarrow$
iff	$\leftrightarrow$

Connectives →

1) Negation (Not) →

If 'P' is a proposition then

not P written as '$\sim P$' is a proposition whose truth value is "false" if P has truth value "true" and viceversa.

Truth table of '$\neg P$' →

p	$\sim p$
T	F
F	T

2) Disjunction (or) →

If p and q are any two propositions then "p OR q" written as "$p \vee q$" is a proposition whose truth value is "false" iff both p and q have truth values false

Truth table of $p \vee q$ →

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

$p \vee q$ is true if at least one of the two propositions is true.

Disjunctive syllogism / elimination rule →
(Rules of inference)

If $\{p \vee q$ is true and p is "false"$\}$.

then q is true

3) Conjunction (and) →

If p and q are any two propositions, then "p and q" written as "$p \wedge q$" is a proposition whose truth value is true only when both p and q have truth value "true"

Truth table of $p \wedge q$ →

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

If $p \wedge q$ is false then at least one of the two propositions is false.

(2) Conjunctive Syllogism →

If {$p \wedge q$ is false and p is true}, then q is false.

(4) Implication / Condition (conditional) →

If p and q are any two propositions, then "p implies q" (or) "if p then q" written as "$p \to q$" is a proposition whose truth value is false only when p has truth value "true" and q has truth value "false".

Truth table of $p \to q$ is

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

If $p \to q$ is true, then either p is false or q is true or both too.

In $p \to q$, p is called "antecedent / hyposis" a q is called "consequent / conclusive".

*Note - 1) whenever p is false, then $p \to q$ is true always

2) whenever q is true, then $p \to q$ is also true always

3) a) The converse of $(p \to q)$ is $(q \to p)$ and viceversa.

b) The inverse of $(p \to q)$ is $(\bar{p} \to \bar{q})$ | $(\sim p \to \sim q)$.

c) The contrapositive of $(p \to q)$ is $(\sim q \to \sim p)$

4) $(p \to q)$ is $\equiv (\sim q \to \sim p)$
T F T F

5) Biconditional / Biimplication (iff) -

If p and q are any two propositions then "p iff q" written as "$p \leftrightarrow q$" is a proposition whose truth value is true only when both p and q have same truth values.

Truth table for $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

$p \leftrightarrow q$ is false <u>iff</u> both p and q have diff truth values.

$$(p \leftrightarrow q) \cong ((p \to q) \wedge (q \to p))$$

$$\sim (p \leftrightarrow q) \cong ((p \wedge \sim q) \vee (\sim p \wedge q))$$

$$\cong (p \oplus q) \text{ (xor)}$$

Tautology →

An atomic proposition cannot be tautology. So, a compound proposition which is always true is called as "tautology".

ex → 1) $p \vee (\sim p)$

2) $p \vee (p \to q)$

3) $p \to (p \vee q)$

Contradiction →

A compound proposition which is always false is called a "contradiction."

ex → $p \wedge (\sim p) \cong F$

Contingency →

A compound proposition which is neither a tautology nor a contradiction is called "contingency."

ex → $p \rightarrow q$, $p \vee q$, $p \wedge q$, $p \leftrightarrow q$

Satisfiable Function / Satisfiability →

A compound proposition which is not a contradiction is called "satisfiable func^n".

(i) A satisfiable func^n can be a tautology also.

Note • Every contingency is satisfiable but a satisfiable func^n can be a tautology also.

Equivalences →

Let P, Q, R be any compound propositions, then

i) Double negation

$\sim(\sim p) \Leftrightarrow p$

2) Commutative laws →

$$(p \vee q) \Leftrightarrow (q \vee p)$$

$$(p \wedge q) \Leftrightarrow (q \wedge p)$$

$$(p \leftrightarrow q) \Leftrightarrow (q \leftrightarrow p)$$

3) Associative laws →

$$(p \vee q) \vee R \Leftrightarrow p \vee (q \vee R)$$

$$(p \wedge q) \wedge R \Leftrightarrow p \wedge (q \wedge R)$$

4) Distributive laws →

$$p \vee (q \wedge R) \Leftrightarrow (p \vee q) \wedge (p \vee R)$$

$$p \wedge (q \vee R) \Leftrightarrow (p \wedge q) \vee (p \wedge R)$$

$$p + (q \cdot R) \Leftrightarrow (p + q) \cdot (p + R)$$

$$p \cdot (q + R) \Leftrightarrow (p \cdot q) + (p \cdot R)$$

⊛ Principle of duality →

The dual of any formula in boolean algebra can be obtained by replacing ?

i) '+' with '·' and viceversa

and

ii) '0' with '1' and viceversa.

5) De-Morgan's laws →

i) $\sim(p \vee q) \Leftrightarrow (\sim p \wedge \sim q)$

(or) $(\overline{p+q}) \Leftrightarrow (\overline{p} \cdot \overline{q})$

ii) $\sim(p \wedge q) \Leftrightarrow (\sim p \vee \sim q)$

(or) $(\overline{p \cdot q}) \Leftrightarrow (\overline{p} + \overline{q})$

6) Idempotent laws →

i) $(p \vee p) \Leftrightarrow p$ (or) $p+p \Leftrightarrow p$

ii) $(p \wedge p) \Leftrightarrow p$ (or) $p \cdot p \Leftrightarrow p$

7) Absorption laws →

i) $p \vee (p \wedge q) \Leftrightarrow p$

ii) $p \wedge (p \vee q) \Leftrightarrow p$

8) More identities →

i) $p \vee T \Leftrightarrow T$ (or) $p+1 \Leftrightarrow 1$

ii) $p \wedge T \Leftrightarrow p$ (or) $p \cdot 1 \Leftrightarrow p$

iii) $p \wedge F \Leftrightarrow F$ (or) $p \cdot 0 \Leftrightarrow 0$

iv) $p \vee F \Leftrightarrow p$ (or) $p+0 \Leftrightarrow p$

v) $p \wedge \sim p \Leftrightarrow F$ (or) $p \cdot \overline{p} \Leftrightarrow 0$

vi) $p \vee \sim p \Leftrightarrow T$ (or) $p + \overline{p} \Leftrightarrow 1$

9) $(p \rightarrow q) \Leftrightarrow (\sim p \vee q)$

10) $\sim(p \rightarrow q) \Leftrightarrow (p \wedge \sim q)$

11) $(p \rightarrow q) \Leftrightarrow (\sim q \rightarrow \sim p)$

12) $(p \leftrightarrow q) \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$

13) $\sim(p \leftrightarrow q) \Leftrightarrow (p \wedge \sim q) \vee (\sim p \wedge q) \rightarrow$ Ex-OR

14) $p \rightarrow (q \rightarrow R) \Leftrightarrow (p \wedge q) \rightarrow R$

LHS $\Rightarrow$ $\sim p \vee (\sim q \vee R)$ (by Eq. 9)

$\Rightarrow (\sim p \vee \sim q) \vee R$ By associative equivalence

$\Rightarrow (p \wedge q) \rightarrow R$ (eq. 9)

$=$ R.H.S.

Q.1. The compound proposition $(p \vee q) \vee \sim p$ is a

✓a) Tautology

b) Contradiction

c) Contingency

d) $\Leftrightarrow q$

e) $\Leftrightarrow p$

$\underset{T}{p} \rightarrow \underset{F}{(q+R)}$

$\Rightarrow (q \vee p) \vee \sim p$ By E2 Commutative

$\Rightarrow q \vee (p \vee \sim p)$ by E3 Associative

$\Rightarrow q \vee T$

$\Rightarrow \boxed{T}$

Q.2. The compound proposition $\sim(p \vee q) \vee (\sim p \wedge q) \vee p$ is

Some options: $\boxed{\text{Tautology}}$

$\Rightarrow (\sim p \wedge \sim q) \vee (\sim p \wedge q) \vee p$ DeMorgan's

$\Rightarrow \sim p \wedge (\sim q \vee q) \vee p$

$\Rightarrow \{\sim p \wedge T\} \vee p$

$\Rightarrow \neg p \vee p \rightarrow \boxed{T}$.

(a) The given formula is equivalent to

$(\overline{p+q}) + (\bar{p} \cdot q) + p$

$\Rightarrow (\bar{p}\,\bar{q}) + (\bar{p}\,q) + p$ DeMorgan's

$\Rightarrow \bar{p} \cdot (\bar{q} + q) + p$

$\Rightarrow \bar{p} \cdot 1 + p \Rightarrow \bar{p} + p \Rightarrow \boxed{1}$.

Q.3. The statement formula

$((p \to q) \leftrightarrow (\sim p \vee q)) \wedge s$ is

a) Tautology

b) $\Leftrightarrow p$

c) $\Leftrightarrow q$

✓d) $\Leftrightarrow s$

Q.4. Consider the following statement formulae

i) $p \vee \sim(p \wedge q)$

ii) $(p \wedge q) \wedge (\sim p \vee \sim q)$

Which of the following options is true?

✓a) S_1 is a tautology and S_2 is a contradicⁿ.

b) S_1 is a contradiction and S_2 is a tautology.

c) Both are valid.

d) Both S_1 and S_2 are not satisfiable

$\rightarrow$ S2: $(p \wedge q) \wedge \sim(p \wedge q)$

Contradiction

S1: $p \vee \sim(p \wedge q) \Rightarrow p \vee (\sim p \vee \sim q)$

$\Rightarrow (p \vee \sim p) \vee \sim q$

$\Rightarrow$ Tautology.

Q.5 The propositional function

$(\sim p \wedge (\sim Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R)$

$\Rightarrow (\sim p \wedge \sim q \wedge \sim R) \vee R \wedge (Q \vee P)$

$\Rightarrow (\sim(p \vee q) \wedge R) \vee (R \wedge (Q \vee P))$

$\Rightarrow R \wedge ((p \vee Q) \vee \sim(p \vee Q)) \Rightarrow R \wedge T$

$\Rightarrow \boxed{R}$

a) $\Leftrightarrow p$

b) $\Leftrightarrow Q$

✓c) $\Leftrightarrow R$

d) a tautology

Q.6. The propositional function

$\{(P \vee Q) \wedge \sim(\sim P \wedge (\sim Q \vee \sim R))\} \vee (\sim P \wedge \sim Q) \vee (\sim P \wedge \sim R)$ is

→ a) tautology

b) contradiction

c) contingency d) none of these.

1st term →

→ $(P \vee Q) \wedge (P \vee (Q \wedge R))$

→ $P \vee (Q \wedge (Q \wedge R))$

$P \vee (Q \wedge R)$.

2nd term → $(\sim P \wedge \sim Q) \Rightarrow \sim(P \vee Q)$

$\vee$

3rd term → $(\sim P \wedge \sim R)$.

$\sim P \wedge (\sim Q \vee \sim R)$.

∴ total → $P \vee (Q \wedge R) \vee (\sim P \wedge (\sim Q \vee \sim R))$

→ $P \vee (Q \wedge R) \vee \overline{(P \vee (Q \wedge R))}$

→ $\boxed{T}$

Tautology.

(F) $\overset{F}{P}\ \overset{P}{T \to T}\ \overset{Q}{T}\ \overset{R}{F}$ $T \to F$

$T \to F$ $\underset{T}{P}\ \underset{FF}{QR} \Rightarrow F \to F^{f} \to$

Q.7. The compound proposition

$$(P \to (Q \vee R)) \to ((P \wedge Q) \to R) \text{ is}$$

✓a) satisfiable but not valid (i.e. contingency).

b) valid (tautology)

c) Not satisfiable (contradiction)

d) valid but not satisfiable (not possible)

→ when $P \to T$, $Q \to T$ and $R \to F$, the given formula is false. Hence, it is not valid.

when $P \to T$, $Q \to F$, $R \to F$, $P \to (Q \vee R)$ is false. So, whole formula true.

∴ The given formula is satisfiable but not valid.

Q.8. The statement formula is

$$(\underset{T}{a} \to (\underset{T}{b} \to \underset{F}{c})) \to ((a \to b) \to (a \to c)) \text{ is}$$

→ tautology i.e. Valid

Here, the given can be false only in one case when $((a \to b) \to (a \to c))$ is false and

$(a \to (b \to c))$ is true. But when earlier is false, later also becomes false. $\boxed{\text{tautology}}$

Quine's method →

case-1 → when a is true.

$$(T \to (b \to c)) \to ((T \to b) \to (T \to c))$$

$$\Rightarrow (b \to c) \to (b \to c)$$

$$\Rightarrow T$$

case-2 → when a is false.

$$(F \to (b \to c)) \to ((F \to b) \to (F \to c))$$

$$\Rightarrow T \to (T \to T)$$

$$\Rightarrow T$$

∴ Both the cases yield T ⇒ ∴ given formula is valid.

Q 9. The statement formula

$$((p \to \sim p) \wedge (\sim p \to p))$$ is

$\sim p \vee \sim p$ F

$$\Rightarrow (\sim p \vee \sim p) \wedge (p \vee p)$$

$\sim p \vee \sim p$ F

$$\Rightarrow \sim p \wedge (p \vee p)$$

$$\Rightarrow \sim p \wedge p \to \boxed{\text{contradiction}}$$

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Q.10 No. of nonequivalent propositional functions (different truth tables) possible with n atomic propositions is = ?
a) 2^n b) n^2 ✓c) $2^{(2^n)}$ d) $2^{(n^2)}$

→ $f(x_1, x_2, x_3, \dots, x_n)$

Note: * A set of connectives is said to be functionally complete if any statement formula can be written using these connectives.

Q.11 Which of the following sets of connectives is not functionally complete?

a) $\{\sim, \vee\}$ b) $\{\sim, \wedge\}$ c) $\{\sim, \rightarrow\}$ ✓d) $\{\vee, \wedge\}$

e) $\{\rightarrow, F\}$

(eq. to NOR gate)

→ a) $\{\sim, \vee\}$ → 1) $p \wedge q$

⇒ $\sim(\sim p \vee \sim q)$

3) $p \leftrightarrow q$

⇒ $(p \rightarrow q) \wedge (q \rightarrow p)$

⇒ $(\sim p \vee q) \wedge (\sim q \vee p)$

⇒ $\sim\{\sim(\sim p \vee q) \vee \sim(\sim q \vee$

2) $p \rightarrow q$

⇒ $(\sim p \vee q)$

We can imply any operation using only $\{\sim, \vee\}$.
So, it is functionally complete.

b) $\{\sim, \wedge\}$ → equivalent to NAND gate.

So, it is also functionally complete.

c) $\{\sim, \rightarrow\}$

→ i) $(p \vee q)$

⇒ $(\sim p \rightarrow q)$

NOT is there and we can perform OR operation so, c) is also functionally complete.

d) $\{\wedge, \vee\}$

→ We cannot perform NOT operation using $\{\vee, \wedge\}$. So, it is not functionally complete.

e) $\{\rightarrow, F\}$

→ i) $\sim p$

⇒ $(p \rightarrow F)$

⇒ $(\sim p \vee F)$

⇒ $\sim p$.

We can perform not operation & is there. So, This set is functionally complete.

Q.12 Which of the following is not valid?

a) $(p \wedge q) \Rightarrow (p \rightarrow q)$

$(T \wedge T) \Rightarrow (T \rightarrow T)$ always true.

The statement is valid because AND operation is TRUE in only one case for which the

$F \leftrightarrow F \quad T.$

biconditional is also true. Hence it is a tautological implication.

b) $(p \leftrightarrow q) \Rightarrow (q \rightarrow p)$

$T \leftrightarrow T \Rightarrow T \rightarrow T$

$F \leftrightarrow F \rightarrow F \rightarrow F$ always.

The statement is valid because whenever biconditional is true, the conditional on the other side is also true.

c) $(p \leftrightarrow \sim q) \Rightarrow (p \rightarrow q)$

$p \quad \sim q \qquad p \quad q$

$T \leftrightarrow T \qquad T \rightarrow F$ ✓ F.

$F \leftrightarrow F \qquad F \rightarrow T$

for one case, when $(p \leftrightarrow \sim q)$ is true, $(p \rightarrow q)$ is coming false

This statement is not valid because when p is T and q is F, the biconditional is false, but the conditionals is false.

d) $\sim (p \rightarrow q) \Rightarrow \sim q$.

it will be false only when $T \rightarrow F$

$\sim q \rightarrow F \quad q \rightarrow T$.

whenever q is True, $\sim (p \rightarrow q)$ can not be true,
so, the statement is valid.

Q.12. which of the following is not a tautology?

a) $p \to (p \to q)$

$\to$ $T \to F$, then it will become false.

$(p \to q) = F$ when $p = T$ $q = F$, possible.

$\therefore$ $p \to (p \to q)$ gives False for $p = T$, $q = F$

$\therefore$ It is not a tautology.

The statement is not a tautology, because when $p = T$ and $q = F$ - given statement is false.

b) $q \to (p \to q)$

$\to$ $T \to T$ always.

The given statement is a tautology because whenever $q = T$, $p \to q$ is always true.

c) $(p \wedge (p \to q)) \to q$

to check whether tautology or not, we have to check for the False condition i.e $T \to F$.

$\therefore$ $p \wedge (p \to q)$
$T \wedge (T \to T)$
T

but whenever L.H.S is T, right side becomes T always.

$\therefore$ the given stmt is a tautology.

$(p \vee q)$ T
$\sim p$
T F F (T)

$T \to F$
$p:T$ $q:F$
$T \wedge F$

d) $(\sim p \wedge (p \vee q)) \to q$ (Disjunctive syllogism)

$\to$ T F

$(\sim p \wedge (p \vee q))$

$\sim p = T$ $p = F$

$T \wedge (F \vee F) \to$ (F) always $\therefore T \to F$ cond^n not possible.

$\therefore$ the given statement is a tautology.

Q.14. Which of the foll arguments is not valid?

a) The conclusion R follows from the premises / hypothesis.

$\{P \to Q,\ Q \to R,\ P\}$

1) $P \to Q$
2) $Q \to R$
3) P

$\therefore R$ is valid.

4) From 1) and 2), $P \to R$ by the rule of transitivity

5) $P \to R$
P
$\therefore R$ by modus-ponens

The argument is valid.

b) The conclusion $\sim p$ follows from the premise $\{p \to q, q \to r, \sim r\}$.

→ 1) $p \to q$

2) $q \to r$

3) $\sim r$.

from 1) and 2) by the rule of transitivity.

4) $p \to r$

3) $\sim r$

$\therefore \sim p$ by modus-tollens.

$\therefore$ argument is valid.

c) The conclusion $\sim r$ follows from the premises $\{(p \to q), (q \to r), \sim p\}$.

→ 1) $p \to q$

2) $q \to r$

3) $\sim p$

4) $p \to r$ from 1, 2

3) $\sim p$

$\therefore$ nothing, r can be anything.

$\therefore$ the given argument is not valid.

$p \to (q \to R)$

$p \wedge q$

$p = T \quad q = T \quad \underset{T}{p} \wedge \underset{T}{q}$ $\quad T \to (T \to R)$ $\quad R$

We cannot derive the conclusion from the premises by any rules of inference ∴ argument is not valid.

* If the given conclusion $\sim R$ on the premises to be valid, we can write in the form of tautological implication as :

$$\underbrace{\{(p \to q), (q \to R), \sim p\}}_{T} \to \underset{F}{\sim R}$$

this condition is satisfied. Hence the above tautological implication is not valid.

d) The conclusion R follows from the premises

$\{ p \to (q \to R), (p \wedge q) \}$

1) $p \to (q \to R)$

2) $p \wedge q \quad : p = q = T$

3) $T \to (T \to R)$

∴ R

∴ the argument is valid.

Q.15 which of the following arguments is not valid ?

a) $[a \to b, \; c \to d, \; a \vee c] \to [b \vee d]$

→ (constructive dilemma)

1) $a \to b$

2) $c \to d$

3) $a \vee c$ (by eq. 9)
$\sim a \to c$

from 3) and 2), by transitivity.

4) $\sim a \to c$
$c \to d$
$\therefore (\sim a \to d)$

5) $(a \to b) = (\sim b \to \sim a)$ (contrapositive)

6) $\sim b \to \sim a$ (Rule of transitivity).
$\sim a \to d$
$\therefore (\sim b \to d)$

7) $(\sim b \to d) = (b \vee d)$

$\therefore \{(a \to b), (c \to d), a \vee c\} \Rightarrow (b \vee d)$ is valid

b) $\{p \to (R \to S), \sim R \to \sim P, P\} \Rightarrow S$

→ 1) $p \to (R \to S)$

2) $\sim R \to \sim p$

3) p

$\sim d \to T$
$d \to F$
$a \to F$
$b \to T$

$a \to F$ $b \to T$

$\sim d$ $d \to F$ $c \to F$

$a \to F$ $b \to T$ $b \to T$ $(c \to T)$

from 1) & 2), by modus-ponens.

4) $p \to (R \to S)$

$$\frac{P}{\therefore R \to S}$$

5) $\sim R \to \sim P$

$$\frac{P}{\therefore R}$$ by modus-tollens.

6) $R \to S$

$$\frac{R}{\therefore S}$$ by modus-ponens.

$\therefore$ argument is valid.

④ $\{a \vee b,\ b \to c,\ a \to d,\ \sim d\} \to c$

→ 1) $a \vee b$

2) $b \to c$

3) $a \to d$

4) $\sim d$.

5) $a \to d$ ③ and ④ by modus-tollens.

$$\frac{\sim d}{\sim a.}$$

$a \to f \quad b \to T$

$T \to (T \to c)$

6) $\dfrac{a \vee b \qquad \sim a}{b}$ from 1) & 5) by Disjunctive syllogism.

7) $\dfrac{b \to c \qquad b}{\therefore c}$ by modus ponens.

$\therefore$ the given argument is valid.

d) $\{(\sim a \wedge b) \wedge (b \to (a \to c))\} \Rightarrow \sim c$

$\Rightarrow$ 1) $\sim a \wedge b$

2) $b \to (a \to c)$

3) $\dfrac{\sim a \wedge b}{\sim a \cdot b}$

4) $\dfrac{b \to (a \to c) \qquad b}{\therefore (a \to c)}$

5) $\dfrac{a \to c \qquad \sim a}{\therefore c}$ (fallacy.) we cant say.

$\therefore$ the argument is invalid.

Q.14. Which of the following is not valid?

a) $(a \wedge (a \to c) \wedge (b \vee \sim c)) \to b$

→ This argument / statement is valid if the foll. argument is valid.

1) a

2) $a \to c$

3) $\dfrac{b \vee \sim c}{\therefore b}$

4) $\dfrac{\begin{matrix} a \to c \\ a \end{matrix}}{\therefore c}$ 1 & 2.

5) $\dfrac{\begin{matrix} b \vee \sim c \\ c \end{matrix}}{\therefore b}$ 3 and 4 Disjunctive syllogism.

∴ the given argument is valid.

b) $[(\sim p \to \sim R) \wedge \sim S \wedge (p \to w) \wedge (R \vee S)] \to w$

→ 1) $\sim p \to \sim R \Rightarrow R \to p$ (contrapositive)

2) $\sim S$

3) $p \to w$

4) $R \vee S$

5) from 1) and 3) $\dfrac{\begin{matrix} R \to p \\ p \to w \end{matrix}}{\therefore R \to w}$ transitivity rule

6) $R \vee S$ 2) and 4)

$$\frac{\begin{array}{c} \sim S \end{array}}{R}$$

7) $R \to W$

$$\frac{R}{\therefore W}$$

$\therefore$ the given statement is valid.

c) $\big(\sim R \to (S \to \sim P)\big) \wedge (\sim R \vee W) \wedge (\sim Q \to S) \wedge \sim W\big) \to (P \to Q)$

$\to$ 1) $\sim R \to (S \to \sim P)$

2) $\sim R \vee W$

3) $\sim Q \to S$

4) $\sim W$

5) $\sim R \vee W$ from 2) & 4).

$$\frac{\sim W}{\therefore \sim R}$$

6) $\sim R \to (S \to \sim P)$ from 1) and 5)

$$\frac{\sim R}{\therefore (S \to \sim P)}$$ by modus ponens.

7) $\sim Q \to S$ from 3) & 6) by transitivity rule

$$\frac{S \to \sim P}{\sim Q \to \sim P}$$

$p \to q$ $\quad \sim\sim \quad (v \to r) \quad \sim p \to T \quad (p \to r)$

$T \to (q \to \sim p) \quad (p \to q)$

$c \to f \quad r \to a \quad b \to T$

8) $\sim q \to \sim p \;\equiv\; p \to q$ by contrapositive.

Given statement is valid.

4) $\big((b \wedge c) \to a\big) \wedge (\sim b \vee a)\big) \to c$

$\Rightarrow$ 1) $(b \wedge c) \to a$

2) $(\sim b \vee a) \Rightarrow b \to a$

3) No relation can be found. So, put some truth values.

$$\underbrace{((b \wedge c) \to a) \wedge (\sim b \vee a))}_{T} \to \underbrace{c}_{F}$$

We get a cond^n $T \to F$. So, the given statement is not valid.

Conditional Proof (C.P) :

If a set of premises $\{P_1, P_2, \ldots, P_n \text{ and } (\wedge)\, Q\} \Rightarrow R$ — ①

then $\{P_1, P_2, \ldots, P_n\} \Rightarrow (Q \to R)$ — ②

Note: To apply C.P Rule for an argument which is given in form ②,

i) first convert the argument into form ① and prove the argument ①.

ii) then by C.P rule, the given argument is also valid.

Indirect Proof (Proof by contradiction) :

To apply this rule, first we assume that the argument is not valid. So, we take negation of the conclusion as a new premise.

When this new premise is combined with other premises, if we get any contradiction, then the argument is valid.

Inconsistency :

A set of premises $(P_1, P_2, \ldots, P_n)$ is said to inconsistent if all the premises cannot be true simultaneously.

i.e. the conjunction of all the premises is a contradiction.

$\{P_1 \wedge P_2 \wedge \ldots \wedge P_n\} \Leftrightarrow F$

q→r
q→r
p→r

* Note: * In any argument if the premises are inconsistent then the argument is valid (always).

$$\underbrace{\underbrace{\{P_1 \wedge P_2 \wedge \dots \wedge P_n\}}_{F\ (\text{always})} \Rightarrow Q}_{\text{always valid.}}$$

Q17 which of the following arguments is not valid ?

a) $\{p \to (q \to s), (\sim R \vee p), q\} \Rightarrow (R \to s)$

* This argument is valid by C.P. Rule if the following argument is valid

1) $p \to (q \to s)$

2) $(\sim R \vee p)$

3) q

4) R ← new premise to apply C.P.

5) $\sim R \vee p$ 2) 4(4) by disjunctive syllogism.

$$\frac{R}{\therefore p}$$

6) $p \to (q \to s)$ 1)(5) by modus-ponens.

$$\frac{p}{\therefore (q \to s)}$$

7) $q \to s$ 2) & (6) by modus ponnens
q
$\therefore s$

$\therefore$ the argument is valid.

and $\therefore$ the given argument is also valid by C.P. Rule.

b) $\{(p \to q), \sim(p \wedge q)\} \Rightarrow \sim p$

1) $p \to q$

negation of $(\sim p)$

2) $\sim(p \wedge q)$

3) p a new premise to apply I.P.

4) $p \to q$ 1) and 3)
p modus ponnens
q

5) $(p \wedge q)$ by 3) and 4)

Here, ② and ⑤ contradict each other. Hence, the given argument is valid.

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Q.18. Which of the following is not valid ?

a) $(\sim(a \wedge b) \wedge (b \vee c) \wedge (c \to d)) \to (a \to d)$

$\to$ 1) $\sim(a \wedge b)$

2) $(b \vee c)$

3) $(c \to d)$

4) $\dfrac{a}{\therefore d}$ → new premise to apply CP rule

the given formula is valid by CP rule if the following argument is valid.

5) $\dfrac{\sim(a \wedge b) \quad a}{\sim b}$ 1) and 4) by conjuctive syllogism.

6) $\dfrac{b \vee c \quad \sim b}{\therefore c}$ 2) and 5) by disjunctive syllogism.

7) $\dfrac{c \to d \quad c}{\boxed{\therefore d}}$ 3) and 6) by modus-ponens ✓

$\therefore$ the given statement is valid.

b) $((a \vee b) \to (c \wedge d)) \to (b \to d)$.

→ This formula is valid by C.P rule if the following argument is valid.

1) $(a \vee b) \to (c \wedge d)$

2) $c \wedge d$ *

3) $\dfrac{b}{\therefore d}$ * new premise to apply C.P.

4) $\dfrac{b}{\therefore (a \vee b)}$

5) $(a \vee b) \to (c \wedge d)$ 1 and 4 modus ponnens

$\dfrac{(a \vee b)}{\therefore (c \wedge d)}$

6) $\dfrac{c \wedge d}{\therefore d}$

∴ the given statement is valid.

c) $\{(a \vee b) \to c,\ a\} \Rightarrow \{c \to b\}$

→ 1) $(a \vee b) \to c$

2) a

3) $\dfrac{c}{\therefore b}$ by (C.P Rule).

$((a \vee b) \to (c \wedge a)) \to (b \to c)$ $\quad b \to T \quad c \to F$

$(T \to F) \quad F$

$F \to T$

$(((a \vee b) \to c) \wedge a) \to (c \to b)$

$F \wedge T \quad (F \to F)$

The given argument is not valid because when $a \to T$, $c \to T$, $b \to F$, the premises are true but conclusion is false.

d) $((\sim x \to y) \wedge (\sim x \wedge \sim y)) \to x$

$\Rightarrow$ 1) $\sim x \to y = x \vee y$

2) $\sim x \wedge \sim y = \sim(x \vee y)$

$\Rightarrow (x \vee y) \wedge \sim(x \vee y) \to x$

$\Rightarrow F \to x$

$\Rightarrow T$

$\therefore$ the given statement is valid

Q.19. Which of the following is not valid?

a) If today is David's birthday then today is 2nd of July.
Today is 2nd of July.
$\therefore$ Today is David's birthday.

$\rightarrow$ p: Today is David's birthday.

q: Today is 2nd of July.

$p \to q$

q

$\therefore p$ (not valid)

the given statement is not valid.

b) If Canada is a country then London is a city.

London is not a city

$\therefore$ Canada is not a country

→ p: Canada is a country.

q: London is a city.

$$\begin{array}{l} p \to q \\ \underline{\sim q} \\ \therefore \sim p \end{array}$$

$\therefore$ the given statement is valid.

c) If A works hard then B or C will enjoy themse[lves]

If B enjoys himself, then A will not work hard.

If C enjoys himself, then D will not enjoy himself

$\therefore$ If A works hard then D will not enjoy himself

→ 1) $a \to (b \vee c)$

2) $b \to (\sim a)$

3) $\underline{c \to \sim d}$

$(a \to \sim d)$ then it will be valid.

$a \to (b \vee c)$ $(a \to (b \vee c)) \wedge$ $b \to T$
$F \to T$ $(b \to \sim a) \wedge$ $a \to d$
$(c \to \sim d)$ $\to$ $A \to \sim d$

4) a ← new premise to apply C.P.
5) $(b \vee c)$ 1 and 4 M.P.
6) $b \to (\sim a)$ from 2) & 4)

$$\frac{a}{\sim b}$$ by modus - tonnens

7) $$\frac{\begin{matrix} b \vee c \\ \sim b \end{matrix}}{c}$$ 5) & 6) by disjunctive syllogism

8) $$\frac{\begin{matrix} c \to \sim d \\ c \end{matrix}}{\sim d}$$ 3) & 7) by modus ponnens

∴ the given statement is valid.

d) If John misses many classes (P), then he fails high school (q)

If John fails high school, then he is uneducated (s)

If John read a lot of books (t), then John is not uneducated (~s)

John missed many classes and read a lot of books

∴ John is educated.

→ 1) $p \to q$
2) $q \to s$
3) $t \to \sim s$
4) $p \wedge t$

5) $p \to s$ 1) & 2) transitivity rule

6) p (2) simp.

7) t 4) simp.

8) s 5) & 6) Modus-ponnens

9) $\sim s$ 3) & 7) modus-ponnens

8) & 9) contradict each other.

The given statement / argument is valid.

Q.20. A binary relation $*$ is defined by the following truth table

p	q	$p * q$
T	T	T
T	F	T
F	T	F
F	F	T

Which of the following is equivalent to $(p \wedge q)$

a) $\sim (p * q)$
b) $(p * \sim q)$
✓ c) $\sim (\sim p * q)$
d) $\sim (p * \sim q)$.

for option c)

p	q	~p	q	~p*q	~(~p*q)
T	T	F	T	F	T
T	F	F	F	T	F
F	T	T	T	T	F
F	F	T	F	T	F

$= (p \wedge q)$

∴ C is correct

(or) $(p * q) \Rightarrow (p \vee \sim q)$

Q.→ A binary operator * is defined by foll. truth table

p	q	p * q
T	T	F
T	F	F
F	T	T
F	F	F

$(p \vee q) = ?$

a) $\sim (p * q)$

b) $(p * \sim q)$

c) $\sim (\sim p * q)$

d) $\sim (p * \sim q)$

→ $(p * q) \Rightarrow (\sim p \wedge q)$

$(p \vee q) \Rightarrow \sim (\sim p \wedge \sim q)$

$\Rightarrow \boxed{\sim (p * \sim q)}$

First Order Logic (Predicate Logic):

Consider,

John is a politician.

subject predicate

let

j : John

P: is a politician

So, we can denote the above statement as P(j).

∴ P(j) : John is a politician.

Consider,

Dhoni is a sportsman.

D: Dhoni

S: is a sportsman

∴ S(D) : Dhoni is a sportsman.

If x and y are any two persons, then

P(x) : x is a politician
S(y) : y is a sportsman

} not propositions because we cannot assign any truth values.

Consider,

x is a friend of y.

F is a friend of

$\therefore F(x,y)$: x is a friend of y.

Let G denote the predicate "is greater than" and let x and y be any two nos.

Then $G(x,y)$: x is greater than y. } not a proposition.

Consider,

B; is in between.

Then $B(x,y,z)$: y is in between x and z.

(4) Let $p(x)$; x is a politician.

$S(x)$: x is a sportsman

where x is any person

1) $\neg p(x)$: x is not a politician.

2) $p(x) \vee S(x)$: Either x is a politician or x is a sportsman.

(Inclusive OR)

3) $p(x) \wedge S(x)$: x is a politician cum sportsman.

2) $\exists x\, p(x)$: At least one is true.

3) $\forall x \sim p(x)$: All are false.

4) $\exists x \sim p(x)$: At least one is false.

5) $\sim[\forall x\, p(x)]$: At least one is false
(or)
Not All are true.

$\Leftrightarrow \exists x \sim p(x)$

6) $\sim[\exists x\, p(x)]$: none is true
(or)
: All are false.

$\Leftrightarrow \forall x \sim p(x)$

7) $\sim[\forall x \sim p(x)]$: At least one is true
(or)
Not all are false.

$\Leftrightarrow \exists x\, p(x)$

8) $\sim[\exists x \sim p(x)]$: All are true
(or)
none is false.

$\Leftrightarrow \forall x\, p(x)$

$\sim\forall x\{p(x)\rightarrow q(x)\}$

$\sim p(x)\vee q(x)$

$p(x)\wedge\sim q(x)$

* **Note:** To negate a statement formula in first order logic, we have to replace $\exists x$ with $\forall x$, $\forall x$ with $\exists x$ and finally, we have to negate the scope of the quantifiers.

Q.1. ex: The negation of

$\exists x\{\sim p(x)\wedge q(x)\}$ is

$\Rightarrow\ \sim\{\exists x\{\sim p(x)\wedge q(x)\}\}$

$\Rightarrow\ \forall x\{\sim(\sim p(x)\wedge q(x))\}$

$\Rightarrow\ \forall x\{p(x)\vee\sim q(x)\}$ (by De-Morgan's law)

Q.2. The negation of

$\forall x\,\exists y\,[p(x,y)\rightarrow\{q(x,y)\wedge R(x,y)\}]$ is

$\Rightarrow\ \exists x\,\forall y\,[\sim\{p(x,y)\rightarrow\{q(x,y)\wedge R(x,y)\}\}]$

$\Rightarrow\ \exists x\,\forall y\,[p(x,y)\wedge\sim\{q(x,y)\wedge R(x,y)\}]$

$\Rightarrow\ \exists x\,\forall y\,[p(x,y)\wedge\{\sim q(x,y)\vee\sim R(x,y)\}]$.

Q.3. The statement formula

$\exists x\{p(x)\wedge\sim q(x)\}$ is equivalent to

Q.3. Which of the following is not valid ?

a) $\forall x \exists y\ P(x,y) \Rightarrow \exists x \exists y\ P(x,y)$

→ Valid.

b) $\exists x \forall y\ P(x,y) \Rightarrow \forall x \exists y\ P(y,x)$

→ Valid

c) $\exists x \forall y\ P(x,y) \Rightarrow \forall x \exists y\ P(x,y)$.

→ not valid.

*As from relationship diagram.

d) $\forall x \forall y\ P(x,y) \Rightarrow \forall x \exists y\ P(x,y)$

→ valid.

Q.4. Let $G(x,y)$ denote a predicate "x is greater than y"

$G(x,y)$: x is greater than y.

where x and y are any positive integers.

Consider the statement.

"For any positive integer there exists a greater positive integer."

Which of the following first order logic sentences correctly represent the above statement

a) $\forall x \exists y\ G(x,y)$

→ $\{1, 2, 3, \dots, \infty\}$

1 is not greater than any positive integer

b) $\forall y \exists x\ G(x,y)$.

→ For every positive integer there exists a great positive integer.

c) $\exists x \forall y\ G(x,y)$.

→ There exists a largest +ve integer

d) $\exists y \forall x\ G(x,y)$.

→ There exists a smallest +ve integer.

$(1 \neq 1)$ False

Q.5 Let $F(x,y)$: x is father of y

where x and y any two persons.

Which of the foll. statements is true ?

a) $\forall x \exists y \ F(x,y)$

→ Everybody is father of someone.

↑ False

b) $\exists x \forall y \ F(x,y)$.

→ There is someone who is father of everybody

↑ False.

c) $\exists y \forall x \ F(x,y)$

→ There exists of someone who is child of everybody. ↑ False

✓d) $\forall y \exists x \ F(x,y)$

→ Everyone is ~~father~~ child of someone.

↑ True

Q6. Let $L(x,y)$: x likes y.

Universe of discourse is set of all people

Consider the statement

"There is someone whom noone likes".

$\forall \exists y \forall x \sim L(x,y)$.

$\sim L(x,y)$: x does not like y.

which of the following first order logic sentence correctly represent the above sentence ?

a) $\forall x \exists y \sim L(x,y)$ ⊗.

→ Everybody hates someone
not same.

b) $\sim [\forall x \exists y \; L(x,y)]$ ⊗

→ $\exists x \forall y \sim L(x,y)$

There exists someone whol hates everyone
not same.

✓c) $\sim [\forall y \exists x \; L(x,y)]$

→ $\exists y \forall x \sim L(x,y)$
same.

There exists some y, whom everybody hates.
same

✓d) $\exists x \forall y \sim L(y,x)$
same.

[illegible] $\exists x\,(C(x) \wedge \sim C(x))$

3.7. Which of the following statements are true if the universe of discourse is set of all integers?

S1) $\forall x\, \exists y\, \underbrace{(x+y=5)}$ — True — $y = 5 - x$

S2) $\exists y\, \forall x\, (x+y=0)$ — False

* It means there exists some y for which $x+y=0$ for all x.

* y is fixed. For any fixed set of integers the given statement is not true.

S3) $\forall y\, \exists x\, (x \cdot y = 1)$ — False

* For it to true, we have to take $x = \frac{1}{y}$ but then x is no more an integer.

* Also, for integer $y=0$, no integer can satisfy the eqn.

$\exists y$ some x for which

64) $\exists x \forall y \, (x^2 \cdot y = y)$

True

* for $x = \pm 1$, $x^2 \cdot y = y$ always for all.

Q.8 Let $G(x)$: x is a graph

and $C(x)$: x is connected.

Consider the statement

" Not every graph is connected."

which of the foll. first-order logic sentences does not represent the above statement?

a) $\sim [\forall x \, G(x) \rightarrow C(x)]$

$\sim$ (All graphs are connected)

same

b) $\exists x [G(x) \wedge \sim C(x)]$

* Some graphs are not connected.

same

c) $\sim [\forall x \{\sim G(x) \vee C(x)\}]$

same

✓ d) $\forall x \{G(x) \rightarrow \sim C(x)\}$
not same

* Every graph is not connected

Q.9. Let $G(x)$: x is gold ornament

$P(x)$: x is platinum ornament

$V(x)$: x is valuable ornament

Consider the statement:

"Gold and Platinum ornaments are valuable"

which of the following first-order logic sentence is most appropriate one to represent above st?

a) $\forall x \{V(x) \rightarrow (G(x) \wedge P(x))\}$
not same

* All valuable ornaments are gold and platinum

cannot use $\rightarrow$ in $\exists x$

b) $\exists x \{G(x) \wedge P(x) \rightarrow P(x)\}$
not same

c) $\forall x \{(G(x) \wedge P(x)) \rightarrow V(x)\}$ → not same

✓ d) $\forall x \{(G(x) \vee P(x)) \rightarrow V(x)\}$
Same

Rules of inference for quantified propositions :

1) Universal instantiation : (specification). (U.S.)

$$\frac{\forall x\ p(x)}{\therefore p(a)}$$

valid for any element 'a' in the universe of discourse.

2) Existential specification (instantiation) : (E.S.)

$$\frac{\exists x\ p(x)}{\therefore p(c)}$$

valid for some elements 'c' in the universe of discourse.

3) Existential Generalization : (E.G.)

If p(c) is valid / true for some element in the univer of discourse

$$\frac{}{\therefore \exists x\ p(x)}$$

4) Universal Generalization (U.G.) :

If p(a) is true for all elements in the universe of discourse

$$\frac{}{\therefore \forall x\ p(x)}$$

Equivalences →

1) $\forall x\,\{p(x) \wedge Q(x)\} \Leftrightarrow \{\forall x\, p(x)\} \wedge \{\forall x\; Q(x)\}$

* Note → If we replace $\forall x$ with $\exists x$, the above statement does not hold good.

i.e. $\exists x\{p(x) \wedge Q(x)\} \not\Leftrightarrow \exists x\, p(x) \wedge \exists x\, Q(x)$.

It is a tautological implication i.e

$$\exists x\{p(x) \wedge Q(x)\} \Rightarrow \{\exists x\, p(x)\} \wedge \{\exists x\, Q(x)\}$$

tautological implication.

2) $\exists x\;\{p(x) \vee Q(x)\} \Leftrightarrow \{\exists x\; p(x)\} \vee \{\exists x\; Q(x)\}$

* Note → If we replace $\exists x$ with $\forall x$, then these above statement does not hold good.

i.e. $\forall x\;\{p(x) \vee Q(x)\} \not\Leftrightarrow \{\forall x\; p(x)\} \vee \{\forall x\, Q(x)\}$

It is a tautological implication i.e.

$$\forall x\,\{p(x) \vee Q(x)\} \not\Rightarrow \{\forall x\; p(x)\} \vee \{\forall x\; Q(x)\}$$

$\Leftarrow$

$1.6 \to 10$ $\forall_x p(x)$. $\forall x \, (p(x) \to q(x))$

Q10. Which of the following is not valid ?

a) $\forall x \{p(x) \to q(x)\} \Rightarrow \{\forall x \; p(x) \to \forall x \; q(x).\}$

Valid.

