NUMBER SYSTEM

A number may be real or complex.

A real number may be

- an integer or a fraction,
- positive, negative or zero,
- rational or irrational.

An integer may be odd or even.

A fraction may be proper or improper.

Even and Odd Integers

- Sum of an odd number of odd integers is odd.
- Sum of an even number of odd integers is even.
- Sum of any number of even integers is even.
- Product of any number of even integers is even.
- Product of any number of odd integers is odd.

Application 1

If x and y are odd integers which of the following cannot be odd?

(a) $x + y$ (b) xy (c) $2x + y$ (d) $x + 2y$

Since sum of two odd integers is always even, $x + y$ cannot be odd. [Answer]

Product of two odd integers is always odd and so xy is odd.

For any values, $2x$ and $2y$ are even and even plus odd is odd.

Hence $(2x + y)$ and $(2y + x)$ are odd.

Application 2

If product of k integers is odd, which of the following must be true?

I: k is odd.

II: All the k integers are odd.

III: At least one of the k integers is odd.

Product of an even number of odd integers is also odd. So, I is not necessary.

Product of an even integer with any number of odd or even integers is always even. So, if the product is odd all the integers must be odd.

=> II must be true.

II also implies III is not sufficient to get the product as odd. Hence III is not true. Therefore, only II is true. [Answer]

Divisibility Criteria

The following table is a ready reckoner for determining whether a given number n is divisible by a number k.

k	n is divisible by k if
2	the last digit of n is divisible by 2,
3	the sum of the digits of n is a multiple of 3,
4	the number formed by the last two digits is divisible by 4,
5	the last digit is 0 or 5,
6	n is divisible by both 2 and 3,
8	the number formed by the last three digits is divisible by 8,
9	the sum of the digits of n is a multiple of 9,
10	the last digit is 0,
11	the difference between the sum of digits in the odd positions and the sum of digits in the even positions is 0 or a multiple of 11.

Application 3

All the following numbers are divisible by 66 EXCEPT

(a) 7326 (b) 21978 (c) 22110 (d) 21786

For a number to be divisible by 66, it must be divisible by each of 2, 3 and 11.

Option (a): Last digit is 6 which is even. Sum of the digits is 18 which is a multiple of 3 and the difference between the sum of the digits in the odd positions and the sum of the digits in the even positions, is (9 – 9) = 0. So (a) is divisible by 66.

Option (b): Last digit is 8 which is even. Sum of the digit is 27 which is a multiple of 3 and the difference between the sum of the digits in the odd positions and the sum of the digits in the even positions is (19 – 8) = 11 which is a multiple of 11. So (b) is divisible by 66.

Option (c): Last digit is 0 which is even. Sum of the digits is 6 which is a multiple of 3 and the difference between the sum of the digits in the odd positions and the sum of the digits in the even positions is (3 – 3) = 0. So (c) is divisible by 66.

Option (d): Last digit is 6 which is even. Sum of the digits is 24 which is a multiple of 3 and the difference between the sum of the digits in the odd positions and the sum of the digits in the even positions is (15 – 9) = 6 which is neither 0 nor divisible by 11. So (d) is not divisible by 66. [Answer]

Prime Numbers

An integer is a prime number if it has only two distinct factors.

2 is the only even prime number.

Application 4

Given x and y are prime numbers, which of the following can also be a prime number?

(a) xy (b) y/x (c) y^x (d) $x + y$

(a) has 1, x, y and xy as factors and so cannot be a prime number.

(b) cannot be an integer since y being a prime number, cannot have x as a factor.

(c) has at least three factors, namely 1, y and y^x and so it cannot be a prime number.

By elimination, (d) is the answer. As confirmation, x = 2 and y = 3 gives ($x + y$) = 5 which is a prime number.

Number of Divisors:

> Steps involved in determining the number of divisors of a number n
>
> 1. Express n as a product of powers of prime numbers, say $n = p^k \times q^m \times r^s$
> 2. The number of divisors $= (k+1)(m+1)(s+1)$

Application 5

If $pq = 60$, what is the maximum number of values p can take?

Given condition, implies that p and q are divisors of 60.

Now, $60 = 2^2 \cdot 3^1 \cdot 5^1$ and hence the number of divisors of

$60 = (2+1)(1+1)(1+1) = 12$

So, p can take any of these 12 values. [Answer]

HCF and LCM

HCF (Highest Common Factor) of two or more numbers is the largest factor that can divide all the given numbers and LCM (Latest Common Multiple) is the smallest number that can be divided by all the given numbers.

