CmpE 343
Introduction to Probability and Statistics for Computer Engineers
Fall 2019
Homework 1

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Axioms of Probability

$$
\begin{aligned}
&(1) \quad P(A) \geq 0 \text{ for all } A \subseteq S \\
&(2) \quad P(S) = 1 \\
&(3) \quad \text{If } A \cap B = \emptyset \text{ then } P(A \cup B) = P(A) + P(B)
\end{aligned}
$$

Solution 1

$$
\begin{aligned}
S_1 &\rightarrow \text{"s" is first character} \\
S_2 &\rightarrow \text{"s" is second character} \\
G_1 &\rightarrow \text{"g" is first character} \\
G_2 &\rightarrow \text{"g" is second character} \\
D_1 &\rightarrow \text{"d" is first character} \\
D_2 &\rightarrow \text{"d" is second character}
\end{aligned}
$$

$$
\begin{aligned}
P(G_1) = P(S_1) = P(D_1) &= \frac{1}{3} \\
P(G_2|S_1) &= \frac{5}{10} \\
P(D_2|S_1) &= \frac{3}{10} \\
P(D_2|G_1) &= \frac{7}{10} \\
P(G_2|G_1) &= \frac{1}{10} \\
P(S_2|D_1) &= \frac{6}{10} \\
P(D_2|D_1) &= \frac{1}{10}
\end{aligned}
$$

1.a

$$
\begin{aligned}
P(S \cap D_1) &= P(D_1) \\
P((S_2 \cup D_2 \cup G_2) \cap D_1) &= P(D_1) \\
P((S_2 \cap D_1) \cup (D_2 \cap D_1) \cup (G_2 \cap D_1)) &= P(D_1) \\
P(S_2 \cap D_1) + P(D_2 \cap D_1) + P(G_2 \cap D_1) &= P(D_1) \qquad \text{Since, axiom (3)} \\
\frac{P(S_2 \cap D_1) + P(D_2 \cap D_1) + P(G_2 \cap D_1)}{P(D_1)} &= 1 \\
P(S_2|D_1) + P(D_2|D_1) + P(G_2|D_1) &= 1 \\
\frac{6}{10} + \frac{1}{10} + P(G_2|D_1) &= 1 \\
P(G_2|D_1) &= \frac{3}{10}
\end{aligned}
$$

$$
\begin{aligned}
P(G_2 \cap D_1) &= P(G_2|D_1).P(D_1) \\
P(G_2 \cap S_1) &= P(G_2|S_1).P(S_1) \\
P(G_2 \cap G_1) &= P(G_2|G_1).P(G_1) \\
P(G_2 \cap D_1) + P(G_2 \cap S_1) + P(G_2 \cap G_1) &= P(G_2|D_1).P(D_1) + P(G_2|S_1).P(S_1) + P(G_2|G_1).P(G_1) \\
P(G_2 \cap D_1) + P(G_2 \cap S_1) + P(G_2 \cap G_1) &= \frac{3}{10}.\frac{1}{3} + \frac{5}{10}.\frac{1}{3} + \frac{1}{10}.\frac{1}{3} \\
P(G_2 \cap D_1) + P(G_2 \cap S_1) + P(G_2 \cap G_1) &= \frac{3}{30} + \frac{5}{30} + \frac{1}{30} \\
P((G_2 \cap D_1) \cup (G_2 \cap S_1) \cup (G_2 \cap G_1)) &= \frac{9}{30} \qquad \text{Since, axiom (3)} \\
P(G_2 \cap (D_1 \cup S_1 \cup G_1)) &= \frac{3}{10} \\
P(G_2 \cap S) &= \frac{3}{10} \\
P(G_2) &= \frac{3}{10}
\end{aligned}
$$

1.b

$$\begin{aligned}
P(D_2|S_1) + P(D_2|D_1) + P(D_2|G_1) &= \frac{3}{10} + \frac{1}{10} + \frac{7}{10} \\
P(D_2|S_1) + P(D_2|D_1) + P(D_2|G_1) &= \frac{11}{10} \\
\frac{P(D_2 \cap S_1)}{P(S_1)} + \frac{P(D_2 \cap D_1)}{P(D_1)} + \frac{P(D_2 \cap G_1)}{P(G_1)} &= \frac{11}{10} \\
\frac{P(D_2 \cap S_1)}{\frac{1}{3}} + \frac{P(D_2 \cap D_1)}{\frac{1}{3}} + \frac{P(D_2 \cap G_1)}{\frac{1}{3}} &= \frac{11}{10} \\
P(D_2 \cap S_1) + P(D_2 \cap D_1) + P(D_2 \cap G_1) &= \frac{16}{10}.\frac{1}{3} \\
P((D_2 \cap S_1) \cup (D_2 \cap D_1) \cup (D_2 \cap G_1)) &= \frac{11}{30} && \text{Since, axiom (3)} \\
P(D_2 \cap (S_1 \cup D_1 \cup G_1)) &= \frac{11}{30} \\
P(D_2 \cap S) &= \frac{11}{30} \\
P(D_2) &= \frac{11}{30}
\end{aligned}$$

$$\begin{aligned}
P(S_1|D_2) &= \frac{P(S_1 \cap D_2)}{P(D_2)} \\
P(S_1|D_2) &= \frac{P(D_2 \cap S_1)}{P(D_2)} \\
P(S_1|D_2) &= \frac{P(D_2|S_1)}{P(D_2)}.P(S_1) \\
P(S_1|D_2) &= \frac{\frac{3}{10}}{\frac{11}{30}}.\frac{1}{3} \\
P(S_1|D_2) &= \frac{3}{10}.\frac{30}{11}.\frac{1}{3} \\
P(S_1|D_2) &= \frac{3}{11}
\end{aligned}$$