* This formula is valid by C.P. Rule if the following argument is valid.

1) $\forall x \{p(x) \to q(x)\}$

2) $\forall x \; p(x)$ * new premise to apply C.P.

$\therefore \forall x \; q(x)$

* If all are universal quantifiers, we can treat it like ordinary argument.

$\therefore$ by modus ponnens, we can get the answer *

3) $p(a) \to q(a)$

4) $p(a)$ by modus ponnens.

$\therefore q(a)$

$\therefore$ we can write $\forall x \; q(x)$

$\therefore$ The given statement is valid.

b) $\{\forall x\ p(x) \to \forall x\ q(x)\} \Rightarrow \{\forall x\ \{p(x) \to q(x)\}\}$

not valid.

for $U = \{$ Modi, Dhoni $\}$.

$p(x)$: x is a politician

$q(x)$: x is a sportsman

then $\{\underbrace{\forall x\ p(x)}_{\text{false}} \to \underbrace{\forall x\ q(x)}_{\text{true}}\}$

and $\underbrace{\forall x\ \{p(x) \to q(x)\}}_{\text{false}}$

$\therefore$ not valid.

c) $\forall x\ \{p(x) \lor q(x)\} \Rightarrow \{\forall x\ p(x)\} \lor \{\exists x\ q(x)\}$

$\{\exists x \sim p(x) \to \exists x\ q(x)\}$

The given formula is valid by C.P rule if the foll. argument is valid.

1) $\forall x\ \{p(x) \lor q(x)\}$

2) $\exists x \sim p(x)$ — new premise to apply C.P

$\therefore \exists x\ q(x)$

3) $\sim p(a)$ from 2) by E.S.

~~4) $p(a) \vee q(a)$~~
4) $p(a) \vee q(a)$ from 1) by U.S.

5) $q(a)$ from 3) and 4) by disjunctive syllogism.

6) $\exists x\, q(x)$ from 5) by E.G.

Valid

d) $\forall x \{p(x) \vee q(x)\} \Rightarrow \{\forall x\, p(x)\} \vee \{\forall x\, q(x)\}$

→ not valid

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Q-11. which of the following arguments is not valid

a) $\{\forall x\{p(x) \to q(x)\}, \exists y\, p(y)\} \Rightarrow \exists z\, q(z)$

valid

1) $\forall x\{p(x) \to q(x)\}$

2) $\exists y\, p(y)$

3) $p(a)$ by E.S. from ②

4) $p(a) \to q(a)$ from ① by U.S.

5) $q(a)$ from 3 and 4 by modus-ponens

6) $\therefore \exists x\, Q(x)$ E.G. from 5)

$\therefore$ the given argument is valid.

b) $\{\exists x\, p(x), \exists x\, Q(x)\} \Rightarrow \exists x\{p(x) \wedge Q(x)\}$

Not valid

→ left side denotes → there exists some politician and there exists some sportsman.

Right side denotes → there exists some who is both politician and sportsman.

$\therefore$ LHS $\neq$ RHS

$\therefore$ Not valid.

c) $\{\{\forall x \forall y \{p(x,y) \rightarrow Q(x,y)\}\}, \sim Q(a,b)\} \Rightarrow \sim p(a,b)$.

1) $\forall x \forall y \{p(x,y) \rightarrow Q(x,y)\}$

2) $\sim Q(a,b)$

3) $\forall y \{p(a,y) \rightarrow Q(a,y)\}$ from 1), by U.S.

4) $p(a,b) \rightarrow Q(a,b)$ from 3) by U.S.

5) $\sim p(a,b)$ from 3) and 2) and 4) by modus-tonnens.

$\therefore$ the given argument is valid.

$\forall(x)(($ …

$\exists x((C(x) \wedge P(x)))$.

d) $\underbrace{\forall x\{P(x) \wedge Q(x)\}}_{A} \rightarrow \underbrace{\exists y\{R(y) \rightarrow W(y)\}}_{B}$

$\forall y\{R(y) \wedge \sim W(y)\} \rightarrow \sim B$

$\therefore \exists x\{P(x) \rightarrow \sim Q(x)\} \rightarrow \sim A$

$\Rightarrow$ 1) $A \rightarrow B$

2) $\sim B$

$\therefore \sim A$

$\therefore \sim \exists x\{P(x) \rightarrow \sim Q(x)\}$.

$\therefore$ the given argument is valid.

Q12. Which of the following arguments is not valid?

a) All computer_science graduates are ~~logical thinkers~~ people.

Some computer science graduates are logical thinkers

$\therefore$ Some people are logical thinkers.

$\rightarrow$ Let $C(x)$: x is a C.S graduate

$P(x)$: x is a person.

$L(x)$: x is a logical thinker.

first stmt → $\forall x\{c(x) \rightarrow p(x)\}$ 1)

Second stmt → $\exists x\{c(x) \wedge L(x)\}$ 2)

conclusion → $\exists x\{p(x) \wedge L(x)\}$ 3)

4) $c(a) \wedge L(a)$ from 2) by E.S.

5) $c(a)$ and $L(a)$ from 4) by simplificat^n

6) $c(a) \rightarrow p(a)$ from 1) by U.S.

7) $p(a)$ from 6) & 5) by modus-ponnens.

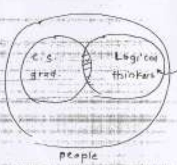
8) $p(a) \wedge L(a)$ from 7) & 5) by conjunction.

9) $\therefore \exists x\{p(x) \wedge L(x)\}$ from 8) by E.G.

∴ the given statement/argument is valid.

(OR)

we can solve it by Venn diagrams.



logical thinker need not be person. It might be a mic.

b) i) Every dog likes people or hates cats.

ii) Rover is a dog.

iii) Rover likes cats.

∴ iv) Some dogs like people.

→ Let $D(x)$: x is a dog.

$L(x)$: x likes people

$H(x)$: x hates cats.

a : Rover

i) → $\forall x \{D(x) \rightarrow \{L(x) \vee H(x)\}\}$ 1)

ii) → $D(a)$ 2)

iii) → $\sim H(a)$ 3)

∴ iv) $\exists x \{D(x) \wedge L(x)\}$

4) $D(a) \rightarrow \{L(a) \vee H(a)\}$ from 1), U.S.

5) $L(a) \vee H(a)$ from 4) and 2), by modus ponens.

6) $L(a)$ from 5) & 3) by disjunctive syllogism.

7) $D(a) \wedge L(a)$ from 5) and 6).
by conjunct^{n}.

$\therefore \exists x\,(D(x) \wedge L(x))$ by E.G.

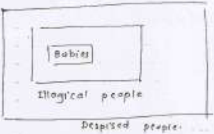
$\therefore$ the given statement is valid.

Q) i) Babies are illogical.

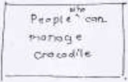
ii) Nobody is despised who can manage a crocodiles.

iii) Illogical people are despised.

$\therefore$ iv) Babies cannot manage crocodiles.



(A)



(B)

(A) and (B) are disjoint sets.
$\therefore$ the given statement is valid.

(OR)

$B(x)$: x is a baby

$I(x)$: x is illogical

$D(x)$: x is despised

$M(x)$: x can manage crocodiles

i) $\forall x \{B(x) \to I(x)\}$ 1)

ii) $\forall x \{M(x) \to \sim D(x)\}$ 2)

iii) $\forall x \{I(x) \to D(x)\}$ 3)

$\therefore$ iv) $\forall x \{B(x) \to \sim M(x)\}$.

4) $D(x) \to \sim M(x)$ from 2), by contrapositive.

5) $B(x) \to \sim M(x)$ from 1), 3) & 4) by transitivity.

$\therefore \forall x \{B(x) \to \sim M(x)\}$.

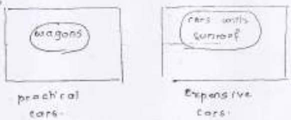
$\therefore$ the given argument is valid.

d) i) No practical car is expensive.

ii) Cars with sun roof are expensive.

iii) All wagons are practical cars.

∴ iv) Some wagons have sunroofs.



∴ No wagon may have sun roof.

∴ the argument is not valid.

Combinatorics

Sum Rule →

Let $E_1, E_2, \ldots, E_n$ are mutually exclusive events which can happen in $e_1, e_2, \ldots, e_n$ ways respectively, then no. of ways in which E_1 or $E_2 \ldots$ or E_n can happen is "$e_1 + e_2 + \ldots + e_n$"

i.e. $\underset{\text{no. of ways}}{n}(E_1 \text{ or } E_2 \text{ or } \ldots \text{ or } E_n) = (e_1 + e_2 + \ldots + e_n)$

Product Rule →

Let $E_1, E_2, \ldots, E_n$ are independent events which can happen $e_1, e_2, \ldots e_n$ ways respectively, then no. of ways in which "E_1 and E_2 and $\cdots$ and E_n" can happen is "$e_1 . e_2 \ldots e_n$".

i.e. $n(E_1 \text{ and } E_2 \ldots \text{ and } E_n) = (e_1 . e_2 \ldots e_n)$.

Q.1. A pair of distinct dice were tossed. No. of ways we get a sum of 7 or 8 is what?

→ $7 \to \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

$8 \to \{(2,6), (3,5), (4,4), (5,3), (6,2)\}$

total no. of ways to get 7 or 8

$= 6+5 = \boxed{11}$

(*) i.e. if a same die is flipped twice Then it won't matter if 1 comes first or last in (6,1)

* In the above example, if the two dice are identical, then the reqd no of ways is

$3+3 = \boxed{6}$

Q.2. How many ternary sequences of length 5 are possible ? (a ternary sequence can have 3 digits, 0, 1 and 2).

→ length - 5

$\underset{3ways}{x}\ \underset{3ways}{x}\ \underset{3ways}{x}\ \underset{3ways}{x}\ \underset{3ways}{x}$

In the sequence each digit we can chose in 3 ways.

By product rule, the reqd no of sequences are

$3^5 = 243$

Q.3. How many integers betn 10^5 and 10^6 have no digits other than 0, 2, 5, 8 ?

→ 6 digit integer

	3ways	4ways	4ways	4ways	4ways	4ways
first digit can't be '0' →	x	x	x	x	x	x
Smallest →	2	0	0	0	0	0
biggest →	8	8	8	8	8	8

∴ By product rule,

no. of digits $= 3\times(4)^5$

$= \boxed{3072}$

Q.4. Let we have 6 different English movies, 8 different Telugu movies and 10 different Hindi movies. How many ways we can chose 2 movies of different languages?

→ E.T. or T.H. or H.E

$(6\times8) + (8\times10) + (10\times6)$

$= 48 + 80 + 60$

$= \boxed{188}$ ways.

Q.5 How many integers in the set $\{1,2,3,\ldots,1000\}$ have distinct digits?

→ $x \to 9$

$x \; x \to = 81$

9 ways (no '0') 9 ways (any 9 but the one in the first place)

$x \; x \; x \to 9.(9)(8).$

Required no. of integers $= 9 + 9\cdot(9) + 9\cdot(9)(8)$

$= 738.$

Q.5 How many integers are there in the set with at least one digit repeating?

set $\rightarrow \{1, 2, 3, \ldots, 1000\}$

$\rightarrow$ at least one digit remaining

$=$ 1000 − (No digit repeated)

$= 1000 - 738$

$= \boxed{262}$

Q.6. How many 4-digit integers are there with digit '6' appearing exactly once?

$\rightarrow$ case-I $\rightarrow$ If the first digit is '6' $\rightarrow$

$$6 \quad \underset{9\text{ ways}}{x} \quad \underset{9\text{ ways}}{x} \quad \underset{9\text{ ways}}{x}$$

then each of the remaining digits we can chose in 9 ways.

$\therefore$ $9 \times 9 \times 9 = 729$ ways.

case II → If the first digit is not 6

$$\underset{8 \text{ ways}}{x} \; x \; x \; x$$

then first digit we can chose in 8 ways and digit '6' can appear in 3 ways and each of the remaining two digits, we can chose in 9 ways

∴ by product rule,

$$8 \times 3 \times 9 \times 9 = 1944$$

∴ By sum rule the required no of digits

$$= 729 + 1944 = \boxed{2673}$$

Q.7. Suppose 4 dice were tossed/rolled, then no. of possible outcomes with atleast one dice shows a 2 ?

→ we will find out no. dice in which no dice shows 2 →

$$\underset{5\text{ways}}{x} \quad \underset{5\text{ways}}{x} \quad \underset{5\text{ways}}{x} \quad \underset{5\text{ways}}{x}$$

$$= 5^4 \text{ ways.}$$

And total no. of outcomes with 4 dice are $= 6^4$.

∴ Reqd. no. of outcomes = $6^4 - 5^4$

$= \boxed{671}$

Q.8. Suppose n players are enrolled in a single elimination tennis tournament. How many matches need to be conducted to decide the winner?

→ Each match eliminates a player.

∴ we have a one-one correspondence b/w no. of matches and no. of losers. Since we have to eliminate (n-1) players, we have to organize $\boxed{(n-1)}$ matches.

Q.9. A set A has n elements. How many ways we can choose subsets P and Q of A so that $P \cap Q$ is empty set.

→ Each element of A can appear like:

i) the element may appear in P()

ii) the element may appear in Q()

iii) the element may appear in none.

∴ $\boxed{3^n}$

Q10. A fruit salad can be made using at least one of the fruits - mango, Apple, Pineapple, watermelon and banana ?

∴ Required no of varieties $= 2^5 - 1$

$= \boxed{31}$

Permutations :

An arrangement of (or) ordered select^n of objects is called a "permutation".

for $\{a, b, c\}$,

ab	bc	ac
ba	cb	ca
aa	bb	cc

} w/o repetition } with repetition.

Permutations w/o repetitions :

$$p(n,k) = {}^nP_k$$

= no. of permutations of 'n' distinct objects taking/taken k at a time w/o repetitions.

$${}^nP_k = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-(k-1))$$

n	$(n-1)$	$(n-2)$		$(n-(k-1))$
$\times$	$\times$	$\times$	$\ldots$	$\times$
1st	2nd	3rd		kth object

$$\therefore {}^nP_k = \frac{n!}{(n-k)!}$$

${}^nP_1 = n$ ${}^nP_3 = n(n-1)(n-2)$

${}^nP_2 = n(n-1)$ and so on.

$^nP_n = n!$

Q.11. How many ways 6 different persons can sit in a room.w (row) ?

→ I II III IV V VI

$^6P_6 = 6! = \boxed{720}$

∴ Required no. of ways $= {}^6P_6 = 6! = 720$

Q.12. How many ways 10 different books can be distributed among 15 persons so that no person can take more than one book and max. no. of books are to be distributed. ?

→ $^{15}P_{10} = 15 \times 14 \times 13 \times 12 \times 11$

$=$

Q.13. How many ways 5 boys and 5 girls can sit in a row so that boys and girls should sit alternately ?

→

case I →
B

1 2 3 4 5 6 7 8 9 1

Boys → $5!$

Girls → $5!$

case-II :

odd numbered places ← Girls

$5!$ ways

even numbered places ← Boys

$5!$ ways

These two cases are mutually exclusive and exhaustive.

∴ By sum rule.

req. no. of ways = $(5! \cdot 5!) + (5! 5!)$

$= 2(5! 5!) = 2(120 \times 120)$

$= \boxed{28800}$

Q.4. How many ways 5 boys and 5 girls can sit in a row so that no 2 boys can sit side by side ?

→ – G_1 – G_2 – G_3 – G_4 – G_5 –

Now, we have 6 places among the girls for the 5 boys to sit.

∴ Boys can sit in $^6P_5$ ways.

Girls can sit in $^5P_5$ ways.

$\therefore$ total reqd ways $= {}^5P_5 \times {}^6P_6$

$= 5! \times 6!$

$= \boxed{86,400}$

30/9/2013

Q 15. No. of | How many signals can be generated using 6 different colored flags ? if any no. of the flags can be hoisted at a time in a row ?

6 $\quad$ $^6P_2$ $\quad$ $^6P_3$ $\quad$ $^6P_6$

$\Rightarrow$ the required no of signals

$= 6 + {}^6P_2 + \dots + {}^6P_6$

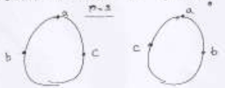
$= 6 + 30 + 120 + 360 + 720 + 720$

$= \boxed{1956}$

Permutation in a circular manner

Number of permutations of 'n' distinct objects around a circle is $(n-1)!$ (taken all at a time)

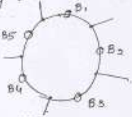
$n=3$



So $(n-1)! = 2! = 2$ ways.

Q.16. How many ways 5 boys and 5 girls can sit around a circular table so that boys and girls should sit alternately ?

→



5 Boys can sit in a circle in $(5-1)! = 4!$ ways.

Girls can sit in $5!$ ways.

∴ Required no. of ways $= 4! \times 5!$

$= 24 \times 120$

$= \boxed{2880}$

a time)

Note Number of garlands (necklaces) possible with n different colored beads is

$$\frac{(n-1)!}{2}$$

In case of necklaces, the above two combinations are same.

Permutations with repetitions

$U(n,k)$ → no. of permutations of 'n' distinct objects, taken 'k' at a time with unlimited repetitions

$$\boxed{U(n,k) = n^k}$$

$$\underbrace{\begin{matrix} \text{n ways} & \text{n ways} & \cdots & \text{n ways} \\ \times & \times & \times \cdots & \times \\ 1 & 2 & 3 & k \end{matrix}}_{(n^k)}$$

Q.17. How many 5-letter permutations are possible with letters a, b, and c ?

→ Required no. of permutations

$= n^k$.

$= 3^5$

$= \boxed{243}$

Q.18 How many ways we can distribute 15 different books among 10 persons ?

→ $\underset{B_1}{\overset{10}{\uparrow}}$ $\underset{B_2}{\overset{10}{\uparrow}}$ $\underset{B_3}{\overset{10}{\uparrow}}$ ..... $\underset{B_{15}}{\overset{10}{\uparrow}}$

$= \boxed{10^{15}}$

Q.19. How many ways 6 children can be admitted to 10 different schools ?

→ $\underset{C_1}{\overset{10}{\uparrow}}$ $\underset{C_2}{\overset{10}{\uparrow}}$ $\underset{C_3}{\overset{10}{\uparrow}}$ $\underset{C_4}{\overset{10}{\uparrow}}$ $\underset{C_5}{\overset{10}{\uparrow}}$ $\underset{C_6}{\overset{10}{\uparrow}}$

$= \boxed{10^6}$

Note → Suppose we have 'n' objects of which 'n_1' objects are alike, 'n_2' objects are alike and ... 'n_k' objects are alike, then

no. of permutations of n objects taken all at a time

$$= \frac{n!}{n_1! \cdot n_2! \cdots n_k!}$$

Q.20. How many 10-letter permutations are possible with the letters {a, a, b, b, b, c, c, c, c, d} if all the letters of the set are used at a time?

→ the required no of permutations

$$= \frac{10!}{2!\cdot 3!\cdot 4!\cdot 1!} = \frac{10\times 9\times 8\times 7\times 6\times 5\times 4!}{2!\cdot 3!\cdot 4!}$$

$$= 90\times 4\times 7\times 5 = \boxed{12600}$$

Q.21. How many different messages of length 5 can be sent with 3 dashes and two dots if all the 5 symbols are used at a time?

→ Required no of permutations

$$= \frac{5!}{3!\times 2!} = \frac{5\times 4\times 3!}{3!\cdot 2!} =$$

$$= \boxed{10}$$

Q.22. How many binary sequences of length 10 are possible with ~~six~~ 1's and four 0's if all the 10 bits are used at a time?

→ $$\frac{10!}{6!\times 4!} = \frac{10\times 9\times 8\times 7\times 6!}{6!\times 4!}$$

$$= \boxed{210}$$

Q.23. A coin is tossed 10 times. No. of outcomes possible with 5 heads and 5 tails is what?

$$\rightarrow \frac{10!}{5!\,5!} = \boxed{252}$$

Q.24. How many ways 10 office buildings can be painted so that 3 are in blue, 2 are in brown, and 5 are in white color?

$$\rightarrow \frac{10!}{3! \times 2! \times 5!} = \boxed{2520}$$

(ordered partitions)

Q.25. How many ways 10 persons can be divided into 3 teams so that first team consists 3 members, 2nd team contains 2 members and 3rd team contains 5 members?

→ Reqwired no of ways

$$= \frac{10!}{3! \cdot 2! \cdot 5!} = \boxed{2520}$$

(Unordered parttions) (No first, second team)

Q.26. " — " How many ways 10 persons can be divided into 5 teams of 2 each?

→ Reqwired no of ways

$$= \frac{10!}{2! \cdot 2! \cdot 2! \cdot 2! \cdot 2! \, \underset{\uparrow \text{ 5 teams}}{5!}} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 3}{2 \times 2 \times 2 \times 2 \times 2 \times 5}$$

$$= \boxed{945}$$

(Unordered)

Combinations →

An unordered selection of objects is called a "combination".

ex → {a, b, c}

2-letter combinations → {a, b}, {b, c}, {a, c}.

Combinations without repetitions →

^nC_k = $C(n, k)$ ⇒ No. of combinations of 'n' distinct objects taken 'k' at a time without repetitions.

$$^nC_k = \frac{n \cdot (n-1) \cdot (n-2) \cdots\cdots (n-(k-1))}{1 \cdot 2 \cdot 3 \cdots\cdots k}$$

$$= \boxed{\frac{^nP_k}{k!}} = \boxed{\frac{n!}{(n-k)!\,(k!)}}$$

$$\boxed{^nC_k = {}^nC_{(n-k)}}$$

ex. $^{10}C_8 = {}^{10}C_2 = \frac{10 \cdot 9}{1 \cdot 2} = 45$

1) $^nC_0 = 1$

2) $^nC_1 = n$

3) $^nC_2 = \frac{n \cdot (n-1)}{1 \cdot 2}$

4) $^nC_3 = \dfrac{n\cdot(n-1)(n-2)}{1\cdot 2\cdot 3}$

5) $^nC_n = 1$.

⑥ $^nC_0 + {^nC_1} + {^nC_2} + \cdots + {^nC_n} = 2^n$

Q.1. How many ways we can distribute 10 similar books among 15 persons so that no person can take more than one book?

→ Here, only 10 persons will get books. The 10 persons we can choose in $^{15}C_{10}$ ways.

∴ Required no. of ways $= {^{15}C_{10}}$

$\boxed{1}$.

Here, we can select the books in one way and can also distribute it in 1 way.

Q.2. How many binary sequences of length 10 are possible with exactly 3 zeros?

→ 7 1's and 3 0's.

x x x x
1 2 3 10

• In the sequence, we have to select 3 positions for placing 3 zero's.

→ $^{10}C_3$ ways.

∴ Required no. of binary sequences

$= {}^{10}C_3 = {}^{10}C_7$

$= \frac{10 \times 9 \times 8}{1 \times 2 \times 3}$

$= \boxed{120}$

Q.3. How many binary sequences of length 10 with exactly 4 0's and no two 0's are consecutive?

→ 1 1 1 1 1 1   0 0 0 0

↑ 1 ↑ 1 ↑ 1 ↑ 1 ↑ 1 ↑ 1 ↑

zeros can appear

∴ No. binary sequences $= {}^7C_4$

$= \frac{7!}{4!\,3!} = \frac{7 \times 6 \times 5 \times 4!}{3 \times 2 \times 4!}$

$= \boxed{35}$

(P)

Q.4. How many 5-digit integers are possible so that in each of these integers, every digit is greater than the digit on its right?

→ To meet the given condn, we have to chose 5 distinct decimal digits and then we have to arrange them in decending order

∴ Required no. digits = $^{10}C_5 \times 1$

$= \boxed{252}$

Q.5 Suppose n couples are in a party. If each person shakes hands with every other person except his/her spouse; then no. of different handshakes possible in the party is?

→ 2n persons

No. of handshakes possible with 2n persons

$= {}^{2n}C_2$

Every person does not shake hands with his/her spouse

total no. of handshakes = $^{2n}C_2 - n$

$= \frac{2n(2n-1)}{1 \cdot 2} - n$

$= \boxed{2n(n-1)}$

Q-6. How many ways six persons {a,b,c,d,e,f} can speak at a convention with b speaking after a ?

→ b speaking after a ⇒ b should not speak before a.

1 2 3 4 5 6

The speeches of a and b can be arranged in $^6C_2$ ways because we can allots 2 slots in $^6C_2$ ways and the remaining 4 speeches can be arranged in 4! ways.

∴ Required no of ways $= {}^6C_2 \times 4!$

$= {}^6C_2 \times 3!$

$= \boxed{120}$

Q-7. How many ways 10 MCQ's which 4 choices for each question can be answered so that exactly 3 answers are correct ?

→ 3 questions can be correctly answered in $^{10}C_3$ ways.

$\frac{10!}{7!\,3!} = \frac{10 \times 9 \times 8}{3 \times 2} = \boxed{120}$

and also remaining 7 questions must be wrongly answered. This can be in 3^7 ways

$\therefore$ By product rule =

$10C_3 \times 3^7 =$

Q.8. How many ways 5 out of 10 persons can sit around a circular table?

→ We can select 5 persons from 10 in $10C_5$ ways.

And those 5 persons can sit around a circular table in $4!$ ways.

$\therefore$ Required no of ways $= 10C_5 \times 4!$

$= 252 \times 24$

$= \boxed{6048}$

Q.9. How many ways can we select a comittee of 5 members out of 5 men and 5 women so that at least 1 women is included in the comittee?

→ $5C_4 \cdot 5C_1 + 5C_3 \cdot 5C_2 + 5C_2 \cdot 5C_3 + 5C_1 \cdot 5C_4 + 5C_5$

at least 1 woman → complement

no woman selected.

10 C 5
5 0 4 +
6 0 4 8

No women selected = all men selected

$= {}^5C_5 = 1$

and total are $= {}^{10}C_5$

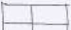
$\therefore$ at least one women selected

$= {}^{10}C_5 - 1$

$= \boxed{251}$

Q.10. How many rectangles are there in a chess-board which are not squares?

$\Rightarrow$ 2×2 chessboard ?



A rectangle can be formed by any two horizontal lines and any two vertical lines.

The chessboard consists of 9 horizontal and 9 vertical lines.

$\therefore$ No. of rectangles (including squares) in the chessboard $= {}^9C_2 \times {}^9C_2$

$= 36 \times 36 = 1296.$

No of squares in a chessboard are :

$\underset{(8\times 8)}{1^2} + \underset{(7\times 7)}{2^2} + \underset{(6\times 6)}{3^2} + \dots + \underset{(1\times 1)}{64\,(8^2)}$

$$= \frac{n(n+1)(2n+1)}{6}$$

$$n = 8$$

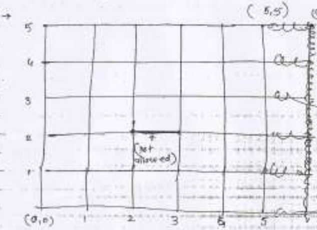
$$= \frac{8 \times 9 \times 17}{6} = \boxed{204} \text{ squares}$$

∴ Total no of rectangles that are not squares

$$= 1296 - 204$$

$$= \boxed{1092}$$

Q.11. A robot is placed at O(0,0). At/In each operation the robot can move 1 unit to the right (+ve x-axis) or 1 unit up (+ve y-axis). How many different paths are there from (0,0) to (5,5) if the robot is not allowed to use the line segment joining (2,2) and (3,2)?



Each path consists of 5 moves along x-axis and 5 moves along y-axis.

$\therefore$ the no. of paths from (0,0) to (5,5)

= no. of binary sequences possible with 5 x's and 5 y's.

$= {}^{10}C_5$

no. of paths from (0,0) to (5,5) via the line segment joining (2,2) & (3,2) =

${}^4C_2 \times {}^5C_2$

= 60 paths.

$\therefore$ Reqd. no. of paths = total no of possible paths − paths via {(2,2) & (3,2)}

$= 252 - 60 = \boxed{192}$

Q.12 There are 12 stations on rail-line. How many ways a special train can stop at 4 of these stations so that no two stops are consecutive stations?

→ Let us denote the stations where train stops by 0's and the other stations by 1's.

then the required no. of ways

= no. of binary seq. possible with 8 1's and 4 0's so that no two zeros are consecutive

$$\Rightarrow \underbrace{-1-1-1-1-1-1-1-1-}_{9 \text{ places}} \mid 0\,0\,0\,0$$

$= {}^{9}C_4 = \boxed{126}$ ways

Combinations with Repetitions →

$V(n,k) = {}^{n}V_k$

= no. of combinations of n distinct objects taken k at a time with unlimited repetitions.

(k can be > n).

the no. of ways = $C(n-1+k, k)$

= no. of ways we can distribute 'k' similar balls into 'n' numbered boxes.

P.13. No. of nonnegative integers solutions to the equation $x_1+x_2+\dots+x_n = k$?

→ let $k=10$, $n=5$.

then $x_1+x_2+x_3+x_4+x_5 = 10$.

⇒ no. of binary sequences / bit strings possible with $(n-1)$ 1s and k 0's.

$C(n-1+k, k)$

Q.14. How many ways we can distribute 10 similar books among 5 persons?

→ Required no of ways = $V(5,10)$

$V(n,k) = C(n-1+k, k)$

$V(5,10) = C(14, 10) = {}^{14}C_{10}$

$= {}^{14}C_4 = \frac{14 \times 13 \times 12 \times 11}{4 \times 3 \times 2} = \boxed{1001}$

Q.15. How many ways we can distribute 16 similar balls in 4 numbered boxes (distinguishable) so that each box contains at least one ball.

→ To meet the condition, let us put one ball in each box in only one way.

B_1	B_2	B_3	B_4
1	1	1	1

Remaining 12 balls, we can distribute into 4 boxes in any way

$V(n,k) = V(4,12)$ ways.

$= C(4-1+12, 12) = C(15, 12)$ ways.

$= \boxed{455}$

possible.

Same as previous example now.

$x_1+x_2+x_3+x_4+x_5+x_6=9$

$\therefore V(6,9) = C(6-1+9, 9) = C(14,9)$

$= \boxed{2002}$

Q.19. No. of ternary sequences possible with 6 1's, 6 2's, and 4 0's if so that each '1' is followed by '2'

→ Consider 6 pairs of '12'. How we have 7 places among these pairs for placing the 4 0's

∴ the reqd. no of ternary sequences

$= V(7,4) = C(10,4)$

$= \frac{10!}{6! \times 4!} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2}$

$= \boxed{210}$

Q.20 Suppose A question paper has 10 questions for 30 marks. The first question should carry 5 marks and each of the rem. quest's carry at least 2 marks. How many ways the marks can be distributed among the 10 questions subjected to above cond's. ?

9 → ques. 25 marks

9 → 7 marks

$V(9,7)$ $C(15,7)$ 15C7

$^{10}C_4$ $\quad$ $^5P_5 =$ $\quad$ $C(10,4)$

To meet the given conditions, let us allot 5 marks to first questn and (2 ea) to 9 questns

So, we are left with 17 marks and to distr among last 9 questn.

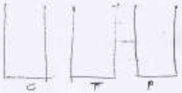
∴ Required no of ways

$= V(9,7) = C(15,7)$

$= \boxed{6435}$

Q. 21. Suppose 10 persons are in a canteen which offers coffee, tea and pepsi. How many ways they can order their drinks (as group) if each person can select or wants only one of the 3 drinks?

→



Required no. of orders

$= V(3,10)$

$= C(12,10) = \dfrac{12 \cdot 11}{1 \cdot 2} = 66.$

Pigeonhole Principle →

If A is the average number of pigeons per a pigeonhole, then

a) some pigeonhole contains at least $\lceil A \rceil$ pigeons.

b) some pigeonhole contains at most $\lfloor A \rfloor$ pigeons.

So, let 10 pigeons and 4 pigeonhols.

$$A = \frac{10}{4} = 2.5$$

a) Some pigeonhole contains at least $\lceil A \rceil = 3$ pigeons.

b) Some pigeonhole contains at most $\lfloor A \rfloor = 2$ pigeons.

If $n+1$ pigeons are distributed / kept in n pigeonhole then $A = \frac{n+1}{n} = 1 + \frac{1}{n}$

$\therefore \lceil A \rceil = 2$. $\quad \lfloor A \rfloor = 1$.

pigeonhole princ.

$\therefore$ By P.H.P, some pigeonhole contains at least 2 pigeons.

and some other pigeonhole contains at most 1 pigeon.

* If $(2n+1)$ pigeons are kept in n pigeonholes, then
$$A = \frac{2n+1}{n} = 2 + \frac{1}{n}$$

$$\lceil A \rceil = 3 \qquad \lfloor A \rfloor = 2$$

a) Some pigeonhole contains at least 3 pigeons.

b) Some pigeonhole contains at most 2 pigeons.

* If k is any positive integer, and if $kn+1$ pigeons are kept in n pigeonholes, then

$$A = \frac{kn+1}{n} = k + \frac{1}{n}$$

$$\lceil A \rceil = k+1$$

$$\lfloor A \rfloor = k$$

∴ a) Some pigeonhole contains at least '$k+1$' pigeons.

b) Some pigeonhole contains at most 'k' pigeons.

Note → Suppose we have 'n' pigeonholes, the min. no. of ~~pigeonholes~~ reqd. to ensure that some pigeonhole contains at least $(k+1)$ pigeons

$= (kn+1)$

81. what is the min no. of persons we have to choose randomly to ensure that at least 10 persons were born in the same month.

→ $n = 12$ months → pigeonholes

$k+1 = 10$

$k = 9$ $n = 12$

$(9 \times 12) + 1 = \boxed{109}$

(OR)

$A = \left\lceil \frac{x}{12} \right\rceil = 10$

$= \left\lceil \frac{109}{12} \right\rceil = 10$

$\boxed{109}$

Q-2. If 410 letters were distributed in 50 apartments which of the foll. st. are necessarily true?

Sol. At least 9 per

✓ S1) Some apartment received at least 9 letters

✓ S2) Some apartment received at most 8 letters

✗ S3) Some apartment received at least 10 letters

✗ S4) Some apartment received at most 7 letters

✓ S5) Some apt received at least 7 letters

54) Some apt received at most 10 letter.

→ $A = \frac{410}{50} = 8.2$

$\lceil A \rceil = 9 \quad \lfloor A \rfloor = 8$

By P.H.P, S1 & S2 follows

S3 is also true coz there are all apts receiving letters ≥ 7

S4) no. letters ≤ 10 always true.

Q.3. A bag contains 6 red balls, 8 blue balls, 10 green balls and 15 white balls and 20 yellow balls. What is the min. no of balls we have to choose randomly from the bag to ensure that we get at least 6 balls of some color.

→ $\underbrace{6\text{-R} \quad 8\text{-B} \quad 10\text{-G} \quad 15\text{-W} \quad 20\text{-Y}}_{5 \text{ pigeon holes}}$ $n=5$

$k+1 = 6$ balls (pigeons).
$\therefore k=5$

Min. no of balls reqd = $kn+1$

$= (5 \times 5) + 1 = 26$

Q.4. In the above example, what is the min no of balls we have to choose from the bag to ensure that we get at least 9 balls of same color ?

→ to find the min no of balls reqd., include

$$\underbrace{\overline{\text{all Red}}\ \ 6 + \overline{\text{all blue}}\ \ 8}_{\uparrow} + \underset{G}{8} + \underset{W}{8} + \underset{Y}{8}$$

(we have to 14 include them). + 24

= 38 + 1

Min no of balls reqd. to be drawn = $\boxed{39}$

Q.5. In the above ex., what is the min no of balls we have to choose from bag to ensure that we get at least 12 balls of same color ?

→ $\left(\underbrace{\underset{R}{6} + \underset{B}{8} + \underset{G}{10}}\ + \underbrace{\underset{W}{\overset{11}{12}} + \underset{Y}{\overset{11}{12}}}\right) + 1$

= (24 + 22) + 1

= 46 + 1

= $\boxed{47}$

Q.6. A CS dept offer 4 yr B-Tech programme. To form a with an intake of 60 students each year. A student's club can be formed in the dept with any
a) any 10 second year students.
(or)
b) any 8 third year students.
(or)
c) any 6 final year students.

What is the min no of students you have to pick randomly from dept. to ensure that the student's club is formed ?

→ $(\underbrace{60 + 9 + 7 + 5}_{\text{I} \quad \text{II} \quad \text{III} \quad \text{IV}}) + 1$ — with these 81 students, we cannot form a club.

$= (81) + 1$

$= \boxed{82}$

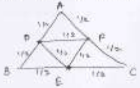
Constructing a pigeonhole

Q 7. 5 points were randomly selected in an equilateral Δ with each side 1 unit. There will be at least 2 points (of those 5 points) such that the distance betn the two points cannot exceed

a) 1/2 b) 1/3 c) 1/4 d) 1/5

* Apply pigeon hole principle by dividing the Δ into 4 parts.

→ * Divide the Δ into 4 equal parts.



By pigeon hole princ. the avg. no. of points in a small Δ is $5/4 = 1.25$

Now $A = 1.25$ ∴ $\lceil A \rceil = \underline{2}$

∴ By P.H.P., at least two of those 5 points lie in one of the small Δs.

(∴) the distance bet^n any two points in a small Δ cannot exceed (1/2)

Q.8 – 10 points are randomly selected in a square with each side 3 units. There will be at least 2 points (of those 10 pts.) such that the distance bet^n them cannot exceed →

a) 1 ✓b) $\sqrt{2}$ c) $\sqrt{2}-1$ d) $2-\sqrt{2}$.

→ * Divide the □ in 9 equal parts.

for a small square

The avg no of points in a small square

$A = \frac{10}{9} = 1.111 \quad \lceil A \rceil = 2$

By PHP, at least two points belong to same pigeonhole.

∵ The max distance betn any 2 points in a square is diagonal = $\sqrt{2}$

∴ ans = $\boxed{\sqrt{2}}$

Q.9. 7 points were randomly selected in a regular hexagon with each side 1 unit. There will be at least 2 points such that dist betn them is

(a) 1

∴ There exist at least 2 points whose dist does not exceed 1

Euler Function → (of n).

If 'n' is a positive integer, then Euler function of 'n' denoted by $\phi(n)$ is defined as,

$\phi(n)$ = no of positive integers which are $\leq n$ and relatively prime (co-prime) to n.

① Relatively Prime / Co-prime →
Two +ve integers a and b are said to co-prime if

G.C.D. (a,b) = 1

Q1. what is Euler function 6.

→ $\phi(6) = \{1, 2, 3, 4, 5, 6\}$.

$= \boxed{2}$.

Q.2 $\phi(7) = \{1, 2, 3, 4, 5, 6, 7\}$.

$= \boxed{6}$

Note → If n is a prime no. then $\boxed{\phi(n) = n-1}$

Q.3 $\phi(8) = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

$= \boxed{4}$

$$\phi(n) = \left\{ \frac{n \times (P_1 - 1)(P_2 - 1) \cdots (P_k - 1)}{P_1 \cdot P_2 \cdot P_3 \cdots P_k} \right\}$$

where $P_1, P_2, \ldots, P_k$ → distinct prime factors of 'n'.

Q-4. No. of +ve integers which are less than 110 and relatively prime to 110 is ?

→ $110 = \underset{P_3}{2} \times \underset{P_2}{5} \times \underset{P_1}{11}$

$$\begin{array}{r|l} 2 & 110 \\ \hline 5 & 55 \\ \hline & 11 \end{array}$$

$$\therefore \phi(n) = \phi(110) = \frac{\cancel{110}^{10} \times (10 \times 4 \times 1)}{\cancel{11} \times 5 \times 2}$$

$$= \boxed{40}$$

Q-5. No. of +ve int less than 180 and < 180

$$180 = $$

$$\to \phi(180) = \frac{\cancel{180}^{\cancel{36}} \times (4)(2)(2)(1)(1)}{\cancel{3} \times \cancel{3} \times 2 \times 2 \times \cancel{5}}$$

$$= 16.$$

we have to take distinct prime nos

$$= \frac{\cancel{180}^{\cancel{36}^{12}} \times 4 \times 2}{\cancel{5} \times \cancel{3} \times 2} = \boxed{48}$$

Q.6. $\phi(323)$

$\rightarrow 323 = 19 \times 17$

$\therefore \phi(323) = 323 \times \frac{(16 \times 18)}{19 \times 17}$

$= \boxed{288}$

Q.7. $\phi(113)$

$\rightarrow$ 113 is prime.

$\therefore \phi(n) = 113 - 1 = \boxed{112}$

Q.8. Let $n = p^2 q$ where p and q are prime nos. No. of +ve integers m such that $1 \le m \le n$ and G.C.D of $\{m, n\} = 1$ is ?

$\rightarrow$ Reqd. no. of +ve integers

$\phi(n) = n \cdot \frac{(p-1)(q-1)}{p \cdot q}$

$= p^2 q \cdot \frac{(p-1)(q-1)}{p \cdot q} = \boxed{p(p-1)(q-1)}$

(*) if suppose a no. n is given
then we have to check if its prime or not
upto $\sqrt{n}$ only......

Derangements (kind of permutations) :

A permutation of 'n' distinct objects in which no object appears at its original/correct place is called a derangement of objects.

No. of derangements of 'n' distinct objects taken all at a time

$$D_n = n!\left\{\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\cdots+\frac{(-1)^n}{n!}\right\}$$

ex :

suppose $n=2$, $D_2 = 2! \times \frac{1}{2!} = 1$

$n=3$, $D_3 = 3! \times \left\{\frac{1}{2!}-\frac{1}{3!}\right\}$

$= \frac{3!}{2!} - \frac{3!}{3!} = 3-1 = 2$

$n=4$, $D_4 = 4! \times \left\{\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right\}$

$= \frac{4!}{2!} - \frac{4!}{3!} + \frac{4!}{4!} = 12-4+1$

$= 9$

$n=5$, $D_5 = 5! \times \left\{\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right\}$

$= 60-20+5-1 = 44.$

1, 2, 3

$n=6, \; D_6 = 265$

Q.1. How many 1-1 functions are possible with 6 elements so that no element is mapped to itself?

→

$$\begin{array}{ccc} A & \to & A \\ 1 & \longleftrightarrow & 1 \\ 2 & \longleftrightarrow & 2 \\ 3 & \longleftrightarrow & 3 \\ 4 & \longleftrightarrow & 4 \\ 5 & \longleftrightarrow & 5 \\ 6 & \longleftrightarrow & 6 \end{array}$$

Required no. of one-one functions is

$D_6 = \boxed{265}$

Q.2. How many ways we can put 5 letters in 5 envelopes (@ 1 letter per envelope) so that

a) no letter is correctly placed (i.e. letter l_i is not in envelope e_i $(i = 1$ to $5)$).

b) at least one letter is correctly placed

c) exactly two letters are correctly placed.

d) atmost one letter is correctly placed.

e) at least one letter is wrongly placed.

f) exactly one letter is wrongly placed.

