Cmpe 300: Homework 1 — Due: October 25th 16:00

Solve the following questions in LaTeX or using a word processor. Keep your answers to the main 3 questions in separate pages, though each may span across multiple pages. Deliver a hard copy of your homework to the assistant's mailbox (in the secretary's office) or to his desk in BM 31.

The purpose of this homework is to familiarize you with the complexity related questions. This is an individual homework, so work on your own. Please do not submit just an answer, but show all your reasoning, and how you arrive at the answers. For any further questions, contact the assistant at utkan.gezer@boun.edu.tr.

1. **(30 pts)** Choose the *most precise (smallest)* complexity class among O, Ω, Θ, and $\sim$ making the following statements true. Prove your answer.

   (a) $5^n \in O(7^n)$

$$lim_{n\to\infty}\frac{5^n}{7^n} = lim_{n\to\infty}(\frac{5}{7})^n$$
$$Since \quad 0 \leq \frac{5}{7} \leq 1,$$
$$lim_{n\to\infty}\frac{5^n}{7^n} = 0$$

   (b) $5^{n+2} \in \Theta(5^n)$

$$lim_{n\to\infty}\frac{5^{n+2}}{5^n} = lim_{n\to\infty}5^2$$
$$lim_{n\to\infty}\frac{5^{n+2}}{5^n} = lim_{n\to\infty}25$$
$$lim_{n\to\infty}\frac{5^{n+2}}{5^n} = 25$$

   (c) $\log(n)\cdot\log(n) \in \Omega(\log\log(n))$

$$L'HospitalRule,$$
$$lim_{n\to\infty}\frac{log^2(n)}{log(log(n))} = lim_{n\to\infty}\frac{\frac{2log(n)}{n.log^2(2)}}{\frac{1}{n.log(2).log(n)}}$$
$$lim_{n\to\infty}\frac{log^2(n)}{log(log(n))} = lim_{n\to\infty}\frac{2log^2(n)}{log(2)}$$
$$lim_{n\to\infty}\frac{log^2(n)}{log(log(n))} = \infty$$

(d) $\log(n) \in \Omega(\log\log(n))$

$$L'Hospital Rule,$$

$$lim_{n\to\infty}\frac{log(n)}{log(log(n))} = lim_{n\to\infty}\frac{\frac{1}{n.log(2)}}{\frac{1}{n.log(2).log(n)}}$$

$$lim_{n\to\infty}\frac{log(n)}{log(log(n))} = lim_{n\to\infty}log(n)$$

$$lim_{n\to\infty}\frac{log(n)}{log(log(n))} = \infty$$

(e) $\sum_{i=1}^{n}\sqrt{i} \in \Theta\left(\frac{n\sqrt{n}}{2}\right)$

$$\int_0^n \sqrt{i}di < \sum_{i=1}^{n}\sqrt{i} < \int_0^{n+1}\sqrt{i}di$$

$$\frac{2n^{\frac{3}{2}}}{3} - 0 < \sum_{i=1}^{n}\sqrt{i} < \frac{2(n+1)^{\frac{3}{2}}}{3} - 0$$

$$\frac{2n^{\frac{3}{2}}}{3} < \sum_{i=1}^{n}\sqrt{i} < \frac{2(n+1)^{\frac{3}{2}}}{3}$$

$$lim_{n\to\infty}\frac{\frac{2n^{3/2}}{3}}{\frac{n^{3/2}}{2}} = lim_{n\to\infty}\frac{2.2}{3}$$

$$lim_{n\to\infty}\frac{\frac{2n^{3/2}}{3}}{\frac{n^{3/2}}{2}} = lim_{n\to\infty}\frac{4}{3}$$

$$lim_{n\to\infty}\frac{\frac{2n^{3/2}}{3}}{\frac{n^{3/2}}{2}} = \frac{4}{3}$$

$$\frac{2n^{3/2}}{3} \in \Theta(\frac{n^{3/2}}{2})$$

$$lim_{n\to\infty}\frac{\frac{2(n+1)^{3/2}}{3}}{\frac{n^{3/2}}{2}} = lim_{n\to\infty}\frac{4(n+1)^{3/2}}{3n^{3/2}}$$

