problem 1

Theorem. *Suppose f is a function defined on $\mathbb{R}$ and that for all $x, y \in \mathbb{R}$*

$$|f(x) - f(y)| \leq (x - y)^2$$

Then f converges.

Proof. We have

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq |x - y|,$$

this means f' exists. Suppose $x < y$, then there is a point $x < a < y$ such that $(x - y)^2 \geq f(x) - f(y) = (x - y)f'(a) \implies x - y \geq f'(a)$. Now we can take x, y arbitrarily close to each other and $f'(a)$ would go to zero. Hence f' is 0 on all of $\mathbb{R}$, meaning f is constant. $\square$

problem 2

Theorem. *Suppose $f' > 0$ on (a, b), and g is its inverse in this range.Show*

$$g'(f(x)) = \frac{1}{f(x)}$$

Proof.

$$g'(f(x)) = \lim_{f(a) \to f(x)} \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} = \lim_{x \to a} \frac{x - a}{f(x) - f(a)} = \frac{1}{f'(x)}$$

Note the continuity of f was used in changing the bound of the limit. $\square$

problem 3

Theorem. *let g be a differentiable function on $\mathbb{R}$ with $|g'| < M$, then $f(x) = x + \epsilon g(x)$ is bijective for sufficiently small epsilon.*

Proof. We have $f' = 1 + \epsilon g'(x) > 1 - \epsilon M$, pick $\epsilon < \frac{2}{m}$ and then $f' > \frac{1}{2}$. Hence f is injective. $\square$

problem 4

Theorem. *Suppose*

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_n}{n+1} = 0,$$

then

$$C_0 + C_1 x + \cdots + C_n x^n = 0$$

for some $x \in [0, 1]$.

Proof. let

$$f(x) = C_0 x + \frac{C_1}{2} x^2 + \cdots + \frac{C_n}{n+1} x^{n+1}$$

Clearly $f(1) = f(0) = 0$. by MVT, there is some $x \in [0, 1]$ with $f(1) - f(0) = f'(x) \to f'(x) = 0$ which is the theorem. $\square$

problem 5

Theorem. *Suppose f is differentiable on $\mathbb{R}^+$, and $f(x) \to 0$ as $x \to \infty$. If $g(x) = f(x+1) - f(x)$, then $g(x) \to 0$.*

Proof. By MVT, there is some $a \in [x, x+1]$ with $f'(a) = f(x+1) - f(x) = g(x)$. Hence $g(x)$ gets arbitrarily close to 0 as $x \to \infty$. □

problem 6

Theorem. *Suppose f is continuous and differentiable on $\mathbb{R}^+ \cup \{0\}$ and $\mathbb{R}^+$ respectively, and that $f(0) = 0$ and that f' is monotonically increasing. Then $g(x) = \dfrac{f(x)}{x}$, then g is monotonically increasing.*

Proof. For every x, there is some $0 \leq a \leq x$ such that $f(x) = xf'(a)$. now

$$g'(x) = \frac{xf'(x) - f(x)}{x^2} = \frac{f'(x) - f'(a)}{x} \geq 0$$

Hence g is monotonically increasing. □

problem 14

Theorem. *A differentiable function is convex $\iff$ f' is monotonuously increasing.*

Proof. Suppose f is convex, and that $x > y$. Then

$$f(y + \lambda') \leq f(y) + \lambda' \frac{f(x) - f(y)}{x - y} \implies \frac{f(y + \lambda') - f(y)}{\lambda'} \leq \frac{f(x) - f(y)}{x - y}$$

with $\lambda' = \lambda(x - y)$ Now this clearly means that

$$x > x' \implies \frac{f(x') - f(y)}{x' - y} \leq \frac{f(x) - f(y)}{x - y}$$

and hence

$$\frac{f(x) - f(x - \epsilon)}{\epsilon} \geq \frac{f(x') - f(x - \epsilon)}{x' - (x - \epsilon)} \geq \frac{f(x') - f(x' - \epsilon)}{\epsilon}$$