→ a) $l_1 \quad l_2 \quad l_3 \quad l_4 \quad l_5$

$e_1 \quad e_2 \quad e_3 \quad e_4 \quad e_5$.

Required no. of ways

$$= D_5 = 5! \left\{ \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} \right\}$$

$$= \boxed{44}$$

b) total No. of ways 5 letter in 5 envelopes

$$= 5! = 120.$$

No. of ways in which no letter is correctly placed $= D_5 = 44$.

$\therefore$ No. of ways in which at least one letter is correctly placed

$$= 120 - 44 = \boxed{76.}$$

c) No. of ways we can select 2 letters out of 5

$$= 5C_2 = \frac{5!}{3!\,2!} = \frac{5 \times 4}{2} = 10$$

And these two letters can be placed in correct envelope in one way.

$\therefore$ No. of ways 2 out of 5 letters can be correctly placed = $10 \times 1 = \underline{10}$

And also, remaining 3 letters should be placed in wrong envelopes. in D_3 ways

$\therefore D_3 = 2.$

Required no of ways = $10 \times 2 = \boxed{20}$

d) $\underline{\text{atmost one}}$ ≤ 1

$= 0$ (or) $= 1$

Here we have two cases where no letter is correctly placed or only one letter is correctly placed.

No. of ways in which no letter is correctly placed is $D_5 = 44$

(+)

and no of ways in which only one letter is correctly placed is $5C_1 \cdot D_4$

$$5C_1 \cdot D_4 = \frac{5!}{4!\,1!} \times 9 = 5 \times 9 = 45$$

By sum rule,

$\therefore D_5 + 5C_1 \cdot D_4 = 44 + 45 = \boxed{89}$

e) → there is only one way, we can put all 5 letters correctly. In all the remaining cases, at least one letter will be wrongly placed.

∴ Required no. of ways = $5! - 1$

$= \boxed{119}$

f) l_1 l_2 l_3 l_4 l_5

e_1 e_2 e_3 e_4 e_5

* It is not possible to put only one letter in wrong envelope because if we put one letter in wrong envelope, then the letter corresponding to that envelope also goes to wrong envelope.

∴ Required no. of ways = $\boxed{0}$

Q3. No. of derangements possible for the sequence {a, b, c, d, e, f, g, h, i, j} so that

a) the first five letters of this sequence are in first five places.

b) None of the first five letters of the sequence are in first 5 places.

→ a) → $\{\underbrace{a, b, c, d, e}_{D_5}, \underbrace{f, g, h, i, j}_{D_5}\}$

$D_5 \times D_5$

* The first 5 letters of the sequence can be deranged in first 5 places in D_5 ways.

* ||^{ly}, the last 5 letters of the seq. can be deranged in last 5 places in D_5 ways.

$\therefore$ Required no. of derangements

$= D_5 \times D_5$

$= 44 \times 44 = 1936$.

b) $\rightarrow$ Any permutation of the given sequence in which the first 5 letters are in last 5 places & last 5 letters are in first 5 places is a derangement.

$\therefore$ Reqd. no. of derangements

$= 5! \times 5! = \boxed{14400}$

$\{\underbrace{a, b, c, d, e}, \underbrace{f, g, h, i, j}\}$.

5! ways

5! ways

Q.4. Suppose 5 diff books are distributed among 5 students @ one book per student. Further suppose that the books were returned by the students and again distributed among the students later on.

How many ways this can be done so that no student can take the same book twice?

First time, the books can be issued to the students in $5!$ ways.

Second time,

$B_1 \quad B_2 \quad B_3 \quad B_4 \quad B_5$

$S_1 \quad S_2 \quad S_3 \quad S_4 \quad S_5$

(Second time)
the books can be distributed in D_5 ways.

$\therefore$ Required no. of ways

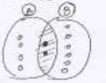
$= 5! \times D_5 = 44 \times 120$

$= \boxed{5280}$

Q.

Principle of Mutual Inclusion and exclusion →

- Let A and B are any two sets.



then no. of elements (A or B)

$= n(A) + n(B) - n(A \text{ and } B)$

i.e. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

$= 6 + 8 - 2 = \boxed{12}$

- Let A, B and C are any 3 sets.

then no. of ele. (A or B or C)

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C)$

$+ n(A \cap B \cap C)$.

* $n(A \cup B \cup C \cup D) = n(A) + n(B) + n(C) + n(D)$

$- n(A \cap B) - n(A \cap C) - n(A \cap D) - n(B \cap C) - n(B \cap D)$

$- n(C \cap D) + n(A \cap B \cap C) + n(A \cap C \cap D) + n(B \cap C \cap D)$

$+ n(A \cap B \cap D) - n(A \cap B \cap C \cap D)$.

54
32
86

$- \cancel{112} + 32 - 4$

$= 130 - 76 + 32$

Q.1. Let A, B, C, D are 4 sets such that

$n(A) = 42$, $n(B) = 36$, $n(C) = 28$, $n(D) = 24$.

Intersection of any two of these four sets contains 12 elements, and

Intersection of any 3 of these 4 sets contains 8 elements, and

$n(A \cap B \cap C \cap D) = 4$. then $n(A \cup B \cup C \cup D) = ?$

$n(A \cup B \cup C \cup D) = n(A) + n(B) + n(C) + n(D) -$

$n(A \cap B) - n(A \cap C) - n(A \cap D) - n(B \cap C) - n(B \cap D)$

$- n(C \cap D) + n(A \cap B \cap C) + n(A \cap C \cap D) + n(A \cap B \cap D)$

$+ n(B \cap C \cap D) - n(A \cap B \cap C \cap D)$

$= 42 + 36 + 28 + 24 - (6 \times 12) + (4 \times 8) - 4$

$= \boxed{86}$

Q.2. In a class of 100 students,

48 students can speak French,

19 students can speak German,

and 24 students can speak both lang.

How many students can speak

a) at least one of the two languages?

b) None of the two lang.?

c) only one of the two lang?

→ a) $n(F \cup G) = n(F) + n(G) - n(F \cap G)$

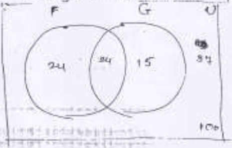
$= 48 + 39 - 24$

$= \boxed{63}$

b) $n(\overline{F \cup G}) = 100 - 63$

$= \boxed{37}$

c) the given data can be represented as



No. of students who can speak only one of the two lang. $= 24 + 15$

$= \boxed{39}$

$\frac{1000}{10}$ $\frac{\cdots}{6}$ $\frac{1000}{15}$ $\frac{1000}{30}$ $\frac{1000}{6}$ 400 333 $\frac{1000}{3}$ 114

Q.3. In the set $\{1,2,3,\ldots,1000\}$, how many integers are not divisible by 2, 3 or 5 ?

→ Let $n(2)$ → no. of integers in the set div by 2

$n(3)$ → no of integers in the set div by 3

$n(5)$ → no of integers in the set div by 5.

$n(2\cup 3\cup 5) = n(2)+n(3)+n(5) - n(2\cap 3) - n(3\cap 5)$

$- n(2\cap 5) + n(2\cap 3\cap 5)$

$= 500+333+200-166-66-100$
$+33$

$= 1033 - 332 + 33$

$= \boxed{734}$

∴ Reqd. no. of integers $= 1000 - 734$

$= \boxed{266}$

Q4 In a class of 100 students,

40 students failed in maths.

30 students failed in physics.

25 students failed in chemistry.

12 students failed in maths & physics.

$$\begin{array}{r} 166 \\ 56 \\ \hline 222 \end{array}$$

15 students failed in physics and chem,

10 students failed in maths and chem, and

6 students failed in all the 3 subjects

a) How many students failed in at least one of the three subjects.

b) " — " none of the three subjects ?

c) " — " only one of the three subjects ?

d) " — " exactly two of the three subjects ?

e) " — " at least two of the three subjects ?

→ Let $n(M)$ = no. of students failed in Maths.

a) n(failed in at least one subject)

$= n(M \cup P \cup C) = n(P) + n(M) + n(C) - n(M \cap P)$

$\qquad - n(P \cap C) - n(M \cap C) + n(P \cap M \cap C)$

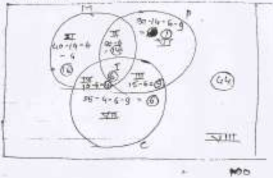
$= 30 + 40 + 25 - 20 - 15 - 10 + 6$

$= \boxed{56}$

b) $n(\overline{M \cup P \cup C}) = 100 - 56$

$= \boxed{44.}$

The given data can be represented by Venn diagram as follows.



c) No of students failed in only one sub

$= 14+1+6 = \boxed{21}$

d) No-of students failed in exactly 2 sub

$= 14+4+9 = \boxed{27}$

e) No of students failed in at least 2 subjects

$= 14+4+9+6 = \boxed{33}$

Q.6 In a competition, 34 students received awards.

18 students received awards in biology.

17 students received awards in chemistry.

21 students received awards in Physics.

3 students received awards in all 3 subjects.

How many students received awards

a) in exactly two subjects.

b) only one of the three subjects.

Sol.
a) →

$n(B \cup P \cup C) = n(B) + n(P) + n(C) - n(B \cap P)$

$- n(P \cap C) - n(B \cap C) + n(B \cap P \cap C)$

$34 = 18 + 17 + 21 - n(B \cap C) - n(P \cap C) - n(B \cap P)$

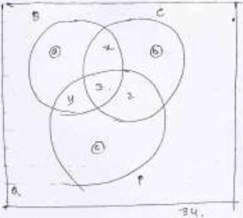
$+ 3$

$\therefore n(B \cap C) + n(P \cap C) + n(B \cap P) = \boxed{20}$

$(x+3) + (y+3) + (z+3) = 20$

$x+y+z = 20 - 9 = \boxed{11}$

Venn diagram representation :



No. of students received awards in exactly two subjects = $\boxed{11}$

b) No. of students receiving awards in exactly one subjects

$= 34 - \underset{x+y+z+3}{14} = \boxed{20}$

Q.6.

$3^{4}-14$

Recurrence Relations →

Let $a_0, a_1, a_2, \ldots, a_n, \ldots$ be a sequence of real numbers.

A formula which relates 'a_n' with one or more of the preceding terms $a_{n-1}, a_{n-2}, \ldots$ is called a "Recurrence Relation"

$$a_n = f\{a_{n-1}, a_{n-2}, \ldots\} \rightarrow \text{Recurrence Relat}^n$$

ex. → For the arithmatic sequence.

$\{a, a+d, a+2d, \ldots\ldots, \}$

the recurrence relation is

$$\underset{(n\geq 1)}{a_n = a_{n-1} + d} \quad \text{where } a_0 = a.$$

ex. → For the geometric sequence.

$\{a, a.r, a.r^2, a.r^3, \ldots\}$

the recurrence relation is

$$a_n = a_{n-1}.r. \quad (n\geq 1) \quad a_0 = a.$$

ex. For the Fibonacci sequence.

$\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$

the recurrence relation is,

$a_n = a_{n-1} + a_{n-2}$ $(n \geq 2)$ where $a_0 = a_1 = 1$.

* Recurrence Relations
  - Linear
  - Non-linear

① Linear Recurrence Relations :

A recurrence relation of the form

$$C_0 \cdot a_n + C_1 \cdot a_{n-1} + \dots + C_k \cdot a_{n-k} = f(n) \quad - ①$$

is a "linear Recurrence Relation of order k".

* Note : If $f(n) = 0$, then eqn ① is said to be "homogeneous linear recurrence relation".

If $f(n) \neq 0$, then eqn ① is said to be "inhomogeneous linear rec. relation".

④ Formation of Recurrence Relations :

ex. →

Q.1. If a_n = no. of binary sequences of length with no consecutive 0's, then recurrence relation for a_n is what ?

→

$x \quad x \quad x \quad \cdots\cdots \quad x$
$1 \quad 2 \quad 3 \quad \cdots\cdots \quad n$

case 1) : If the first bit of the sequence is '1', then the remaining '$n-1$' bits we can choose in 'a_{n-1}' ways.

∴ No. of binary seq. possible $= a_{n-1}$ (if we take first bit '1').

Case 2) : If the first bit is '0', then second bit should be '1' and the remaining bits we can choose in 'a_{n-2}' ways.

∴ No. of binary seq. possible $= a_{n-2}$ (if we take first bit '0').

∴ By sum rule, the recurrence relation for 'a_n' is

$$a_n = a_{n-1} + a_{n-2} \quad (n \geq 2).$$

Q-2. Using the above recurrence relation find the value of a_5.

how to find initial values : For the above recurrence relation, the initial values are

$a_1 = 2$ (binary seq of length one with no consec. 0's).

$a_2 =$ 0 1, 1 0, 1 1 $= \underline{3}$ (binary seq. of length 2 with no consec 0's).

∴ initial values

$a_1 = 2$

$a_2 = 3$

for

∴ $n = 3$, $a_3 = a_2 + a_1 = 5$

$n = 4$, $a_4 = a_3 + a_2 = 5 + 3 = 8$

$n = 5$, $a_5 = a_4 + a_3 = 8 + 5 = \boxed{13}$

Q.8. If a_n = no. of ternary sequences of length 'n' with even no. of 0's then recurrence relation for 'a_n' is ?

→ case 1) 2 ways

$$\underset{1}{x}\ \underbrace{\underset{2}{x}\ \underset{3}{x}\ \cdots\ \underset{n}{x}}_{a_{n-1}}$$

case 1) If the first digit of the sequence is not zero, then we can choose the first digit in 'two ways' and the remaining n-1 digits we can choose in a_{n-1} ways.

∴ The no. of ternary seq. $= 2(a_{n-1})$ (ternary seq. of length n with first digit not 0.)

Case 2) → If the first digit is zero, then the remaining sequence should contain odd no of 0's.

∴ The no. of ternary sequences

$$= \underbrace{3^{n-1}}_{\text{total no of sequences}} - \underbrace{(a_{n-1})}_{\text{no of ternary seq. with even no of seq. 0's}}$$

∴ Reccurence relation for a_n is

$$a_n = 2(a_{n-1}) + (3^{n-1} - a_{n-1})$$

$$\boxed{\therefore a_n = (a_{n-1}) + 3^{(n-1)}} \quad (n \geq 1) \text{ and } a_1 = 2, \; a_2 = 5$$

Q.4. If a_n = no of ways we can arrange a pile (or) stack of 'n' chips using Red, white, Green, Gold and Blue chips so that no two gold chips are together then find the reccurrence relat^n for a_n.

→ case 1) → Let the first chip is not gold.

n chips.

So, the first chip we can select in 4 ways.

and the remaining (n-1) chips we can arrange in a_{n-1} ways.

∴ By product rule,

No of ways $= 4 \cdot (a_{n-1})$

(if first chip is not Gold)

Case 2) : Let the first chip is Gold, then second chip we can choose in 4 ways.

And remaining $n-2$ chips we can arrange in 'a_{n-2}' ways.

∴ By product Rule,

No. of ways $= 4 \cdot (a_{n-2})$

(if first chip is Gold)

∴ Now, by Sum Rule,

$a_n =$ case 1) + case 2)

$= 4 \cdot a_{n-1} + 4 \cdot a_{n-2}$

$$\boxed{\therefore a_n = 4(a_{n-1} + a_{n-2}).} \quad (n \geq 2)$$

$a_1 = 5$ $a_2 = 24$

$= 20 + 4 = 24$

Q.5 If a_n = no. of ways a person can climb a flight of 'n' steps if he/she can skip at most two steps at a time.

Then recurrence relat^n for a_n = ?

→ The person has 3 options in first move →

case 1) → In his first move, if the person covers one step. (skips no step) then the remaining steps he can climb in $\underline{a_{n-1}}$ ways.

case 2) → In his first move, if the person covers two steps by (skipping first), then the rem. steps he can climb in $\underline{a_{n-2}}$ ways.

case 3) → In his " - - -, if the person covers 3 steps (skips 2), then the rem. steps " - - in $\underline{a_{n-3}}$ ways.

$\therefore a_n = a_{n-1} + a_{n-2} + a_{n-3}$. $n \geq 3$.

$a_1 = 1$ $a_2 = 2$

and $a_3 = 4$.

Q. $1+2+2^2+\dots+2^n$ $= \boxed{\dfrac{2^{n+1}-1}{2-1}}$

Solution of Recurrence Relations :

Normally, $a_n = f(a_{n-1}, a_{n-2}, \dots)$ — (A)

A function "$a_n = f(n)$" denoted by — (1) is called a solution if the funcⁿ $f(n)$ satisfies (A).

3 Methods :

1) Substitution method :

In this method, the given recurrence relations is repeatedly used for $n = 1, 2, 3, \dots$ and then an appropriate algebraic simplification is used to get the reqd. solⁿ.

Q.1. The solution of the rec-relation

$a_n = n \cdot a_{n-1}$ where $a_0 = 1$ is ?

→ put $n = 1, \dots$
$a_1 = 1 \cdot a_0 = 1 = 1!$
$a_2 = 2 \cdot a_1 = 2 = 2!$
$a_3 = 3 \cdot 2 = 6 = 3!$
$a_4 = 4 \cdot 6 = 24 = 4!$

$\therefore \boxed{a_n = n!}$

$n \cdot (n-1) + (n-1)$

$a_n = 1 + n \cdot 3 \qquad 2 + (3^1 + 3^2 + \cdots + 3^{n-1})$

$a_2 = 2 + 3^{1} = 5$

$a_3 = 5 + 3^2 = 14$

$a_4 = 14 + 3^3 = 41$

Q.2. The solⁿ of the rec relatⁿ $a_n = a_{n-1} + 3^{n-1}$ where $a_1 = 2$

$\Rightarrow \quad a_1 = 2 \qquad a_2 = 2 + 3^1$

$a_3 = a_2 + 3^2 = 2 + 3^1 + 3^2$

$a_4 = a_3 + 3^3 = 2 + 3^1 + 3^2 + 3^3$

$a_n = 2 + 3 + 3^2 + 3^3 + \cdots 3^{n-1}$

$= 1 + 1 + 3 + 3^2 + 3^3 + \cdots + 3^{n-1}$

$= 1 + \dfrac{3^n - 1}{3 - 1} = \dfrac{3^n + 1}{2}$

$\therefore \boxed{a_n = \dfrac{3^n + 1}{2}}$

Q.3. The solⁿ of the rec relatⁿ

$a_n = a_{n-1} + (2n+1)$ where $a_0 = 1$ is ?

$\rightarrow \quad a_1 = a_0 + 3 = 4 = 2^2$

$a_2 = a_1 + 5 = 9 = 3^2$

$a_3 = a_2 + 7 = 9 + 7 = 16 = 4^2$

$a_n = (n+1)^2$

$a_n = (n+1)^2 \Rightarrow$ sum of first $(n+1)$ odd nos.

Q.4. $a_n = a_{n-1} + n$ where $a_0 = 1$

$\Rightarrow a_1 = a_0 + 1 = 1 + 1 = 2$

$a_2 = a_1 + 1 = 2 + 1 = 3 = a_0 + 1 + 2$

$a_3 = a_2 + 1 = 3 + 1 = 4 = a_0 + 1 + 2 + 3$

$a_4 = a_3 + 1 = 4 + 1 = 5 = a_0 + 1 + 2 + 3 + 4$

$$a_n = a_0 + \frac{n(n+1)}{2} = 1 + \frac{n(n+1)}{2}$$

$$\boxed{a_n = \frac{n^2 + n + 2}{2}}$$

Q.5. $a_n = a_{n-1} + n^2$ where $a_0 = 3$

$\Rightarrow a_1 = a_0 + 1$

$a_2 = a_1 + 4 = a_0 + 1 + 4$

$a_3 = a_2 + 9 = a_0 + 1^2 + 2^2 + 3^2$

$\therefore a_n = a_0 + (1^2 + 2^2 + 3^2 + \cdots + n^2)$

$$= a_0 + \frac{n(n+1)(2n+1)}{6}$$

$= 3 + \ldots$

(5) $1^3 + 2^3 + 3^3 + \cdots + n^3$

$= \dfrac{n^2(n+1)^2}{4}$

$a_0 + \{1 - (1/n+1)\}$

Q-6 The soln of the rec. relat^n

$a_n = a_{n-1} + \dfrac{1}{n(n+1)}$ where $a_0 = 1$ is ?

$\rightarrow \quad a_n = a_{n-1} + \left(\dfrac{1}{n} - \dfrac{1}{n+1}\right)$

$n = 1$ so $a_1 = a_0 + (1 - 1/2)$

$n = 2 \Rightarrow a_2 = a_1 + (1/2 - 1/3)$

$= a_0 + (1 - 1/2) + (1/2 - 1/3)$

$= a_0 + (1 - 1/3)$

$n = 3 \Rightarrow a_3 = a_2 + (1/3 - 1/4)$

$= a_0 + (1 - 1/2) + (1/3 - 1/4)$

$= a_0 + (1 - 1/4)$

$\therefore a_n = a_0 + \left\{1 - \left(\dfrac{1}{n+1}\right)\right\}$

$= 1 + 1 - \left\{\dfrac{1}{n+1}\right\}$

$\boxed{a_n = \left(2 - \dfrac{1}{n+1}\right)}$

Shift operator (E) :

$E(a_n) = a_{n+1}$

$E^2(a_n) = a_{n+2}$

$E^3(a_n) = a_{n+3}$

for any +ve integer k.

$\boxed{E^k(a_n) = a_{n+k}}$ (for $k \geq 1$)

Method of characteristic roots :

Consider the linear recurrence relation

lambda — $\lambda_0 a_n + \lambda_1 a_{n-1} + \dots + \lambda_k a_{n-k} = f(n)$. — (1)

Replacing 'n' with 'n+k', we have,

$\Rightarrow \lambda_0 a_{n+k} + \lambda_1 a_{n+k-1} + \dots + \lambda_k a_n = f(n+k)$

$\Rightarrow \lambda_0 E^k a_n + \lambda_1 E^{k-1}(a_n) + \dots + \lambda_k a_n = f(n)$

(k is a const.)

$\Rightarrow (\lambda_0 E^k + \lambda_1 E^{k-1} + \dots + \lambda_k) a_n = f(n)$

$\Rightarrow \phi(E) a_n = f(n)$ — (2)

The characteristic eq^n is

$\boxed{\phi(t)=0}$

The roots of this eq^n are called "characteristic roots".

Let $t=t_1, t_2, \ldots, t_k$ be the characteristic roots.

Complementary function: (C.F)

This is solution of eq^n ① when $f(n)=0$, i.e. the solution of homogeneous part of eq^n ①.

Rules for finding complementary Func^n (C.F):

characteristic roots	Complementary Func^n
1) Roots are real and distinct	$c_1 \cdot t_1^n + c_2 \cdot t_2^n + \cdots + c_k \cdot t_k^n$
2) Roots are real and 2 roots are equal. say, $t_1, t_1, t_3, \ldots, t_k$	$(c_1 + c_2 \cdot n) t_1^n + c_3 t_3^n + \cdots + c_k \cdot t_k^n$
3) Roots are real and 3 roots are equal. say, $t_1, t_1, t_1, t_4, \ldots, t_k$	$(c_1 + c_2 \cdot n + c_3 n^2) t_1^n + c_4 \cdot t_4^n + \cdots + c_k \cdot t_k^n$

4) If all the roots are Same: $(c_1 + c_2 n + c_3 n^2 + \dots + c_k n^{k-1})$

5) A pair of roots are complex say $(\alpha \pm i\beta)$ $r^n\{c_1 \cos(n\theta) + c_2 \sin(n\theta)\}$

where $r = \sqrt{\alpha^2+\beta^2}$

and $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$

Particular solution ? (P.S.) :-

From eq^n ①.

$$\text{Particular sol}^n \text{ (P.S.)} = \frac{1}{\phi(E)}\{f(n)\}$$

complete of eq^n ①
The sol^n is

$$a_n = \underset{(C.F.)}{\text{complementary func}^n} + \underset{(P.S.)}{\text{particular sol}^n}$$

Rule to find Particular Solution (P.S.) :-

When R.H.S. of eq^n ② i.e. $f(n)$

$f(n) = b^n$ where b is constant

then particular sol^n is $\boxed{\frac{b^n}{\phi(E)}}$

$\Rightarrow \boxed{\frac{b^n}{\phi(b)}}$ $(\because \phi(b) \neq 0)$

$a_n = 2 \cdot a_{n-1} - 1$
$a_1 = 2 \cdot a_0 - 1$ ①

Case of failure i.e. when $\phi(b) = 0$,

$$\frac{b^n}{(E-b)^k} = C(n,k) \cdot b^{n-k} \qquad [\because k = 1,2,3,\ldots]$$

Q.1. The solution of the recurrence relation $a_n = 2 \cdot a_{n-1} - 1$ where $a_1 = 2$ is what ?

→ Replacing 'n' with 'n+1',

$\Rightarrow a_{n+1} - 2 \cdot a_n = -1$

$\Rightarrow E(a_n) - 2a_n = -1$

$\Rightarrow (E-2) \cdot a_n = -1$ ——— ①

The characteristic eq^n is

$t - 2 = 0 \qquad \boxed{\therefore t = 2}$

For $t = 2$, the complementary func^n is

$= \boxed{c_1 \cdot 2^n} = C.F.$

From eq^n ①, P.S. is

$$\left(\frac{-1}{E-2}\right) = -\left(\frac{1^n}{E-2}\right)$$

$\Rightarrow -\left(\frac{1^n}{1-2}\right)$

$$\left(\frac{b^n}{\phi(E)} = \frac{b^n}{\phi(b)}\right)$$

$\Rightarrow \boxed{1}$ is P.S.

$\therefore$ the solution is

$a_n =$ CF + PS

$a_n = c_1 \cdot 2^n + 1$ — ②
for $n=1$
$\therefore 2 = 2 \cdot c_1 + 1$

$\therefore c_1 = 1/2$

Substitute c_1 in eq^n ②, we have

$a_n = 1/2 \cdot 2^n + 1$

$a_n = 2^{n-1} + 1$

Q.2. The solution of the recurrence relation

$T(2^k) = 3T(2^{k-1}) + 1$ where $T(1) = 1$

is what?

$\rightarrow$ Let $T(2^k) = a_k$.

$\Rightarrow a_k = 3 \cdot a_{k-1} + 1$

Replace 'K' with 'K+1'

$\Rightarrow a_{K+1} = 3 \cdot a_K + 1$

$\Rightarrow E(a_K) - 3a_K = 1$

$\Rightarrow \therefore (E-3) \cdot a_K = 1$ —— ①

Now, the characteristic eqn is

$(t-3) = 0 \quad \therefore t = 3.$

$\therefore$ The complementary function is

$\Rightarrow c_1 \cdot 3^K$

from ①, particular soln is

$\frac{1}{E-3} \Rightarrow \frac{(1)^K}{(E-3)} \Rightarrow \frac{(1)^K}{1-3}$

replace E by 1

$\therefore$ p.s. $\Rightarrow -\left(\frac{1^K}{2}\right) = -1/2.$

$\therefore$ The solution is

$a_K = T(2^K) = c_1 \cdot 3^K - 1/2$ —— ②.

$a_0 = 1 = c_1 - 1/2.$

$\therefore c_1 = 3/2$

$\therefore T(2^K) = \left(\frac{3^{K+1} - 1}{2}\right)$

Q.3. The solution of the recurrence relation

$a_n - 2 \cdot a_{n-1} = 2^n$ where $a_0 = 1$.

is what ?

$\Rightarrow$ put $n = n+1$

$\therefore a_{n+1} - 2a_n = 2^{n+1}$

$\Rightarrow E(a_n) - 2a_n = 2^{n+1}$

$\Rightarrow (E-2)\, a_n = 2(2^n)$ — ①

characteristic eqn is

$(t-2) = 0 \quad \therefore t = 2$

$\therefore$ the complementory funcn is

$c_1 \cdot 2^n$

From ①, P.S. is

$$\frac{2(2^n)}{E-2} = 2\,C(n,1)\,2^{n-1} = n \cdot 2^n$$

cant replace E by 2 here.

$$\left(\frac{b^n}{(E-b)^k} = C(n,k) \cdot b^{n-k} \right)$$

$a_n = n!$

$\therefore$ The solution is

$a_n = CF + PS$

$= C_1 2^n + n \cdot 2^n \quad — (2)$

$a_0 = 1$ given $\quad \therefore 1 = C_1 \cdot 1 + 0 \cdot 1$

$1 = C_1 + 0 \quad \therefore C_1 = 1 \quad \boxed{C_1 = 1}$

$a_n = (1-n) \cdot 2^n + n \cdot 2^n$

$= (1-n+n) \cdot 2^n$

$= \boxed{2^n}$

$\therefore a_n = 2^n(n+1)$.

Q.4. The solution of the rec. relat^n

$a_n - 2a_{n-1} + a_{n-2} = 0$ where

$a_0 = 1$ and $a_1 = 2$ is what?

$\rightarrow a_{n+2} - 2a_{n+1} + a_n = 0$

$\Rightarrow (E^2 - 2E + 1) a_n = 0$.

The char. eq^n is

$t^2 - 2t + 1 = 0 \quad t = 1, 1$.

The solution is

$$a_n = (c_1 + c_2 n) \cdot t_1^n \quad -(1).$$

$$= (c_1 + c_2 n) \cdot 1^n$$

put $n = 0$ in eq^n (1),

$$a_0 = 1 = c_1.$$

$n = 1$ $\quad a_1 = 2 = 1 + c_2$

$$c_2 = 1.$$

$$\therefore \boxed{a_n = (1+n).}$$

Q.15. The soln of the rec relatn

$$a_n - 7a_{n-1} + 12 \cdot a_{n-2} = 0.$$

$$\Rightarrow a_{n+2} - 7a_{n+1} + 12 a_n = 0.$$

$$\Rightarrow (E^2 - 7E + 12) a_n = 0$$

The char. eq^n is

$$t^2 - 7t + 12 = 0.$$

$$t = 3, 4.$$

The soln is $a_n = c_1 3^n + c_2 \cdot 4^n$

$a_n = 7a_{n-1} - 12a_{n-2}$ $\quad a_0 = 2$, $a_1 = 5$

$a_2 = 35 - 24 = 11$

$a_3 = 77 - 60 = 17$

$a_4 =$

$n = 0, \Rightarrow 2 = c_1 + c_2$

$n = 1 \Rightarrow 5 = 3c_1 + 4c_2$

$c_1 = 3 \quad c_2 = -1$

$$\therefore \boxed{a_n = 3^{n+1} - 4^n}$$

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Q.6 The solution of the recurrence relation

$a_n - a_{n-1} + a_{n-2} = 0$ where $a_0 = a_1 = 1$ is ?

→ The characteristic eq^n (directly)

$t^2 - t + 1 = 0$

$\therefore t = \dfrac{1 \pm \sqrt{3}\, i}{2} \Big\}$ complex $= 1/2 \pm \frac{\sqrt{3}}{2} i$

$r = \sqrt{\alpha^2 + \beta^2} = 1$

$\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right) = \pi/3.$

$\therefore a_n = c_1 \cdot \cos\left(\frac{n\pi}{3}\right) + c_2 \cdot \sin\left(\frac{n\pi}{3}\right)$ — ①.

$n = 0 \Rightarrow 1 = c_1$

$n = 1 \Rightarrow 1 = c_1/2 + c_2 \cdot \sqrt{3}/2$

$c_2 = \frac{1}{\sqrt{3}}$

$\therefore a_n = \cos(n\pi/3) + \frac{1}{\sqrt{3}} \sin(n\pi/3)$.

Q.7 The solution of the rec relation

$a_n - 3 \cdot a_{n-1} + 2 \cdot a_{n-2} = 2^n$ is ?

→ put $n = n+2$.

$\therefore a_{n+2} - 3a_{n+1} + 2a_n = 2^{n+2}$

$(E^2 - 3E + 2) a_n = 4(2^n)$ — ①

The characteristic eqn is

$t^2 - 3t + 2 = 0$.

$\therefore t = 2, t = 1$. (Roots are reals and distinct).

$\therefore$ complementary funcn is

$c_1 \cdot 1^n + c_2 \cdot 2^n$

$\Rightarrow c_1 + c_2 2^n$

from eqn ①

The particular soln is

$\frac{4 \cdot (2^n)}{E^2 - 3E + 2} \Rightarrow \frac{4 \cdot (2^n)}{(E-2)(E-1)}$

$$\Rightarrow 4 \cdot \frac{2^n}{(E-2)(E-1)}$$

$$\Rightarrow \frac{4}{(E-2)} \cdot \frac{(2^n)}{(E-1)} \Rightarrow \frac{4}{(E-2)} \cdot \frac{2^n}{(2-1)}$$

$$\Rightarrow \frac{4}{E-2} \cdot 2^n \Rightarrow \frac{4 \cdot 2^n}{E-2} \rightarrow \frac{4 \cdot 2^n}{(E-2)}$$

$$\Rightarrow 4 \cdot {}^nC_1 \cdot 2^{n-1} \quad \left(\because \frac{b^n}{(E-b)^k} = C(n,k) \cdot b^{n-k} \right)$$

$$= 2n \cdot 2^n$$

The solution is

$$\boxed{a_n = c_1 + c_2 \cdot 2^n + 2n \cdot 2^n}$$ (General soln).

Q.8. The soln of

$a_n - 6a_{n-1} + 9a_{n-2} = 8^n$ is ?

(no initial condns given).

$\rightarrow$

$$a_{n+2} - 6a_{n+1} + 9a_n = 8^{n+2}$$

$$\therefore (E^2 - 6E + 9) \cdot a_n = 8^{n+2}$$

$$\Rightarrow (E^2 - 6E + 9) \cdot a_n = 8^n (64) \quad - (1)$$

The characteristic eqn is

$$t^2 - 6t + 9 = 0. \qquad t = 3.$$

∴ Complementary function is

$(c_1 + c_2 n) \cdot 3^n$

Particular soln is

$$\frac{9 (3^n)}{E^2 - 6E + 9}$$

$$\Rightarrow \frac{9 (3^n)}{(E-3)(E-3)} \Rightarrow \frac{9 \cdot (3^n)}{(E-3)^2}$$

Here, $\frac{b^n}{(E-b)^k} = C(n,k)\, b^{n-k}$

∴ $k = 2$ here

$$\therefore \Rightarrow 9 \cdot \frac{3^n}{(E-3)^2} \Rightarrow 9 \cdot {}^nC_2 \cdot 3^{n-2}$$

$$\Rightarrow \frac{n(n-1)}{2} \, 3^n$$

The soln is

$$\boxed{a_n = 3^n \left(c_1 + c_2 n + \frac{n(n-1)}{2} \right).}$$

(**) Recurrence relations reduceable to linear form by substitution.

Q.9. The soln of the rec relatn

$a_n^2 - 2 \cdot a_{n-1}^2 = 1$ where $a_0 = 2$ is what?

→ Let $a_n^2 = x_n$, then

$x_n - 2x_{n-1} = 1$

(now same as prev.)

put $n = n+1$

$\therefore x_{n+1} - 2x_n = 1$

$\therefore E(x_n) - 2x_n = 1$

$\therefore (E-2) \cdot x_n = 1$ — ①

∴ Complementary funcn is

$c_1 \cdot 2^n$

Particular soln is

$\frac{1}{E-2} \Rightarrow \frac{1^n}{E-2} \Rightarrow \frac{1^n}{1-2}$

$\Rightarrow -1$

The solution is

$x_n = a_n^2 = c_1 \cdot 2^n - 1$

Given $a_0 = 2$

$\therefore$ $4 = c_1 \cdot 1 - 1$

$\therefore$ $c_1 = 5$

$\therefore$ $\boxed{a_n^2 = 5 \cdot 2^n - 1}$

$\Rightarrow$ $\boxed{a_n = \sqrt{5 \cdot 2^n - 1}}$

Q.10. The solution of the recurrence relation

$\sqrt{a_n} - \sqrt{a_{n-1}} - 2 \cdot \sqrt{a_{n-2}} = 0$ ← because of this, complementary funcn

where $a_0 = a_1 = 1$ is what? is the soln.

→ Let $x_n^2 = a_n$.

$\therefore$ $x_n - x_{n-1} - 2 \cdot x_{n-2} = 0$

$\therefore$ put $n = n+2$

$\therefore$ $x_{n+2} - x_{n+1} - 2 \cdot x_n = 0$

$\therefore$ $(E^2 - E - 2) \cdot x_n = 0$

$\therefore$ The characteristic eqn is

$t^2 - t - 2 = 0$

$\therefore$ $t^2 - 2t + t - 2 = 0$

$\therefore$ $t(t-2) + 1(t-2) = 0$

$\therefore \quad t = 2, -1.$

$\therefore$ The complementary func^n is

$c_1 \cdot 2^n + c_2 (-1)^n$

$\therefore$ The sol^n is

$x_n = c_1 \cdot 2^n + c_2 \cdot (-1)^n$

put $n=0 \Rightarrow 1 = c_1 + c_2$

put $n=1 \Rightarrow 1 = 2c_1 - c_2$

$\therefore c_1 = 2/3, \quad c_2 = 1/3.$

$\therefore \quad x_n = \sqrt{a_n} = 2/3 \cdot 2^n + 1/3 \cdot (-1)^n.$

$$\boxed{\therefore a_n = \left(\frac{2^{n+1} + (-1)^n}{3}\right)^2}$$

Divide-And-Conquer Relations :

A rec relat^n of the form

$$\boxed{T(n) = c \cdot T\left(\frac{n}{d}\right) + f(n)}$$

is called "Divide-And-Conquer Relation."

where 'c' and 'd' are constants.

This rec relatⁿ can be reduced to linear form by substituting

$n = 2^k$.

Q.11 The solⁿ of the rec. relatⁿ

$T(n) = 2 \cdot T(n/2) + (n-1)$ where $T(1) = 0$ is ?

→ put $n = 2^k$

So. $T(2^k) = 2 \cdot T(2^{k-1}) + (2^k - 1)$

Let $T(2^k) = a_k$.

$\therefore a_k - 2a_{k-1} = 2^k - 1$

put $k = k+1$

$\therefore a_{k+1} - 2a_k = 2^{k+1} - 1$

$\therefore (E-2) \cdot a_k = 2^{k+1} - 1$

$\therefore (E-2) \cdot a_k = 2 \cdot (2^k) - 1$ — ①

∴ The characteristic eqⁿ is

$t - 2 = 0 \quad \therefore t = 2$.

∴ The complementary funcⁿ is

$c_1 \cdot 2^k$

The particular sol^n is

$$2 \cdot \frac{2^K}{E-2} - \frac{1}{E-2}$$

$\Rightarrow$ DSR $\cdot b$

$$\Rightarrow 2 \cdot K \cdot 2^{K-1} - \left(\frac{1^K}{1-2}\right) \Rightarrow K \cdot 2^K + 1$$

The soln is

$$a_K = C_1 \cdot 2^K + K \cdot 2^K + 1 \quad \text{— (2)}$$

put $K=0$, $\therefore a_0 = 0 = C_1 + 1$

$$\therefore C_1 = -1$$

$$\therefore T(2^K) = 2^K (K-1) + 1.$$

$\therefore$ replace 2^K with n.

$$\boxed{T(n) = n \cdot (\log_2 n - 1) + 1}$$

Q.12. The solution of the rec. relat^n

[illegible]

$T(n) = 7T(n/3) + 2n$ where, $T(1) = 5/2$ is ?

$\therefore$ put $n = 3^k$.

$\therefore T(3^k) = 7 \cdot T(3^{k-1}) + 2 \cdot 3^k$.

$\therefore$ put $T(3^k) = a_k$.

$\therefore a_k - 7 \cdot a_{k-1} = 2 \cdot 3^k$.

put $k = k+1$

$\therefore a_{k+1} - 7 a_k = 2 \cdot 3^{k+1}$

$\therefore (E - 7) \cdot a_k = 2 \cdot 3^{k+1}$

$\therefore$ Characteristic eqn

$E = 7$.

$\therefore$ Complementary funcn is

$C_1 \cdot 7^k$.

Particular soln is

$$\frac{2 \cdot 3^{k+1}}{E - 7} = \frac{6 \cdot 3^k}{(E - 7)}$$

$$= \frac{6 \cdot 3^k}{-4} = -\frac{3 \cdot 3^k}{2}$$

$$= -\frac{3^{k+1}}{2}$$

Master's theorem →

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$

where $f(n) = \theta(n^k)$.

1) if $a < b^k$ then $T(n) = \theta(n^k)$
2) if $a = b^k$ then $T(n) = \theta(n^k \log n)$
3) if $a > b^k$ then $T(n) = \theta(n^{\log_b a})$

Q.13. The solution of

$T(n) = 2T(n/2) + (n-1)$ is ?

a) $\theta(n)$ ✓b) $\theta(n \log n)$ c) $\theta(n^2)$ d) $\theta(\log n)$.

→ $a = 2$, $b = 2$, $k = 1$

$f(n) = n - 1$ it is of order 'n^k' where $k = 1$

we have,

$a = b^k$

By Master's theorem, the sol^n is $n^k \log n$

$\therefore \boxed{\theta(n \log n)}$

$*3/ \left(n^{\log_3 4} = 7^{\log_3 n} \right)$

Q-14. The sol^n of the rec relat^n

$T(n) = 7 \cdot T(n/8) + 8n$ is ?

$a = 7$, $b = 8$, and $k = 1$.

we have $a > b^k$

$\therefore$ By master's theorem,

$T(n) = \Theta\, \cancel{n \log} \left(n^{(\log_8 7)} \right).$

Q-15. The sol^n of the rec relat^n

$T(n) = 6 \cdot T(n/3) + 3 \cdot n^4$ is ?

$\rightarrow a = 6,\ b = 3,\ k = 4.$

$a < b^k.$

$\therefore \boxed{T(n) = \Theta(n^4).}$

Method of undetermined coefficients :

Consider the linear rec relat^n

$\phi(E) \cdot a_n = f(n)$

The characteristic eq^n is

$\phi(t) = 0.$

* multiplicity → no. of times 1 appears as char. root

So, we get characteristic roots

$t =$

The complementory $func^n$ → same.

@Rules to find particular $soln$ →

$f(n)$	$\phi(t)$	Particular $soln$
(i) n^k ($k = 1, 2, 3, \ldots$) (ii) any polynomial funcn of degree 'k'	$\phi(1) \neq 0$ i.e. 1 is not a characteristic root	$A_0 \cdot n^k + A_1 n^{k-1} + \cdots + A_k$ where $A_0, A_1, \ldots, A_k$ are undetermined coefficients.
(iii) same $cond^n$ as above	$\phi(1) = 0$ i.e. 1 is a characteristic root with multiplicity 'm'	$(A_0 \cdot n^k + A_1 n^{k-1} + A_k) \cdot n^m$
(i) $b^n \cdot n^k$ ($k = 1, 2, 3, \ldots$)	$\phi(b) \neq 0$ i.e. 'b' is not a characteristic root	$b^n (A_0 \cdot n^k + A_1 \cdot n^{k-1} + \cdots + A_k)$
(ii) same condn as above	$\phi(b) = 0$ i.e. 'b' is a char. root with multiplicity 'm'.	$b^n \cdot n^m (A_0 \cdot n^k + A_1 \cdot n^{k-1} + \cdots + A_k)$

Q.16. The solution of

$a_n - 2a_{n-1} = (n+2)$ is ?