$$lim_{n\to\infty}\frac{\frac{2(n+1)^{3/2}}{3}}{\frac{n^{3/2}}{2}} = \frac{4}{3}lim_{n\to\infty}\frac{(n+1)^{3/2}}{n^{3/2}}$$

$$lim_{n\to\infty}\frac{\frac{2(n+1)^{3/2}}{3}}{\frac{n^{3/2}}{2}} = \frac{4}{3}\left(lim_{n\to\infty}\frac{(n+1)}{n}\right)^{3/2}$$

$$L'Hospital Rule,$$

$$lim_{n\to\infty}\frac{\frac{2(n+1)^{3/2}}{3}}{\frac{n^{3/2}}{2}} = \frac{4}{3}\left(lim_{n\to\infty}\frac{(1+0)}{1}\right)^{3/2}$$

$$lim_{n\to\infty}\frac{\frac{2(n+1)^{3/2}}{3}}{\frac{n^{3/2}}{2}} = \frac{4}{3}.1^{3/2}$$

$$lim_{n\to\infty}\frac{\frac{2(n+1)^{3/2}}{3}}{\frac{n^{3/2}}{2}} = \frac{4}{3}$$

$$\frac{2(n+1)^{3/2}}{3} \in \Theta(\frac{n^{3/2}}{2})$$

$$Since \quad \frac{2n^{3/2}}{3} \in \Theta(\frac{n^{3/2}}{2}), \text{ there are } c_0, c_1, n_0 \text{ exists such that:}$$

$$c_0.\frac{n^{3/2}}{2} \leq \frac{2n^{3/2}}{3} \leq c_1.\frac{n^{3/2}}{2}$$

$$\text{for } n_0 \leq n$$

$$Since \quad \frac{2(n+1)^{3/2}}{3} \in \Theta(\frac{n^{3/2}}{2}), \text{ there are } c_2, c_3, n_1 \text{ exists such that:}$$

$$c_2.\frac{n^{3/2}}{2} \leq \frac{2(n+1)^{3/2}}{3} \leq c_3.\frac{n^{3/2}}{2}$$

$$\text{for } n_1 \leq n$$

$$\frac{2n^{3/2}}{3} < \sum_{i=1}^{n} \sqrt{i} < \frac{2(n+1)^{3/2}}{3}$$

$$For \quad max(n_0, n_1) = n_2, \quad n_2 \leq n$$

$$c_0.\frac{n^{3/2}}{2} \leq \frac{2n^{3/2}}{3} < \sum_{i=1}^{n} \sqrt{i} < \frac{2(n+1)^{3/2}}{3} \leq c_3.\frac{n^{3/2}}{2}$$

$$c_0.\frac{n^{3/2}}{2} \leq \sum_{i=1}^{n} \sqrt{i} \leq c_3.\frac{n^{3/2}}{2}$$

$$\text{Therefore, } \sum_{i=1}^{n} \sqrt{i} \in \Theta(\frac{n^{3/2}}{2})$$

2. **(35 pts)** Function to the right checks whether n is a prime number in a naive manner.

Assume that the input n also has the size n. So, for example, if $n = 13$, take size of input as 13.

Note *Usually, the size of* n *is taken as* $\log(n)$*, the number of digits, but you should take it as* n *for this question.*

Assume that the **mod** operation takes constant time. Then, give the *most precise* O-class for the worst-case time complexity of this algorithm for when;

Algorithm 1 Primality check (naive)

```
function IsPrime(n)
    for i ← 2 to (n − 1) do
        if (n mod i) = 0 then
            return(.false.)
        end if
    end for
    return(.true.)
end function
```

(a) n is even,

$O(1)$;
If n is an even number, it just takes 1 control to understand if the n is prime or not.
Because it first checks mod 2.

(b) n is prime,

$O(n)$;
If n is an prime number, all the mods are checked from 2 to n-1.
That makes n-2 operations and that leads us to $O(n)$.

(c) n is composite (not prime),

$O(\sqrt{n})$;
If n is composite, worst-case is n is being square of a prime number.
So program has to make $\sqrt{n} - 1$ mod operations. (Since it starts from 2)
That means answer is $O(\sqrt{n})$.