Solution 2

2.a

$$\begin{aligned} A \cup A' &= S \\ P(A \cup A') &= P(S) \\ P(A) + P(A') &= P(S) && \text{Since, axiom (3)} \\ P(A) + P(A') &= 1 && \text{Since, axiom (2)} \\ P(A') &= 1 - P(A) \end{aligned}$$

2.b

$$\begin{aligned} P(A') &\geq 0 && \text{Since, axiom (1)} \\ 1 - P(A) &\geq 0 && \text{Since, P(A') = 1-P(A)} \\ 1 &\geq P(A) \\ P(A) &\leq 1 \end{aligned}$$

Solution 3

$$\begin{array}{rcl} \text{k} = 1, & H \rightarrow \frac{1}{4} & = \dfrac{1}{4} \\ \text{k} = 2, & TH \rightarrow \frac{3}{4} \cdot \frac{1}{4} & = \dfrac{3}{16} \\ \text{k} = 3, & TTH \rightarrow \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} & = \dfrac{9}{64} \\ \text{k} = 4, & TTTH \rightarrow \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} & = \dfrac{27}{256} \\ \text{k} = 5, & TTTTH \rightarrow \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} & = \dfrac{81}{1024} \\ \text{k} = ..., & ... \rightarrow ... & = ... \end{array}$$

$$\begin{aligned} P(A') &= \frac{1}{4} + \frac{3}{16} + \frac{9}{64} + \frac{27}{256} \\ P(A') &= \frac{175}{256} \\ 1 - P(A) &= \frac{175}{256} \\ P(A) &= 1 - \frac{175}{256} \\ P(A) &= \frac{81}{256} \end{aligned}$$

Solution 4

4.a

For x < -2

$$\begin{aligned}
F(x) &= \int_{-\infty}^{x} f(x)dx \\
F(x) &= \int_{-\infty}^{x} 0dx \\
F(x) &= 0
\end{aligned}$$

For -2 $\leq x \leq 3$

$$\begin{aligned}
F(x) &= \int_{-\infty}^{x} f(x)dx \\
F(x) &= \int_{-\infty}^{-2} f(x)dx + \int_{-2}^{x} f(x)dx \\
F(x) &= \int_{-\infty}^{-2} 0dx + \int_{-2}^{x} \frac{1}{5}dx \\
F(x) &= \int_{-2}^{x} \frac{1}{5}dx \\
F(x) &= \frac{x}{5} - (-\frac{2}{5}) \\
F(x) &= \frac{x+2}{5}
\end{aligned}$$

For 3 < x

$$\begin{aligned}
F(x) &= \int_{-\infty}^{x} f(x)dx \\
F(x) &= \int_{-\infty}^{-2} f(x)dx + \int_{-2}^{3} f(x)dx + \int_{3}^{x} f(x)dx \\
F(x) &= \int_{-\infty}^{-2} 0dx + \int_{-2}^{3} \frac{1}{5}dx + \int_{3}^{x} 0dx \\
F(x) &= \int_{-2}^{3} \frac{1}{5}dx \\
F(x) &= \frac{3}{5} - (-\frac{2}{5}) \\
F(x) &= \frac{3+2}{5} \\
F(x) &= 1
\end{aligned}$$

4.b

$$\begin{aligned}
P(X>0) &= F(3)-F(0)\\
P(X>0) &= \frac{3+2}{5}-\frac{0+2}{5}\\
P(X>0) &= \frac{5}{5}-\frac{2}{5}\\
P(X>0) &= \frac{3}{5}
\end{aligned}$$

4.c

$$\begin{aligned}
P(|X|<\frac{2}{3}) &= P(-\frac{2}{3}<X<\frac{2}{3})\\
P(|X|<\frac{2}{3}) &= F(\frac{2}{3})-F(-\frac{2}{3})\\
P(|X|<\frac{2}{3}) &= \frac{\frac{2}{3}+2}{5}-\frac{-\frac{2}{3}+2}{5}\\
P(|X|<\frac{2}{3}) &= \frac{\frac{8}{3}}{5}-\frac{\frac{4}{3}}{5}\\
P(|X|<\frac{2}{3}) &= \frac{8}{15}-\frac{4}{15}\\
P(|X|<\frac{2}{3}) &= \frac{4}{15}
\end{aligned}$$