→ The characteristic eqn is

$(t-2) = 0 \quad \therefore t = 2$

The complementary funcn is

$c_1 \cdot 2^n$

Let particular soln is

$a_n = $ P.S. $= (cn + d)$ —① where c and d are undetermined coefficients.

Substituting in given rec relatn, we have,

$(cn+d) - 2\{c(n-1)+d\} = n+2$

equating the coeff of 'n' ⇒

$-c = 1 \Rightarrow \underline{c = -1}.$

equating constants ⇒

$2c - d = 2.$

$d = 2c - 2 = \underline{-4} = -d.$

∴ particular sol^n is $-n \cdot 4$.

The sol^n is

$$\boxed{a_n = c_1 \cdot 2^n \cdot n - 4}$$

* 2nd order ... rec. relatⁿ

Q.17 The solution of the rec relatⁿ

$a_n - 2a_{n-1} + a_{n-2} = (3n+5)$ is ?

→ The char. eqⁿ is

$t^2 - 2t + 1 = 0.$

∴ $t = 1, 1$

∴ complementary funcⁿ is

$(c_1 + c_2 n) \cdot 1^n$

By rule 2, let particular sol^n

$P.S. = (cn+d) \cdot n^2$
$(n^m \because m=2)$

$= (cn^3 + dn^2)$

substituting in the given rec relatⁿ, we have,

$(cn^3 + dn^2) - 2(c(n-1)^3 + d(n-1)^2) + c(n-2)^3 + d(n-2)^2$

$= 3n+5$

put $n=1$, $c+d-c+d = 3(1)+5$

$$2d = 8$$

$$\boxed{d=4}$$

$n=2$, $8c+(-2c) = 6c+2d = 11$
$+4d-2d$

$$\boxed{c = 1/2}$$

$\therefore$ P.S. is

$$P.S. = 1/2\, n^3 + 4n^2$$

The solⁿ is

$$= (c_1 + c_2 n) + 4n^2 + \frac{n^3}{2}$$

Q.18. The solⁿ of the rec. relatⁿ

$$a_n - 2a_{n-1} = 3^n \cdot (n+2)$$

Ans.

→ The characteristic eqⁿ is

$$t-2=0 \qquad \therefore t=2.$$

Complementary funcⁿ is

C.F. $c_1 \cdot 2^n$

By rule 3.14
Particular solⁿ $P.S. = 3^n (An+B)$ ... ($\because k=1$)

or

$$a_n = P.S. = 3^n (cn+d). \quad \text{---} \; (1)$$

Substituting in the given rec. relatn

$3^n c(n+d) - 2 \cdot 3^{n-1}\{c \cdot (n-1) + d\}$

$= 3^n \cdot (n+2)$

Dividing the eqn by 3^n,

$(n+d) - 2/3 \cdot \{c \cdot (n-1) + d\} = (n+2)$

comparing coeff of n on both sides,

$c - 2/3 c = 1$

$\therefore 1/3 c = 1 \quad \boxed{c = 3}$

equating the constants.

$\frac{2c}{3} + d/3 = 2.$

$\therefore \boxed{d = 0.}$

$\therefore$ P.S $= 3^n \cdot 3n = n \cdot 3^{(n+1)}$

$\therefore$ The soln is

$\boxed{a_n = c_1 \cdot 2^n + n \cdot 3^{(n+1)}}$

Q-13. The solⁿ of the recurrence relatⁿ (an -

i) $a_n - 2 \cdot a_{n-1} = \underline{2^n \cdot n}$
$b^n \cdot n^k$ $(b = 2, k = 1)$.

→ characteristic eqⁿ is

$t - 2 = 0$ $\therefore t = 2$

Complementary funcⁿ is

$c_1 \cdot 2^n$

By rule 4, let particular solⁿ

$P.S. = 2^n (cn + d) \cdot n^1$ (by rule 4)

$P.S. = 2^n (cn^2 + dn)$. — ①

Substituting in the given rec relatⁿ,

$\underbrace{2^n \cdot (cn^2 + dn)}_{a_n} - \underbrace{2 \cdot \{2^{n-1} (c \cdot (n-1)^2 + d(n-1))\}}_{a_{n-1}}$
$= 2^n \cdot n$.

Div the eqⁿ by 2^n,

$cn^2 + dn - \{c \cdot (n-1)^2 + d \cdot (n-1)\} = n$

put $n = 1$. → $c + d = 1$

put $n = 0$ → $-c + d = 0$.

$\therefore c = d = 1/2$

$\therefore$ P.S. is

$$P.S. = \frac{2^n(n^2+n)}{2} = 2^{n-1}.(n^2+n).$$

The sol^n is

$$\boxed{a_n = c_1 2^n + 2^{n-1}.(n^2+n)}$$

Generating Functions

Let $\{a_0, a_1, a_2, \ldots, a_n, \ldots\}$ be a sequence of real nos., then a function $f(x)$ defined by

$$f(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \cdots + a_n \cdot x^n + \cdots$$

is called "Generating function" of the sequence.

If the sequence contains infinitely many terms, then

$$f(x) = \sum_{n=0}^{\infty} (a_n \cdot x^n)$$

Q.1. For the sequence

$C_0, C_1, C_2, C_3, \ldots, C_n$ where $C_k = C(n,k)$, the generating func^n is ?

→ $f(x) = C_0 + C_1 \cdot x + C_2 \cdot x^2 + C_3 \cdot x^3 + \cdots + C_n \cdot x^n$

Generating function

= Binomial expansion of $(1+x)^n$

$$\boxed{f(x) = (1+x)^n}$$

Q.2. For the sequence

$C_0, -C_1, C_2, -C_3, \ldots, (-1)^n \cdot C_n$

→ The Generating function is

$f(x) = C_0 - C_1 \cdot x + C_2 \cdot x^2 - C_3 \cdot x^3 + \cdots + (-1)^n \cdot C_n \cdot x^n$

- Binomial expansion of $(1-x)^n$

$$\boxed{f(x) = (1-x)^n}$$

Q.5. For the sequence.

$\{1,1,1,\cdots,1\}$ (n terms)

Gen. func^n = ?

→ The Generating Func^n is

$f(x) = 1 + x + x^2 + \cdots + x^{n-1}$

$$\boxed{f(x) = \left(\frac{1-x^n}{1-x}\right)}$$

If $n = \infty$,

$f(x) = 1 + x + x^2 + \cdots\cdots$

- Binomial expansion of $(1-x)^{-1}$

$$\boxed{f(x) = 1/(1-x)}$$

Q.6. " – "

$\{1, -1, 1, -1, \cdots, (-1)^n, \cdots \infty\}$

→ The generating func^n is

$f(x) = 1 - x + x^2 - x^3 + \cdots + \infty = (1+x)^{-1}$

$$\boxed{f(x) = \frac{1}{1+x}}$$

Note: $(1-ax)^{-k} = \sum_{n=0}^{\infty} \{c(n-1+k, n) \cdot a^n \cdot x^n\}$

where $(k = 1, 2, 3, \dots)$

and a is any constant

1) $(1-x)^{-1} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots \infty$

$a = 1,\ k = 1$

2) $(1+x)^{-1} = \sum_{n=0}^{\infty} (-1)^n \cdot x^n$

$a = -1,\ k = 1$

$= 1 - x + x^2 - x^3 + \dots \infty$

3) $(1-ax)^{-1} = \sum_{n=0}^{\infty} (a^n \cdot x^n)$

$k = 1$

$= 1 + ax + a^2 \cdot x^2 + a^3 x^3 + \dots + a^n x^n + \dots \infty$

4) $(1-x)^{-2} = \sum_{n=0}^{\infty} (n+1) \cdot x^n$

$a = 1,\ k = 2$

$= 1 + 2x + 3x^2 + \dots + (n+1)x^n + \dots + \infty$

5) $(1-x)^{-3} = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n$

$a = 1,\ k = 3$

$= 1+3x+6x^2+10x^3+\dots \infty$

6) $(1-x)^{-4} = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)(n+3)}{6} x^n$

Q.5. The generating funcn of the sequence $\{1,1,3,1,1,1,1,\dots\infty\}$ is what?

→ Gn funcn $f(x) =$
$(1+x+3x^2+x^3+\dots+\infty)$

$= (1+x+x^2+x^3+\dots\infty) + 2x^2$

$= (1-x)^{-1} + 2x^2$

$$\boxed{f(x) = \frac{1+2x^2-2x^3}{1-x}}$$

Q.6. The generating funcn of the sequence $\{1,2,3,4,(n+1),\dots\infty\}$.

$= 1+2x+3x^2+4x^3+\dots\infty$

$= (1-x)^{-2}$

Q.7. The generating funcn of the sequence $\{0,1,2,3,\dots,n,\dots\infty\}$.

$= 0+x+2x^2+3x^3+\dots\infty$

$= x(1+2x+3x^2+4x^3+\dots\infty)$ Binomial expansion of $(1-x)^{-2}$ ← prev.

$$= x \cdot (1-x)^{-2}$$

Q.8. The generating function seq. of

$$\{0^2, 1^2, 2^2, 3^2, 4^2, \dots, n^2, \dots \infty\} \text{ is}$$

$$f_1(x) = \frac{x}{(1-x)^2} = 0 + 1 \cdot x + 2 \cdot x^2 + 3 \cdot x^3 + 4 \cdot x^4 \dots \infty$$

we know,

$$x \frac{d(f_1(x))}{dx} = \frac{x^2+x}{(1-x)^3} = x\{0^2 \cdot 1 + 4x + 9x^2 \}$$

$$= 0^2 + 1^2 \cdot x + 2^2 \cdot x^2 + 3^2 \cdot x^3 + \dots \infty$$

$\therefore$ Required Generating Function $= \dfrac{x^2+x}{(1-x)^3}$

Q.9. The Generating functⁿ of the sequence

$$\{0^3, 1^3, 2^3, \dots n^3, \dots \infty\} \text{ is ?}$$

$\therefore$ Required G.F. $= x \cdot \dfrac{d}{dx} \{f_2(x)\}$

$$= x \cdot \frac{d}{dx} \left\{ \frac{x^2+x}{(1-x)^3} \right\}$$

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Q.10. The generating funcn of the sequence

$\{a_0, a_1, a_2, \ldots\ldots, a_n, \ldots \infty\}$ where $a_n = (n+1).(n+2)$

is what?

$\rightarrow$ $f(x) = 2.\sum\limits_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} . x^n$

$= 2.(1-x)^{-3} = \boxed{\frac{2}{(1-x)^3}}$

Q.11. " — "

$\{a_0, a_1, a_2, \ldots, a_n, \ldots \infty\}$ where $a_n = n.(n+1)$ is ?

$\rightarrow$ $f(x) = \sum\limits_{n=0}^{\infty} n.(n+1)\ x^n$

$= x.\sum\limits_{n=0}^{\infty} n(n+1).x^{n-1}$

Replace 'n' with 'n+1'.

$\therefore f(x) = 2.x.\sum\limits_{n=0}^{\infty} \frac{(n+1)(n+2)}{2}.x^n$

$= 2x\ (1-x)^{-3} = \boxed{\frac{2x}{(1-x)^3}}$

Q.12. Generating funcn of the sequence

$\{a_0, a_1, a_2, \ldots, a_n, \ldots \infty\}$

where, $a_n = (n+1)(n+2)(n+3)$ is what ?

$\rightarrow f(x) = 6 \sum_{n=0}^{\infty} \frac{(n+1)(n+2)(n+3)}{6} \cdot x^n$

$= 6 \cdot (1-x)^{-4} = \frac{6}{(1-x)^4}$

$\boxed{f(x) = \frac{6}{(1-x)^4}}$

Q.13. Generating funcn of the sequence

$\{1, -2, 4, -8, \ldots, (-2)^n, \ldots, \infty\}$ is ?

$\rightarrow f(x) = \sum_{n=0}^{\infty} (-2)^n \cdot x^n$

$= (1+2x)^{-1}$

$\boxed{f(x) = \frac{1}{1+2x}}$

Q.14. The coefficient of x^{20} in the expansion of

$(x^3 + x^4 + x^5 + \ldots \infty)^5$ is ?

→ We know,

$$(1-ax)^{-k} = \sum_{n=0}^{\infty} C(n-1+k, n) \cdot x^n$$

Given,

$$\Rightarrow x^{15} \cdot (1 + x^{2} + x^{3} + \cdots \infty)^5$$

$$\Rightarrow x^{15} (1-x)^{-5}$$

$$\Rightarrow x^{15} \sum_{n=0}^{\infty} C(n+4, n) \cdot x^n$$

The coefficient of x^{20} = ~~[illegible]~~ $C(n+4, n)$

taking $n=5$

$= C(9,5)$

$= \boxed{126}$.

To find no. of nonnegative integer solutions to the equation

$$x_1 + x_2 + x_3 + \cdots + x_n = k$$

let the generating soln be
$f(x) = f_1(x) \cdot f_2(x) \cdot f_3(x) \cdots f_n(x)$

where $f_i(x) = 1 + x + x^2 + \cdots \infty \quad (i = 1, 2, \cdots, n)$

$= (1-x)^{-1}$

The coefficient of 'x^k' in the expansion of $f(x)$ is the answer to our problem

$$\boxed{(1-x)^{-n} = \sum_{n=0}^{\infty} \frac{C(n-1+k,k)\ x^k}{V(n,k)}}$$

$\therefore$ coefficient of x^k in the expansion of $f(x) =$

$\boxed{V(n,k)}$.

In the above example, if we have constraints on the variables $\textcircled{2} \le x_i \le \textcircled{6}$, then we have to choose

$f(x) = x^{2} + x^3 + x^4 + x^5 + x^{6}$

if suppose, $1 \le x_i \le 5$

then $f_1(x) = x^{1} + x^2 + x^3 + x^4 + x^{5}$

Q.15. No. of non-negative integer solutions to the equation

$x_1 + x_2 + x_3 + \cdots + x_5 = 15$

where $1 \le x_1 \le 5$, $2 \le x_2 \le 5$, $x_3 \ge 2$,

$x_4 \ge 2$, $x_5 \ge 2$.

$\rightarrow$ The generating funcn for the problem is

$f(x) = (x + x^2 + x^3 + x^4 + x^5) \cdot (x^2 + x^3 + x^4 + x^5) \cdot$

$(x^2 + x^3 + x^4 + \cdots \infty)^3$

$$= x\cdot(1+x+x^2+x^3+x^4)\cdot x^2(1+x+x^2+x^3+x^4)$$

$$(x^2)^3\cdot(1+x+x^2+x^3+\dots\infty)^3$$

$$= x^9\cdot\left(\frac{1-x^5}{1-x}\right)^2\cdot(1-x)^{-3}$$

$$= x^9\cdot(1-2x^5+x^{10})(1-x)^{-5}$$

$$= (x^9-2x^{14}+x^{19})\cdot\sum_{n=0}^{\infty} C(n+4,n)\cdot x^n$$

$$\left(\because (1-x)^{-k} = \sum_{k=0}^{\infty} \frac{C(n-1+k,n)\cdot x^n}{U(n,k)}\right)$$

Required no. of nonneg integer solutions to the given problem i.e. coeff of x^{15} in the expansion of $f(x)$.

$$= C(10,6) - 2C(5,1)$$

$$= \boxed{200}$$

Q.16. How many ways we can choose a commitee of 9 members from 3 political parties so that no party has absolute majority in the comittee.

→ Required no. of ways =

no. of non negative integer sol^ns to the eq^n

$$x_1+x_2+x_3 = 9$$

where $1 \le x_i \le 4$

where x_i = no. of representatives from ith party P_i

the generating func^n of this problem is

$f(x) = f_1(x) * f_2(x) * f_3(x)$

where $f_i(x) = x + x^2 + x^3 + x^4$

$\therefore f(x) = (x + x^2 + x^3 + x^4)^3$

$= x^3 (1 + x + x^2 + x^3)^3$

$= x^3 \cdot \left(\frac{1 - x^4}{1 - x}\right)^3$

$= x^3 \cdot (1 - 3x^4 + 3x^8 - x^{12}) \; (1 - x)^{-3}$

$= (x^3 - 3x^7 + 3x^{11} - x^{15}) \cdot \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n$

when x^3, $n = 6$
x^7, $n = 2$

Coeff of x^9 = The required no of ways

$= \frac{7 \times 8}{2} - \frac{3 \times 3 \times 4}{2}$

$= 28 - 18$

$= \boxed{10}$.

GRAPHS →

A graph G is defined as a pair of sets

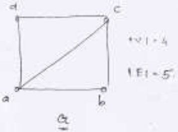
$G = (V, E)$

where V → set of all vertices/nodes/points.

& E → set of all edges in the graph.

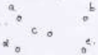
|V| → no. of vertices in G.
OR
Order of graph G.

|E| → no. of edges in the graph.
OR
Size of the graph G.



* NULL Graph *

A graph no edges is called a "Null graph"



* Trivial graph *

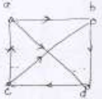
A null graph with only one vertex is called "trivial graph"

* Non directed Graph → (Undirected Graph)

In a non-directed graph, each edge is represented by a set of two vertices $\{V_i, V_j\}$

$\{V_i, V_j\}$ = an edge betn V_i and V_j.

* Directed Graph (Di-Graph) →

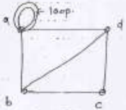


In a digraph, each edge is represented by an ordered pair of two vertices V_i and V_j

(V_i, V_j) = An edge from V_i to V_j

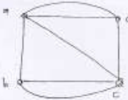
* Loop →

An edge drawn from a vertex to itself is called a "loop".



* Parallel edges →

In a graph, if a pair of vertices are allowed to join by more than one edge, then those edges are called "Parallel edges" and the resulting or corresponding graph is called a "multigraph".



* Simple Graph →

A graph with no loops and no parallel edges is called a "Simple Graph".

max. no of edges here = 6.

{a, b, c, d}. : 6

Max. no. of edges possible in a simple graph with 'n' vertices

$$= {}^{n}C_2 = \boxed{\frac{n(n-1)}{2}}$$

No of simple graphs possible with n vertices

$$= 2^{nC_2} = 2^{\left(\frac{n(n-1)}{2}\right)}$$

no. of edges.

Q.1. No. of simple graphs poss with 5 vertices & 4 edges ?

→ Max no. of edges possible with 5 vertices

$= 5C_2 = 10.$

No. of ways we can choose we can choose any 4 edges.

$= 10C_4 = \frac{10 \times 9 \times 8 \times 7}{4!} = 210.$

∴ Required no. of graphs = 210

Q.2. No. of simple graphs poss. with n vertices & m edges ?

→ With n vertices, max. no. of edges poss.

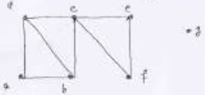
$$= nC_2 = \frac{n \cdot (n-1)}{2}$$

∴ we can select m edges from $\frac{n(n-1)}{2}$ edges in $\boxed{\frac{n \cdot (n-1)}{2} C_m}$ ways.

$$\boxed{C\left(\frac{n \cdot (n-1)}{2}, m\right)}$$

Connected Graph :

A graph 'G' is said to be "connected" if there exists a path betn every pair of vertices.



A graph which is not connected will have two or more connected components

(OR)

One or more connected components and an isolated vertex.

Degree of a vertex (V) →

denoted by deg (v)

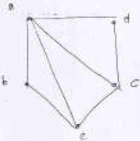
→ number of edges incident with the vertex V.

A vertex with degree '0' is called an 'isolated vertex'.

A vertex with degree '1' is called 'pendant vertex'.

In a simple graph with 'n' vertices, degree of any vertex V is less than or equal to (n-1)

i.e. $\deg(v) \le (n-1)$ (for all vertices).
$\forall v \in G$



In an undirected graph, a loop at a vertex is counted as 2 edges.

In a digraph:

Indegree of a vertex $v = \deg^+(v)$

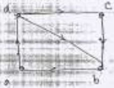
= no. of edges incident to the vertex

(or) no. of incoming edges.

Outdegree of a vertex $v = \deg^-(v)$

= no. of edges incident from the vertex

(or) no. of outgoing edges.



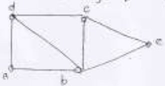
$\deg^+(a) = 1$

$\deg^-(a) = 1$

* Loops are not allowed in a "digraph"

⊛ Adjacency →

*In a graph, two vertices are said to be adjacent (neighbours) if there exists an edge betn the two vertices.



a and c are not adjacent

*In a graph, two edges are said to adjacent, if there exists a common vertex for the two edges.

⊛ Degree Sequence →

If the degrees of all the vertices in the graph G are arranged in ascending or descending order, then the sequence so obtained is called "degree sequence" of the graph.

{ 4, 3, 3, 2, 2 }

⊛ $\delta(G)$ → min of the degrees of all vertices in G

⊛ $\Delta(G)$ → max of the degrees of all vertices in G

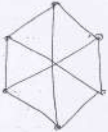
Regular Graph →

A graph is said to "Regular" if all the vertices have same degree.

In a graph,
"If degree of each vertex is 'k', then the graph is called "k-regular graph".

Ex:- A polygon is a 2-regular graph.

3-regular graph →



Complete Graph →

A simple graph with 'n' mutually adjacent vertices is called a complete graph and it is denoted by 'K_n'.

ex → K_n = complete graph with two mutually adjacent vertices

K_2

K_3

K_4

K_4

K_5

In a complete graph K_n, degree of each vertex is $(n-1)$.

(***) - every complete graph is a "Regular Graph".

No. of edges in K_n = $^nC_2 = \frac{n(n-1)}{2}$

Every complete graph is a simple graph with max no of edges.

* Cycle Graph →

A simple graph with 'n' vertices and n edges $(n \geq 3)$ is called a "cycle graph" if all the edges form a cycle of length 'n'. A cycle graph with n vertices is denoted by 'C_n'

(**)

C_3

→ the only cycle graph which is also complete is C_3.

C_4

C_5

C_6.

⊛ Wheel Graph:

A wheel graph with n vertices $(n \geq 4)$ can be obtained from a cycle graph C_{n-1} by adding a new vertex (called hub) which is adjacent to all vertices in C_{n-1}.

It is denoted by 'W_n'.

Ex.

$W_4 = K_4$

* The only wheel graph which is also complete graph.

W_5

W_6.

W_7.

In a wheel graph, degree of hub is $(n-1)$

No. of edges in W_n = $2(n-1)$

* Cyclic graph :

A graph with at least one cycle is called as 'cyclic graph'.

Every cycle graph and wheel graph and complete graph are cyclic graphs too.

* Acyclic graph :

A graph with no cycles is called acyclic graph.

* Tree :

A connected acyclic graph is called a 'tree'

(OR)

A connected graph with no cycles is called a 'tree'.

A tree with 'n' vertices has 'n-1' edges

Every tree has <u>at least two</u> vertices with degree '1'

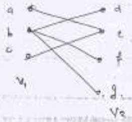
④ <u>Forest</u> :

A disconnected acyclic graph is called a '<u>forest</u>'

* <u>Bipartite Graph</u> :

A simple graph $G = (V,E)$ with vertex partition $V=\{V_1,V_2\}$ is called a bipartite graph if every vertex

A simple graph $G=(V,E)$ with vertex partition $V=\{V_1,V_2\}$ is called 'bipartition graph' if every edge of E, joins a vertex in V_1 to a vertex in V_2.



In a biparted graph, no 2 vertices in V_1 and V_2 are adjacent.

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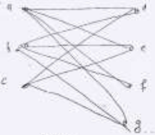
* Complete Bipartite graph →

A bipartite graph G with vertex partition, $V = \{V_1, V_2\}$ $(G = (V, E))$, is said to be a "complete bipartite graph" if every vertex in V_1 is adjacent to every vertex in V_2.



$K_{3,4}$.

In general, if $|V_1| = m$ and $|V_2| = n$, then a complete bipartite graph is denoted by "$K_{m,n}$".

$K_{m,n}$ has '$m+n$' vertices and 'mn' edges.

$K_{m,n}$ is a regular graph iff $m = n$

In general, a complete bipartite graph is not a complete graph.

Exception → $K_{1,1}$ is the only complete graph which is also complete bipartite graph.

Max. no. of edges possible in a complete bipartite graph with 'n' vertices is $\left\lfloor \frac{n^2}{4} \right\rfloor$.

ex-

if $n=10$, then max. no of edges

$$= \frac{100}{4} = 25$$

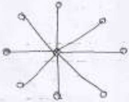
using $K_{5,5}$.

(**) G is a bipartite graph iff G has no cycles of odd length.

A special case of bipartite graph → "star graph"

* Star Graph →

A bipartite graph of the form $K_{1,n-1}$ is a star graph with n vertices. $(n \geq 2)$.

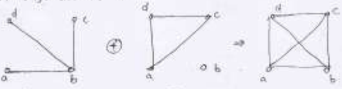


Complement of a graph :-

Let G be a simple graph with 'n' vertices.

Complement of G, denoted by $\bar{G}$ is also a simple graph with same vertices as that of G, but an edge $\{u,v\}$ is present in $\bar{G}$ iff the edge is absent in G i.e. $\{u,v\} \notin G$.

i.e. two vertices are adjacent in $\bar{G}$ iff they are not adjacent in G.



G $\quad\quad$ $\bar{G}$ $\quad\quad$ K_4

If G is any simple graph, then no. of edges in G + no. of edges in $\bar{G}$ is always equal to no. of edges in complete graph K_n, when $n = |V(G)|$.

Q.1. Let G be a simple graph with 9 vertices and 12 edges. Find no. of edges in $\bar{G}$.

→ we have,

$n(E(G)) + n(E(\bar{G})) = n(E(K_n))$

∴ no. of edges in $K_n = \frac{n \cdot (n-1)}{2} = \frac{9 \times 8}{2} = 36$

$n(n-1) = 156$ $\quad$ $n^2 - n + 156 = 0$

$n^2 - n = 178$ $\quad$ $-116-$

$\therefore$ no. of edges in $\bar{G} = 36 - 12 = \boxed{24}$

Q2. G is a simple graph with 40 edges and $\bar{G}$ has 38 edges. Find no. of vertices in graph.

→ let no. of vertices in graph = n.

we have, $|E(G)| + |E(\bar{G})| = |E(K_n)|$

$= 40 + 38 = 78.$

$\therefore \frac{n(n-1)}{2} = 78 \quad \therefore \frac{n(n-1)}{2} = 156$

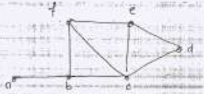
$n(n-1) = 13 \times 12$

$\therefore \boxed{n = 13}.$

Distance betⁿ two vertices u and v ?

$= d(u,v)$

* no. of edges in the shortest path betⁿ u and v.



(1) Eccentricity of a vertex $V \to e(V) \to$

$e(V) = \max. d(u,v) \quad \forall u \in G$

from prev. graph,

$e(a) = 3$
$e(b) = 2$
$e(c) = 2$
$e(d) = 3$
$e(e) = 3$
$e(f) = 2.$

(3)
2
3
3
3
2

(2) Radius of a connected graph $\to r(G)$.

$r(G) =$ min. of the eccentricities of all vertices

$\therefore$ For the graph previous,

$r(G) = 2.$

(3) Diameter of a connected graph $\to d(G) \to$

$d(G) =$ max of eccentricities of all vertices

$\therefore d(G) = 3.$

(4) Central point $\to$

If $e(v) = r(G)$ then v is called as a "central point of G"

For the previous graph, b, c and f are central points of G.

* **Centre of a graph** :

Set of all central points of a graph is called "Centre of the graph".

* **Circumference / perimeter of G** :

The no. of edges in a longest cycle of G is called circumference of G.

For the prev. graph, circumference is 5.

* **Girth of a graph G** :

No. of edges in a smallest cycle of G is called Girth of the graph G and is denoted by $g(G)$.

For the prev. graph, girth of the graph is 3.

Sum of degrees of vertices theorem :

If $G = (V, E)$ be a nondirected graph with vertices $V = \{v_1, v_2, \ldots, v_n\}$, then

$$\sum_{i=1}^{n} \deg(v_i) = 2 * \text{no. of edges}$$

$$= 2 * |E|.$$

* Corollary 1) :-
Let G be a directed graph with $V = \{V_1, V_2, \ldots, V_n\}$ then

$$\sum_{i=1}^{n} deg^{+}(V_i) = |E|$$

and

$$\sum_{i=1}^{n} deg^{-}(V_i) = |E|.$$

* Corollary 2) :-
In any non-directed graph, the no of vertices with odd degree is always even.

$$\sum_{i=1}^{n} deg(V_i) = 2 \times E$$

* because of this the sum of degrees of all vertices is always even.
So, no. of vertices with odd degree can not be odd in no.

* Corollary 3) :-
In a nondirected graph if degree of each vertex is 'k', then sum of degrees of all vertices is $k \cdot |V|$.

$$k \cdot |V| = 2 \cdot |E|$$

Corollary 4) →

In a nondirected graph, if degree of each vertex is atleast k ($\geq k$), then

$k \cdot |V| \leq 2 \cdot |E|$.

$\{ 3, 2, 3, 2 \}$
d b a c

∴ $k = 2$ here. ∴ $k \cdot |V| \leq 2 \cdot |E|$

∴ $2(4) \leq 10$.

* k is denoted by $\delta(G)$.

Corollary 5) →

In a nondirected graph, if degree of each vertex is atmost k ($\leq k$) then

$k \cdot |V| \geq 2 \cdot |E|$.

∴ In above graph, $k = \Delta(G) = 3$.

∴ $3(4) \geq 5 \times 2$. ⇒ $12 \geq 10$.

* Here, k is denoted by $\Delta(G)$.

* For any graph G,

$\delta(G) \cdot |V| \leq 2 \cdot |E|$ and —(cor 4)

$\Delta(G) \cdot |V| \geq 2 \cdot |E|$ —(cor 5)

$d(G)\cdot|V|$ $\quad \delta(G)|V| \le 2E$ $\quad = \frac{14}{5}$

Q.1. Let G be a simple nondirected graph with 5 vertices and 7 edges. Which of the foll. statements are true?

✓a) $\delta(G) \le 2$

b) $\delta(G) \ge 2$ $\qquad$ a) and d) both are correct.

c) $\Delta(G) \le 3$

✓d) $\Delta(G) \ge 3$

$\Rightarrow \delta(G)\cdot|V| \le 2\times 7$

$\therefore \delta(G) \le 14/5$

$\therefore \delta(G) \le 2.8$

$\boxed{\delta(G) \le 2}$

Also,

$\Delta(G)\cdot|V| \ge 7\times 2$

$\therefore \Delta(G) \ge 14/5 \Rightarrow \Delta(G) \ge 2.8$

$\boxed{\Delta(G) \ge 3}$

Q.2. Let G be a ~~simple~~ graph with 21 edges, 3 vertices with degree 4 and rem. vertices with degree 2. Find no. of vertices in the graph

$21 \times 2 = \frac{42}{12}$ $12 + 2x = 42$, $2x = 30$, $x = 15$ $E|V| = 2|E|$

$\rightarrow \sum_{i=1}^{n} deg(V_i) = 2|E|$

$\therefore 3 \times (4) + x(2) = 21 \times 2$

$\therefore 12 + 2x = 42 \Rightarrow \therefore 2x = 42 - 12$

$\Rightarrow \therefore 2x = 30 \quad \therefore x = 15$

$\therefore$ total no. of vertices $= 15 + 3 = \boxed{18}$.

Q-3 Let G be a simple graph with 38 edges and degree of each vertex is 4. Then $|V(G)| = ?$

$\rightarrow$ By cor 3, $k \cdot |V| = 2 \cdot |E|$.

$\therefore 4|V| = 2 \times 38$

$|V| = \frac{2 \times 38}{4} = \boxed{19}$.

Q-4. Let G be a simple graph with 24 edges and degree of each vertex is k. which of the foll. is possible no. of vertices.

a) 6

b) 10

c) 12 ✓

d) 15

by cor 8), degree of each vertex is 'k',

$k.|V| = 2.|E|$

$\Rightarrow k.|V| = 48$

$\Rightarrow |V| = \frac{48}{k}$ $(k = 1, 2, 3, 4, 6)$

$k = 5$ not possible
$k = 7, 8$ not possible.

$\therefore$ possible no. of vertices are
(k1) (k2) (k3) (k4)
48, 24, 16, 12, 8

Q.5. Max. no. of ~~edges~~ vertices possible in a simple graph with 35 edges and degree of each vertex is 3 is ?

$\rightarrow$ by corollary 4), if degree of each vertex is at least k,

then $k.|V| \leq 2.|E|$

$\therefore 3.|V| \leq 2 \times 35$

$\therefore |V| \leq 70/3$

$\therefore |V| \leq 23$.

$\therefore$ max. no. of vertices possible $= \boxed{23}$.

$\Rightarrow |V| \geq \frac{70}{3}$ $= 23$ $|V| \cdot 6 \geq 34$
$|V| \geq \frac{34}{6}$ 7

Q.6. Min no of edges necessary in a simple graph with 13 vertices and degree of each vertex is at least 4.

By cor. 4) $k \cdot |V| \leq 2 \cdot |E|$.

$\Rightarrow 13 \times 4 \leq 2 |E|$

$\therefore |E| \geq \frac{13 \times 4}{2}$

$\therefore |E| \geq 26.$

$\therefore$ min no of edges $= \boxed{26.}$

Q.7. Min no. of vertices necessary in a simple graph with 17 edges and degree of each vertex at most 5.

by cor. 5) $\rightarrow$
$\Rightarrow 5|V| \geq 34.$

$\Rightarrow |V| \geq 34/5$

$|V| \geq 6.8$

$|V| \geq 7$

$\therefore$ Min. no. of vertices $\boxed{7}$

Q.8. Which of the foll. degree sequences represent a simple nondirected graph?

a) $\{5, 3, 3, 4, 4, 5\}$
a b c d e f

→ sum of degrees of all vertices is odd.

$\boxed{\text{not possible}}$

* (or) no. of odd degree vertices is not even)

b) $\{2, 3, 4, 4, 5\}$

* The graph has 5 vertices. So, degree of each vertex ≤ 4 always.

∴ $\boxed{\text{not possible.}}$

c) $\{1, 3, 3, 4, 5, 6, 6\}$
a b c d e f g.

→ Cannot represent a simple nondirected graph because in a simple graph with 7 vertices if we have 2 vertices with deg. 6 then deg. of every other vertex should be ≥ 2.

∴ a vertex with deg 1 not possible.

∴ $\boxed{\text{not possible}}$

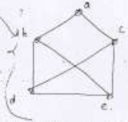
d) $\{0, 1, 2, \ldots, n-1\}$.
$\quad v_1 \quad v_2 \quad v_3 \quad v_n$

→ Cannot represent a simple nondirected graph, because in a simple graph with n vertices, if there is a vertex with deg '$n-1$', then that vertex is adjacent to all other vertices.

∴ a vertex with deg. 0 is not possible.

Note: In a simple graph with n vertices, $(n \geq 2)$, at least two vertices should have same degree.

e) $\{2, 3, 3, 3, 3\}$.
$\quad a \quad b \quad c \quad d \quad e$



∴ the given degree seq. is possible.

f) $\{3, 3, 3, 1\}$.

→ deg. of 4th vertex cannot be 1.

∴ not possible.

* Havel-Hakimi's Result *

Consider the foll. degree sequences I and II and assume that sequence I is in descending order

$\mathrm{I}: \{s, t_1, t_2, \ldots, t_s, d_1, d_2, \ldots, d_n\}$

$\mathrm{II}: \{t_1-1, t_2-1, \ldots, t_s-1, d_1, d_2, \ldots, d_n\}$

(*) I is graphic (simple nondirected graph) iff II is graphic

Q.9 Which of the foll. deg sequences represent a simple nondirected graph?

$$S_1)\ \{\underset{a}{6}, \underset{b}{6}, \underset{c}{6}, \underset{d}{6}, \underset{e}{4}, \underset{f}{3}, \underset{g}{3}, \underset{h}{0}\} \leftarrow \text{isolated}$$

deg (f) and deg (g) cannot be 3.

$$\rightarrow S_1)\ \{\cancel{6}, \underbrace{6, 6, 6, 4, 3, 3}_{6}, 0\}$$

$$\Downarrow$$

$$S_2)\ \{\cancel{5}, \underbrace{5, 5, 3, 2, 2}_{5}, 0\}$$

$$\Downarrow$$

$$S_3)\ \{\cancel{4}, \underbrace{4, 2, 1, 1}_{4}, 0\}$$

$$\Downarrow$$

$$S_4)\ \{\cancel{3}, \underbrace{1, 0, 0}_{3}, 0\}$$

a b c d e

can't reduce any further.

The reduced sequence cannot be represented by a simple non-directed graph.

∴ the given sequence also cannot be represented by a simple nondirected graph.

8e) $\{6,5,5,4,3,3,2,2,2\}$

→ 1) $\{6,5,5,4,3,3,2,2,2\}$

$\Downarrow$

2) $\{4,4,3,2,2,1,2,2\}$

$\Rightarrow \{4, \underbrace{4,3,2,2}, 2,2,1\}$

$\Downarrow$

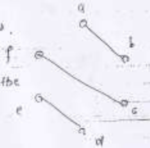
3) $\{3, \underbrace{2,1,1}, 2,2,1\}$

$\Downarrow$

$\Rightarrow \{3, \underbrace{2,2,2}, 1,1,1\}$

$\Downarrow$

4) $\{1,1,1,1,1,1\}$



So, we can get a simple nondirected graph from the last seq.

We can also draw a simple nondirected graph from the given sequence.

Isomorphic graphs ⇒

Two graphs G_1 and G_2 are said to be isomorphic if there exists a funcn $f : V(G_1) \to V(G_2)$ such that

i) f is a bijection (one-one onto funcn) and

ii) funcn f preserves adjacency of vertices. i.e. if any two vertices are adjacent in graph G_1 then the images of these vertices should be adjacent in G_2.

i.e. if the edge $\{u, v\} \in G_1$ then

edge $\{f(u), f(v)\} \in G_2$.

then $G_1 \cong G_2$
isomorphic to

* If G_1 and G_2 are isomorphic, then the foll. condns must hold good ⇒

1) No. of vertices in graph G_1 must be equal to no. of vertices in graph G_2.

i.e. $|V(G_1)| = |V(G_2)|$

2) $|E(G_1)| = |E(G_2)|$

and

3) The degree sequences of G_1 and G_2 are same.

4) If the vertices $\{v_1, v_2, \ldots, v_k\}$ form a cycle of length 'k' in G_1, then the images of these vertices (i.e. $\{f(v_1), f(v_2), \ldots, f(v_k)\}$ should form a cycle of length 'k' in G_2.

Note: All the above conditions are necessary for graphs G_1 and G_2 to be isomorphic. But these cond^ns are not sufficient to prove that graphs are isomorphic.

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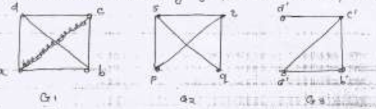
* $(G_1 \cong G_2)$ iff $(\overline{G_1} \cong \overline{G_2})$

* $(G_1 \cong G_2)$ iff the adjacency matrices of G_1 and G_2 are same.

* $(G_1 \cong G_2)$ iff the corresponding subgraphs of G_1 and G_2 obtained by deleting some vertices in G_1 and their images in G_2 are isomorphic.

Q1. Which of the following graphs are isomorphic?



* In graph G_3, we have only 4 edges.

$\therefore$ G_3 is not isomorphic to G_1 or G_2.

Taking the complements of G_1 and G_2, we have,

$\overline{G_1}$:

$\overline{G_2}$ =

$\overline{G_1} \cong \overline{G_2}$

$\therefore \boxed{G_1 \cong G_2}$

(OR) by using adjacency matrices :

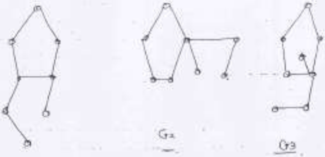
G_1	a	b	c	d
a	0	1	0	1
b	1	0	1	1
c	0	1	0	1
d	1	1	1	0

G_2:

	p	q	r	s
p	0	0	1	1
q	0	1	0	1
r	1	1	1	0
s	1	0	1	1

$\therefore G_1$ & G_2 are isomorphic.

Q.2. Which of the foll. graphs are isomorphic?



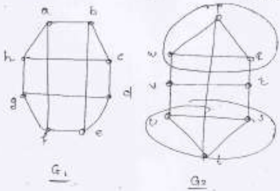
G_1 G_2 G_3

→ G_1 is not isomorphic to G_2, because degree sequences of G_1 and G_2 are not same.

But G_2 and G_3 are isomorphic

$\boxed{G_2 \cong G_3}$

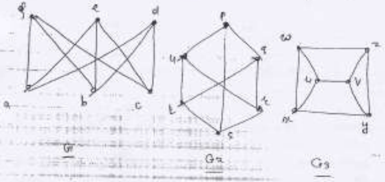
Q.3. Which of the foll. graphs are isomorphic ?



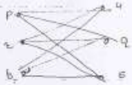
In G_1, no cycle of length 3 present. In G_2, cycle of length 3 is there.

$\therefore$ G_1 and G_2 are not isomorphic.

Q.4. Which of the foll. graphs are isomorphic ?

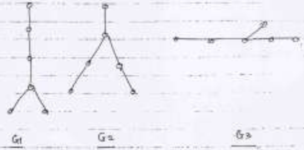


→ G1 and G2 are isomorphic because G2 can be drawn as an bipartite graph.



G3 is not isomorphic to G1 or G2, in G3, we have cycles of odd length. So, it cannot be a bipartite graph.

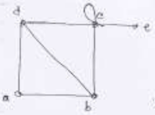
Q5 Which of the foll. graphs are isomorphic?



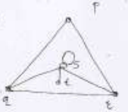
→ G2 and G3 are isomorphic.

G1 and G2 are not isomorphic bcoz, in the graph G1 the vertex with degree 3 has two neighbours with degree 1. Whereas in graph G2 the vertex with deg. 3 has only one neighbour with deg 1.

Q.6. Find whether the foll graphs are isomorphic?



G_1

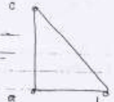


G_2

* Comparing vertices of degree 5, we have, image of c = s.

Further c and s have similar neighbours.

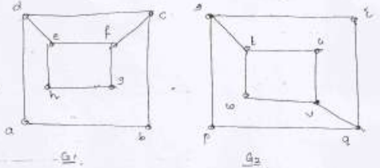
∴ Deleting c and s from G_1 and G_2, we get



The subgraphs are isomorphic.

∴ $\boxed{G_1 \cong G_2}$

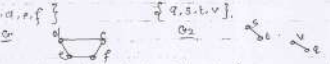
Q.7. Find whether the foll. graphs are isomorphic.



→ In G1, all the vertices form a cycle of length 8 whereas, in G2 there is no cycle of length 8.

∴ G1 and G2 are not isomorphic.

{r, d, e, f} {q, s, t, v}

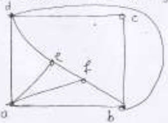


Here, the 4 vertices of deg 3 form a cycle in G1 whereas, no cycle is formed by the vertices of deg 3 in G2.