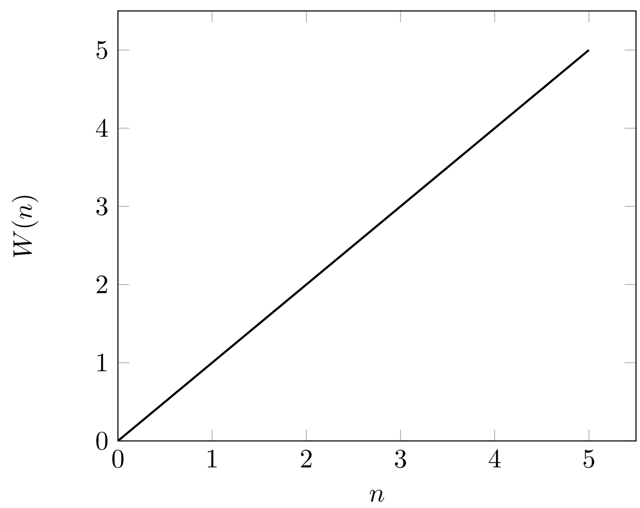
(d) and when $n = p \cdot q$, where p and q are primes.

$O(\sqrt{n})$;
Worst-case is p=q.
So program has to make $\sqrt{n} - 1$ mod operations. (Since it starts from 2)
That means answer is $O(\sqrt{n})$.

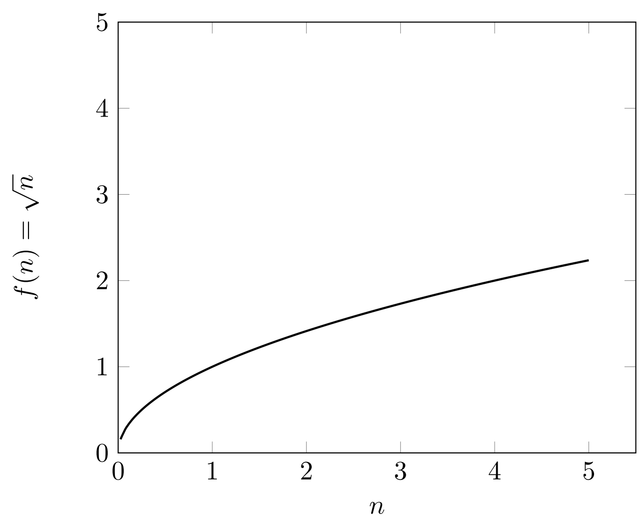
(e) Give the *most precise* O-class in general (call this $W(n)$).

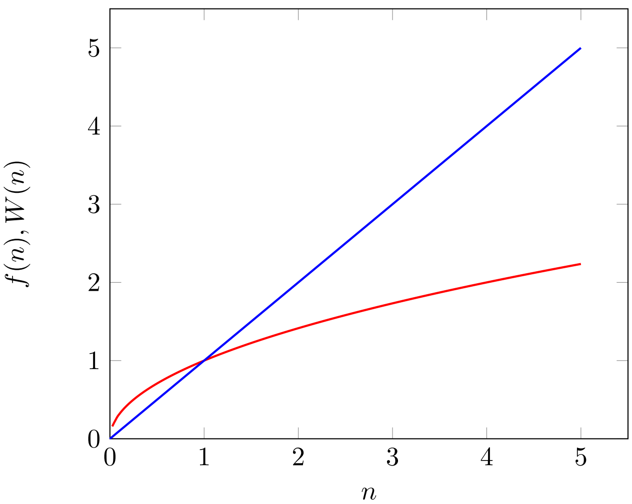
$O(n)$;
General worst-case is the case n is a prime number. Therefore answer is $O(n)$.

(f) Plot the function $W(n)$ against n.



(g) Plot the function $f(n) = \sqrt{n}$. What is the asymptotic relation between $W(n)$ and $f(n)$?





Asymtotic relation between f(n) and W(n) is $f(n) \in W(n)$

Because while $n \geq 1, \quad 1.n \geq f(n)$

3. **(35 pts)** Draw the sets $O(n^2)$, $o(n^2)$, $\Omega(n^2)$, $\omega(n^2)$, $\Theta(n^2)$, and $\sim(n^2)$ in a single Venn diagram. Make the boundaries clear for each class. If it the boundary of a class remains uncertain, state which areas in the Venn diagram belongs to that class verbally for clarification.

   Find and write one example member into each region of your diagram. Mark the empty regions with the ∅ symbol.