> Steps for finding the HCF
>
> 1. Factorise each given number into prime factors.
> 2. Identify the factors common to all the given numbers.
> 3. HCF is the product of all factors identified in Step 2.

> Steps for determining the LCM
>
> 1. Divide each of the given numbers by prime factors successively till all the quotients become 1.
> 2. LCM is the product of all the prime factors involved.
>
>   [The prime factors used for division need not be factors of all the given numbers. Even if the factor divides just one number division is to be performed, the number itself being the quotient for such divisions.]

> For any two numbers, their product is equal to the product of their HCF and LCM.

Application 6

Find the HCF and LCM of 42, 91 and 154.

$42 = 2 \times 3 \times 7$; $\quad 91 = 7 \times 13$; $\quad 154 = 2 \times 7 \times 11$

The only common factor is 7 and hence the HCF = 7 [Answer 1]

$$\begin{array}{r|ccc} 2 & 42 & 91 & 154 \\ 7 & 21 & 91 & 77 \\ 3 & 3 & 13 & 11 \\ 11 & 1 & 13 & 11 \\ 13 & 1 & 13 & 1 \\ & 1 & 1 & 1 \end{array}$$

$LCM = 2 \times 7 \times 3 \times 11 \times 13 = 6006$ [Answer 2]

Application 7

What is the value of p if q is 119 and the LCM and the HCF of p and q are 595 and 17 respectively?

$LCM \times HCF = 595 \times 17 = pq$

Given q = 119, p = (595 × 17)/119 = 85 [Answer]

Application 8

A P.T. Master wants to make a formation with 299 students – 39 from Primary, 65 from Secondary and 195 from Higher Secondary sections, fulfilling all the following conditions.

1. Number of students in each row should be the same.
2. The front rows can have students from Primary section only.
3. The last rows can have students from Higher Secondary section only.
4. Students of Primary section cannot be accommodated in any row other than the front rows and the Higher Secondary students can be only in the last rows.
5. No student should be left behind.

If the P.T. Master wishes to have the formation with as few rows as possible, what should be the number of students per row?

Conditions (2), (3) and (4) imply that each row can have students from one section only. This in conjunction with (1) and (5) means that the number of students in each section should be a multiple of the number of students per row. Or in other words, the number of students per row should be a factor of 39, 65 and 195.

More the number of students per row, less the number of rows. Thus the last condition requires the number of students per row to be as large as possible. Combining this with the earlier implication, the number of students per row should be the largest factor of 39, 65 and 195, which is nothing but the HCF.

Now,

$39 = 3 \times 13; 65 = 5 \times 13; 195 = 15 \times 13$

The HCF is 13. [Answer]

Application 9

A coin collector has 111 coins. She wants to enrich her collection so that the total number of coins is an integral multiple of the number of coins of each type. She has in her collection 17 coins of Rs.50, 35 of Rs.10 and 59 of Rs.5. Which of the following combination ensures that she can achieve her target with minimum addition to her collection?

(a) 16 of Rs.5, 10 of Rs.10, 3 of Rs.50

(b) 11 of Rs.5, 5 of Rs.10, 3 of Rs.50

(c) 1 of Rs.5, 1 of Rs.10, 7 of Rs.50

(d) 1 of Rs.5, 5 of Rs.10, 3 of Rs.50

Let (x, y, z) be the right combination. Then, the total collection ($111 + x + y + z$) must be a multiple of each of ($17 + x$), ($35 + y$) and ($59 + z$). Since the addition ($x + y + z$) must be minimum, ($111 + x + y + z$) must be the least multiple or the LCM of ($17 + x$), ($35 + y$) and ($59 + z$).

Solving the above mathematically would turn out to be tedious. Inserting the given answer choices would be a better strategy.

Step 1: Check the total addition in each case – 29, 19, 9, 9.

Step 2: Opting for the minimum addition, check if the given conditions are satisfied for options (c) and (d).

Option (c) gives a total of 60, 36 and 24 and a grand total of 120. 36 is not a factor of the grand total. So, (c) cannot be the answer.

Option (d) gives a total of 60, 40 and 20 and a grand total of 120. All the totals are factors of the grand total. So, (d) is the answer.

Division Properties

When 45 is divided by 7, the quotient is 6 and the remainder is 3. So, the following statements can be made.

1. $45 = (7 \times 6) + 3$
2. (45 – 3) is a multiple of 7
3. 45 + (7 – 3) = 49 is also a multiple of 7

Or in general,

When a number n is divided by another number m, if the remainder is r, then

1. $n = (k \times m) = r$
2. $(n - r)$ is a multiple of m
3. $n + (m - r)$ is also a multiple of m

Application 10

When a number is divided by 9 the remainder is 3 and when the same number is divided by 5 the remainder is 3. What is the largest 3-digit number which satisfies this condition?