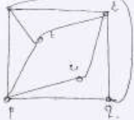
∴ $G_1 \not\cong G_2$.

In graph G1, a pair of vertices of deg 2 are adjacent, whereas, in G2, no two vertices of deg 2 are adjacent.

Q.8 Find whether the foll. graphs are isomorphic



G1



G2.

→ Comparing vertices of deg 4.

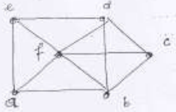
{a, b, d}   { p, q, s }.

G1.   G2.

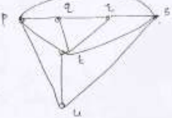
In G_1, the three vertices of deg 4 form a cycle, whereas, in the second graph, no cycle is formed by the vertices of deg 3.

$\boxed{G_1 \ncong G_2}$

29. Find whether the foll. graphs are isomorphic?

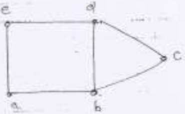


G_1

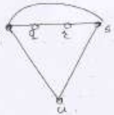


G_2

→ Deleting f and t from G_1 and G_2.



H_1



H_2

Image of c = u.

Deleting c and u from H_1 and H_2.

H_3

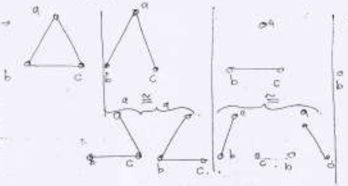
H_4

$\therefore H_3 \cong H_4$.

$\therefore \boxed{G_1 \cong G_2}$.

Q.10 How many simple nonisomorphic graphs are possible with 3 vertices?

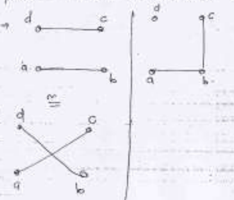
→



$\therefore \boxed{4}$

Q.11. How many simple nonisomorphic graphs are possible with 4 vertices and 2 edges.

→

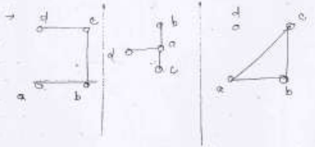


In a simple graph with 4 vertices and 2 edges, the two edges may be adjacent or nonadjacent.

∴ only 2 are possible.

Q. 12- How many simple nonisomorphic graphs are possible with 4 vertices and 3 edges?

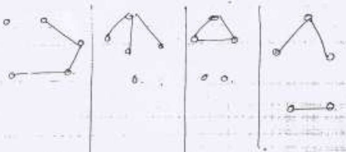
→



∴ only $\boxed{3}$ possible.

Q. 13- How many " —" are poss. with 5 vertices & 3 edges?

→



∴ $\boxed{4}$ nonisomorphic graphs possible.

Q.14. " — " with 4 vertices ?

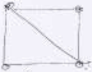
⑥

4V
6E

≅ K_4

⑤

4V
5E



≅

④

4V
4E

③

4V
3E

②

4V
2E

①

4V
1E

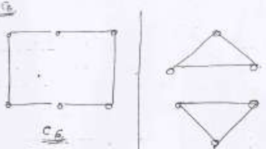
4V
0E

∴ total = 1+1+2+3+2+1+1

= $\boxed{11}$

Q.15 How many simple nonisomorphic graphs poss with 6 vectices, 6 edges and deg. of each vectex is 2?

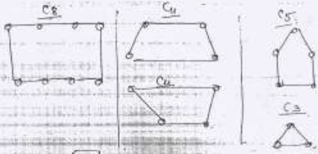
→



Ans. $\boxed{2}$

Q.16 " —" with 8 vectires, 8 edges and deg. of each vectex 2. ?

→



Ans $\boxed{3}$

16. " — " 10 vertices & 10 edges and deg of each vertex 2 ?

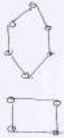
→ C_{10}

C_5

C_5

C_6

C_4



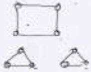
C_7

C_3

C_4

C_3

C_3



17. How many simple non isomorphic trees are possible with 3 vertices ?

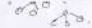
→

ans $\boxed{1}$

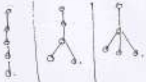
Q.18. " — " 4 vertices ?

→

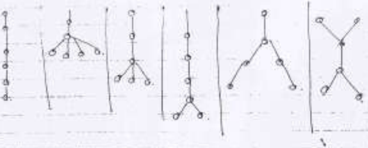
ans. $\boxed{2}$

Q.19. " -" 5 vertices?

→  ∴ ans $\boxed{3}$.

Q.20 " -" 6 vertices?

→ 

∴ ans $\boxed{6}$

Note →

\# If a simple graph G is isomorphic to its complement $\bar{G}$, then G is said to be "Self-Complementary" Graph.

\# Results →

Let G be a simple graph with 'n' vertices.

If G is isomorphic to $\bar{G}$ i.e. $G \cong \bar{G}$, then

i) No. of edges in the graph $= \frac{n(n-1)}{4}$

i.e. $\boxed{|E(G)| = \frac{n(n-1)}{4}}$

ii) No. of vertices in the graph

$$\boxed{|V(G)| = 4k \text{ (or) } (4k+1)} \quad (k = 1, 2, 3, \ldots)$$

Q.21. If a cycle graph C_n is isomorphic to $\overline{C_n}$ ($C_n \cong \overline{C_n}$), then $n = ?$

→ By the above result, $|E(C_n)| = \frac{n(n-1)}{4}$

$\Rightarrow n = \frac{n(n-1)}{4}$ (By defn of C_n). $(\because C_n \cong \overline{C_n})$

$\therefore n - 1 = 4$

$\boxed{n = 5}$

Note: ∴ C_5 is the only cyclic graph isomorphic to itself.

Q.22. If a tree T with n vertices is isomorphic to $\overline{T}$, then $n = ?$

→ By the above result, no. of edges in tree $= \frac{n(n-1)}{4}$ $(\because T \text{ is } \cong \overline{T})$. i.e. $|E(T)| = \frac{n(n-1)}{4}$

$\Rightarrow (n-1) = \frac{n(n-1)}{4}$

$\boxed{n = 4}$

* Note: A tree with 4 vertices is isomorphic to its complement.

Q.23. If G is a simple graph with n vertices and $G \cong \bar{G}$, then which of the foll. is not true?

a) $n=6$ b) $n=8$ c) $n=9$ d) $n=13$.

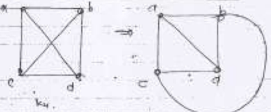
→ Given that $G \cong \bar{G}$. So, no of vertices $= 4k$ (or) $4k+1$

∴ 8 possible 9 ✓ 13 ✓.

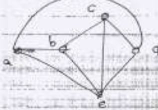
but 6 not possible.

Planar Graph :

A graph 'G' is said to be planar graph if it can be drawn on a plane (sphere) so that no two edge cross each other at a nonvertex point.



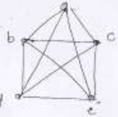
So, planar graph.



planar graph.

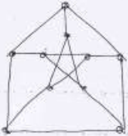
The following are some examples of non planar graphs :

1) K_5



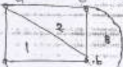
2) $K_{3,3}$

3)

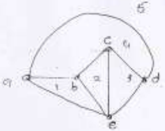


Peterson's Graph.

Regions : Every planar graph divides the plane into connected areas called "Regions".



(4) ← unbounded region



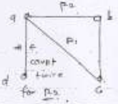
deg·(1) = 8

deg·(2) = 3

deg·(3) = 3

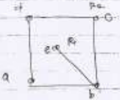
deg·(4) = 4

deg·(5) = 3.



deg·(R1) = 3

deg·(R2) = 5· (count ad twice;



deg·(R1) = 6. ('be' twice)

deg·(R2) = 6.

Degree of a bounded region → deg(R) →

No of edges enclosing that region R.

(or) exterior

Degree of an unbounded region R → deg(R) →

No. of edges exposed to the region R.

$\{a, b, c, d, e, f\}$

b should speak after a

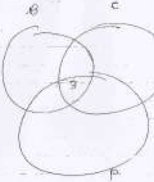
(6 $\times$ 4!)

(6 $-$ 3 $\times$ 2) $=$

(6 $-$ 3 $\times$ 2 $\times$ 1) $=$

6 - people . Speeches of a & b $\Rightarrow$ $^6C_2$ ways

300



FCFS

SRTF

Tushar V. Kokane
Data Structures
8087586479.

Books →

1) Sahni
2) Weiss
3) Kruse

$\Bigg\}$ Level-1

1) Coreman
2) Goodrich & Tamassia
3) Drozdek

$\Bigg\}$ Level-2

(interviews).

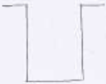
1) Data structures - Forouzan.

What is a Data Structure →

Data Structure →

A mathematical or logical model

Data structure



Stack

Physical Structure →

Implementation of Data structure in physical memory.

Array stack

linked list stack.

Why do we need Data Structures → ?!!

Solving a problem (Fastly) by occupying optimum memory

time —— Analysis —— space

Q1. Calculate the locatⁿ of an element A[0] in an array of [-5 to +5] where the starting locatⁿ is 1000 and each element occupies two memory locations.

→ $\boxed{1010}$

Array is a contiguous and homogeneous.

A: array [lb ... ub] of elements.

lb → lower boundary, ub → upper boundary.
c → count → ele. size.
Lo → starting location.

∴ $\boxed{loc\ A(i) = Lo + (i - lb) * c}$

∴ applying this formula →

$loc\ A(0) = 1000 + (0+5)*2$
$= \boxed{1010}$

Correlation with 'C' →

1000 1002 1004 1006 1008 1010

Correlation with 'C' →

loc $a[i]$ = $L_0 + (i - lb) * c$

= $\& a_0 + (i-0) * c$

$\& a[i]$ = $(a+i)$ (scalar arithmetic)

$\therefore a[i] = *(a+i)$

	10	12	14	16
a →	a_0	a_1	a_2	a_3

a = name of the array.

= address of very first element.

$\therefore a = (\& a_0)$

char $10 + (1) \cdot 1 \rightarrow 11$

int $10 + (1) \cdot 2 \rightarrow 12$

float $10 + (1) \cdot 4 \rightarrow 14$

double $10 + (1) \cdot 8 \rightarrow 18$

long-double $10 + (1) \cdot 10 \rightarrow 20.$

Properties of planar graph →

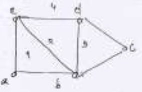
1) In a planar graph with 'n' vertices sum of degrees of all vertices

i.e. $\boxed{\sum_{i=1}^{n} \deg(v_i) = 2 \times \text{No. of edges in the graph} = 2|E|}$

2) Sum of degrees of regions theorem →

In a planar graph with 'm' regions, sum of degrees of regions i.e.

$$\boxed{\sum_{i=1}^{m} (r_i) = 2 \cdot |E|}$$

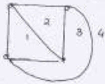


$8+3+3+5 = 14 = 7 \times 2$

Corollary 2.1 →

In a planar graph if degree of each region is 'k' then sum of degrees of regions becomes $k \cdot |R|$.

$\therefore \boxed{k \cdot |R| = 2 \cdot |E|}$



$\therefore 3 \times 4 = 2 \times 6$

rollary 2.2 →

In a planar graph, if degree of each region is at least 'K' (greater than or equal to K), then

$$\boxed{K \cdot |R| \le 2 \cdot |E|}$$

$3 \cdot (3) \le 2 \cdot (6)$.

Corollary 2.3 →

Simple planar graph →

In a simple planar graph with at least two edges degree of each region is greater than or equal to 3. (at least 3).

$\boxed{3 \cdot |R| \le 2 \cdot |E|}$ ← For a simple planar graph.

Corollary 2.4 →

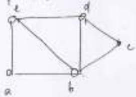
In a planar graph, if degree of each region is at most 'K' (i.e., $\le K$) then

$$\boxed{K \cdot |R| \ge 2 \cdot |E|}$$

Euler's Formula →

If 'G' is a connected planar graph, then

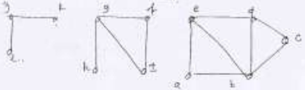
$$\boxed{|V| + |R| = |E| + 2}$$



Corollary 3.1 →

If G is a planar graph with 'k' components, then

$\boxed{|V| + |R| = |E| + (k+1)}$



12+5 = 13 + (3+1).

4) Edge-vertex Inequality :-

If G is a connected planar graph, with degree of each region at least 'k' ($\geq k$), then

No. of edges $\leq \frac{k}{k-2} \times$ ((no. of vertices) − 2))

i.e. $\boxed{|E| \leq \frac{k}{k-2}(|V| - 2)}$

5) If G is a simple connected planar graph, then

5.1) No. of edges i.e. $|E| \leq (3 \cdot |V| - 6)$

5.2) using Euler's formula,

$\{|V| + |R| - 2\} \leq \{3|V| - 6\}$

1 In a complete bipartite graph, cycle of length odd not possible

$\therefore |E| \leq \{2 \cdot |V| - 4\}$.

5.B) There exists at least one vertex $V \in G$, such that

$deg(v) \leq 5$.

6) If G is a simple connected planar graph (with at least two edges) (and no triangles), then

$|E| \leq (2 \cdot |V| - 4)$

Note :

1) $K_{3,3}$ is not a planar graph.

2) In K_5, triangle (i.e. cycle of odd length) exist.

So. K_5 is not a planar graph.

7) Kurtowski's thm :

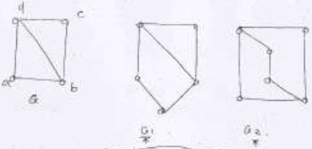
A graph G is non-planar iff G has a subgraph which is homeomorphic to K_5 or $K_{3,3}$.

Note :

Homeomorphic graphs :

Two graphs 'G_1' and 'G_2' are said to be homeomorphic if each of these graphs can be obtained from the same graph. G. by dividing some edges of G with more vertices.

ex:



can be obtained from G by putting extra node

Note →

1) If any two graphs G1 and G2 are isomorphic, then they are homeomorphic also.

2) The converse of the above statement need not be true.

Corollary 7.1) → Any graph with 4 or fewer vertices is planar.

Corollary 7.2) → Any graph with 8 or fewer edges is planar.

Corollary 7.3) → The complete graph 'K_n' is planar iff n is ≤ 4.

i.e. $n \leq 4$

Corollary 7.4) → The complete bipartite graph '$K_{m,n}$' is planar iff $m \leq 2$ (or) $n \leq 2$.

~~ollary 7.5~~ → The simple nonplanar graph with min. no. of vertices is the complete graph $\underline{K_5}$.

~~ollary 7.6~~ → The simple non-planar graph with min no. of vertices is $\underline{K_{3,3}}$ (complete bipartite graph).

i) Polyhedral Graph →

A simple connected planar graph is called a "polyhedral graph" if

$\boxed{deg(v) \geq 3}$ $\underline{\forall\, v \in E}$.

The following inequalities must hold good.

i) $3 \cdot |V| \leq 2 \cdot |E|$.

ii) $3 \cdot |R| \leq 2 \cdot |E|$

Q.

1. Let G be a connected planar graph with 25 vertices, 60 edges. Find no. of bounded regions in G is?

→ By Euler's formula,

$|V| + |R| = |E| + 2$

$\therefore 25 + |R| = 60 + 2$ $\quad \therefore |R| = 37$.

Of these 37 regions, we have one unbounded region

$\therefore$ No. of bounded Regions = $\boxed{86}$

Q.2. Let G be a connected planar graph with 10 vertices with 10 vertices 15 edges and 8 components. No. of regions in G is ?

→ By Euler's formula,

$$|V| + |R| = |E| + (k+1)$$

$$\therefore 10 + |R| = 15 + (8+1)$$

$$\therefore \boxed{|R| = 9}$$

Q.3. Let G be a connected planar graph with 20 vertices and degree of each vertex is 8. Find no. of regions in graph.

→ By sum of degrees of vertices thm,

$$\sum_{i=1}^{20} \deg(v_i) = 2 \cdot |E|$$

$$\therefore 20 \cdot (8) = 2 \cdot |E| \quad \therefore \boxed{|E| = 80}$$

$\therefore$ By Eulers formula,

$$|V| + |R| = |E| + 2$$

$$\therefore |R| = 82 - 20$$

$$\boxed{|R| = 62}$$

14. Let G be a connected planar graph with 35 regions and degree of each region is 6. $|V| = ?$

- by sum of degrees of regions rule,

$k \cdot |R| \leq 2|E|$ here $k \cdot |R| = 2 \cdot |E|$

$\therefore 6 \times 35 = 2 \times |E|$

$\therefore \underline{|E| = 105}$

By Euler's formula,

No of vertices $(|V|)$ + No. of regions $(|R|)$

$= 2 + |E|$

$\therefore |V| = 2 + |E| - |R| =$ [illegible]

$\therefore |V| = \boxed{72}$

15. Let G be a polyhedral graph with with 20 vertices, 30 edges and degree of each region is 'k'. then $\underline{k = ?}$

- By Euler's formula, ...

$|V| + |R| = 2 + |E|$

$\therefore 20 + |R| = 2 + |E|$

$\therefore |R| = \underline{12}.$

By sum of degrees of regions,

If degree of each region is 'k', then

$k \cdot |R| = 2 \cdot |E|$,

$\therefore$ $k = \frac{2 \times 30}{12} = \boxed{5}$

$\boxed{k = 5}$.

Q.6. Max no. of edges possible in a simple connected planar graph with 8 vertices is ?

→ By theorem 5.1, no of edges; $|E| \leq (3 \cdot |V| - 6)$

$\therefore$ $|E| \leq (3 \times 8) - 6$ $\therefore$ $|E| \leq 18$.

$\therefore$ max. no. of edges possible is $\boxed{18}$.

Q.7. Min. no. of vertices necessary in a simple connected planar graph with 11 edges is ?

→ $11 \leq 3 \cdot |V| - 6$. → add 6 both sides.

$\therefore$ $17 \leq 3 \cdot |V|$. → divide by 3.

$\therefore$ $17/3 \leq |V|$.

$\therefore$ $|V| \geq 5.66$.

$\therefore$ $|V| \geq \boxed{6}$. $\therefore$ Min. no. of vertices $= \boxed{6}$.

8. Min. no. of vertices necessory with a simple connected planar graph with 20 edges and degree of each region ≥ 5.

→ By edge-vertex inequality (thm 5.)

$|E| \leq \frac{k}{k-2}(|V|-2)$.

where, $k=5$.

$\therefore$

$20 \leq \frac{5}{3}(|V|-2)$.

$\therefore 60 \leq 5(|V|-2)$.

$\therefore |V| \geq \frac{60}{5}+2$

$\boxed{|V| \geq 14}$

1.9. min no. of vertices necessory in a simple connected planar graph with 15 regions is - ?

→ thm. 6.2.

$|R| \leq (2.|V|-4)$

$\therefore 15 \leq 2.|V|-4$.

$\therefore |V| \geq \frac{19}{2} \geq 9.5$

$\boxed{|V| \geq 10}$

9.10. If G is a simple connected planar graph, then $\delta(G)$ (min. of the degrees of all degrees in G) cannot be

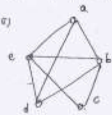
For any simple connected planar graph, there exists at least one vertex such that $\deg(v) \le 5$.

a) 3 b) 4 c) 5 ✓d) 6.

$\boxed{\delta(G) \le 5}$ $\therefore$ $\underline{\delta(G) \ne 6}$.

9.11. Which of the following is (are) not a planar graph?

a)



By Kuratowski's thm, cor. 7.5, this graph is planar because the only simple nonplanar graph is $\underline{K_5}$.

b)

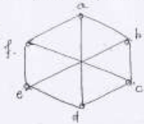
It is $K_{4,2}$. Corolary 7.4. The complete bipartite graph is planar if any suffix $\underline{\le 2}$.

c)

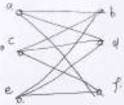
$\cong$

So, it's a planar graph.

d)



$\cong$ $K_{3,3}$ $\rightarrow$



It is not a planar graph.

e)

By edge-vertex inequality, we have

$\Rightarrow |E| \leq \frac{k}{k-2}(|V|-2)$.

where, $k=5$.

$\therefore 15 \leq \frac{5}{5-2}(8)$.

$\Rightarrow 15 \cancel{\leq} \frac{5}{3} 8$

$\therefore$ edge-vertex inequality is not satisfied.

$\therefore$ It's not a planar graph.

8.12 Which of the following is/are true ?

S1) A polyhedral graph with 30 edges and 11 regions does not exist.

By euler's formula,

Suppose a polyhedral graph with 30 edges and 11 regions exist. Then by euler's formula,

$|V| + |R| = |E| + 2$

$\therefore |V| = 21$

By thm 8.1, for polyhedral graph, the foll. inequalities must hold good.

$3|V| \leq 2 \cdot |E|$ and $3 \cdot |R| \leq 2 |E|$

$\therefore 3(21) \leq 2 \cdot (30)$

$\therefore 63 \nleq 60$

$\therefore$ Our assumption is not true.

$\therefore$ S1 is true.

S2) A polyhedral graph with 7 edges does not exist.

→ Suppose, a polyhedral graph with 7 edges exist.

For polyhedral graph, the foll. inequalities must hold good

$3 \cdot |V| \leq 2 \cdot |E|$ and $3 \cdot |R| \leq 2 \cdot |E|$

$\therefore 3 \cdot |V| \leq 2 \times 7$ $\quad$ $3 \cdot |R| \leq 2 \times 7$

$3|V| \leq 14$ $\quad$ $\therefore |R| \leq 4.66$.

$\therefore |V| \leq 4.66$ $\quad$ $\therefore |R| \leq 4$.

$\therefore |V| \leq 4$

By Euler's formula,

$|V| + |R| = |E| + 2$.

$\therefore 4 + 4 \geq 7 + 2$

$\therefore 8 \not\geq 9$

$\therefore$ it is not possible.

$\therefore$ our assumption is not true.

$\therefore$ such a graph does not exist.

$\therefore S_2$ is true. i.e. polyhedral graph with 7 edges does not exist.

13. Let G be a simple nonplanar graph with minimum of vertices, then G has

a) 5 vertices, 9 edges.

b) 5 vertices. 10 edges.

c) 6 vertices, 9 edges.

d) 8 vertices, 8 edges.

→ The simple nonplanar graph with min no of vertices is K_5 and K_5 has 5 vertices and 10 edges.

Q.14. Let G be a simple nonplanar graph with min no edges, then G has

a) 5V, 9E

b) 5V, 10E

c) 6V, 9E → ($K_{3,3}$)

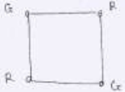
d) 6V, 8E.

→ The simple nonplanar graph with min no of edges is $K_{3,3}$. $K_{3,3}$ has 6 vertices and 9 edges.

Colorings :-

Vertex coloring :-

An assignment of colors to the vertices of the graph G so that no two adjacent vertices have same color is called as "vertex coloring of G".



chromatic number of a graph :-

The min. no. of colors reqd. for vertex coloring of graph 'G' is called "Chromatic number of G". and is denoted by "$\chi(G)$".

1) chromatic no of G = 1 iff G is a null graph.

2) If G is not a null graph, then chromatic no of G is ≥ 2.

3) A graph 'G' is said to be n-colorable, if there is a vertex coloring that uses at most n-colors. i.e.

$$\boxed{\chi(G) \leq n.}$$

* Any planar graph does not need more than 4 colors for vertex-coloring so that no two adjacent vertices have same color.

22/10/13

Four-color theorem :-

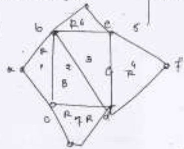
Every planar graph 'G' is "four-colorable"

i.e. $\chi(G) \le 4$.

The above thm is valid for 'map-coloring' / 'Region-coloring' also.

(Assigning colors to the regions of a graph so that no two adjacent regions have same color.)

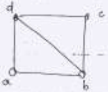
Two regions are said to be 'adjacent' if they have a common edge.



Welch-Powel's Algorithm → (It gives upper limit of chromatic number)

* (This algorithm is used for "vertex-coloring")

1) Arrange vertices in the descending order of their degrees.



$\{ \underset{3}{b}, \underset{3}{d}, \underset{2}{a}, \underset{2}{c} \}$

If two (or) more vertices have same degree then arrange these vertices in alphabetical / numerical order.

2) Assign colors to the vertices in the above order, so that no two adjacent vertices have same color.

Q.1. The chromatic number of K_n = ?

✓a) n   b) n-1   c) $\lfloor n/2 \rfloor$ d) $\lceil n/2 \rceil$

→ $\underline{K_5}$

(every vertex is adjacent to every other vertex in complete graph.)

∴ chromatic number is $\underline{n}$ for complete graph.

$\boxed{\chi(K_n) = n}$

Q.2. The chromatic no. of a complete bipartite graph $K_{m,n}$ is ?

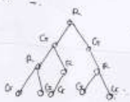
→ $\underline{K_{3,3}}$

(No two vertices in the same group are adjacent. So, one color for each group).

$\therefore \chi(K_{m,n}) = \underline{2}$.

Q.3. The chromatic number of a tree T with 'n' vertices ($n \geq 2$) is ?

→ Every tree can be represented as a bipartite graph and chromatic number of any bipartite graph is '2'.



$\therefore \chi(T) = \underline{2}$.

Q.4. If G is a cyclic graph with 'n' vertices ($n \geq 3$) and G has no cycles of odd length, then what is the chromatic no. of G ?

→ G has no cycles of odd length. That's why G is a "bipartite graph". (by thm)

$\therefore \chi(G) = \underline{2}$

5. Chromatic number of a cycle graph C_n ($n \geq 3$) is what ?

a) 2 b) 3 c) $n - 2\left\lfloor n/2 \right\rfloor + 2$ d) $n - 2\left\lceil \frac{n}{2} \right\rceil + 1$

→ for even no. of vertices → 2 colors needed.

for odd no. of vertices → 3 colors needed.

$$\therefore \chi(C_n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor + 2.$$

Q.6 What is chromatic no. of wheel graph W_n $(n \geq 4)$?

√a) $n - 2\left\lceil n/2 \right\rceil + 4$

b) $n - 2\left\lfloor \frac{n}{2} \right\rfloor + 4$

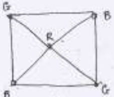
c) 4

d) 3.

W_4

$\chi(W_n)$ $n=4$ i.e even $= \underline{4}$.

$n = \text{odd} = 5$



W_5

$\chi(W_n)$ $(n = \text{odd}) = \underline{3}$.

$\therefore \chi(C_n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor + 2.$

Q.4. what is chromatic no. of wheel graph W_n $(n \geq 4)$.?

✓a) $n - 2.\left\lceil n/2 \right\rceil + 4$

b) $n - 2\left\lfloor \frac{n}{2} \right\rfloor + 4$

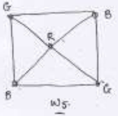
c) 4

d) 3.

→ 

→ $\chi(C_n)$ n=4 i.e even = 4.

n = odd = 5



$\chi(W_n)$ (n=odd) = 3.

7.7. Chromatic no. of a star graph with n vertices (n≥2)

→

A star graph can be represented as a bipartite graph of $K_{1,n-1}$.

$S_8 \rightarrow K_{1,7} \rightarrow$

$\therefore \chi(G) = 2$
star graph

7.8. Chromatic no. of the graph given below = ?

Applying Welch-Powell's algom,

Vertex	a	b	c	d	e	f
color	c_1	c_2	c_1	c_2	c_1	c_2

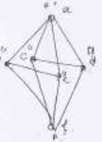
$\chi(G) = 2$ -①.

G is not a null graph

$\therefore$ chromatic no. $\chi(G) \geq 2$. – ②

From ① and ②, $\underline{\chi(G) = 2}$

Q.3. chromatic no. of the graph shown below is ?



Applying Welsh-Powell's algorithm.

vertex	a	f	b	c	d	e
color	c_1	c_1	c_2	c_2	c_3	c_3

$\therefore \chi(G) \leq 3$ – ①

Further, we have 3 mutually adjacent vertices $\{a, c, d\}$
And also they form a cycle of odd length

$\therefore$ we require min 3 colors.

$\therefore \chi(G) \geq 3$ – ②

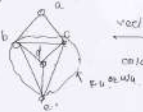
$\therefore$ From ① and ②,

$\chi(G) = \underline{3}.$

* Every 2-colorable graph is bipartite graph.

Q.10. Chromatic no. of the graph shown below.

→



vertex	b	c	d	e	a
color	C_1	C_2	C_3	C_4	C_3

$\therefore \chi(G) \le 4$. — ①

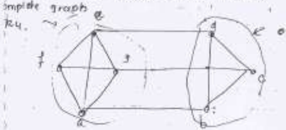
Further we have four mutually adjacent vertices $\{b, c, d, e\}$

$\therefore \chi(G) \ge 4$. — ②

$\therefore$ from ① and ②,

$\chi(G) = \underline{4}$.

Q.11. For the graph shown below,



G is a planar graph because we can draw w/o crossovers.

$\therefore$ So, by four-color thm.

$$\chi(G) \le 4 \quad -①$$

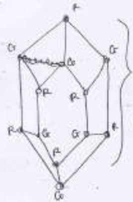
Further, we have four mutually adjacent vertices $\{a, e, f, g\}$ forming complete graph K_4.

$$\chi(G) \ge 4 \quad -②.$$

$\therefore$ from ① and ②,

$$\chi(G) = 4.$$

12. Chromatic number of the graph shown below is ?

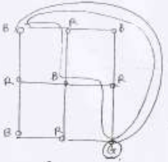


(G is a cyclic graph in which all the cycles are of even length.)

So, $\chi(G) = 2.$

$\because$ no odd length cycle.

Q.13 Chromatic number of the graph shown below is ?



G

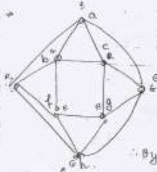
G has a cycle of length 5.

$\therefore \chi(G) \geq 3$. — ①

By welch-powell algom, 3 coloring is possible.

$\therefore \chi(G) = 3$.

Q.14. What is chromatic no. of graph shown below?



G

G has cycles of length '3'.

$\therefore \chi(G) \geq 3$ — ①.

G is a planar graph.

$\therefore$ By 4-color thm,

$\therefore \chi(G) \leq 4$ — ②.

$\therefore \chi(G) = 4$.

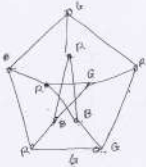
Applying welch-powell's algom.

vertex	a	b	c	d	e	f	g	h
color	c_1	c_2	c_3	c_3	c_2	c_1	c_4	c_5

$\therefore \chi(G) \le 5$ — ③.

15. chromatic number of the graph shown below ?

G.



G.

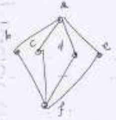
G has cycles of odd length.

$\therefore \chi(G) \ge 3$. — ①

3- coloring is possible

$\therefore \chi(G) = 3$.

Q.18. Chromatic no. of the graph shown below ?



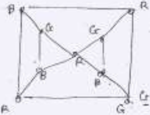
G

The graph has no cycles of odd length

$\therefore \chi(G) = \underline{2}$

(It is a bipartite graph.)

Q.19 chromatic no. of the graph shown below ?



The graph is a planar graph

$\therefore \chi(G) \leq 4.$

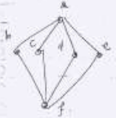
$\underline{G}$ has cycles of odd length

$\therefore \chi(G) \geq 3.$

$\therefore$ 3-coloring is possible

$\therefore \chi(G) = \underline{3}.$

Q.18. Chromatic no. of the graph shown below. ?



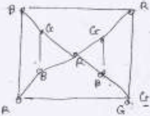
The graph has no cycles of odd length

$\therefore \chi(G) = \underline{2}$

(It is a bipartite graph.)

$\underline{G}$.

Q.19. chromatic no. of the graph shown below ?



The graph is a planar graph

$\therefore \chi(G) \le 4$.

$\underline{G}$ has cycles of odd length

$\therefore \chi(G) \ge 3$.

$\therefore$ 3-coloring is possible

$\therefore \chi(G) = \underline{3}$.

Matchings And Coverings :

Matching :

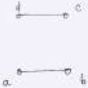
Let 'G' = (V,E) be a graph. A subgraph 'M' of G is called a 'matching of G', if each vertex of G is incident with at most one edge in M.

i.e. in a matching 'M', $\deg(v) \leq 1$, $\forall v \in G$.

ex :

G.

$\rightarrow$



M_1

(or)

M_2

(or)

M_3

(or)

M_4

(or)

Null graph

M_5

Note :

1) In a matching, if $\deg(v) = 1$, then the vertex 'v' is said to "matched."

2) If $\deg(v) = 0$, then 'v' is not matched.

3) In a matching, no two edges are adjacent.

Maximal matching →

① A matching (M) of a graph G (or $G(z)$) is said to be maximal if no other edges of G can be added to M.

From the previous matchings,
M_1, M_2 and M_3 are maximal matchings of G.

(and) M_4 and M_5 are not maximal.

② and also M_1, M_2 and M_3 are the only maximal matchings for this graph.

Maximum (Largest) matching →

A maximal matching with max no. of edges is called "maximum matching".

→ The no. of edges in the maximum matching of G is called "matching number" of G.

Ex. → From prev., M_1 and M_2 are "maximum matchings"

∴ Matching no. of the graph = $\underline{2}$.

Perfect matching.

A matching M of graph G is said to be perfect matching if every vertex of G is matched in M.

i.e. in a perfect matching, $\deg(v) = 1$; $\forall v \in G$.

For the prev. ex. M_1 and M_2 both are "Perfect matchings" of G.

Note:

1) Every perfect matching of a graph is also a "maximum matching" of graph.

2) A maximum matching of a graph need not be perfect.

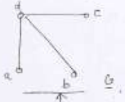
(maximum matching but not a perfect matching.)

3) If a graph G has a perfect matching, then no. of vertices in the graph is even.

(A graph with odd no. of vertices cannot have perfect matching).

The converse of 3), need not be true.

ex

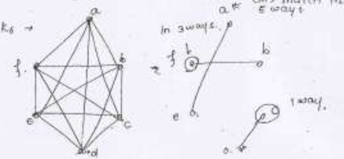


(This graph has even no of vertices, but no perfect matching possible.)

4) Complete graph 'K_n' has a perfect matching iff n is even. (no. of vertices are even).

Number of perfect matchings in a complete graph with 2n vertices i.e. K_{2n} is
(n = 1, 2, 3, ...)

let K_6 →



∴ No of perfect matchings in K_6 = $5 \times 3 \times 1 = 15$

Now, for no. of perfect matchings in K_{2n} is

$(2n-1).(2n-3).(2n-5) \ldots 5.3.1$

$$= \frac{2n!}{2n.(2n-2).(2n-4)\ldots 6.4.2}$$

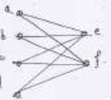
$$= \boxed{\frac{2n!}{2^n.\,n!}}$$

Q.1. How many perfect matchings in K_8 ?

$\rightarrow \quad \frac{2n!}{2^n.\,n!} = 105$ $\qquad 2n=8$ $\qquad n=4$

Note →

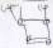
The complete bipartite graph has a perfect matching iff $m=n$.



Here, perfect matching is not possible for $K_{4,2}$.

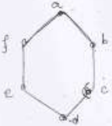
Q. How many perfect matching possible in $K_{m,n}$ $(m=n)$

→ $\boxed{n!}$



Q.2. $n \geq 4$ and n is even in a cycle graph C_n

* In a cycle graph, only 2 perfect matchings are possible.



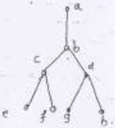
101 13

* Note *

A tree can have at most one perfect matching.

i.e. no. of perfect matchings in a tree ≤ 1. } (***)

Q3. No. of perfect matchings in the tree shown below is



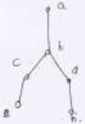
* In a perfect matching, degree of every vertex of the graph is 1.

matching

Therefore, we have to delete two edges at vertex c. By deleting any two edges at vertex c, perfect matching is not possible.

∴ No. of perfect matchings in the tree = $\boxed{0}$

Q.6. No. of perfect matchings in the tree below?

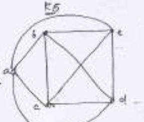


No. of perfect matchings = $\boxed{1}$

5. Matching number of the complete graph K_n = ?

→ Matching number of $K_n = \frac{n}{2}$ (if n is even).

If n is odd.



Here perfect matching is not possible. So, we can match only n-1 vertices.

∴ matching number $= \frac{(n-1)}{2}$ (if n is odd).

Matching number of $K_n = \left\lfloor \frac{n}{2} \right\rfloor$

Q.6 Matching number of C_n ($n \geq 3$) = ?

if n is even :

matching number $= \frac{n}{2}$

if n is odd :

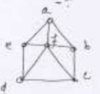
matching number $= \frac{n-1}{2}$

$\therefore$ Matching no of $C_n = \lfloor n/2 \rfloor$.

Q.7 Matching number of W_n ($n \geq 4$) = ?

if n is even

matching no. $= \frac{n}{2}$.



$\Rightarrow$

max matching

if n is odd

matching no. $= \frac{n-1}{2}$.

W_7

$\therefore$ matching no. of $W_n = \lfloor n/2 \rfloor$.

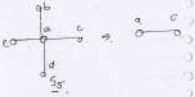
Q-8. Matching no of complete bipartite graph $K_{m,n}$ is ?

→ ∴ Matching no. of

$\underline{K_{m,n}} = \underline{\min(m,n)}$

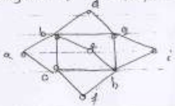
Q.9. Matching number of a star graph with 'n' vertices is ?

→ every star graph S_n is a bipartite graph of the form $K_{1,n-1}$.



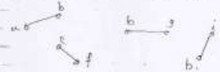
∴ $\underline{\text{matching}}$ $\underline{\text{no.}}$ $\underline{\text{of}}$ $\underline{S_n} = \boxed{1}$.

Q.10. Matching no. of the graph shown below is ?



→ no of vertices in graph = 9.

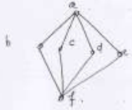
∴ Max no of vertices we can match = 8.



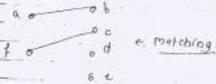
∴ max no of edges = 4.

∴ matching number of graph = 4.

Q.4. What is matching number of the graph shown below. ?

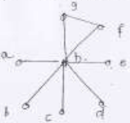


→ It is a complete bipartite graph $K_{2,4}$.



∴ matching no = $\boxed{2}$.

2. Matching no. of the graph shown below is ?



⇓ max matching

∴ matching no. = 2

13. No of maximal matchings in the graph shown below ?

perfect and maximum matchings

maximal matching

∴ total no. of maximal matchings = $\boxed{3}$.

matchings.

∴ total no. of matchings in graph = $5+3 = \boxed{8}$.

Q.14. No. of maximal matchings in the graph given below is ?

K_4

→ no. of maximal matchings $= (n-1)(n-3)$.

$= 3\times1 = \boxed{3}$.

Total no. of matchings $= 3+6+1 = \boxed{10}$.

Coverings →

* **Line covering / Edge covering →**

Let $G = (V, E)$ be a graph. A subset C of E is called a "line covering of G" if every vertex of G is incident with at ~~most~~ least one edge in C.
i.e. in a line covering, degree of every vertex is ≥ 1 for every $v \in G$.

i.e. $\deg_C(v) \geq 1 \quad ; \forall v \in G$.

ex →

$C_1 = \{\{a,b\}, \{c,d\}\}$. (minimal) ✓ minimum) ✓

$C_2 = \{\{a,d\}, \{b,c\}\}$. (minimal) ✓ (minimum)

$C_3 = \{\{a,b\}, \{b,c\}, \{b,d\}\}$ (minimal) ✓ (not minimum) ✗

$C_4 = \{\{a,b\}, \{b,c\}, \{c,d\}\}$ ✗ (not minimal)

Note →
Line covering of a graph G does not exist iff G has an isolated vertex.

Line covering of a graph with n vertices has at least $\boxed{\lceil n/2 \rceil}$ edges.

Minimal line covering -

A line covering 'c' of a graph G is said to be minimal if no edge can be deleted from 'C'.

For the graph given in above example C_1, C_2, C_3 are minimal line coverings whereas C_4 is not a minimal line covering.

Minimum line covering (smallest minimal line covering)

A minimal line covering with minimum no. of edges is called "minimum ~~of~~ line covering of G"

(2) No of edges in minimum line covering of G is called "line covering number of G"

(1) It is denoted by "α_1"

(3) For the graph given in above example, C_1 and C_2 are minimal line coverings of G and $\alpha_1 = 2$.

Properties -

(2) Every line covering contains a minimal line covering.

(4) but every line covering contains a minimum line covering is (not true)

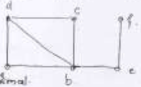
(3) The line covering C_3 does not contain any minimum line covering.

(i) No minimal line covering contains a cycle.

(ii) If a line covering contains no paths of length 3 or more, then C is a minimal line covering because all the components of C are star graphs and from a stargraph, no edge can be deleted.

Independent Line Set :-

Let $G=(V,E)$ be a graph. A subset L of E is called an independent line set of G if no two edges in L are adjacent.



$L_1 = \{\{a,b\}\}$ not maximal

$L_2 = \{\{b,d\},\{e,f\}\}$ ✓ maximal

$L_3 = \{\{a,b\},\{c,d\},\{e,f\}\}$ ✓ maximal L_4 not possible.

$L_3 = \{\{a,d\},\{b,c\},\{e,f\}\}$ ✗ not minimal.

Maximal Independent Line set :-

A independependt line set of a graph G is maximal if no other edge of G can be added to it.

For the graph given in the above example, L_2 and L_3 are maximal indep line sets.

maximum indep line set →

A maximal indep line set with max. no. of edges is called "maximum independent line set" of G.

No. of edges in a maximum independent line set of G is called "line independence number" of G.

Denote it by β_1 = matching number of graph.

For the graph above L_3 is a maximum independent line set of G and $\beta_1 = 3$.

Note → (line cov no.) (line independence no)

For any graph G, $\alpha_1 + \beta_1$ = no of vertices in G.

(with no isolated vertex)

Q.1. Line covering number of $K_n = \alpha_1 = ?$

→ line independence no. = $\beta_1 = \lfloor \frac{n}{2} \rfloor$ (matching no).

and $\alpha_1 + \beta_1 = n$ —①.

$\therefore \boxed{\alpha_1 = \lceil n/2 \rceil}$

K_4

K_5

cov no. = $n/2$. (n is even)

cov no. = $\frac{n-1}{2} + 1 = \frac{n+1}{2}$ (n is odd)

$$\therefore \boxed{\alpha_1 = \text{line covering no} = \lceil n/2 \rceil}$$

Note →

1) Line covering no α of C_n $(n \geq 3)$

$= \boxed{\lceil n/2 \rceil}$

2) Line covering no of wheel graph W_n $(n \geq 4)$

$= \boxed{\lceil n/2 \rceil}$

3) Line covering no of complete bipartite graph $K_{m,n}$

$= \underline{\max(m,n)}$.

$\alpha_1 + \beta_1 = (m+n) \rightarrow$ total no of nodes

$\alpha_1 = \max(m,n)$, β_1 = minimum matching no

ex for $\underline{K_{1,4}}$.

