

Solution - Exercise [2]

Introduction to Computer Graphics - B-IT Master Course

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Exercise 1

Given are two points p_1, p_2 on the unit sphere in \mathbb{R}^3 . Using a quaternion the point p_1 is to be rotated onto the point p_2 .

- a. Give a formula to determine the angle of rotation α

$$\alpha = \arccos \left\{ \frac{p_1 \cdot p_2}{\|p_1\| \cdot \|p_2\|} \right\}$$

- b. Give a formula to determine the rotation axis v

$$v = p_1 \times p_2$$

- c. Write down the quaternion q which performs the rotation with angle α around v

a = rotation angle

v = rotation axis

$$q = \cos\left(\frac{a}{2}\right) + i\left(x \sin \frac{a}{2}\right) + j\left(y \sin \frac{a}{2}\right) + k\left(z \sin \frac{a}{2}\right)$$

- d. Write down the relationship between p_1 and p_2 using quaternion multiplication

$$(0, p_2) = q(0, p_1)q^{-1}$$

Exercise 2

Given a point $p \in \mathbb{R}^3$ in homogenous coordinates $(x \ y \ z \ 1)^T$:

Rotation Matrix

$$R_z = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Translation Matrix

$$T_t = \begin{pmatrix} 1 & 0 & 0 & t_1 \\ 0 & 1 & 0 & t_2 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

a. Derive a matrix M_1 which first rotates the point α degrees (α is given in radians) around the axis $(0 \ 0 \ 1)^T$ and then performs a translation with an offset of $(t_1 \ t_2 \ t_3)^T$

$$\begin{aligned} M_1 &= \begin{pmatrix} 1 & 0 & 0 & t_1 \\ 0 & 1 & 0 & t_2 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 & t_1 \\ \sin(\alpha) & \cos(\alpha) & 0 & t_2 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

b. Derive a matrix M_2 which first performs a translation with an offset of $(t_1 \ t_2 \ t_3)^T$ and then rotates the point α degrees (α is given in radians) around the axis $(0 \ 0 \ 1)^T$

$$\begin{aligned} M_2 &= \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & t_1 \\ 0 & 1 & 0 & t_2 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 & t_1 \cos(\alpha) - t_2 \sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) & 0 & t_1 \sin(\alpha) + t_2 \cos(\alpha) \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

c. In which way affects the order of operations the respective final transformation matrix in this case
if the point translates before doing the rotation, the final transformation matrix has to rotate this translation offset too. On the other hand, if it rotates first, then we simply need to add the translation offset into the final transformation matrix.