Let n be the number. Then by rule (2) above, (n – 3) is a multiple of 9 and also 5. So, the least possible value of (n – 3) is the LCM of 9 and 5, which is 45 and all other values (n – 3) can take must be multiples of 45.

Now, the largest 3-digit number is 999. On dividing 999 by 45, the remainder is 9 and again applying rule (2) above, (999 – 9) = 990 is a multiple of 45. Thus, the largest possible 3-digit value for (n – 3) is 990 or the largest possible value for n is 993. [Answer]

Application 11

What is the largest 4-digt number that leaves a remainder of 6 when divided by 15?

The largest 4-digit number is 9999. On dividing 9999 by 15, the quotient is 666 with a remainder of 9. So, 9999 = (15×666) + 9, which in turn implies (9999 – 9) = 9990 is the largest 4-digit multiple of 15. Hence (9990 + 6) = 9996 must be the largest 4-digit number yielding a remainder of 6 when divided by 15. [Answer]

Application 12

A number leaves a remainder 5 when divided by 7 and 7 when divided by 9. What is the least possible number that has this property?

Noting (7 – 5) = 2 = (9 – 7), if n is the least possible number, then (n + 2) is a multiple of both 7 and 9. Hence least possible value for (n + 2) is the LCM of 7 and 9, viz. 63. Therefore, the least possible value for n is 61.

5 divided by 3 leaves a remainder of 2, 4 divided by 3 leaves a remainder of 1 and 7 divided by 3 leaves a remainder of 1.

Now, 11 divided by 3 leaves a remainder of 2 and 20 divided by 3 leaves a remainder of 2.

11 = 4 + 7 and the remainder of 11/3 = remainder of 4/3 + remainder of 7/3

Also, $20 = 4 \times 5$ and the remainder of 20/3 = remainder of 4/3 × remainder of 5/3

Or in general, if R(n/k) denotes the remainder of n when divided by k,

$$R\{(n+m)/k\} = R(n/k) + R(m/k)$$

$$R\{(n \times m)/k\} = R(n/k) + R(m/k)$$

And extending the second rule,

$$R\{(n^m)/k\} = \{R(n/k)\}^m$$

Application 13

What is the remainder when 4^{29} is divided by 63?

4^3 is 64 which is just 1 more than 63 and hence $R(4^3/63) = 1$

Now, $4^{29} = (4^3)^9 \times 4^2$ and so $R(4^{29}/63) = R[\{4^3)^9 \times 4^2\}/63]$

$= R[\{(4^3)^9\}/63] \times R[(4^2)/63] = [R(4^3/63)]^9 \times 16 = 1^9 \times 16 = 16$ [Answer]

In general, to find the remainder when n^m is divided by k, a step by step rule would be

Step 1: Identify a power t of n so that n^t is very close to k.

Step 2: Divide m by t and get the quotient and the remainder - say q and r respectively.

Step 3: Determine $R(n^t/k)$ and $R(n^r/k)$.

Step 4: Raise the first quantity of Step 3 to power q and multiply this by the second quantity to get the final answer.

In Application 13, n = 4, m = 29, k = 63, t = 3, q = 9 and r = 2.

Application 14

If the remainder is 75 when a number is divided by 85, what is the remainder when the same number is divided by 17?

Let the number be n. Then $n = 85k + 75$.

Now, $R(n/17) = R\{(85k + 75)/17\} = R(85k/17) + R(75/17) = 0 + 7 = 7$ [Answer]

Digit in the Units Position

The method to find the digit in the units position of a number n raised to a power k is given below in a step-by-step algorithmic style.

If the digit in the units position of n is 0, 1, 5 or 6, the required digit is 0, 1, 5 or 6 as the case may be irrespective of the value of k.

If the digit in the units position is 2, 3, 7 or 8, divide k by 4 and get the remainder, say r. If r > 0 then the required digit is the unit digit of $2^r, 3^r, 7^r$ or 8^r as the case may be and if r = 0, the required digits will be 6, 1, 1, 6 respectively.

If the digit in the units position is 4 or 9 and k is odd, the required units digit is 4 or 9 as the case may be, and if K is even the answer is 6 for 4 and 1 for 9.

Application 15

Determine the digit in the units position of $13^{23} \times 27^{37} \times 19^{45}$.

13 ends in 3. So, divide the power 23 by 4 to get the remainder 3. 3^3 ends in 7 and hence 13^{23} ends in 7.