4 edges are reqd in minimum line covering.

4) Line covering no of a star graph with 'n' vertices $(n \geq 2)$ is

$\boxed{(n-1)}$

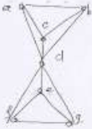
* Every star graph S_n can be represented as a complete bipartite graph $K_{1,n-1}$

So, the matching no (β_1) of $K_{1,n-1}$ is $\beta_1 = \underline{1}$.

$\therefore \alpha_1 + \beta_1 = n$

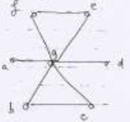
$\boxed{\alpha_1 = n-1}$ (for a star graph).

Q.2. Line covering of the graph shown below is ?



$\rightarrow$ $|V| = n = 7$. $\alpha_1 \geq \lceil n/2 \rceil = \boxed{4}$

3. Line covering of the graph show below is ?



$\Rightarrow$

$\therefore \boxed{\alpha_1 = 4}$

9.4. Give covering of the " — " ?

→

$K = 4$

∴ $\alpha_1 = 6 - 2 = \boxed{4}$

(no. of vertices) (matching no.)

5. " —" for the graph shown below

i) No. of min line coverings in graph

→

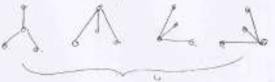
① ② ③

∴ no. of minimum line coverings = $\boxed{3}$

ii) No of minimal line coverings

→ 3 minimum line coverings

+



4

∴ total minimal line coverings = 3+4 = 7

iii) which of the foll. is a minimal line covering of G?

a) {(b,d) (a,b) (c,d)}
(can be deleted)

b) {(a,c), (b,c), (b,d)}
can be deleted

✓ c) { (a,d) , (d,c) , (b,a) } ✓ minimal.

d) { (a,d), (b,c), (a,c) } not minimal
can be deleted.

iv) Number of total line coverings in G

$= 3+4+12+C(6,4)+C(6,5)+1.$

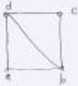
(3 min × 4 ways)

$= \boxed{41}$

Vertex-coverings →

Let $G = (V, E)$ be a graph. A subset K of V is called as "Vertex-covering" of G if every edge of G is covered by / incident with a vertex in K.

ex →



G

$K_1 = \{b, d\}$ minimal ✓

$K_2 = \{a, b, c\}$ minimal ✓

$K_3 = \{b, c, d\}$ not minimal ✗

$K_4 = \{a, d\}$ ✗ not a vertex covering.

Minimal vertex covering →

A vertex covering 'K' of a graph G is said to be minimal if no more vertex can be deleted from 'K'.

for the graph given above, K_1 and K_2 are minimal vertex coverings of G.

Minimum vertex covering → (smallest minimal vertex covering).

* A minimal vertex covering of G with min no. of vertices is called 'minimum vertex covering'

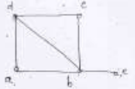
* No. of vertices in minimum vertex covering is called "vertex-covering number" of a graph, G.

It is denoted by 'α_2' (say).

For the graph given in above example, K_1 is a minimum vertex covering of G and $\boxed{\alpha_2 = 2}$

⑧ Independent vertex set →

Let $G = (V, E)$ be a graph. A subset 'S' of 'V' is called an "independent vertex set" of G if no two vertices in S are adjacent.



G.

$S_1 = \{b\}$ maximal ✓

$S_2 = \{d, e\}$ maximal ✓

$S_3 = \{a, c, e\}$ maximal ✓ maximum

$S_4 = \{c, e\}$ not maximal ✗ (a can be added).

⑨ Maximal Independent Vertex Set →

Let $G = (V, E)$ be a graph. An independent vertex set of G is said to be maximal if no other vertex of G can be added to it (S).

For the above ex., S_1, S_2 and S_3 are maximal indep. vertex sets of G.

Maximum Independent Vertex Set (this set need not be a vertex covering of G).
(Largest maximal indep. vertex set)

A maximal independent vertex set of G with max no. of vertices is called as "maximum indep. vertex set" of G.

* No. of vertices in maximum indep vertex set of a graph is called "vertex independence no. of G."

Let it denoted by 'β_0' (say).

For the graph given in above ex., S_3 is maximum independent vertex set of G and $\beta_0 = 3$.

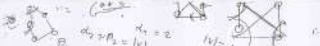
Q1. Vertex covering no for the complete graph K_n = ?

* In a complete graph, each vertex is adjacent to rem. (n-1) vertices.
∴ a maximum indep. set of K_n contains only 1 vertex.

Note: For any graph G, $G=(V,E)$.

i) $\alpha_0 + \beta_0 = |V|$.

ii) if S is an indep. vertex set of G, then $|V-S|$ is a vertex covering of G.



$\alpha_2 + \beta_2 = |V|$ $\quad \alpha_1 = 2$ $\quad |V| = 4$

$\beta_2 = 2$

$\therefore \quad \beta_2 = \boxed{1}$

$\therefore \alpha_2 = 4 - \beta_2$

$= 4 - 1$

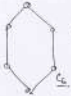
$= \boxed{3}$

$$\boxed{\begin{array}{l} \text{For } K_n, \quad \beta_2 = 1. \\ \quad\quad \alpha_2 = n-1 \end{array}}$$

Q.2 For the cycle graph C_n, $(n \geq 3)$, what is α_2, β_2

→ For C_n, (n even)

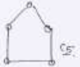
$\alpha_2 = \frac{n}{2}$



$\alpha_2 = 3 = \frac{6}{2}$

(n odd)

$\alpha_2 = \frac{(n-1)}{2} + 1$



$5/2 = \boxed{3}$

$= \frac{4}{2} + 1$

$\therefore$ For a cyclic graph C_n,

$\boxed{\alpha_2 = \lceil n/2 \rceil}$

$\therefore \beta_2 = n - \lceil n/2 \rceil = \lfloor n/2 \rfloor$

$\boxed{\beta_2 = \lfloor n/2 \rfloor}$

$$\boxed{\begin{array}{l} \text{For } C_n \\ \alpha_1 = \alpha_2 \\ \beta_1 = \beta_2 \end{array}}$$

1. For the wheel graph W_n, $(n \geq 4)$. Find α_2 and β_2 ?

→ $\alpha_2 = \begin{cases} \frac{n+2}{2}, & (n \text{ is even}) \\ \frac{n+1}{2}, & (n \text{ is odd}) \end{cases}$

W_5

$\boxed{\cancel{\alpha_2 = \lfloor n/2 \rfloor + 1}}$ (for W_n)

$\therefore \alpha_2 = 3$.

$\alpha_2 = 4$ W_6

$\boxed{\alpha_2 = \left\lceil \frac{n+1}{2} \right\rceil}$

$\boxed{\beta_2 = \left\lfloor \frac{n-1}{2} \right\rfloor}$

For a complete bipartite graph $K_{m,n}$, find α_2 and β_2 ?

→ $\boxed{\alpha_2 = \min.(m,n)}$

$K_{2,4}$

$\alpha_2 + \beta_2 = m+n$

$\therefore \boxed{\beta_2 = \max(m,n)}$

$n+1$ n - $\lfloor n/2 \rfloor + 1$ $n = \lceil n/2 \rceil - 1$ $\frac{n}{2}+1$ $\lfloor n/2 \rfloor - 1$

$\beta_2 = \lceil n/2 \rceil - 1$ $n \geq 3$ 275

$\alpha_0 + \beta_0 = n$

$n - \lfloor n/2 \rfloor - 1$

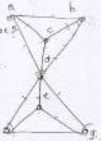
Q.5. For a star graph with n vertices, ($n \geq 2$)

$\alpha_2 = ?$ $\beta_2 = ?$

→ $S_n = K_{1,n-1}$

$\boxed{\alpha_2 = 1 \quad , \quad \beta_2 = n-1}$

Q.6. For the graph shown below, which of the following statements is/are not true?



S1) The set {b, f} is a max. independent set.

S2) $\beta_2 = 2$ S3) $\alpha_2 = 5$

S4) {a, b, c, e, g} is a minimum vertex cover.

→ S1) → The indep vertex set of 3 vertices is not possible because there is cycles of length 3. So, this is maximum.

So, S1 is true.

S2) From S1, S2 is true. ∴ $\beta_2 = 2$ ∴ S2 true

S3) $\alpha_2 + \beta_2 = 7$ ∴ $\alpha_2 = 5$ ∴ S3 true.

S4) These vertices cannot cover all the edges.

So, not true

Q.7 For the graph shown below, which of the foll. are true

$S_1)$ α_2 is 4.

$S_2)$ $\beta_2 = 2$

$S_3)$ $\{a, b, c, f\}$ is a minimum vertex cover.

$S_4)$ $\{a, c\}$ is maximum independent vertex set.

→ $\alpha_2 =$ (for a wheel graph)

$$= \left\lceil \frac{n+1}{2} \right\rceil \quad (n=6)$$

$$= \left\lceil \frac{7}{2} \right\rceil = \boxed{4}$$

$\therefore S_1)$ is true.

$\therefore \alpha_2 + \beta_2 = n.$ $\therefore \boxed{\beta_2 = 2}$

$\therefore S_2)$ is true.

$S_3)$ → $\{a, b, c, f\}$ cannot cover the edge 'ed'. So, $S_3)$ is not true.

$S_4)$ true. (Indep. vertex set with 3 vertices is not possible)

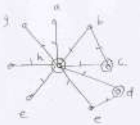
11a. For graph given below,

→

$\boxed{\alpha_2 = 3}$

$\beta_2 = 9 - 3 = 6$

$\boxed{\beta_2 = 6}$



{d, a, h} ← minimum vertex covering.

{c, a, a, g, f, e} ← maximum indep. vertex set.

Complete Matchings :-

Let $G = (V, E)$ be a bipartite graph with vertex portition $V = \{V_1, V_2\}$

A matching from V_1 to V_2 is called a "complete matching" if every vertex in V_1 is matched

i.e. in a complete matching,

$$\deg(v) = 1 \; ; \; \forall v \in V_1$$

Ex. :-

complete matching from V_1 to V_2

$\Leftarrow$ (complete matching from V_1 to V_2

G

Note :-

1) Every complete matching in a bipartite graph is a maximum matching.

2) But the converse of the above statement need not be true. i.e a maximum matching of a bipartite graph from V_1 to V_2 need not be a complete matching

ex. :-

In the max. matching from V_1 to V_2 only two vertices can be matched. So, it is not complete matching.

⇒ In a bipartite graph a complete matching from V_1 to V_2 exists only when no of vertices in V_1 should be less or equal to than no of vertices in V_2.

⇒ The converse of the above statement need not be true.

ex:-

One vertex among a, b, c cannot be mapped. So, the complete matching from V_1 to V_2 does not exist even if $m \leq n$.

a, b, c are collectively adjacent to 'd' and 'e'. So, only two of a, b, c can be mapped.

Hall theorem :-

let G be a bipartite graph with vertex partition $V = \{V_1, V_2\}$.

A complete matching from V_1 to V_2 exists iff every subset of 'k' vertices in V_1 are collectively adjacent to at least 'k' vertices in V_2.

$(k = 1, 2, 3, \ldots, |V_1|)$.

* Note :

1) In a bipartite graph with vertex partition $\{V_1, V_2\}$, a complete matching from V_1 to V_2 exists if

$\delta(V_1) \geq \Delta(V_2)$.

i.e. if min. degree of all vertices in V_1 is greater than or equal to max degree of all vertices in V_2.

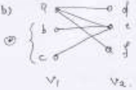
Q: Which of the following bipartite graphs has a complete matching from V_1 to V_2?

a) $K_{6,4}$

Here, no. of vertices in V_1 is greater than no. of vertices in V_2.

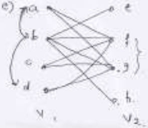
∴ Complete matching is not possible.

b)



Complete matching is not possible because in set V_1, the two vertices b and c are adjacent to same vertex e. So, only one among b and c can be mapped.

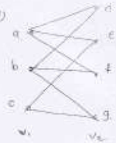
c)



In set V_1, we have 3 vertices a, c and d which are collectively adj to only f and g in V_2.

∴ Complete matching is not possible.

d)



$\delta(V_1) \geq \Delta(V_2)$

$\therefore$ complete matching is possible.

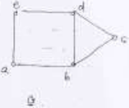
Spanning Trees →

① Let G be a connected graph. A subgraph 'H' of 'G' is called a spanning tree of G if

i) H is a tree and H

ii) H contains all vertices of G

ex. →



② Circuit Rank of graph G :

Let 'G' be a connected graph with 'n' vertices and 'm' edges.

A spanning tree 'T' of 'G' contains (n-1) edges
∴ the no of edges we have to delete from G in order to get a spanning tree is equal to $(m-n+1)$ which is called "circuit-Rank" of the graph G.

∴ Circuit Rank of G = $|E| - |V| + 1$.

For the graph given in above ex,
circuit Rank of G = $7-5+1 = \boxed{3}$.

1. Let G be a connected graph with 6 vertices and degree of each vertex 3. Find circuit Rank of G

→ By sum of degrees of vertices thm,

$\sum_{i=1}^{6} \deg(v_i) = 2 \times$ no. of edges

$\therefore 6 \times 3 = 2 \times$ no. of edges.

no. of edges $= |E| = 9$.

$\therefore$ Circuit-Rank of $G = |E| - |V| + 1$

$= 9 - 6 + 1 = \boxed{4}$

3-2 Let G be a connected graph with 7 vertices, no cycles of odd length and max no. of edges. Find circuit graph of G.

→ If a graph has no cycle of odd length, then it is a bipartite graph.

$K_{3,4}$ i.e. no. of edges will be maximum $= 4 \times 3 = \boxed{12}$

or max no. of edges in bipartite graph with 'n' vertices

$= \left\lfloor \frac{n^2}{4} \right\rfloor = \left\lfloor \frac{49}{4} \right\rfloor = \boxed{12}$

$\therefore$ Circuit Rank of $G = 12 - 7 + 1 = \boxed{6}$.

$= n+2$

$n/2 \cdot n-1$

Q.3 Circuit rank of a complete graph K_n?

→ $= |E| - |V| + 1$

$= \frac{n \cdot (n-1)}{2} - n + 1$

$= \boxed{\frac{(n-1) \cdot (n-2)}{2}}$

Q.4 Circuit Rank of a wheel graph W_n?

→ $= |E| - |V| + 1$

$= 2(n-1) - (n-1)$

$= \boxed{(n-1)}$

Q.5 Circuit rank of cycle graph C_n?

$= \underline{1}$. (if we delete one edge we get a tree).

Q.6. Circuit Rank of Graph shown below.

Circuit Rank $= 15 - 10 + 1$

$= \boxed{6}$

7. Number of spanning trees in the graph shown below

→ No. of spanning trees = 3.

I and III are isomorphic

∴ for the graph, no. of nonisomorphic spanning trees = $\boxed{2}$.

1.8. No. of spanning trees in the graph shown below,

→ No. of spanning trees = $\boxed{4}$

But all are isomorphic

∴ No. of nonisomorphic spanning trees = $\boxed{1}$

Q-9. Number of spanning trees in the graph G is ?

→ No. of spanning trees in complete graph K_n = $\boxed{(n)^{n-2}}$ (coley's formula).

∴ No. of spanning trees in K_4

$= 4^{(4-2)} = 4^2 = \boxed{16}$

Q-10. No of spanning trees in the graph shown below is

→ No of spanning trees = $\boxed{8}$

For a complete graph K_n.

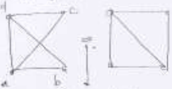
Imp. To get the no of spanning trees.

Kirchoff's theorem ?

- Let 'A' be the adjacency matrix of the connected graph 'G'
- Let 'M' be the matrix obtained from 'A' by replacing each '1' with '-1' and replacing each '0' in the principle diagonal of A with the degree of the corresponding vertex.
- The cofactor of any element of M = no. of spanning trees in G.

11. No. of spanning trees in the graph shown below?

$$A = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 1 & 1 \\ b & 1 & 0 & 0 & 1 \\ c & 1 & 0 & 0 & 1 \\ d & 1 & 1 & 1 & 0 \end{array}$$



$$M = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

cofactor of 3 = $2\left[6-1\right] - 0 + (-1)(+2)$

$= 10 - 2 = \boxed{8}$

1.12 No. of spanning trees in the graph shown below?

$$A = \begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 0 & 1 & 0 & 0 & 1 \\ b & 1 & 0 & 1 & 1 & 1 \\ c & 0 & 1 & 0 & 1 & 0 \\ d & 0 & 1 & 1 & 0 & 1 \\ e & 1 & 1 & 0 & 1 & 0 \end{array}$$

G

6 2 + = 10

3 - 4 - 12 - 6

5+9+4

18 (18)

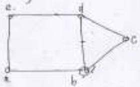
$$M = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ -1 & -1 & 0 & -1 & 3 \end{bmatrix}$$

cofactor of M_{51} = cofactor (-1)

$$= (-1)^{5+1} \begin{vmatrix} -1 & 0 & 0 & -1 \\ 4 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 3 & -1 \end{vmatrix}$$

$= \boxed{21}$

9.13. No of spanning trees in the graph shown below, is?

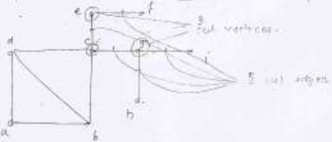


Ans = (11)

Connectivity →

→ A graph is G is connected if there exists a path betn every pair of vertices.

→ A graph G is connected iff G has a "spanning tree".



① Cut vertex / Articulation point →

Let 'G' be a connected graph. A vertex $v \in G$ is called a "cut vertex of G" if $G-\{v\}$ results in a disconnected graph. then V

For the graph given above, c, e and g are cut vertices.

→ A connected graph G with n vertices can have at most '(n-2)' cut vertices.

ex → a tree

→ cut vertices are not necessary to be there in a graph.

ex →

5) Cut edge / Bridge :-

Let 'G' be a connected graph. An edge $E \in G$ is called a "cut edge" if $(G-E)$ results in a disconnected graph.

Note :-

- Let 'G' be a connected graph and edge $E \in G$ is a cut-edge iff the edge E is not a part of any cycle in G.

- If 'G' is a connected graph with 'n' vertices, then max. no. of cut edges possible is $(n-1)$

- In a connected graph, wherever cut edge exists, cut vertex also exists because at least one vertex of a cut edge is a cut vertex.

- In a connected graph, if cut vertex exists, then a cut-edge may or may not exist.

ex →

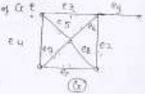
(cut vertex but no cut edge)

6) Cut set :-

Let 'G' be a connected graph $G=(V,E)$. A subset E' of E is called a "cut set of G" if deletion of all the edges of

E' from G makes G disconnected and deletion of
proper
no ~~global~~ subset of E' can disconnect G.

Q.1. ex- For the graph shown below, which of the foll. are cut sets of G?



1) $E_1 = \{e_1, e_3, e_5, e_7\}$
* It is a cut-set

2) $E_2 = \{e_1, \underbrace{e_7, e_8, e_6}\}$
(proper subset)
So, E_2 is not a cut set

E_2 is not a cutset because it has proper subset $\{e_7, e_8, e_6\}$ whose deletion can disconnect the graph.

3) $E_3 = \{e_9\}$ ✓ cut set.

4) $E_4 = \{e_3, e_4, e_5\}$ ✓ cut set

5) $E_5 = \{e_1, e_3, e_5, e_8\}$ * not a cut set

Edge connectivity of graph G $\lambda(G)$ (Imp) →

Let G be a connected graph. The min. no of edges whose removal makes G disconnected is called edge connectivity of G.

If G has a cut edge then $\boxed{\lambda(G)=1}$.

(10)13

* Vertex connectivity * $K(G)$

→ Let G be a connected graph. The minimum no. of vertices whose removal makes G either disconnected or reduces G into a trivial graph is called "Vertex connectivity" of G

Denoted by $K(G)$ (say).

* If G has a cut vertex, then $K(G)=1$.

Note →
For any connected graph G, vertex connectivity of G

$\boxed{K(G) \leq \lambda(G)}$
edge connectivity.

and $\boxed{\lambda(G) \leq \delta(G)}$
min. degree of vertices.

$\therefore \boxed{K(G) \leq \lambda(G) \leq \delta(G).}$

1. For the graph shown below, what is vertex connectivity and edge connectivity ?

→ c is a cut vertex of graph G.

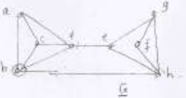
∴ $K(G)$ (vertex connectivity) $= \boxed{1}$

The graph G has no cut edge.

$\therefore \lambda(G) \geq 2$ and also $\lambda(G) \leq \delta(G)$

$\therefore \boxed{\lambda(G) = 2}$

Q.2 For the graph shown below, find $k(G)$ and $\lambda(G)$.



→ The graph has no cut edge.

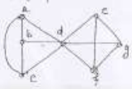
$\therefore \lambda(G) \geq 2$. $\therefore \boxed{\lambda(G) = 2}$
(by deleting edges d-e and bh).

The graph has no cut vertex

$\therefore k(G) \geq 2$. $\therefore \boxed{k(G) = 2}$
(by deleting vertices d and b (or) e and h).

(As there are two edge conn = 2 cut edges,)
Vertex conn should be ≥ 2.

Q.3: For the graph shown below, find $k(G)$ and $\lambda(G)$



G has a cut vertex d

$\therefore \boxed{\kappa(G)=1}$

Min deg. $\delta(G)=3$.

$\lambda(G) \le \delta(G)=3$ $\therefore \lambda(G) \le 3$. — (1)

and G has no cut edge

$\therefore \boxed{\lambda(G)=3}$

Q.4 For complete graph K_n, $\kappa(G)$ and $\lambda(G)$?

→ We can reduce the complete graph into trivial graph (as a complete graph cannot be reduced to disconnected graph) by deleting (n-1) vertices

K_4

$\lambda(G)=3$ $(n-1)$ for K_n

$\therefore \boxed{\kappa(G)=n-1}$

Also, $\boxed{\lambda(G)=n-1}$

For a complete graph ($\kappa(G) \le \delta(G)=\lambda(G)$).

Q.5 For a cycle graph, $\kappa(G)$ and $\lambda(G)$?

→ $\lambda(G)=2$ $\kappa(G)=2$ $\delta(G)=2$.

For a cycle graph, $\lambda(G)=\kappa(G)=\delta(G)=2$.

Q.6 For a wheel graph, $\lambda(G)$ and $K(G)$?

→ $\lambda(G) = K(G) = \delta(G) = 3$.

Q.7 For a complete bipartite graph, $\lambda(G)$ and $K(G)$?

→ $\lambda(G) = K(G) = \min(m, n)$

$K = 4$

Q.8 For a star graph with n vertices $(n \geq 2)$, $\lambda(G)$ and $K(G)$?

→ $\lambda(G) = K(G) = 1$.

Every star graph is a tree.
For any tree with n vertices,
$\lambda(G) = 1$, $K(G) = 1$.

$S_5 = K_{1,4}$

Note :

1) A simple graph with 'n' vertices is necessarily disconnected if $\boxed{|E| < (n-1)}$ because a simple connected graph with min. no. of edges is a tree and no. of edges in a tree with n vertices is $(n-1)$.

2) A simple graph with 'n' vertices is necessarily connected if $\boxed{|E| > \frac{(n-1)(n-2)}{2}}$

no. of edges in $K_{n-1} = \frac{(n-1)(n-2)}{2}$

and suppose we have n vertices, to make a graph connected we need one more edge.

Q.9 Min no. of edges necessary in a simple connected graph to ensure connectedy is ?

→ $|E| > \frac{(n-1)(n-2)}{2}$

$> 36.$ $\boxed{|E| = 37}$

Q.10. Which of the foll. graphs is necessarily connected ?

a) simple graph with 7 vertices and 14 edges

$\left(\frac{6 \times 5}{2} = \frac{30}{2} = 15 \right)$ $(14 < 15)$

∴ not necessarily connected.

b) A simple graph with 8 vertices and 21 edges

$\frac{7 \times 6}{2} = 21.$ $21 = 21$ ∴ not necessarily connected.

✓c) A simple graph with 9 vertices and 29 edges

$\frac{8 \times 7}{2} = \frac{56}{2} = 28$ ∴ $29 > 28.$

∴ necessarily connected.

10-3

Note:

1) A simple graph with 'n' vertices and 'k' edges has at least $(n-k)$ components.

2) A simple graph with 'n' vertices and 'k' components at least $(n-k)$ edges.
   i.e. no. of edges $\geq (n-k)$.

Q.11 Minimum no of edges necessary in a simple graph with 10 vertices and 3 components is?

→ $|E| = n-k = 10-3 = \boxed{7}$

Q.12 ~~A simple graph with~~

Note:

3) A simple graph with 'n' vertices and k components has at most $\frac{(n-k).(n-k+1)}{2}$ edges.

$$\therefore \boxed{|E| \leq \frac{(n-k).(n-k+1)}{2}}$$

Q.12 Max no. of edges possible in a simple ~~connected~~ graph with 10 vertices and 3 components is?

→ $|E| = \frac{(n-k).(n-k+1)}{2} = \frac{7\times 8}{2} = \boxed{28}$ (max no. of edges)

3. Let G be a simple graph with 'n' vertices and 'k' components and 'm' edges. If we delete an edge in G then the no. of components in G is ?

✓a) (k) or (k+1) b) (k-1) or (n-k)

c) (k+1) or (m-k) d) (n-k) or (m-k)

→ 

* If the edge we are deleting is not a cut edge for any component then the no. of components remain same as 'k'.

* On the other hand, if the edge we are deleting is a cut edge for any component then the no. of components become (k+1).

1.14. Let G be a simple graph with n vertices, m edges and k components. If we delete a vertex in G, then the no. of components in G should be between

a) k and n-1 b) k-1 and n-k.

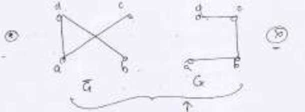
c) k and n-k. ✓d) k-1 and n-1.

* If the vertex we are deleting from the graph is an isolated vertex, which a component itself, then the no of components become (K-1).

* If the given graph is a star graph with n vertices, then by deleting the cut vertex of star graph, we get (n-1) components.

Q.15 Which of the foll. statements is/are true?

S1) If A simple graph G is connected then, then its complement $\overline{G}$ is not connected. (FALSE)



we have a counter example

✓S2) If a simple graph is not connected, then $\overline{G}$ is connected.

$\Rightarrow$

$\bar{G}$ (disconnected graph)

So, always, there will exist a path betn the vertices and no vertex will be left isolated.

14. which of the foll. is fare here?

(S1) A simple graph with n vertices is connected if $\delta(G) = \frac{n-1}{2}$

→ Suppose G is not connected.

Let G_1 and G_2 are two connected components of G

Let $v \in G_1$, $\deg(v) \geq \frac{n-1}{2}$ $\left(\because \delta(G) = \left(\frac{n-1}{2}\right) \right)$

⇒ No. of vertices in $G_1 \geq \left(\frac{n-1}{2}\right) + 1$

$= \left(\frac{n+1}{2}\right)$

$\therefore |V(G_1)| \geq \left(\frac{n+1}{2}\right)$

Similarly, no. of $|V(G_2)| \geq \left(\frac{n+1}{2}\right)$

Now, $|V(G)| \geq |V(G_1)| + |V(G_2)|$

$|V(G)| \geq \left(\frac{n+1}{2}\right) + \left(\frac{n+1}{2}\right)$

$|V(G)| \geq (n+1)$ (which is a contradiction to the our hypothesis.)

$\therefore$ The given statement is true.

5c) If a simple graph 'G' has exactly two vertices of odd degree then there exists a path betⁿ the two vertices of odd degree.

$\rightarrow$

By sum of degrees thm, if a graph has exactly two vertices with odd degree, because a component with only one odd degree is not possible.

Traversible Graphs -

A connected graph 'G' is said to be traversible if there exists a path which contains each ~~vertex~~ edge of G exactly once. and each vertex of G atleast once.

Such a path is called "Euler path".

ex. → b-c-d-b-a-d

b-c-d-a-b-d

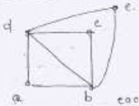
G1

So given graph is traversible.

} Not traversible.

Euler circuit ?

In a Euler path, if the starting vertex is same as ending vertex, then it is called "Euler circuit".



e-d-a-b-d-c-b-e

So, euler circuit.

each edge is used exactly once.

Theorem →

A connected graph 'G' is traversible iff no of vertices with odd degree in G is exactly 2 (or) 0.

case 1) → In a connected graph G, if no of vertices with odd degree is exactly 2, then Euler path exists but Euler circuit does not exist.

→ This Euler path begins with a vertex of odd degree and ends with the other vertex of odd degree.

case 2) → In a connected graph G, if no of vertices with odd degree is 0, then Euler circuit exists. But Euler circuit is also a Euler path. So, both Euler circuit and Euler path exist.

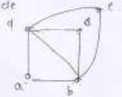
Hamiltonian Graph :

* A connected graph 'G' is said to be Hamiltonian if there exists a cycle which contains each vertex of G exactly once.

Note :

① Every cycle is a circuit but not every circuit is a cycle

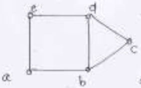
ex



a-b-d-c-b-e-d-a it is not a cycle

* The cycle in hamiltonian graph is called "Hamiltonian cycle".

* A connected graph G is said to be "Semi-Hamiltonian" if there exists a path which contains each vertex of G exactly once. Such a path is called "Hamiltonian path".



a-b-c-d-e-a
Hamiltonian cycle

a-b-c-d-e
Hamiltonian path.

Note :-
Euler circuit contains each edge of the graph exactly once, whereas in hamiltonian circuit some edges can be skipped.

Q-1. For the graph given below, denoted by G, which of the foll statements are/is true ?

S_1) ✗ Euler path exists (traversible)

S_2) ✗ Euler circuit exists

S_3) ✓ Hamiltonian cycle exists.

S_4) ✓ Hamiltonian path exists.

→

G has 4 vertices with odd degree. ∴ It is not traversible.

∴ S_1 and S_2 are false.

By skipping internal edges, the graph has hamiltonian cycle passing th' all vertices. (a-b-c-d-e-f-a).

a-b-c-d-e-f ← Hamiltonian path.

Q.2. For the graph shown below, which of the foll. is/are true? (same options).

→ G has no vertices with odd degree.

∴ S_1 and S_2 are true.

Note →

If G has a cut vertex, then Hamiltonian cycle is not possible (Hamiltonian path may exist)

So, by deleting 2 edges at C → hamiltonian path exists but hamiltonian cycle is not possible.

Q.3. " — " (same options)

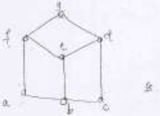
→ G has no vertices with odd degree.

∴ S_1 and S_2 are true.

In hamiltonian cycle, degree of each vertex should be 2. ∴ we have to delete two edges at vertices a and c. If we delete any vertex at a, hamiltonian cycle is not possible. Also, hamiltonian path is not possible.

If we delete two edges each at a and c, we are left with 6 vertices and 4 edges. ∴ Hamiltonian path is also not possible.

Q.4. " – " (same options)



G has 4 vertices with odd degree.
∴ G is not traversable ∴ S1 and S2 are false

To construct a Hamiltonian cycle, we have to delete one edge each at b, d and f. Then we are left with 7 vertices and 6 edges.

However, with 6 edges and 7 vertices path exists

∴ S3 false. S4 true

Sets - Theory

Topics →

* Sets
* Relations
* Functions
* Partial orders
* Lattices
* Boolean Algebras
* Groups

I) Set →

A set can be defined as a well-defined unordered collection of distinct elements.

ex → $A = \{1, 2, 3, 4, \ldots, \infty\}$. (Repitition not allowed and order not important)

$S = \{x \mid (x \text{ is a positive integer}) \text{ and } 1 \le x \le 10\}$

$\downarrow$

$S = \{1, 2, 3, \ldots, 10\}$.

Null set (empty set) →

A set with no elements is called a "Null/empty set". Denoted as 'ϕ'

ex → $A = \{x \mid x \text{ is a prime no. and } 8 < x < 10\}$.
↑ null set

Subset →

If every element of A is also an element of B, then A is subset of B.

ex → $A = \{a, b, c\}$ So, $A \subseteq B$

$B = \{a, b, c, \dots\}$

* Note → For any set A, A and ϕ are called trivial subsets of A.

Proper Subset →

Any subset of A which is not a trivial subset of A is called proper subset of A.
It is denoted by '$\subset$'

ex → $A = \{1, 2, 3, 4\}$, $B = \{2, 4\}$.

$\therefore B \subset A$ ← proper subset

* Note → If ($A \subseteq B$ and $B \subseteq A$), then $A = B$.

Power set of a set →

If A is a finite set then set of all subsets of A is called "power set of A".
It is denoted by $P(A)$

ex. If $A = \{a,b,c\}$, then $P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$

Trivial subsets: ϕ, A

* Note → If $|A| = n$, then $|P(A)| = 2^n$

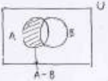
Universal set →
Set of all objects under discussion. It is denoted as 'U'

Complement of a set →
If A is any set then complement of A, denoted by $\bar{A}$ or A^c is defined as

$$A^c = \{x \mid x \notin A \text{ and } x \in U\}.$$

Set Difference →
If A and B are two sets, then

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}.$$



ex: If $A = \{1,2,3,4,5\}$ and $B = \{1,3,5,7,9\}$,
then $A - B = \{2,4\}$.

Set Intersection →

If A and B are two sets, then
$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

ex: $A \cap B = \{1, 3, 5\}$ (prev. ex.)

Set Union →

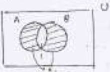
If A and B are two sets then
$A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ or } x \in A \cap B\}$.

ex: $A \cup B = \{1, 3, 5, 2, 4, 7, 9\}$.

Note → If $A \cap B$ is empty set, then A and B are called disjoint sets.

Symmetric Difference / Boolean sum →

$A \Delta B / A \oplus B = \{x \mid x \in A \text{ or } x \in B \text{ but } x \notin A \cap B\}$.



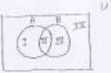
$A \oplus B / A \Delta B$

ex →
$A \oplus B = \{2, 4, 7, 9\}$.

Note → The symmetric difference of A and B →

$A \oplus B = (A - B) \cup (B - A)$
$= (A \cup B) - (A \cap B)$

b)



I. $(A-B) = (A \cap B^c)$

II. $(A \cap B)$

III. $(B-A) = (B \cap A^c)$

IV. $(A \cup B)^c = (A^c \cap B^c)$

c) For any 3 sets A, B, C, the following properties hold good:

1) If $A \subseteq B$, then $A \cup B = B$ and $A \cap B = A$

2) $(A^c)^c = A$

3) Commutative laws:

i) $(A \cup B) = (B \cup A)$

ii) $(A \cap B) = (B \cap A)$

iii) $(A \oplus B) = (B \oplus A)$

4) Associative laws:

i) $(A \cup B) \cup C = A \cup (B \cup C)$.

ii) $(A \cap B) \cap C = A \cap (B \cap C)$.

iii) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$.

5) Distributive laws :

i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

6) De Morgan's laws :

i) $(A \cup B)^c = (A^c \cap B^c)$

ii) $(A \cap B)^c = (A^c \cup B^c)$

* iii) $A - (B \cup C) = (A - B) \cap (A - C)$

* iv) $A - (B \cap C) = (A - B) \cup (A - C)$

7) Idempotent laws :

i) $A \cup A = A$

ii) $B \cap B = B$

8) Absorption laws :

i) $A \cup (A \cap B) = A$

ii) $A \cap (A \cup B) = A$

9) Modular laws :

i) $(A \cup B) \cap C = A \cup (B \cap C)$ iff $A \subseteq C$

ii) $(A \cap B) \cup C = A \cap (B \cup C)$ iff $C \subseteq A$

10) $A \cup \phi = A$, $A \cap \phi = \phi$

$A \cup U = U$, $A \cap U = A$

$A \cup A^c = U$, $A \cap A^c = \phi$

$A = \{a, b, c, d\}$ $P(A) = \{\phi,$

Q.1 which of the following is not true ?

✓ a) $A - (A - B) = B$ not true

b) $A - (A - B) = (A \cap B)$ true

c) $(A \cap B) \cup (A \cap B^c) = A$ true

d) $B \cap (A \cup B) = B$ true

Q.2 which of the following is not true ?

a) If $(A \subset B)$, then $(B^c \subset A^c)$ true

b) $A \cap P(A) = \phi$ true

✓ c) $A \cap P(A) = A$ not true

d) $P(A) \cap P(P(A)) = \{\phi\}$ not true

$A^c = II + III$, $B^c = III$

$\therefore B^c \subset A^c$

$A = \{a, b\}$

$P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

∴ no common element

Q.3. If $A = \phi$ then $|P(P(A))| = ?$

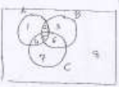
→ $A = \{\ \}$

$P(A) = \{A\}$.

$P(P(A)) = \{A, P(A)\}$ $\therefore |P(P(A))| = \boxed{2}$.

Q.4 Which of the following is not true ?

a) $(A-B)-C = (A-C)-B$. true



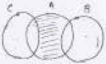
$\{1,4\} - \{4,5,6,7\}$

$= \{1\} =$ LHS.

$\{1,2\} - \{2,3,5,6\} = \{1\} =$ RHS.

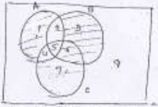
$\therefore$ LHS = RHS

b) $(A-B)-C = (A-C) - (B-C)$. true



$(A-C)-(B-C)$

c) $A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C)$ not true



LHS $= \{1,2,4,5\} \oplus \{2,3,4,5,6,7\}$

$= \{1,3,6,7\}$

R.H.S = $\{1,4,3,6\} \cup \{1,2,6,7\}$

$= \{1,2,3,4,6,7\}$.

$\therefore$ L.H.S $\neq$ RHS.

d) $A - (B \cup C) = (A-B) \cap (A-C)$. [illegible]

LHS = $\{1,2,4,5\} - \{2,3,4,5,6,7\}$

$= \{1\}$.

R.H.S = $\{1,4\} \cap \{1,2\}$

$= \{1\}$

L.H.S = R.H.S.

(11)13

Reflexive Relation :-

A relation R on a set A is said to be reflexive, if $(x R x)\ \forall x \in A$

i.e. $(x, x) \in R \quad \forall x \in A$.

Note : The diagonal relation on set A is reflexive and any superset of diagonal relation is also reflexive

Ex.:- Let A be $A = \{a, b, c\}$.

✓ $R_1 = \{(a,a), (b,b), (c,c)\}$ ← (The smallest reflexive relation on A is diagonal relation)

✓ $R_2 = \{(a,a), (b,b), (c,c), (a,b), (b,a), (a,c)\}$.

✓ $R_3 = A \times A$ ← (The largest reflexive relation on set A)

Ex. : Let $A = \{1, 2, 3, \dots n\}$, then no of reflexive relation possible on A is ?

→ No. of non diagonal elements in $|A \times A| = n^2 - n$

$= n(n-1)$

∴ No. of subsets possible with $n(n-1)$ elements is $2^{n(n-1)}$.

∴ No. of reflexive relations $= \boxed{2^{n(n-1)}}$.

Q.2 No. of relations on A which are not reflexive is?

→ Total no. of relations on A = $2^{(n^2)}$

No. of reflexive relations on A = $2^{n(n-1)}$

∴ No. of nonreflexive relations on A = $2^{n^2} - 2^{n(n-1)}$

Note :-

1) The relation '≤' is reflexive on any set of real nos.

2) The relation 'is a divisor of' denoted by '|' is reflexive on any set of non-zero real nos.
(Every number is a divisor of itself)

3) The relation 'is a subset of' denoted by '⊆' is reflexive on any collection of sets.
(Every set is a subset of itself).

Irreflexive Relation :-

The relation 'R' on a set A is called irreflexive if x is not related to x i.e. $x \not R x \; \forall x \in A$ i.e. the ordered pair $(x, x) \notin R \; \forall x \in A$

ex :- Let $A = \{a, b, c\}$

✓ $R_1 = \{\ \}$ ← (The smallest irreflexive relation on A is empty relation).

✓ $R_2 = \{(a,b), (b,c), (c,b)\}$

✓ $R_3 = \underbrace{A \times A - \Delta_A}_{}$ ← (The largest irreflexive relat^n on A)

diagonal relation

(ii) $R_3 = \{(a,b), (b,a), (b,c), (c,b), (a,c), (c,a)\}$

Note :

If A is a set with 'n' elements, then no. of irreflexive relations possible on A = $\boxed{2^{n(n-1)}}$

Q.1 Let A be a set with n elements, then no. of relations on A which are reflexive or irreflexive ?

$\rightarrow$ $= 2^{n(n-1)} + 2^{n(n-1)}$

$= 2 \cdot 2^{n(n-1)} = \boxed{2^{n(n-1)+1}}$

Q.2. " — " no. of relations on A which are neither reflexive nor irreflexive ?

$\rightarrow$ $= \boxed{2^{n^2} - 2^{n(n-1)+1}}$

(or)

The diagonal elements pairs can be selected as $2^n - 2$ ways.

$\{(1,1), (2,2), \ldots (n,n)\}$

$\therefore$ no. of relations which are neither reflexive nor irreflexive $= \boxed{(2^n - 2) \cdot 2^{n(n-1)}}$

Note : 1) The relation '$<$' is irreflexive on any set of real nos.

2) The relation '$\in$' is irreflexive on any collection of sets.

Symmetric Relation :-

A relation R on a set A is said to be symmetric if $(x R y)$ then $(y R x)$ $\forall x, y \in A$ i.e. if the ordered pair $(x, y) \in R$ then $(y, x) \in R$. $\forall x, y \in A$

ex :- Let $A = \{a, b, c\}$, then

✓ $R_1 = \{\ \}$ ← (The smallest symmetric relation on A is an empty relation)

✓ $R_2 = \{(a, a), (c, c)\}$

✓ $R_3 = \{(a, b), (b, a)\}$

✓ $R_4 = A \times A$ ← (The largest symmetric relation on A).

Ex :- Let A be a set with 'n' elements, no. of symmetric relations possible on A is :-

No. of symmetric relations with diagonal pairs $= 2^n$

No. of symmetric relations with nondiagonal pairs $= 2^{\frac{n(n-1)}{2}}$

∴ total no. of symmetric relations on A

$= 2^n \cdot 2^{\frac{n(n-1)}{2}}$

$= \boxed{2^{\frac{n(n+1)}{2}}}$

$\therefore\ 2^{\frac{n(n+1)}{2}} = 2^{n(n+1)}$

2) Let A be a set with 'n' elements then no. of relations on A which are reflexive and symmetric are →

$$= 2^{{}^nC_2} = \boxed{2^{\frac{n(n-1)}{2}}}$$

3) No. of relations which are reflexive but not symmetric

→ $$= \underbrace{2^{n(n-1)}}_{\text{no. of refl. relations}} - 2^{\frac{n(n-1)}{2}}$$ ← no. of both refl. and symm. relations

4) No of relations which are symmetric but not reflexive

→ $$\underbrace{2^{\frac{n(n+1)}{2}}}_{\text{no. of symm. relations}} - 2^{\frac{n(n-1)}{2}}$$

5) No. of relations which are neither reflexive nor symmetric →

$$2^{(n^2)} - \left(2^{\frac{n(n+1)}{2}} + 2^{n(n-1)} - 2^{\frac{n(n-1)}{2}}\right)$$

Q-1. The relation 'x' is a brother of 'y' is (symmetric or not symmetric) on any set of men.

