Uladzislau Bohdan

MS program “Data Science”

Ordered Sets in Data Analysis

Home assignment #1

Task 1

By defition, $f : U \to U$ is a surjective function if $\forall b \in U$, $\exists a \in U$ and $b = f(a)$.

Let's assume the statement is wrong and f is not necessarily injective. That means $\exists a_1, a_2 \in U$, $a_1 \neq a_2$, such that $f(a_1) = f(a_2) = b_0$.

Then, because f is surjective, $\forall b \in U/\{b_0\}$, $\exists a \in U/\{a_1, a_2\}$ such that $b = f(a)$.

Because cardinality of a set $U/\{a_1, a_2\}$ is less than such of a set $U/\{b_0\}$, then by definition of a function and by pigeonhole principle we have a contradiction.

Therefore, if function $f : U \to U$ is surjective, than it is also injective.

Task 2

In this task I'll be talking about binary relations R as about matrices 5×5, where element $(i, j) = 1$ if $a_i R a_j$, otherwise $(i, j) = 0$. Here (i, j) is an element of the matrix.

a) Asymmetric and transitive

The relations are asymmetric if $(i, j) \neq (j, i)$.

The relations are transitive if $(i, j) = 1$ and $(j, k) = 1 \Rightarrow (i, k) = 1$.

Given such definitions, it follows that:

- We should only analyze a triangle matrix with no diagonal included. Without loss of generality, we work with upper right triangular matrix. Lower left triangular matrix is then uniquely defined due to asymmetric relations.
- Diagonal is generated independently from the rest of the matrix and diagonal with any values is valid.
- Not all triangular matrices are valid due to transitive relations.

The code below calculates the number of valid triangular matrices:

```
total = 0
for binnum in range(1024): # 1024 = 2^10 - we have 10 binary cells to fill.
    s = '{0:010b}'.format(binnum)
    m = [[0 for x in range(5)] for y in range(5)]
    m[0][1] = s[0]
    m[0][2] = s[1]
    m[0][3] = s[2]
    m[0][4] = s[3]
    m[1][2] = s[4]
    m[1][3] = s[5]
    m[1][4] = s[6]
    m[2][3] = s[7]
```

```
    m[2][4] = s[8]
    m[3][4] = s[9]
    valid = True
    for i in range(0,4):
        for j in range(i+1, 5):
            for k in range (j+1, 5):
                if m[i][j] == "1" and m[j][k] == "1" and m[i][k] == "0":
                    valid = False
    if valid == True:
        total += 1
print(total)
```

The output is 357.

Each valid triangular matrix can coexist with any valid diagonal. Total number of valid diagonals is $2^5 = 32$.

This gives us a total number of relations of $357 * 32 = 11424$.

b) Antisymmetric and antireflexive

The relations are antisymmetric if $(i, j) = 1$ and $(j, i) = 1 \Rightarrow i = j$.

The relations are antireflexive if $(i, i) = 0$.

It follows that:

- Diagonal elements are all zeros.
- Elements upper (WLG) then a diagonal (any) may be any values; for elements lower then the diagonal, if $(i, j) = 1$, then $(j, i) = 0$, if $(i, j) = 0$, then (j, i) is either 0 or 1.

The following code does the calculations:

```
total = 0
for binnum in range(1024):
    s = '{0:010b}'.format(binnum)
    m = [[0 for x in range(5)] for y in range(5)]
    possible_lower_matrices = 1
    for x in range(10):
        if s[x] == "0":
            possible_lower_matrices *= 2
    total += possible_lower_matrices
print(total)
```

The output is 59049 - a total number of matrices which satisfy the conditions.