27 ends in 7. So, divide the power 37 by 4 to get the remainder 1. 7^1 ends in 7 and hence 27^{37} ends in 7.

19 ends in 9. So, divide the power 45 by 2 to get the remainder 1. 9^1 ends in 9 and hence 19^{45} ends in 9.

Now, $7 \times 7 \times 9$ ends in 1 and so $13^{23} \times 27^{37} \times 19^{45}$ ends in 1. [Answer]

Maximum Power of a Factor in Factorial

The method to find the maximum power of a prime factor, say p, in a factorial, say n, consists of

- divide n by p and get the quotient, say $q1$.
- divide $q1$ by p and get the quotient, say $q2$.
- divide $q2$ by p and get the quotient, say $q3$.
- continue the above process till the quotient obtained is less than p.

The required power is the sum of the successive quotients.

Application 16

What is the maximum power of 7 in 1024!?

7)1024
7) 146
7) 20
2

Since 2 < 7, the process is stopped and the maximum power

= 146 + 20 + 2 = 168 [Answer]

CLASS WORK

1. How many odd factors of 12 are there?

   (a) 1 (b) 2 (c) 3 (d) 4

2. Find the sum of factors of $7^4 \times 3^2 \times 2^3$?

   (a) 550000 (b) 546195
   (c) 557125 (d) 532758

3. Find the digit in unit's place of the product $81 \times 82 \times 83 \times ... \times 89$.
(a) 8 (b) 0 (c) 2 (d) 6

4. How many even factors of 12 are there?
(a) 1 (b) 2 (c) 3 (d) 4

5. Find the least number by which 14175 be divided to make it a perfect square.
(a) 3 (b) 5 (c) 7 (d) 15

6. Which of the following is the least number exactly divisible by 24, 28, 36 and 48?
(a) 1,004 (b) 1,008 (c) 1,012 (d) 1,016

7. What is the unit digit in 27^{20}?
(a) 1 (b) 5 (c) 2 (d) 0

8. A number when divided by 6 leaves a remainder of 3. When the square of the number is divided by 6 the remainder is
(a) 0 (b) 1 (c) 2 (d) 3

9. What is the unit digit in the product $(3547)^{153} \times (251)^{72}$?
(a) 1 (b) 3 (c) 5 (d) 7

10. What is the number of even factors of 36000 which are divisible by 9 but not by 36?
(a) 20 (b) 4 (c) 10 (d) 12

11. What is the unit digit of the sum of first 111 whole numbers?
(a) 4 (b) 6 (c) 5 (d) 0

12. The unit digit of the sum 1289 + 2541 + 8215 + 6137 is
(a) 1 (b) 2 (c) 3 (d) 4

13. The unit digit of the product $1022 \times 729 \times 889 \times 971$ is:
(a) 2 (b) 9 (c) 1 (d) 8

14. How many factors of 36288 are perfect cubes?
(a) 9 (b) 4 (c) 6 (d) 8

15. How many factors of 1080 are perfect squares?
(a) 4 (b) 6 (c) 8 (d) 5

16. The sum of the first 100 natural numbers, 1 to 100 is divisible by
(a) 2, 4 and 8 (b) 2 and 4
(c) 2 (d) 100

17. Sum of squares of two positive integers is 100 and the difference of their squares is 28. The sum of the numbers is
(a) 15 (b) 14 (c) 13 (d) 12

18. What is the unit's digit of $2008^{1993!}$?
(a) 0 (b) 2 (c) 4 (d) 6

19. How many zeros are contained in 100?!
(a) 13 (b) 24 (c) 100 (d) 97

20. A positive number is divided by 100 to get a remainder thrice as the quotient. If the number is divisible by 11, then how many such numbers are possible that are less than 100000?
(a) 1 (b) 2 (c) 3 (d) 4

21. Find the unit digit of the following expression
$888^{9235!} + 222^{9235!} + 666^{2359!} + 99^{99999!}$
(a) 5 (b) 9 (c) 3 (d) 7

22. What is the remainder when $1044 \times 1047 \times 1050 \times 1053$ is divided by 33?
(a) 3 (b) 27 (c) 30 (d) 21

23. How many zeros will be there in the value of 25!?
(a) 25 (b) 8 (c) 6 (d) 5

24. How many integral divisors does the number 120 have?
(a) 14 (b) 16 (c) 12 (d) 20

25. A number when divided by a divisor leaves a remainder of 24. When twice the original number is divided by the same divisor, the remainder is 11. What is the value of the divisor?
(a) 13 (b) 59 (c) 35 (d) 37