The relation is not symmetric on set of all people, because if x is a brother of y, y can be a sister of x.

2. The relation ' ' is a complement of ' ' is symmetric on a boolean algebra.

Anti-Symmetric Relation :

A relation 'R' on a set 'A' is said to be antisymmetric if (xRy and yRx) then ($x=y$) $\forall\ x, y \in A$

Ex :- 1) The relation '$\leq$' is antisymmetric on any set of real nos.
(if $a \leq b$ and $b \leq a$, then $a=b$)

2) The relation '<' is antisymmetric on any set of real nos.
($\underbrace{\text{if } (b<a) \text{ and } (a>a)}_{F}$, then $a=b$) (always true)
→ whatever

3) The relation 'is a divisor of' denoted as '|' is antisymmetric on any set of +ve real nos.
(same logic as above)

4) The relation $\subseteq$ (set inclusion) is antisymmetric on any collection of sets.

(also, i) proper subset of '$\subset$'
ii) super set of '$\supseteq$').

Let $A = \{a, b, c\}$.

$R_1 = \{\ \}$ ← (The smallest antisymmetric relation on A)

$R_2 = \{(b,b),(a,a),(c,c)\}$

$R_3 = \{(a,b),(c,b)\}$

$R_4 = \{\underbrace{(a,a),(b,b),(c,c)}_{\text{All diagonal pairs}},\underbrace{(a,b),(b,c),(c,a)}_{\text{half of nondiagonal pairs}}\}$ ← (A largest antisymmetric relation on A)

Note: If A is a set with 'n' elements, then

1) No. of elements in a largest antisymmetric relation on A

$$= n + \frac{n(n-1)}{2} = \boxed{\frac{n(n+1)}{2}}$$

2) Let A be a set with 'n' elements. then the no. of antisymmetric relations possible on A =

$$= \underbrace{2^n}_{\substack{\uparrow \\ \text{no. of antisymmetric} \\ \text{relations with diagonal pairs}}} \cdot \left. 3^{\frac{n(n-1)}{2}} \right\} \leftarrow \text{No. of antisymmetric relations with nondiagonal pairs} = 2^n \cdot 3^{{}^nC_2}$$

(Since each nondiagonal combination can appear in 3 ways).

For

$(1,2) \to (1,2)$ only

$(1,2) \to (2,1)$ only.

$(1,2) \to$ None.

$$= \boxed{2^n \, 3^{\frac{n(n-1)}{2}}}$$

3) Let $A = \{1, 2, 3, \ldots, n\}$. No of relations on A which are symmetric and antisymmetric are $= 2^n$

$$\{(1,1), (2,2), (3,3), \ldots, (n,n)\} \to (2^n)$$

4) No of relations on A which are symmetric or antisymmetric

$$= \boxed{\underbrace{2^{\left(\frac{n(n+1)}{2}\right)}}_{\text{symm}} + \underbrace{2^n \, 3^{\frac{n(n-1)}{2}}}_{\text{antisymm}} - 2^n}$$ symm and antisymm.

5) No of relations on A which are symmetric but not antisymmetric

$$= \boxed{2^{\left(\frac{n(n+1)}{2}\right)} - 2^n}$$

6) No. of relations on A which are irreflexive and and antisymmetric

$$= 3^{nC_2} = \boxed{3^{\frac{n(n-1)}{2}}}$$

antisymm. relation with nondiagonal pairs

7) Any subset of antisymmetric relation is also antisymm

Asymmetric Relation :

A relation 'R' on a set 'A' is said to be asymmetric if (xRy) then $(y \not R x)$ $\forall x, y \in A$

1e-1) Every asymmetric relation is antisymmetric.

1) In a asymmetric relation, diagonal pairs are not allowed whereas, in antisymmetric relation, diagonal pairs can be present.

2) Every asymmetric relation is also irreflexive.

ex → Let $A = \{a, b, c\}$

✓ $R_1 = \{\ \}$ ← (The smallest asymmetric relation on A is empty relation.)

✓ $R_2 = \{(a,b), (a,c)\}$

✓ $R_3 = \{(a,b), (a,c), (b,c)\}$ ← (A largest asymmetric relation on A).

Note →

1) If A is a set with 'n' elements, then no. of asymmetric relations possible on A

$$= 3^{nC_2} = \boxed{3^{\left(\frac{n(n-1)}{2}\right)}}$$ ← (Diagonal element pairs are not allowed).

Here also, 3 possibilities... 1,2 → 1,2 or 2,1 or none

2) The only relation on A which is symmetric and asymmetric = 1 i.e. empty relation.

3) The no. of relations which are asymmetric and reflexive $= 0$. So, there is no relation which is asymmetric and reflexive.

The no. of relations on A which are irreflexive but not asymmetric is

$$= \boxed{2^{n(n-1)} - 3^{\left(\frac{n(n-1)}{2}\right)}}$$

total irreflexive

irreflexive relations that are asymmetric.

4) The relations '<' is asymmetric on any set of real nos. (if $a<b$, then $b \nless a$)

$(>, \subset, \supset)$

Transitive Relation :-

A relation 'R' on a set 'A' is said to be transitive if (xRy) and (yRz), then (xRz) $\forall x,y,z \in A$

ex:- If $A = \{a, b, c\}$, then

✓ $R_1 = \{\ \}$ → (The smallest transitive relation on A is empty relation).

✓ $R_2 = \{(a,a),(c,c),(b,b)\}$

✓ $R_3 = \{(a,b),(c,c)\}$

✓ $R_4 = \{(a,b),(a,c)\}$

✓ $R_5 = \{(a,b),(b,c),(a,c)\}$

$(a,b) \mid (b,a)$

$R_6 = \{(a,b), (b,a), (a,a), (b,b)\}$

$R_7 =$ $A \times A$ → (The largest transitive relation on A)

Note →

1) If $A = \{a, b\}$, no. of transitive relations possible on A

→ The following relations on A are not transitive →

$R_1 = \{(a,b), (b,a)\}$

$R_2 = \{(a,b), (b,a), (a,a)\}$

$R_3 = \{(a,b), (b,a), (b,b)\}$

Required no. of relations are $(2^{n^2} - 3)$ where $n = 2$

$= 2^4 - 3 = 16 - 3 = \boxed{13}$

2) The relation '$\leq$' is transitive on any set of real nos. (a ≤ b, b ≤ c then a ≤ c).

$(\geq, <, >, \subset, \supset, \subseteq, \supseteq)$

3) The relation 'is a divisor of' is transitive on any set of real nos (If 2|4, 4|8; then 2|8).

(set inclusion relation)

4) The relation 'is a subset of $\subseteq$' is transitive on any collection of sets. (If $A \subseteq B$, $B \subseteq C$ then $A \subseteq C$).

Equivalence Relations :-

A relation 'R' on a set 'A' is said to be an equivalence relation on 'A' if 'R' is :

(i) Reflexive,
(ii) Symmetric and
(iii) Transitive

ex :- Let $A = \{a, b, c\}$, then how many equivalence relations are possible on A ?

→ ✓ $R_1 = \{(a,a), (b,b), (c,c)\}$ ← (The smallest equivalence relation on A (i.e. diagonal relation))

✓ $R_2 = \{(a,a), (b,b), (c,c), (a,b), (b,a)\}$

✓ $R_3 = \{(a,a), (b,b), (c,c), (b,c), (c,b)\}$

✓ $R_4 = \{(a,a), (b,b), (c,c), (a,c), (c,a)\}$

✓ $R_5 = \{(a,a), (b,b), (c,c), (a,b), (b,c), (a,c), (c,a), (c,b), (b,a)\}$ ← (The largest equivalence relation).
<u>A×A</u>

<u>Note</u> :-
1) total no. of equivalence relations for $(n=3) = \boxed{5}$.

2) total no. of equivalence relations for $(n=4) = \boxed{15}$

$10-5 \quad 5-\frac{5}{10-5} \quad a-c \quad b-c \quad 10-10$

11/1/17

Q. Which of the following is not an equivalence relation on set of all real no. A?

a) $R_1 = \{(a,b) \mid 'a-b' \text{ is an integer}\}$. equivalence relaⁿ ✓

i.e. $aR_1b \Leftrightarrow (a-b)$ is an integer.

→ 1) we have, 'a-a' is an integer i.e. $\underline{0}$.

$\therefore aR_1a \quad \forall a \in A$

$\therefore R_1$ is reflexive relation

2) Also, if a-b is an integer, b-a is also an integer.

$\therefore$ if aR_1b, then bR_1a.

$\therefore R_1$ is symmetric.

3) if aR_1b, so, a-b is an integer.

also, bR_1c, so, b-c is an integer.

$\therefore$ a-c is also an integer.

$\therefore aR_1c$.

$\therefore R_1$ is transitive.

$\therefore R_1$ is an equivalence relation.

b) $R_2 = \{(a,b) \mid 'a-b' \text{ is divisible by } 5\}$. equivalence relaⁿ ✓

→ 1) 0 is divisible by 5 $\therefore$ a-a = 0 div by 5.

$\therefore aR_2a$ ✓

$\therefore R_2$ is reflexive.

2) a-b div by 5, then b-a also div by 5

(10-5) (5-10)

$\therefore R_2$ is symmetric ($\because$ if aR_2b exists, bR_2a exists).

3) $a-b$ div by 5 ; $b-c$ div by 5
(10-5) (5-15)

$\therefore$ $a-c$ also div by 5
(10-15)

$\therefore$ if aR_2b, bR_2c exists, aR_2c also exists).

$\therefore$ R_2 is transitive

c) $R_3 = \{(a,b) \mid 'a-b'$ is an odd no.$\}$. not equivalence relatn.

→ 1) if $a-a=0$, 0 is not an odd no.
$\therefore$ R_3 is not reflexive.

2) if $a-b =$ odd no., then $b-a =$ odd no.
$\therefore$ R_3 is symmetric

3) $a-b =$ odd no. , $b-c =$ odd no.
$\therefore$ $a-c$ = need not be odd.
$\therefore$ R_3 is not transitive.

$\therefore$ Given relation is not an equivalence relation.

d) $R_4 = \{(a,b) \mid 'a-b'$ is an even no.$\}$. equivalence relatn

Reflexive → $a-a=0$.

Symmetric → $a-b =$ even ; $b-a =$ even.

transitive → $a-b =$ even, $b-c =$ even; $a-c =$ even.

Partial Ordering Relation (Partial order) →

A relation 'R' on a set 'A' is said to be a partial ordering relation (partial order) if R is reflexive, antisymmetric and transitive.

Partially ordered set (poset) →

A set 'A' with a partial order 'R' defined on 'A' is called Partially ordered set (poset) and it is denoted by $[A; R]$

ex →1) The relation '$\leq$' is a partial order on any set of real nos. A i.e. the set A w.r.t '$\leq$' is a 'poset'
$[A; \leq]$

2) The relation 'is a divisor of' (|) is a partial order relation on any set of +ve integers
i.e. $[A; |]$ is a poset.

3) The relation 'is a subset of ($\subseteq$)' is a partial order relatn on any collection of sets 'S'
i.e. $[S; \subseteq]$ is a poset.

4) Let $A = \{a, b, c\}$.

✓ $R_1 = \{(a,a), (b,b), (c,c)\}$ ← (The smallest partial order on A)

Note → and also, the only relation on set A which is both an equivalence relation and a partial order is the diagonal relatn of (A) given above.

$R_2 = \{(a,a),(b,b),(c,c),(a,b),(b,c),(a,c)\}$.
← (Largest partial order on A).

* 19 different partial orders are possible on this set A.

Totally Ordered Set → (Linearly ordered set / chain)
A poset [A; R] is called a "totally ordered set" if every pair of elements in A are comparable
i.e. $a R b$ or $b R a$ $\forall a,b \in A$

Ex: If A is any set of real nos then the poset [A; ≤] is a totally ordered set

2) If $A = \{1,2,3,\dots,10\}$ then the poset [A; |] is not a totally ordered set.

because 2 is not related to 3 and also 3 is not related to 2. So, 2 and 3 are not comparable.

3) If $A = \{5,2,6,30,60,300\}$, then [A; |] is a totally ordered set because each pair is comparable.

4) If $S = \{\phi, \{a\}, \{b\}, \{a,b\}\}$, then [S; ⊆] is not a totally ordered set because singleton sets {a} and {b} are not comparable.

$\therefore \{a\} \not\subseteq \{b\}$ and $\{b\} \not\subseteq \{a\}$.

5) If $S = \{\phi, \{a\}, \{a,b\}, \{a,b,c\}\}$ is, then $[S; \subseteq]$ is a totally ordered set.

Q.1 Let $A = \{a,b,c\}$, which of the following is not true?

a) $R_1 = \{(a,a),(c,c)\}$ is symmetric, antisymmetric and transitive on A. (correct)

b) $R_2 = \{(a,b),(b,a),(a,c)\}$ is symmetric and antisymmetric (false). (c,a) should be there.

The relation R_2 is neither symmetric nor antisymm.
The relation R_2 is only irreflexive.

c) $R_3 = \{(a,b),(b,a),(c,c)\}$ is symmetric but not antisymmetric (correct).

d) $R_4 = \{(a,b),(b,c),(c,c)\}$ is antisymmetric but not symmetric (correct).

Q.2 Let $A = \{a,b,c,d\}$ and a relation on set A is defined as $R = \{(a,a),(b,a),(b,b),(b,c),(b,d),(c,a),(c,b),(c,c),(c,d)\}$ is

which of the foll. is true?

a) R is an equivalence relat^n.
→ There is no pair (d,d) ∴ it is not reflexive.
∴ It is not an equivalence relat^n.

b) R is an irreflexive or antisymm. relat^n.
→ Not irreflexive: (a,a), (b,b), (c,c) are present.
Not antisymm: (b,c) and (c,b) are present.

(a,b) (b,c) (a,c).

c) R is symmetric or asymmetric relatn.

→ Not symmetric → (b,a) ∈ R but (a,b) ∉ R

Not asymmetric → diagonal elements are present.

d) R is transitive

3. Let A = set of all real nos.

$R = \{(a,b) \mid b = a^k$ for some integer $k\}$.

i.e. $a R b \Leftrightarrow b = a^k$

e.g. $2 R 8 \Rightarrow 8 = 2^3$.

a) R is an equivalence relation.

b) R is partial order.

c) R is reflexive and symmetric but not transitive.

d) R is a total order.

→ $a = a^1$ $\therefore aRa$. ∴ R is <u>reflexive</u>.

R is <u>not symmetric</u> →

$2 = 4$ i.e. $4 = 2^2$ $\therefore 2R4$.

but $4 \neq 2^?$ ∴ $4 \not R 2$.

(a, b)

R is not a total order →

$2 \not R 3$ and also $3 \not R 2$. ∴ Not every pair comparable.

R is antisymmetric →
if aRb and bRa then $a = b$.

R is transitive →
if aRb and bRc then aRc
$(2R8)$ $(8R64)$ ∴ $(2R64)$

∴ Given relation is Reflexive, antisymmetric and transitive.

Q.4. which of the foll. statements is <u>not</u> true ?

(a) If a relation R on a set A is symmetric and transitive then R is reflexive. (<u>false</u>)
→ Let $A = \{a, b, c\}$
$R_1 = \{\}$ ← symmetric and transitive but not reflexive.
$R_2 = \{(a,b), (b,a), (a,a), (b,b)\}$ ← symmetric and transitive but not reflexive.

b) If a relation R on a set A is irreflexive and transitive then R is antisymmetric. (<u>true</u>)
→ Suppose, the given statement is false.
Let R be a relation on A which is irreflexive and transitive but not antisymmetric.

$a \le b$, $\neg(b \le a)$, $(a \ne b)$

Here, let $(a,b) \in R$ and $(b,a) \in R$ ($\because$ R is not antisymmetric).

$(a,a) \in R$ ($\because$ R is transitive)

⇒ R is not irreflexive because a diagonal pair exists, which is a contradiction to our hypothesis i.e. R is irreflexive.

∴ the given statement is true.

c) If R and S are antisymmetric relations on a set A then $(R \cup S)$ and $(R \cap S)$ are also antisymmetric. (false)

→ Let $A = \{a,b,c\}$. $R = \{(a,b)\}$, $S = \{(b,a)\}$ antisymmetric.

Here $R \cup S = \{(a,b),(b,a)\}$ ← not antisymmetric.

$R \cap S = \{ \}$ ← antisymmetric (always).

Any subset of antisymmetric relatn is antisymm. and $R \cap S$ is a subset of R. Hence it is always antisymm.

Note: If R is antisymmetric relation, then $R \cap S$ is antisymm. for any relation S on A.

d) If R and S are transitive relations, then $(R \cup S)$ need not be transitive but $(R \cap S)$ is always transitive. (True).

→ Let $A = \{a,b,c\}$ $R = \{(a,b)\}$, $S = \{(b,c)\}$ transitive.

$R \cup S = \{(a,b),(b,c)\}$ ← not transitive

$\{(a,b)\,(b,c)\}$ a, b   b, a   a, b
$\{(a,b)\,(b,c)\}$.   a, b.

$R \cap S \rightarrow$ always transitive

Transitive Closure of A Relation →

Let 'R' be any relation on a set 'A' then transitive closure of R denoted as R^* is defined as the smallest transitive relation on A which contains 'R'.

Note: If R is transitive, then $R^* = R$ (or R)
R is transitive iff $R = R^*$

ex. Let $A = \{a, b, c\}$
and $R = \{(a,b), (b,c)\}$.

$\therefore R^* = \{(a,b), (b,c), (a,c)\}$.

Reflexive closure of A Relation →

Smallest reflexive relation on A which contains R is called 'Reflexive closure of R' and denoted as '$R^{\#}$'.

ex →. Let $A = [a, b, c]$. $R = \{(a,b), (b,c)\}$.

then $R^{\#} = \{(a,b), (b,c), (a,a), (b,b), (c,c)\}$.

δA, diagonal relatn

$\boxed{R^{\#} = R \cup \delta(A)}$

Symmetric closure of a relation →

Smallest symmetric relation on A which contains R is called symmetric closure of R and denoted as 'R^+'.

$\boxed{R^+ = (R \cup R^{-1})}$

ex → Let $A = \{a, b, c\}$ and $R = \{(a,b), (b,c)\}$.

$R^+ = \{(a,b), (b,a), (b,c), (c,b)\}$.

Reflexive symmetric closure of R →

Reflexive symmetric closure of R
= Symmetric Reflexive closure of R.

i.e. $(R^+)^\# = (R^\#)^+$

ex → $A = \{a, b, c\}$ $R = \{(a,b), (b,c)\}$.

$R^+ = \{(a,b), (b,c), (b,a), (c,b)\}$.

ans $= (R^+)^\# = \{(a,b), (b,c), (b,a), (c,b), (a,a), (b,b), (c,c)\}$

ans $= (R^\#)^+ = \{(a,b), (b,c), (a,a), (b,b), (c,c), (b,a), (c,b)\}$

Note ?

i) Reflexive trans. closure of R

= trans. reflexive closure of R

i.e. $(R^*)^\# = (R^\#)^*$

ii) Symmetric transitive closure of R
≠ (need not be) transitive symmetric closure of R

Let $A = \{a, b, c\}$ $R = \{(a,b), (b,c)\}$.

$(R^*)^+ = \{(a,b), (b,c), (a,c), (b,a), (c,b), (c,a)\}$.

$(R^+)^* = \{(a,b), (b,c), (b,a), (c,b), (a,a), (b,b), (c,c), (a,c), (c,a)\}$

$= A \times A$

$\therefore (R^*)^+ \neq (R^+)^*$

P.1) Let $A = \{a, b, c\}$, $R = \{(a,a), (a,c), (b,b), (c,a), (c,b)\}$.
Find transitive closure of R (R^*)

→ The matrix corresp. to given relat^n is

$$\begin{array}{c|ccc|} & a & b & c \\ \hline a & 1 & \cancel{0}\,1 & 1 \\ b & 0 & 1 & 0 \\ c & 1 & 1 & \cancel{0}\,1 \end{array}$$

To find R^*, applying column by row cartesian prod

	I	II	III
column	$\{a,c\}$	$\{b,c\}$	$\{a,c\}$
Row	$\{a,c\}$	$\{b\}$	$\{a,b,c\}$
	(c,c)	~~(b,b)~~ (c,b)	(a,b)

$\therefore R^* = \{(a,a), (a,c), (b,b), (c,a), (c,b), (a,b), (c,c)\}$

$= (A \times A) - \{(b,a), (b,c)\}$

(Warshall's algorithm) →

Q.1 Let $A = \{a, b, c, d\}$

$R = \{(a,d), (b,a), (b,c), (c,a), (c,d), (d,c)\}$

Find the transitive closure of R i.e. R^*

→

$$
\begin{array}{c c}
 & \begin{array}{cccc} a & b & c & d \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \end{array} &
\begin{bmatrix}
\cancel{0}1 & 0 & \cancel{0}1 & 1 \\
1 & 0 & 1 & \cancel{0}1 \\
1 & 0 & \cancel{0}1 & 1 \\
\cancel{0}1 & 0 & 1 & \cancel{0}1
\end{bmatrix}
\end{array}
$$

	I	II	III	IV
Column	$\{b, c\}$	ϕ	$\{b, d\}$	$\{a, b, c, d\}$
Row	$\{d\}$	$\{a, c, d\}$	$\{a, d\}$	$\{a, c, d\}$

$\{b, d\}\{c, d\}$

$\{b, a\}, \{b, d\}$ $\{d, a\}$ $\{d, d\}$

$\{a, a\}\ \{a, c\}\ \{c, c\}$ $\{a, d\}\ \{b, a\}\ \{b, c\}\ \{b, d\}$ $\{c, a\}\ \{c, d\}\ \{d, a\}\ \{d, c\}$ $\{d, d\}$.

$\therefore R^* = \{(a,d), (b,a), (b,c), (c,a), (c,d), (d,c), (b,d), (d,a),$
$(d,d), (a,a), (a,c), (c,c)\}$

Q.2 Let $A = \{a, b, c, d\}$. The relation R on the set A is defined by $R = \{(a,a), (b,a), (b,b), (b,c), (b,d), (c,a), (c,b), (c,d)\}$

$R^* = ?$

→

$$
\begin{array}{c c}
 & \begin{array}{cccc} a & b & c & d \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \end{array} &
\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
1 & 1 & \cancel{0}1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

	I	II	III	IV
column	{a, b, c}	{b, c}	{b, c}	{b, c}
Row	{a}	{a, b, c, d}	{a, b, c, d}	ϕ

(a, a), (b, a), (c, a)

{b, a} ·

So, no new pairs are added
$\therefore \boxed{R^* = R}$

$\therefore$ R is transitive relation.

Equivalence classes →

Let R be an equivalence relation on set A ; then for any element $x \in A$, equivalence class of x denoted by $[x]$ is defined as

$$\boxed{[x] = \{ y \mid y \in A \text{ and } (x, y) \in R \}}$$

Note: 1) We can have $[x] = [y]$ even though $x \neq y$, $; x, y \in A$

2) Set of all distinct equivalence classes of the elements of set A define a partition of A w.r.t relation R.

E.g. If $A = \{a, b, c, d, e\}$ and an equivalence relation R on set A is

$R = \{(a,a), (b,b), (c,c), (d,d), (e,e), (a,e), (e,a), (b,d), (d,b)\}$

Find the partition of A w.r.t R.

→ eq. class of $a = [a] = \{a, e\}$
$[b] = \{b, d\}$ $[c] = \{c\}$ $[d] = \{b, d\}$.
$[e] = \{a, e\}$.

So, the partitions are distinct equivalence classes.

∴ Required partition of A w.r.t R

$= \{[a], [b], [c]\}$

$= \{\{a,e\}, \{b,d\}, \{c\}\}$

Q.5 If $A = \{a,b,c,d,e\}$ and a partition of A is given by $\{\{a,d\}, \{b,c,e\}\}$, then find the equivalence relation on A w.r.t Partition.

→ Required equivalence relation on A
= cartesian product of $\{a,d\}$ with itself.
cartesian product of $\{b,c,e\}$ with itself.

$= \{\{a,d\} \times \{a,d\}, \{b,c,e\} \times \{b,c,e\}\}$

$= \{\{a,a\}, \{a,d\}, \{d,a\}, \{d,d\}, \{b,b\}, \{b,c\}, \{b,e\}, \{c,b\}, \{c,c\}, \{c,e\}, \{e,b\}, \{e,c\}, \{e,e\}\}$

Q.6 If A is set of all real nos. and an equivalence relat^n R on A is given by aRb iff $(a-b)$ is an integer. then.

i) Find $[1]$

ii) Find $[1/2]$

iii) Find $[\sqrt{2}]$.

→ $[x] = \{y \mid y \in A \text{ and } (x-y) \text{ is an integer}\}$.

i) $[1] = \{y \mid y \in A \text{ and } (1-y) \text{ is an integer}\}$.

= set of all integers.

ii) $[1/2] = \{y \mid y \in A \text{ and } (1/2-y) \text{ is an integer}\}$

$(1-y) = 1-(n+1/2)$ where n is any integer

$\therefore [1/2] = \{(n+1/2) \mid n \text{ is an integer}\}$.

$= \{\pm 1/2, \pm 3/2, \pm 5/2, \ldots\}$

iii) $[\sqrt{2}] = \{y \mid y \in A \text{ and } (\sqrt{2}-y) \text{ is an integer}\}$

$\therefore \sqrt{2}-y = \sqrt{2}-(n+\sqrt{2})$ where n is any integer

$\therefore [\sqrt{2}] = \{(n+\sqrt{2}) \mid n \text{ is any integer}\}$.

Q.7 Let $A = \{\text{set of all integers}\}$ and an equivalence relation R on A is defined by aRb iff $(a-b)$ is divisible by 3.

Then find
i) $[0]$
ii) $[1]$
iii) $[2]$.
iv) How many distinct equivalence classes are possible ?

i) $[0] = \{ y \mid y \in A$ and $(0-y)$ is div by $3 \}$.

$= \{ \ldots -9, -6, -3, 0, 3, 6, 9, \ldots \}$

ii) $[1] = \{ y \mid y \in A$ and $(1-y)$ is div by $3 \}$

$\therefore 1-y = 1-(3n+1)$, n is any integer

$\therefore [1] = \{ \ldots -8, -5, -2, 1, 4, 7, 10, \ldots \}$.

iii) $[2] = \{ y \mid y \in A$ and $(2-y)$ is div by $3 \}$

$\therefore 2-y = 2-(3n+2)$, n is any integer

$\therefore [2] = \{ \ldots -7, -4, -1, 2, 5, 8, 11, \ldots \}$.

iv) If we take $3n+3 \rightarrow$ first set gets repeated.
$3n+4 \rightarrow$ second set gets repeated.
$3n+5 \rightarrow$ third set gets repeated.

The given eq. relation divides the given set into 3 distinct equivalence classes.

Q.8 Let A = set of all people. An equivalence relation R on A is defined as aRb iff a and b were born in the same month. How many distinct equivalence classes are possible?

a) 331 b) 12 c) 53 d) 366 e) cannot be determined.

→ Since we have 12 months

$\boxed{12}$

Least Upper Bound (LUB or Join or Supremum) →

Let [A;R] be a poset for $a,b \in A$, if there exists an element $c \in A$ such that

i) aRc and bRc

ii) if there exists any other element d such that (aRd) and (bRd) then (cRd), then c is called least upper bound of a and b.

Greatest Lower Bound (GLB or Meet) →

Let [A;R] be a poset. For $a,b \in A$, if there exists an element $c \in A$ such that

i) (cRa) and (cRb)

and

ii) if there exists any other element d such that (dRa) and (dRb) then (dRc),

then c is called Greatest Lower Bound (GLB) of a and b.

Q.9 If A is any set of real nos., then [A; ≤] is a poset. For $a,b \in A$ find i) LUB of a and b. $= a \cup b$.
ii) GLB of a and $b = a \cap b$.

→ i) LUB of a and b = max. of $\{a,b\}$.

ii) GLB of a and b = min. of $\{a,b\}$

If A is any set of positive integers, then $[A, |]$ is a poset for $a, b \in A$.

$\rightarrow$ LUB of a and b $= a \cup b =$ L.C.M of (a, b).

GLB of a and $b = a \cap b =$ G.C.D of (a, b).

If S is any collection of sets, then $[S; \subseteq]$ is a poset. For $A, B, \in S$

$\rightarrow$ LUB of A and $B =$ $A \cup B$

GLB of A and B $= A \cap B$

* In a poset, the least upper bound of any two sets/elements if exists, is unique.

The above statement is true for GLB also.

Join Semi Lattice → ([A; R])

A poset 'A' w.r.t relation 'R' is called a join semi lattice if least upper bound exists for every pair of elements in A.

Meet Semi Lattice → ([A; R])

A poset 'A' w.r.t relation 'R' is called a meet semi lattice if GLB exists for every pair of elements in A.

Lattice →

A poset 'A' w.r.t 'R' is ([A; R]) is called a lattice if LUB and GLB exists for every pair of elements in 'A'.

7.
Ex • Suppose if $A = \{1, 2, 3, \dots, 10\}$ with the divisibility, then the poset $[A; |]$ is a meet semilattice but not a join semilattice.

→ the given statement is true because the GLB of any two nos is GCD of the two nos.
The GCD of any two nos in the given set exists in the set.

However the given poset is not a join semilattice.
ex: LUB of 3 and 4 is equal to LCM of 3 & 4 = 12 which is not present in set.

If $S = \{\{a\}, \{b\}, \{a, b\}\}$ then the poset $[S; \subseteq]$ is a join semilattice but not a meet semilattice.

→ The LUB for $\subseteq$ operation on sets is $A \cup B$. The union of all pairs exists in set so, it is join semilattice.

However, the GLB of $\{a\}$ and $\{b\}$ is ϕ which is not present in S ∴ it is not meet semilattice.

If A is any set of real nos, then poset $[A; \leq]$ is a lattice.

→ $3, 4 \rightarrow$ LUB = 4 always exists
GLB = 3 for any pair of real nos.

∴ It is a lattice.

*The poset given in this example is a totally ordered set and every totally ordered set is a lattice.

If $A = \{1, 2, 6, 12, 36\}$. The poset $[A; |]$ is a lattice.

→ The given poset is a totally ordered set so true.

If A = set of all +ve integers, then poset $[A; |]$ is a lattice.

→ LCM of any two +ve integers is +ve integer $\in A$.

GCD of any two +ve integers is not <1.
$\therefore \in A$.

$[A; |]$.
∴ It is a lattice.

(imp) lattice

Note :
→ If 'n' is a +ve integer then
D_n = set of all +ve divisors of n.

$D_6 = \{1, 2, 3, 6\}$
$D_{12} = \{1, 2, 3, 4, 6, 12\}$
$D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

If n is a +ve integer, then the poset $[D_n; |]$ is a lattice.

Note :
Let ; $A = \{a, b, c\}$.

$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$
then $[P(A); \subseteq]$ is always a lattice.

If A is a finite set, then poset $[P(A); \subseteq]$ is a lattice.

A lattice 'A' is denoted by the $[L, \vee, \wedge]$
LUB GLB

The following properties hold good in a lattice.

(for any 3 element $a, b, c \in L$),

1) Commutative laws →

$a \vee b = b \vee a$
$a \wedge b = b \wedge a$

2) Associative laws →

$(a \vee b) \vee c = a \vee (b \vee c)$
$(a \wedge b) \wedge c = a \wedge (b \wedge c)$

3) Idempotent laws →

$a \vee a = a$
$a \wedge a = a$

4) Absorption laws →

$a \vee (a \wedge b) = a$
$a \wedge (a \vee b) = a$

Note → In a lattice L, $(a \vee b) = b$ iff $(a \wedge b) = a$, $\forall\, a, b \in L$

Sublattice →
Let L be lattice $[L, \vee, \wedge]$. A subset M of L is called a sublattice of L, if

i) M is a lattice i.e. $[M, \vee, \wedge]$.

ii) for any pair of elements $a, b \in M$, the LUB (and GLB) are same in M and L.

Distributive Lattice →
A lattice $[L, \vee, \wedge]$ is said to be distributive if the following distributive laws hold good →

i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ $\Big\}$ $\forall a, b, c \in L$.

Bounded Lattice →

Let L be a lattice w.r.t R. If there exists an element $I \in L$ such that $(a R I)$ $\forall a \in L$ then I is called upper bound of the lattice L.

Similarly, if there exists an element $O \in L$, such that $(O R a)$ $\forall a \in L$, then O is called lower bound of the lattice L.

In a lattice, if upper bound and lower bound exists then it is called a bounded lattice.

Note → In a bounded lattice, the upper bound (and also lower bound) is unique.

In a bounded lattice, the foll. properties hold good →

1) LUB of a and I i.e. $a \vee I = I$

2) GLB of a and I, i.e. $a \wedge I = a$.

3) LUB of a and O i.e. $a \vee O = a$

4) GLB of a and O i.e. $a \wedge O = O$.

$\Rightarrow$

In a lattice or poset,
if $a \leq b$ then $(a \vee b) = b$ $\in$ LUB
and $(a \wedge b) = a$ $\in$ GLB.

Complement of an element (in a lattice) →

Let L be a bounded lattice, for any element $a \in L$,
if there exists an element $b \in L$ such that
$(a \vee b) = I$
and $(a \wedge b) = 0$,
then b is called complement of a written as $\bar{a} = b$
i.e. a and b are complements of each other.

Note: 1) In a lattice, complement of an element may or may not exist.

If it exists, it need not be unique.

2) In a distributive lattice, complement of an element if exists, is unique. In a distributive lattice, each element has atmost one complement.

Complemented Lattice →

Let L be bounded lattice, if each element of L has a complement in L, then L is called a complemented lattice.

In a complemented lattice, each element has ~~at maximum~~ at least one complement.

Boolean Algebra →

A lattice 'L' is said to be a Boolean Algebra if 'L' is distributive and complemented.

In a boolean algebra, each element has an unique complement.

Hasse Diagram (finite Poset diagram) →

Let [A; R] be a finite poset. On the poset diagram of A,

i) There is a vertex corresponding to each element of A.

ii) An edge between the elements 'a' and 'b' is not present in the diagram if there exists an element $x \in A$ such that $(a R x)$ and $(x R b)$

iii) An edge between the elements 'a' and 'b' is present iff $a R b$ and there is no element 'x' $\in A$ such that $(a R x)$ and $(x R b)$.

Q.1. If $A = \{-1, 0, 1.2, \sqrt{2}, 2\}$ then no. of edges in the Hasse diagram of the poset $[A; \leq]$.

-1 is Lower Bound.

and since every element is related & it is upper bound.

$\therefore$ No. of edges $= \boxed{4}$.

Hint:- a totally ordered set, the complement exists only for upper bound (I) and lower bound (O).

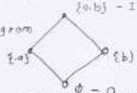
∴ the given poset is not a complemented lattice. However, the given poset is a bounded and distributive lattice.

3) Every totally ordered set is a distributive lattice.

Q.2 If $S = \{\phi, \{a\}, \{b\}, \{a,b\}\}$. then number of edges in the poset diagram of the poset $[S; \subseteq]$ is ?

→ No. of edges in Hasse diagram

$= \boxed{4}$



LUB and GLB exists for every pair of elements. Hence, the given poset is distributive.

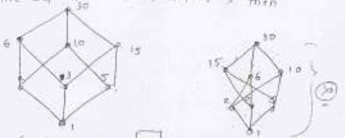
for $\{a\}$ and $\{b\}$, GLB is ϕ and LUB is $\{a,b\}$.
Also, given poset is a complemented lattice.

The poset given in the example is a boolean algebra.

Q.3. If $A = \{1, 2, 3, 6, 9, 18\}$, then no. of edges in the poset diagram of poset $[A; /]$ is ?

→ ∴ No. of edges $= \boxed{7}$

Q.4. The Let $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$ then



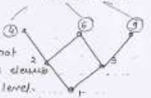
∴ no. of edges = $\boxed{12}$

complement of 2 → 15 (∵ GLB of 2 and 15 is 1 i.e. Lower bound

and LUB of 2 and 15 is 30 i.e. upper bound)

Q.5. Let $A = \{1, 2, 3, 4, 6, 9\}$ then no. of edges in the hasse diagram of poset $[A; |]$ is

→ No of edges = $\boxed{6}$



Here upper bound does not exist as more than one element exist at the same top level.

3)

Maximal elements →

If in a poset, an element is not related to any other element then it is called maximal element.

For any two maximal elements, LUB does not exist.

The given poset is not a lattice.

→ However, here for every pair of elements in the poset, GLB exists. Hence, the given poset is a meet semilattice but not a join semilattice.

1.6. Let $A = \{2, 3, 4, 6, 12\}$ then no. of edges in the Hasse diagram of the poset $[A; |]$ is ?

→ ∴ No. of edges = $\boxed{5}$

Here, lower bound does not exist as there are two minimal elements 2 and 3.

Note →

Minimal elements →

In a poset, an element is called minimal, if no other element of the poset is related to it.

For any two minimal elements, GLB does not exist.

→ The given poset is not a lattice. However, for every pair of elements in the poset, LUB exists. Hence, the given poset is a join semilattice and not a meet semilattice.

1.7. The poset $[\{2, 3, 4, 9, 12, 18\}; |]$ is

a) a join semilattice but not a meet semilattice.

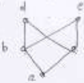
b) a meet semilattice but not a join semilattice.

c) a lattice

✓d) neither a join semilattice nor a meet semilattice.

If the hasse diagram [the given poset] has one of the above structure it is a lattice

1. The poset diag. of a poset $P = \{a,b,c,d,e\}$ is shown below.



Which of the foll statements is not true?

a) P is not a lattice. true

b) The subset $\{a,b,c,d\}$ of P is a lattice. true

c) The subset $\{b,c,d,e\}$ of P is a lattice. not true

d) The subset $\{a,b,c,e\}$ of P is a lattice. true

• The given poset is neither join semilattice and meet sublattice because for d and e GLB does not exist

9. The hasse diag. of a Lattice $L = \{x,a,b,c,d,e,y\}$ is shown below.

which of the foll subsets of L are sublattices of L?

a) $\{x, a, b, y\}$

→ It's a lattice. But the LUB of a and b is not same as in original. In L, LUB of a and b is c, here it is y.

Hence, this is not a sublattice of L.

b) $\{x, a, c, y\}$

Here, LUB of a and c = c

GLB of a and c = a

(same as in original)

∴ It is a sublattice of L.

c) $\{x, a, e, y\}$

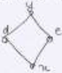
LUB of a and e = y

GLB of a and e = x.

(Same as in given L).

∴ It is a sublattice of L.

d) $\{x, d, e, y\}$

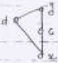


LUB of d and e = y

GLB of d and e = x

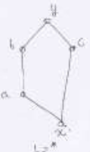
(Same as in given L).

e) $\{x, c, d, y\}$



It is not a sublattice of L because GLB of c and d here is x and a in L.

18. Which of the following lattices is not distributive?



i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
LHS RHS
$a \vee x = y \wedge y$
$\Rightarrow a \neq y$

$\therefore$ The given L_1^* is not distributive.

ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$\Rightarrow a \vee x = b \wedge y$

$\Rightarrow a \neq b$

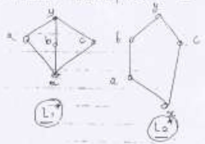
$\therefore$ The given L_2^* is not distributive.

Note: In a distributive lattice, each element can have at most one complement. If it is not true, the lattice is not distributive.

In L_1^*, for the element a, we have two complements b and c. Hence, L_1^* is not a distributive lattice.

In L_2^*, similarly, for the element c, we have two complements a and b. Hence, L_2^* is not a distributive lattice.

Note: A lattice L is not distributive iff L has a sublattice which is isomorphic to L_1^* or L_2^*.



Q11. Which of the following statements is not true

a) A lattice with 4 or fewer elements is distributive.
→ The statement follows from above thm. true

b) Every totally ordered set is a distributive lattice.
→ The statement follows from thm as a totally ordered set is a chain and a chain cannot have a sublattice isomorphic to L_1^* or L_2^*. true

c) Every sublattice of a distributive lattice is also distributive. true.
→ This statement follows from above thm.

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d) Every distributive lattice is a bounded lattice. **false**

→ Let $N = \{1, 2, 3, 4, 5, \dots\}$

the poset $[N; \leq]$ is a totally ordered set and hence a distributive lattice. But it is not bounded lattice because the upper bound of the lattice does not exist. Hence, it is false.

12. Which of the following is not a distributive lattice ?

a) $[P(A); \subseteq]$ where $A = \{a, b, c, d\}$. ✓

→ The elements of the power set are sets, and for any sets, distributive laws hold good.

$\therefore [P(A); \subseteq]$ is a distributive lattice.

b) $[D_{81}; |]$ ✓

→ $D_{81} = \{1, 3, 9, 27, 81\}$

the given lattice It is a totally ordered set and hence, a distributive lattice.

c) $[D_n; |]$ ($\because$ n is any positive integer) ✓

→ $[D_n; |]$ is a distributive lattice

e.g. → $[D_{12}; |] = [\{1, 2, 3, 4, 6, 12\}; |]$ is a totally ordered set.

d) $[\{1, 2, 3, 5, 30\}; |]$ not distributive

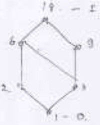
Q.13 For the lattice $[D_{18}; |]$, which of the foll. are not true?

a) The complement of $1 = 18$ true

b) " — " of $2 = 9$ true

✓c) " — " of $3 = 6$. false

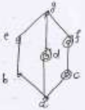
d) " — " of 6 and 3 does not exist true



→ The lattice given in this example is a distributive lattice but not a complemented lattice.

Q.14 For the lattice shown below, how many complements does the element e have?

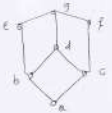
a) 1 b) 2 ✓c) 3 d) 4.



→ f, c and d are complements of e.

The lattice given in this example is a complemented lattice (as each element has at least one complement) but not a distributive lattice (as each element has more than one complement).

15. The lattice shown below is



→ a) dist lattice but not a complemented lattice.

b) complemented but not a dist lattice.

c) Boolean Algebra (both dist. and complemented)

✓d) Neither distributive nor complemented lattice.

→ e has two complements f and c. So, not a distr. lattice.

d has no complement. So, not a complemented lattice.

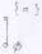
16. If A is a finite set then the poset [P(A); ⊆] is a Boolean Algebra.

→ Complement of X = (A−X) ∀ X ∈ P(A)

17. If A = {a,b,c} then the poset [P(A); ⊆] is a Boolean algebra.

→ complement of {a,c} = {b}.

If $A = \{a\}$ → Hasse diagram



If $A = \{a, b\}$ → Hasse diagram

If $A = \{a, b, c\}$ → Hasse diagram

1) Every Boolean algebra is isomorphic to one of the power sets given in the above ex.

2) So, in the hasse diagram of Boolean algebra, we have a^n vertices and $n \cdot 2^{n-1}$ edges.

3) A lattice with two elements is a totally ordered set and also a Boolean algebra. A totally ordered set with n elements is a Boolean algebra iff $n = 2$.

Square-Free Integer →

A positive integer 'n' is said to be square free if the set D_n has no perfect squares (except 1).

(OR)

A positive integer n is square free if n is a product of distinct prime nos

Theorem →

The poset $[D_n ; |]$ is a boolean algebra iff n is a squarefree number/integer.

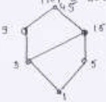
If the poset $[D_n ; |]$ is a boolean algebra then complement of $x = \boxed{\frac{n}{x}}$ $\forall x \in D_n$

Q. which of the following is not a boolean algebra?

a) $[D_{110}, |]$ → $(1,2,5,11)$ 110 is a squarefree integer. So, given poset is a Boolean algebra

b) $[D_{91} ; |]$ → $\{1,7,13\}$ 91 is a squarefree integer. So, given poset is a Boolean algebra.

c) $[D_{45} ; |]$ → $\{1,3,5,3\}$ 45 is not a squarefree integer. So, it is not a boolean algebra. (distributive but not a complemented lattice).



d) $[D_{64} ; |]$ → 64 is not a squarefree no. So, not a boolean algebra.

Q.19. In the boolean algebra $[D_{110}; 1]$, complement of $22 = ?$

→ complement of $22 = \frac{110}{22} = 5$.

complement of $5 = \frac{110}{5} = 22$.

Groups

Algebraic Structure :-

A nonempty set S is called an algebraic structure wrt the binary operation $*$ if $(a*b) \in S$ $\forall a,b \in S$ ie $*$ is a closure operation on S.

The algebraic structure is denoted by $(S,*)$.

$N = \{1,2,3,4,\dots \infty\}$

Z = set of all integers
$= \{0, \pm1, \pm2, \pm3, \dots \infty\}$

Q = set of all rational numbers
R = set of all real nos
C = set of all complex nos.

ex:- 1) $(N,+)$ is an algebraic structure.
$(a+b) \in N$ $\forall a,b \in N$.

2) $(N,\cdot)$ is an algebraic structure.
$(a\cdot b) \in N$ $\forall a,b \in N$.

3) $(N,-)$ is not an algebraic structure.
$(2-3) \notin N$

4) $(Z,-)$ is an algebraic structure.
$(2-3) \notin N$ but $(2-3) \in Z$.

5) $(Z,\div)$ is not an algebraic structure.
$(4 \div 5) \notin Z$.

6) $(\mathbb{Q}, \div)$ is not an algebraic structure.
$(1 \div 0) \notin \mathbb{Q}$

So, Let $\mathbb{Q}^* = (\mathbb{Q} - \{0\})$ = set of all nonzero rational nos.

7) $(\mathbb{Q}^*, \div)$ is an algebraic structure.
$(a \div b) \in \mathbb{Q}^*$

Semi-Groups →

An algebraic structure $(S, *)$ is called a semigroup if $(a * b) * c = a * (b * c)$ $\forall a, b, c \in S$
i.e. $*$ is associative on S

ex → 1) $(N, +)$ is a semigroup
$(a + b) + c = a + (b + c)$ $\forall a, b, c \in N$
i.e. $+$ is associative on N.

2) $(N, \cdot)$ multiplication $(\cdot)$ is a semigroup.
$(a \cdot b) \cdot c = a \cdot (b \cdot c)$ $\forall a, b, c \in N$.

3) $(Z, -)$ is not a semigroup.
$(a - b) - c \neq a - (b - c)$

4) $(\mathbb{Q}^*, +)$ is not a semigroup. (where $\mathbb{Q}^* = \mathbb{Q} - \{0\}$)
$a + (-a) = 0 \notin \mathbb{Q}^*$ $\forall a \in \mathbb{Q}^*$

5) $(\mathbb{Q}^*, \cdot)$ is a semigroup.

Monoid →

A semigroup $(S, *)$ is called a monoid if there exists an element $e \in S$ such that

$(a * e) = (e * a) = a \quad \forall a \in S$

The element e is called identity element of S w.r.t $*$

ex→ 1) $(N, \cdot)$ is a monoid, with identity element $1 \in N$.
$(a \cdot 1) = a \quad \forall a \in N$

2) $(N, +)$ is not a monoid, as $0 \notin N$.

3) $(Z, +)$ is a monoid, as $0 \in Z$.

Group →

A monoid $(S, *)$ with identity element e is called a group if to each element $a \in S$, there exists an element $b \in S$, such that
$(a * b) = (b * a) = e$

then b is called inverse of a denoted as a^{-1}
$\therefore a^{-1} = b$ and $b^{-1} = a$.

ex → 1) $(Z, +)$ is a group
$a + (-a) = 0 \quad$ So, $a^{-1} = (-a)$.

2) $(Q, \cdot)$ is not a group

$a \cdot (1/a) \neq 1$ for $a = 0 \in Q$.
So, inverse does not exist for 0.

3) $(Q^*, \cdot)$ is a group. ($\because Q^* = Q - \{0\}$)
(with identity element 'e')

Let In a group $(G, *)$ the following properties, w.r.t. the following properties must hold good →

1) The identity element of G is unique.

2) The inverse of any element in G is unique.

3) The inverse of identity element e is e itself.

4) Cancellation laws

$(a * b) = (a * c) \Rightarrow b = c$

$(a * c) = (b * c) \Rightarrow a = b$

5) $(a * b)^{-1} = (b^{-1} * a^{-1})$ $\forall a, b \in G$.

Abelian Group (Commutative Group) →

A group $(G, *)$ is said to be abelian if $(a * b) = (b * a)$ $\forall a, b \in G$.

ex →1) $(Z, +)$ is an abelian group.
$(a + b) = (b + a)$ $\forall a, b \in Z$

2) $(R^*, \cdot)$ is an abelian group (where $R^* = R - \{0\}$)

1) Set of all nonsingular matrices of order $n \times n$ is a group w.r.t. matrix multiplication, but not an abelian group because matrix multiplication is not commutative.

$(A \cdot B)$ and $(B \cdot A)$ may or may not be equal.

2) Set of all bijections on a finite set A is a group w.r.t. function composition O, but not an abelian group because function composition is not commutative.

$I: A \to A$ so, $(f \circ I) = f$ so identity function exists.

$(f \circ g) = A \to A$ it is associative also.

But, in general, $\boxed{(f \circ g) \neq (g \circ f)}$ so, it is not commutative.

1. Which of the following is/are true?

S1) In a group $(G, *)$ with an identity element 'e', if $a * a = a$, then $a = e$ true

→ suppose, $a * a = a$

$\Rightarrow a * a = a * e$ $(\because a = a)$

$\therefore \boxed{a = e}$ (by left cancellation law).

S2) In a group $(G, *)$, if $x^{-1} = x$ $\forall x \in G$, then G is abelian group true.

→ we have, $(a * b)^{-1} = (b^{-1} * a^{-1})$ $\forall a, b \in G$

$(a * b) = (b * a)$ $(\because x^{-1} = x \ \forall x \in G)$.

7/11/13

$\therefore$ G is abelian group.

Q. In a group $(G, *)$, if $(a*b)^2 = a^2*b^2$ $\forall a,b \in G$, then G is abelian group. true

$\rightarrow$ $a^2 = a*a$, $a^3 = a*a*a$

Given that $(a*b)^2 = a^2*b^2$ $\forall a,b \in G$

$\Rightarrow (a*b)*(a*b) = (a*a)*(b*b)$

$\Rightarrow \cancel{a}*(b*a)*\cancel{b} = \cancel{a}*(a*b)*\cancel{b}$ (by associative law)

$\Rightarrow (b*a) = (a*b)$

$\therefore$ G is an abelian group.

Q.2. If $A = \{1,3,5,7,8,\ldots\infty\}$ and $B = \{2,4,6,8,10,\ldots\infty\}$ which of the following is not a semigroup?

✓a) $(A,+)$ b) $(A,\cdot)$ c) $(B,+)$ d) $(B,\cdot)$

(ex: $1+3 = 4 \notin A$)

$\rightarrow$ closure property fails for a. i.e. $(a+b) \notin A$ $\forall a,b \in A$.

$\therefore$ $(A,+)$ is not a semigroup and also not a algebraic structure.

Q.3 Let $A = \{1,2,3,4,5,6,\ldots\infty\}$, then and a binary operation $*$ is defined by $a*b = a^b$ $\forall a,b \in A$. which of the foll. is true?

a) $(A,*)$ is a semigroup but not a monoid.

b) $(A,*)$ is a monoid but not a group.

c) $(A,*)$ is a group.

✓d) $(A,*)$ is not a semigroup. ✓(Associative prop. fails).

$\Rightarrow$ we have, $(a*b) = a^b \; \forall a,b \in A$
and $*$ is a closed operation on A.

$(a*b)*c = (a^b)*c = (a^b)^c = a^{bc}$ $\quad$ $a*(b*c) = a*(b^c)$
$= d^c \quad (a^b = d)$ $\quad$ $= (a)^{(b^c)} = a^{b^c}$
$= a^e \quad (\because e = b^c)$

$\therefore (a*b)*c \neq a*(b*c)$

$\therefore *$ is not associative operation on A

$\therefore (A,*)$ is not a semigroup.

3.4. Let $A = \{x \mid 0 < x \leq 1$ and x is a real no.$\}$, then A w.r.t multiplication is

a) A semigroup but not a monoid.

✓b) A monoid but not a group

c) A group.

d) Not a semigroup

$\Rightarrow$ $\cdot$ is a closed operation on A.
$\cdot$ is an associative operation on A.
1 is an identity element in A
$\Big\}$ $\therefore$ (a monoid but not a group)

w.r.t. multiplication, inv. of any element of A except 1 does not exist

5. Let A = set of all integers and a binary operation $*$ is defined by $(a*b)$ = $\min(a,b)$. Then $(A,*)$ is

some options.

→ $*$ is a closed operation on A.
also $*$ is an associative operation on A.
→ $\because (a*b)*c = a*(b*c)$

Let e be the identity element
$\therefore a*e = a$ (by defn of identity element).
$\min(a,e) = a \quad \forall a \in A$
Here, e should be a greatest integer. But the greatest integer does not exist.

$\therefore (A,*)$ is a semigroup but not a monoid.

6. Let S = set of all bit strings including the null string ϵ. '+' denotes the string concatenation. $(S,+)$ is

a) a semigroup but not a monoid.

b) a monoid but not a group.

c) a group.

d) not a semigroup.

→ $+$ is a closed operation on S. ($\because$ 101 + 1101 = 1011101.)

$a+(b+c) = (a+b)+c$ $\left(\because \begin{array}{l} 101 + 1101 = 1011101 + 11 = \underline{10111011} \\ 101 + (1101\,11) = 101 + 110111 = \underline{10111011} \end{array} \right\}$ same

$\therefore +$ is associative on S.

$\in$ is the identity element in S.

But inverse does not exist w.r.t '+' (for nonempty bit string)

($\because$ String + String $\neq \in$ for any string)

$\therefore (S, +)$ is a monoid but not a group.

7. Let A = set of all +ve rational nos. and binary operation $*$ is defined by $(a*b) = \frac{ab}{3}$ $\forall a, b \in A$. then which of the foll. statements are true?

✓ S_1) $(A, *)$ is a group.

✓ S_2) The identity element of A w.r.t $*$ is 3

✓ S_3) The inverse of $a = \frac{3}{a}$ $\forall a \in A$.

A. We have, $(a*b) = \frac{a \cdot b}{3} \in A$ $\forall a, b \in A$.

$\therefore *$ is a closed operation on A.

$a*(b*c) = a * \frac{bc}{3} = \frac{abc}{9}$ $\qquad (a*b)*c = \frac{ab}{3} * c = \frac{abc}{9}$

$\therefore *$ is an associative operation on A.

Let e be the identity element.

$\therefore a*e = a$ $\forall a \in A$. $\therefore \frac{a \cdot e}{3} = a$ $\therefore \boxed{e = 3}$

$a * a^{-1} = e$

$\frac{a \cdot a^{-1}}{3} = 3$ $\boxed{a^{-1} = \frac{9}{a}} \in A$

$\therefore (S,*)$ is a group.

9. Let A = set of all real nos. $*$ be a binary operation $(a*b) = a+b+ab$ then which of the foll. are true?

1) $(A,*)$ is a group

2) The identity element of A w.r.t $*$ is -1

3) The inverse of $a = -2/3$.

$*$ is a closed operation on A.

$(a*b)*c = (a+b+ab)*c$ $\qquad a*(b*c)$

$= a+b+c+ab+ac+bc+abc \qquad = a+(b+c+bc)$

$\qquad = a+b+c+bc+ab+ac+abc.$

$\therefore *$ is an associative operation on A.

Let e be the identity element.

$\therefore a*e = a \quad \forall a \in A. \quad a+e+ae = a \quad \forall a \in A$

$e(1+a) = 0.$

$\boxed{e = 0}$

Let a^{-1} = inv. of $a \quad \forall a \in A.$

$\therefore a * a^{-1} = e \quad ; a * a^{-1} = 0.$

$\therefore a + a^{-1} + a \cdot a^{-1} = 0.$

$\therefore \boxed{a^{-1} = \frac{-a}{a+1}}$ But (i) is not true for $a = -1$) (inv. for -1 does not exist).
$\therefore (A,*)$ is not a group.

Q. Which of the following is not a group?

a) $\{0, \pm 2, \pm 4, \pm 6, \ldots \infty\}$ w.r.t '+'

→ The set is abelian group w.r.t '+'

b) $\{0, \pm k, \pm 2k, \pm 3k, \ldots \infty\}$ w.r.t '+'

→ This set is an abelian group w.r.t '+'

c) $\{2^n \mid n$ is an integer $\}$ w.r.t multiplication $(G, \cdot)$

→ $2^a \cdot 2^b = 2^{a+b} = 2^c$ ∴ is closed on given set

$a^a \cdot (b \cdot a^c) = (a^a \cdot 2^b) \cdot 2^c$ ∴ is associative on given set.

$2^0 = 1$ → identity element ∈ set.

$2^a \cdot 2^{-a} = 1$ ∀ a ∈ set, inverse exists.

∴ The given set $(S, \cdot)$ is a group

Also, $(2^a \cdot 2^b) = (2^b \cdot 2^a)$ ∴ S is an abelian group.

d) Set of all complex nos. w.r.t multiplication.

→ S = $\{a + ib \mid a$ and b are real nos.$\}$

0 is also complex no. Inv. of $(a+ib) = \frac{1}{(a+ib)}$

∴ Inv. of 0 does not exist.

∴ S is not a group

Finite Groups →

A group with finite no of elements is called a finite group.

Order of a finite group $(G,*)$ denoted by $o(G)$ is "no of elements in (G)".

Ex1) If a group has only one element, then it is the identity element of the group.

ex → $S=\{0\}$ is a group of order 1 w.r.t addition operation because 0 is identity element w.r.t addition.

2) The only finite group of real nos w.r.t addition is the $S=\{0\}$.

3) $S=\{1\}$ is a group of order 1 w.r.t multiplication because 1 is identity element w.r.t multiplication.

Also, $S=\{1,-1\}$ is a finite group w.r.t multiplication.

4) composition table →

$\cdot$	1	-1
1	1	-1
-1	-1	1

Identity element

entries.

Inv. of $1=1$
Inv. of $-1=-1$

5) In general, For a group of order 2, $G=\{a,b\}$

composition table →

$\cdot$	a	b
a	a	b
b	b	a

Identity element.

Inv of $a=a$
Inv. of $b=b$.

$\therefore$ For a group G of order 2, Inv of $a=a$ $\forall a\in G$.

* The only finite groups of reals non-zero w.r.t multiplication are $\{1\}$ and $\{1,-1\}$.

ex:1) The cuberoots of unity, $G=\{1,\omega,\omega^2\}$ is an abelian group w.r.t multiplication.

→ $\omega=\left(\frac{-1+\sqrt{3}i}{2}\right)$ $\omega^2=\left(\frac{-1-\sqrt{3}i}{2}\right)$

$\omega^3=1$

composition table →

$\cdot$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	ω^3 1
ω^2	ω^2	1	ω

Identity element → 1

Inv of $1=1$
Inv of $\omega=\omega^2$
Inv of $\omega^2=\omega$

2) The fourth roots of unity $G=\{1,-1,i,-i\}$ is a group w.r.t multiplication.

→

composition table →

$\cdot$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	$i^2=-1$	$-i^2=1$
$-i$	$-i$	i	$-i^2=1$	$i^2=-1$

identity element → 1

$i^2=-1$
$-i^2=1$

∴ Inv. of $1=1$
Inv. of $-1=-1$
Inv. of $i=-i$
Inv. of $-i=i$

* Note: The nth roots of unity is a group w.r.t multiplication ∴ $n=1,2,3,\dots$

Addition Modulo m ($+_m$) →

$+_m$, where 'm' is +ve integer.

If a and b are any two +ve integers, then $a +_m b$ is given as, 6.1.7 = $a \oplus_m b$.

$a \oplus_m b = a+b$, if $(a+b) < m$

$= r$, if $(a+b) \geq m$ where r is the remainder obtained by $\frac{(a+b)}{m}$

Ex → $m = 6$

1) $2 \oplus_6 3 = 5$ 2) $4 \oplus_6 3 = 7/_6 = 1$ 3) $5 \oplus_6 1 = 6/_6 6 = 0$.

4) $4 \oplus_6 5 = 9/_6 6 = 3$.

Th → Set $S = \{0, 1, 2, \ldots, m-1\}$ is a group w.r.t $\oplus_m$.

Multiplication modulo m $\otimes_m$ →

If a and b are any two +ve integers, then $a \otimes_m b$,

$a \otimes_m b = a \cdot b$, if $a \cdot b < m$

$= r$, if $a \cdot b \geq m$ where $r = \frac{a \cdot b}{m}$

The set $S = \{1, 2, 3, 4, 5, 6\}$ is a group w.r.t $\otimes_7$

If n is a +ve integer, then the set 'S_n' is defined as,

S_n = set of +ve integers which are $\underline{<n}$ and relatively prime to n.

GCD of $\{a, b\} = 1$ → a, b are relatively prime.

Ex →

$S_6 = \{1, 5\}$ $S_8 = \{1, 3, 5, 7\}$, S_{15} $\{1, 2, 4, 7$

Euler function of

If n is a positive integer then S_n is a group w.r.t $\otimes_n$

Q. If $G = \{1, 3, 5, 7\}$ is a group w.r.t $\otimes_8$, which of the following is not true?

a) The inv of 1 is 1. true

b) The inv of 3 is 3. true

✓c) The inv of 5 is 7 false

d) The inv of 7 is 7. true.

If a and
given as.

19. Which of the following is a group?

a) $\{1, 2, 3, 4, 5\}$ w.r.t $\otimes_6$.

→ It's not a group because the binary operation is not a closure operation. $2 \otimes_6 3 = \underline{0} \notin$ set.

b) $S = \{0, 1, 2, 3, 4, 5\}$ w.r.t $\otimes_6$.

→ In the given set, inv of 0 does not exist w.r.t $\otimes_6$.
∴ S is not a group.

c) $S = \{\cancel{0}, 1, 2, 3, 4, 5, 6\}$ w.r.t $\otimes_7$.

→ It's a group.

d) $S = \{1, 2, 3, 4, 5, 6\}$ w.r.t $\oplus_7$.

→ not a group. No identity element.

Order of an element of a group →

Let $(G, *)$ be a group and $a \in G$, then

order of element $a = O(a) =$ the smallest +ve integer n such that $a^n =$ identity element

In a group, order of identity element is always 1

Ex. → 1) $G = \{1, -1\}$ is a group w.r.t multiplication.

→ $O(1) = 1$ $\quad 1^n = 1$

$O(-1) = 2$ $\quad (-1)^n, (-1)^2 = 1$

2) $G = \{1, \omega, \omega^2\}$ is a group w.r.t multiplication.

→ $O(1) = 1$

$O(\omega) \Rightarrow (\omega)^n = 1 \quad \therefore n = 3$

$O(\omega^2) \rightarrow (\omega^2)^n = 1 \quad \therefore n = 3.$

3) $G = \{1, -1, i, -i\}$ is a group w.r.t multiplication

→ $O(1) = 1$

$O(-1) = 2$

$O(i) \Rightarrow (i)^n = 1 \quad \therefore n = 4.$

$O(-i) = 4.$

$\therefore \boxed{O(G) = 4.}$

Note → 1) In a finite group $(G, *)$, $O(a)$ is a divisor of $O(G)$ $\forall a \in G$.

2) $O(a) = O(a^{-1}) \quad \forall a \in G.$

7.14. Set $S = \{0, 1, 2, 3\}$ is a group w.r.t $\oplus_4$.

→ $o(0) = 1$

$o(1) = 4$ $\quad 1 \oplus_4 1 \oplus_4 1 \oplus_4 1$

$o(2) = 2$ $\quad 2 \oplus_4 2$

$o(3) = 4$ $\quad 3 \oplus_4 3 \oplus_4 3 \oplus_4 3$

7.15. Set $G = \{1, 2, 3, 4\}$ is a group w.r.t. $\otimes_5$.

→ $o(1) = 1$ $\quad 1^n = 1 \quad n = 1$

$o(2) = 4$ $\quad 2^n = 1 \quad 2 \otimes_5 2 \otimes_5 2 \otimes_5 2 = 1$

$o(3) = 4$

$o(4) = 2$

7.16. Which of the foll. statements is/are true?

S1) In the group $(Z, +)$, the order of any element except 0 does not exist. true.

→ $a^n = 0$ i.e. $\underbrace{a + a + a + \cdots + a}_{n \text{ times}} = 0$. ← not possible for any $a \neq 0$.

S2) In the group $(Q^*, \cdot)$ where Q^* is a set of all nonzero rational nos i.e. $Q^* = Q - \{0\}$, the order of any element except 1 does not exist. False

→ $o(-1) = 2$ it exists. So, the given statement is false.

Subgroups →

Let $(G, *)$ be a group. A subset H of G is called a subgroup of G if $(H, *)$ is a group.

Note: Let $(G, *)$ be a group with identity element e, then $\{e\}$, and G are the trivial subgroups of G.
(which is not a trivial subgroup)
Any other subgroup of G is called proper subgroup of G.

Ex: $G = \{1, -1, i, -i\}$ w.r.t multiplication is a group, then $H = \{1, -1\}$ is a proper subgroup of G.

Theorem 1 →

Let H be a nonempty subset of a group $(G, *)$. H is a subgroup of G iff $(a * b^{-1}) \in H$ $\forall a, b \in H$

Theorem 2 →

Let H be a nonempty finite subset of a group $(G, *)$. H is a subgroup of G iff $(a * b) \in H$ $\forall a, b \in H$.

Theorem 3 → (Lagrange's theorem) →

If H is a subgroup of a finite group $(G, *)$, then $O(H)$ is a divisor of $O(G)$.

The converse of the above theorem need not be true.

Q.17 Let $G = \{0, 1, 2, 3, 4, 5\}$ is a group w.r.t $\oplus_6$. Which of the foll. is/are subgroups of G?

i) $H_1 = \{1, 3\}$

$\oplus_6$	1	3
1	2	4 $\notin H$.
3		

$\therefore$ closure property fails. $\therefore$ Not a subgroup

17.

b) $H_2 = \{1, 5\}$

$\oplus_6$	1	5
1	2	0 $\notin H$
5		

$\therefore H_2$ is not a subgroup.

c) $H_3 = \{0, 3\}$

$\oplus_6$	0	3
0	0	3
3	3	0

$\in H$. $\therefore H_3$ is a subgroup.

d) $H_4 = \{0, 2, 4\}$

$\oplus_6$	0	2	4
0	0	2	4
2	2	4	0
4	4	0	2

$\in H$ $\therefore H_4$ is a subgroup.

e) $H_5 = \{0, 2, 3, 5\}$ $2 \oplus_6 2 = 4 \notin H$ $\therefore H_5$ is not a subgroup.
H_5 is not a subgroup as $O(H_5)$ is not a divisor of $O(G)$.

18. Let $G = \{1, 2, 3, 4, 5, 6, 7\}$ is a group w.r.t $\otimes_7$. Which of the foll. are subgroups of G?

a) $H_1 = \{1, 6\}$

$\otimes_7$	1	6
1	1	6
6	6	1

$\in H_1$. $\therefore H_1$ is a subgroup.

b) $H_2 = \{1, 2, 4\}$

$\otimes_7$	1	2	4
1	1	2	4
2	2	4	1
4	4	1	2

$\in H_2$ $\therefore H_2$ is a subgroup.

c) $H_3 = \{1, 3, 5\}$. $3 \otimes_7 3 = 2 \notin H_3$. $\therefore H_3$ is not a subgroup.

d) $H_4 = \{1,2,3,5\}$ → A subset with 4 elements cannot be a subgroup of G. (Lagrange's thm).

19. Let $(G, *)$ be a group of order p where p is a prime no. No. of proper subgroups in G is ?

→ Let H be a proper subgroup of G with n elements.

By Lagrange's thm, n is a divisor of p, which implies

⇒ $n = 1$ or $n = p$ (∵ p is a prime no)

$$\underbrace{H = \{e\} \text{ or } H = G}_{\text{trivial subgroups}}$$

∴ The only subgroups of G are trivial subgroups. Therefore, no. of proper subgroups in G = 0

20. Which of the foll. statements is not true?

✓a) The union of any two subgroups of a group G is also a subgroup of G. False

→ We have counter example

For group $G = \{1,3,5,7\}$ w.r.t $\otimes_8$

$$\left.\begin{array}{l} H_1 = \{1,3\} \text{ w.r.t } \otimes_8 \\ H_2 = \{1,5\} \text{ w.r.t } \otimes_8 \end{array}\right\} \text{subgroups.}$$

$H_1 \cup H_2 = \{1,3,5\}$ w.r.t $\otimes_8$ as $3 \otimes_8 5 = 7 \notin H_1 \cup H_2$.

∴ $H_1 \cup H_2$ is not a subgroup of G.

b) The intersection of any two subgroups of a group G is also a subgroup of G. True

$\Rightarrow \quad a, b \in (H_1 \cap H_2)$

$(a * b^{-1}) \in H_1, (a * b^{-1}) \in H_2$

$\therefore (a * b^{-1}) \in (H_1 \cap H_2)$

$\therefore$ $H_1 \cap H_2$ is also a subgroup of G.

c) The union of two subgroups H_1 and H_2 of a group $(G, *)$ is also a subgroup of G. true

$\Rightarrow$ true when $(H_1 \subseteq H_2)$ or $(H_2 \subseteq H_1)$

d) Every subgroup of an abelian group is also an abelian group.

$\Rightarrow$ $(G, *)$, H is a subgroup. true

$a, b \in H$

$\Rightarrow a, b \in G$

$\Rightarrow (a * b) = (b * a)$ $\therefore H$ is also abelian.

Cyclic Groups $\rightarrow$

A group $(G, *)$ is said to be cyclic if there exists an element $a \in G$ such that every element of G can be written as a^n for some integer n; then a is called generating element / Generator of group G.

ex $\rightarrow$ 1) $G = \{1, -1\}$ is a cyclic group of order 2 w.r.t multiplication.

The generator of $G = -1$

2) $G = \{1, \omega, \omega^2\}$ is a cyclic group w.r.t multiplication

$\therefore$ The generator of $G = \omega, \omega^2$ $\leftarrow$ two generator.

1) $G = \{1, -1, i, -i\}$ is a cyclic group w.r.t multiplication. The generators are i and $-i$.

Ex. If $(G, *)$ is a cyclic group with generator a then

i) a^{-1} is also a generator of G

ii) $\boxed{\text{the order of the generator} = O(G)}$ for a cyclic group G.

1) $G = \{0, 1, 2, 3\}$ is a cyclic group with resp. to $\oplus_4$. The generators are 1 and 3.

→ 2 cannot be a generator as $O(2) \neq O(G) = 4$.

2. $G = \{1, 2, 3, 4\}$ is a cyclic group w.r.t $\otimes_5$. The generators are 2 and 3.

→ $4^2 = 1$ ∴ $O(4) \neq O(G) = 4$.

∴ 4 is not a generator.

Theorem →

Let $(G, *)$ be a cyclic group of order n with generator a then

i) the no of generators in $G = \phi(n)$. ← Euler function of n

ii) a^m is also a generator of G if $\gcd(m, n) = 1$

$a \quad a^3 \cdot a^5 \cdot a^7$ $\quad S_8 \quad 1, 3, 5, 7 \quad 1, (2, 3, 6, 8)$

$4 \quad 3, 7, 1, 6^2 \quad 4 \quad 3^2 \quad 1, 7, 7, 8, 8$

Q) Let $(G, *)$ be a cyclic group of order 8 with generator a

i) Number of generators in G = ?

ii) Which of the foll. is not a generator of G ?

(a) a^2 b) a^3 c) a^5 d) a^7

→ $S_8 = \{1, 3, 5, 7\}$.

∴ There are 4 generators of G → a, a^3, a^5, a^7.

a^2 is not a generator of G

Q4 " " " of order 84.

→ $S_{84} = [\ ?\ ]$.

$$\begin{array}{r|l} 2 & 84 \\ 2 & 42 \\ 3 & 21 \\ & 7. \end{array}$$

∴ $= 84 \left\{ \frac{(2-1)(3-1)(7-1)}{2 \times 3 \times 7} \right\}$

= 24 generators are there

Q.25. $G = \{1, 2, 3, 4, 5, 6\}$ is a cyclic group w.r.t. $\otimes_7$. How many generators ? what are they ?

→ $o(G) = 6$. ∴ No. of generators $= S_6 = \{1, 5\}$

$= 2$

$3 \rightarrow$ $3^2 = 2$ } $9 \div 7$ and 5 is an inverse of

$3^3 = 6$ } $6 \times 3 = 18$ 3

$3^4 = 4$ } $6 \times 3 = 18$ ∴ 5 is also generator.

$3^5 = 5$ } $4 \times 3 = 15$

$3^6 = 1$.

3 is a generator.

28. Let $G = \{0,1,2,3,4\}$ is a cyclic group w.r.t. $+_5$. How many generators are there and what are they?

→ $S_5 = \{1,2,3,4\}$.

∴ No. of generators $= \phi(5) = 4$.

(1,2,3,4) are generators.

29. Let $G = \{1,3,5,7\}$ is a group w.r.t. $\times_8$ but not a cyclic group because there is no generator for G.

28. The composition table of a cyclic group $G = \{a,b,c,d\}$ w.r.t. $*$ is shown below.

*	a	b	c	d
a	b	d	a	c
b	d	c	b	a
c	a	b	c	d
d	c	a	d	b

identity element(e) → c

→ We can generate all other elements from a. So, a is generator and d is inv of a.

∴ d is also a generator.

29. The incomplete composition table of a group $G = \{a,b,c,d\}$ w.r.t. $*$ is shown below.

*	a	b	c	d
a	b	d	a	c
b	d	c	b	a
c	a	b	c	d
d	c	a	d	b.

The last row is

c a d b

Any group of order ≤ 5 is abelian group and in the composition table of abelian group, the corresp rows & columns are identical.

#113.

* For cyclic groups, following properties hold good →

1) Every cyclic group is abelian.

Let $(G,*)$ be a cyclic group and x be the generator.

$\therefore \boxed{a*b} \Rightarrow x^m * x^n = x^{m+n} = x^{n+m} = x^n * x^m$

$= \boxed{b*a}$

2) Every group of prime order is cyclic. and so, every group of prime order is abelian group.

3) Every subgroup of a cyclic group is also cyclic. but the generator of the subgroup need not be same as that of the cyclic group.

ex → $G = \{1, -1, i, -i\}$; $H = \{1, -1\}$. So, H is subgroup of G

Generators of G: $i, -i$; Generator of $H = -1$.

4) Let $(G, *)$ be a group of even order, then there exists at least one element $a \in G$ $(a \neq e)$ such that $a^{-1} = a$

ex → 1) $G = \{1, -1\}$, inv. of $1 = 1$, inv. of $-1 = -1$.
2) $G = \{a, b, c, d\}$, inv. of $a = a$ (identity ele.), inv. of $b = c$
So, inv. of $d = d$.

Functions

Function →

A relation 'f' from a set A to a set B is called a function if to each element $a \in A$, we can assign an unique element B.

It is denoted by $f : A \rightarrow B$.

ex →

✓ Function

A : Domain of f
B : Codomain of f.

Range (Ran) of the function : $\{y \mid y \in B \text{ and } (x, y) \in f\}$
So, range of function f is always a subset of the codomain.

$\therefore \text{Ran } f \subseteq B$.

A function $f : A \rightarrow A$ is called a function on A.

n^m 4^3 $\frac{184}{25}$ $\frac{65535 \; 418}{65275}$ $2^{?}$ $8^{?}$ $3^?$ ... 27

If no of elements in A, $|A| = m$ and $|B| = n$, then no of functions possible from A to B are n^m

If no of elements in A, $|A| = n$, then no of functions possible on A = n^n

Q.1. If $A = \{a, b, c, d\}$, then no of relations on A which are not functions ?

→ No. of functions $= n^n$
No of relations $= 2^{(n^2)}$

∴ No of relations which are not functions $= (2^{16}) - (4^4)$
$= \boxed{65279}$

Q.2. If there are exactly 81 functions possible from set A to set B then which of the following statements is not true?

a) $|A| = 4$ $|B| = 3$ true $3^4 = 81$

b) $|A| = 2$ $|B| = 9$ true $9^2 = 81$

c) $|A| = 1$ $|B| = 81$ true $(81)^1 = 81$

d) $|A| = 9$ $|B| = 9$ false $(9)^9 \neq 81$

3. Which of the following is a function if domain is set of all real nos.

a) $f(x) = 1/x$ $f: R \to B$ ∵ element 0 in R is not mapped to any no in B.
$0 \to ?$

∴ Not a function.

* b) $g(x) = \sqrt{x}$ It is not a function because the -ve real nos. in the domain are not mapped to any element in codomain.

* c) $h(x) = \pm\sqrt{x^2+1}$ It is not a function because for each real no. we have two images (+/-).

d) $\phi(x) = |x|$. It is a function

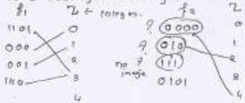
2. Consider the following relations from set of all bit strings to set of all integers

$f_1(s)$ = The no. of 1 bits in the bit string s

$f_2(s)$ = The position of a 0 bit in the bit string s.

Which of the above relations are functions?

* f_1 is a function because the no. of 1 bits in a bitstream is a nonnegative integer.



f_2 is not a function because a bit string with 2 or more 0s will have two or more images.

5. which of the foll. relations on set A is a function.
$A = \{a, b, c\}$.

a) $R_1 = \{(a, b), (b, c), (a, c)\}$ → a has 2 images
∴ Not a function.

b) $R_2 = \{(b, a), (b, c)\}$ → the ele. c has no image
∴ Not a function

c) $R_3 = \{(a, c), (b, c), (c, a)\}$ → It is a function.

6. which of the foll. statements is/are true?

5.1) There exists an equivalence relation which is also a function.
true.

$$\begin{array}{ccc} A & \to & A \\ 1 & \longrightarrow & 1 \\ 2 & \longrightarrow & 2 \\ 3 & \longrightarrow & 3 \end{array}$$

Diagonal relation on A. Δ_A
∴ the statement is true

equival. relation as well as a function.

5.2) The functions $f(x) = x$, $g(x) = \sqrt{x^2}$ are identical.
→ two functions are identical if they have same domain and codomain. false

Range of $f(x)$ = set of all real nos i.e. $-\infty$ to $+\infty$
whereas, range of $g(x)$ is only 0 to ∞. (+ve)

∴ these two functions are not identical.

5.3) $f(x) = \log_e(x^2)$ and $g(x) = 2\log_e(x)$ are identical. false
→ Domain of $f(x) = R - \{0\}$, Domain of $g(x) = (0, \infty)$
∴ not identical

64) The domain of $f(x) = \frac{1}{\sqrt{|x|-x}}$ is $(-\infty, 0)$

→ $|x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$ ∴ the statement is true

case 1: when $x \geq 0$, $|x| - x = x - x = 0$.
∴ $f(x) = \frac{1}{0}$ } ∴ f is not defined ∵ $x \geq 0$

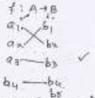
case 2: when $x < 0$, $|x| - x = -x - x = -2x > 0$ ($\because x < 0$)

5.7)

One-to-one Function → (injection)
A function f from a set A to set B is said to be one-to-one if no two elements in A are mapped to same element in B.

note 1) A function $f: A \to B$ is one-to-one if $f(a) = f(b)$, then $a = b$
$a, b \in A$

eg $f: A \to B$



If A and B are finite sets, then a one-to-one function $f: A \to B$ is possible iff $\boxed{|A| \leq |B|}$

2) If no. of elements in $|A| = m$ and $|B| = n$, $(m \leq n)$ then no. of one-to-one functions possible from A to B = $({}^nP_m)$

3) If $|A| = |B| = n$, then no. of one-to-one functions possible from A to B is $\underline{{}^nP_n = n!}$

If there are exactly 120 one-to-one functions possible from A to B, then which of the foll. is not true?

✓ a) $|A| = 5$ and $|B| = 5$ true ${}^5P_5 = 5! = 120$

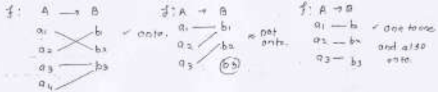
✓ b) $|A| = 4$ and $|B| = 5$ true ${}^5P_4 = 5\times4\times3\times2\times1 = 120$

✓ c) $|A| = 3$ and $|B| = 6$ true ${}^6P_3 = 6\times5\times4 = 120$

d) $|A| = 5$ and $|B| = 4$ false $\underline{|A| > |B|}$

Onto Function (Surjection) →

A function $f: A \to B$ is said to be onto if each element of B is mapped by at least one element of A. i.e. range of the function = B.



* If A and B are finite sets, then an onto function from A to B is possible iff $\underline{|B| \leq |A|}$

* If $|B| = |A|$, then every onto function from A to B is also one to one and viceversa.

* If $|B| = |A| = n$, then no. of onto functions possible from A to B $= \underline{(n!)}$

$(m>n)$

- If $|A| = m$, and $|B| = n$, then no. of onto functions possible from A to B, is

$$\boxed{n^m - {}^nC_1 (n-1)^m + {}^nC_2 (n-2)^m - {}^nC_3 (n-3)^m + \cdots + (-1)^{n-1}\, {}^nC_{n-1} (1)^m}$$

Q8. If no of elements in A, $|A| = 6$, and $|B| = 3$, then no. of onto functions possible from A to B are ?

→ $m = 6, \quad n = 3$

$= 3^6 - {}^3C_1 (2)^6 + {}^3C_2 (1)^m - {}^3C_3 (0)^m$

$= 729 - (3 \times 64) + (3 \times 1)$

$= \boxed{540}$

Q9. If $|A| = n$, $|B| = 2$, $(n > 2)$, then no. of onto functions possible from A to B is ?

→ $2^n - {}^2C_1 (1)^n$

$= \boxed{2^n - 2}$

Q10. In how many ways we can assign 5 employees to 4 projects so that every employee is assigned to only one project and every project is assigned by at least one employee. ?

→ e_1 P_1
e_2 P_2
e_3 P_3
e_4 P_4
e_5

$m = 5 \quad n = 4$

Reqd no of ways

$= 4^5 - {}^4C_1(3)^5 + {}^4C_2(2)^5 - {}^4C_3(1)^5 + 0$

$= 1024 - 4(243) + 6(32) - 4(1)$

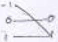
$= 1024 - 972 + 192 - 4 = \boxed{240}$

11. Consider the foll. functions on set of all integers.
$f(x) = x^2$, $g(x) = x^3$ and $h(x) = \lceil \frac{x}{2} \rceil$

which of the foll. true?

S1) f is one-to-one - false
S2) f is on-to - false
S3) g is one-one - true
S4) g is on-to - false
S5) h is one-one - false
S6) h is on-to - ~~false~~ true

→ S1) $f : Z \to Z$



So, it is not one-one ex: $f(-1) = f(+1) = 1$

S2) $f : Z \to Z$. It is not on-to function ex. the negative integers in codomain are not mapped by any integer of the domain.

S3) Let $g(a) = g(b)$. $\because a, b \in Z$.
$a^3 = b^3$.
$\therefore \underline{a = b}$.

So, g is one-one funcⁿ.

54) g is not on-to because elements like 2, 3 in codomain are not mapped by any int in domain.
i.e. $g(x) = x^3 - 2$ is not possible.

55) h is not one-one ex. $h(1) = h(2) = 1$

56) h is on-to because every integer in the codomain is mapped by at least one integer in the domain.

<u>Bijection</u> → (1-1 correspondence)
A function $f : A \to B$ is called a bijection if f is one-one as well as on-to.

If A and B are finite sets then a bijection from A to B is possible iff.
$|A| = |B|$

If $|A| = |B| = n$, then no. of bijections possible from A to B = $\underline{n!}$

<u>Q.12</u> Let $A = R - \{3\}$ and $B = R - \{1\}$. A function $f : A \to B$ is defined by $f(x) = \frac{x-2}{x-3}$.

Which of the foll. statements is true?

a) f is one-one but not onto
b) f is onto but one-one
c) f is a bijection.
d) f is neither one-one nor on-to.

$\Rightarrow$ Let $f(a) = f(b)$

$\Rightarrow \dfrac{a-2}{a-3} = \dfrac{b-2}{b-3} \Rightarrow (a-2)(b-3) = (b-2)(a-3)$

$\Rightarrow \boxed{a=b}$ $\therefore f$ is one-one.

Let $f(x) = \dfrac{x-2}{x-3} = y \Rightarrow x-2 = (x-3)y$

$\Rightarrow x - xy = 2-3y \Rightarrow x(1-y) = 2-3y$

$\Rightarrow x = \dfrac{2-3y}{1-y}$ For each y, there exists x such that $f(x)=y$.

$\therefore f$ is on-to.

<u>Inverse of a function</u> $\rightarrow$

Let $f: A \to B$. If the inverse relation $f^{-1}: B \to A$ is a function then it is called inverse of f and is denoted by <u>$f^{-1}: B \to A$</u>.

Theorem $\rightarrow$

Inverse of $f: A \to B$ exists <u>iff</u> f is a bijection.

Q.13 Which of the foll. functions have inverse defined on their ranges ?

a) $f(x) = x^2,\ x \in R$ — It is not one-one $\therefore$ not bijection $\therefore$ Inverse does not exist.

b) $f(x) = x^3,\ x \in R$ — Here f is bijection. Inv is defined.

$f^{-1}(x) = x^{1/3}$.

[illegible]

c) $g(x) = \sin x$, $x \in (0, \pi)$

→ g is not one-one in the interval $(0, \pi)$

<u>ex</u> $g(\pi/4) = g(3\pi/4) = \frac{1}{\sqrt{2}}$

Inv. of $g(x)$ does not exist.

d) $h(x) = 2^{x}$, $x \in R$.