This means $f'(x) \geq f'(x')$ whenever $x > x'$. □

problem 20

Theorem. *Suppose $\boldsymbol{f}$ is n differentiable on (a, b), and let $\beta > \alpha$ with both in the domain. Let*

$$\boldsymbol{P}_f(t) = \sum_{k=1}^{n-1} \frac{\boldsymbol{f}^{(k)}(\alpha)}{k!}(\beta - \alpha)^k,$$

Then

$$|\boldsymbol{f}(\beta) - \boldsymbol{P}_f(\beta)| \leq \frac{(\beta - \alpha)^n}{n!} \sup_{\alpha \leq x \leq \beta} \left|\boldsymbol{f}^{(n)}(x)\right|$$

Proof. let $\mathbf{z} = \mathbf{f}(\beta) - \mathbf{P}(\beta)$, and then let $\psi(x) = \mathbf{z} \cdot \mathbf{f}(x)$. Then $P_\psi = \mathbf{z} \cdot \mathbf{P}_f$. By taylors theorem, there is some $x \in [\alpha, \beta]$ such that

$$\psi(\beta) - P_\psi(\beta) = \frac{(\beta - \alpha)^n}{n!} \psi^{(n)}(x)$$

This means

$$|\mathbf{z}|^2 = \left|\mathbf{z}.\frac{(\beta-\alpha)^n}{n!}\mathbf{f}^{(n)}(x)\right| \leq \frac{(\beta-\alpha)^n}{n!}|\mathbf{z}|\left|\mathbf{f}^{(n)}(x)\right| \implies |\mathbf{z}| \leq \frac{(\beta-\alpha)^n}{n!}\left|\mathbf{f}^{(n)}(x)\right|$$

The theorem follows. □

problem 22

Problem. *Let f be a function on $\mathbb{R}$, call x a fix point of f if $f(x) = x$*

Theorem (a)**.** *Suppose f' is never 1, then f has at most 1 fixed point.*

Proof. Let $g(t) = f(t) - t$. Now suppose g has 2 zeroes. Since g is continuous and differentiable, g is either constant or has a max/min between these two zeroes. But both of this imply $g' = 0 \implies f' = 1$ which is a contradiction. □

Theorem (c)**.** *Suppose $|f'| \leq A$ with $A < 1$. Then f has exactly one fixed point x and the sequence defined by $x_{n+1} = f(x_n)$, then $x_n \to x$.*

Proof. let $g(t) = f(t) - t$, so $g' \leq A - 1 < 0$ and hence g has exactly one zero. Now let x be this zero, then

$$|x - x_{n+1}| = |f(x) - f(x_n)| \implies |x - x_{n+1}| = |(x - x_n)f'(a)| \leq |x - x_n|A$$

With $a \in [x, x_n]$.This means $|x - x_{n+1}| \leq A^n|x - x_1|$ and hence $x_n \to x$. □

problem 25 (Newton-Raphson)

Theorem. *Let f be twice differentiable on $[a, b]$, with $f(a) < 0, f(b) > 0, f'(x) \geq \delta > 0$ and $0 \leq f''(x) \leq M$.Let f have a unique root ξ in this range. Then the sequence defined by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ converges to ξ. Bound its error term.*

Proof. Clearly $\frac{f(x)}{f'(x)} > 0$ on (ξ, b), as if f were negative on this range, by IMV it would have another zero. Hence $x_{n+1} < x_n$. Also supposing $x_n \geq \xi$, we would have $x_{n+1} = x_n - (x_n - \xi)\frac{f'(a)}{f'(x_n)} \geq \xi$, where MVT was used and the fact that f' is monotonuously increasing.Hence $x_{n+1} \in (\xi, x_n)$.Since ξ is the only fixed point of this recursion, and the sequence is monotonously decreasing towards ξ, it converges to ξ.

By Taylors theorem we know there is some $t_n \in (\xi, x_n)$ such that

$$f(\xi) = f(x_n) + f'(x_n)(\xi - x_n) + \frac{f''(t_n)}{2}(x_n - \xi)^2 \implies x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

From this the bound

$$0 \leq x_{n+1} - \xi \leq \frac{2\delta}{M}\left[\frac{M}{2\delta}(x_1 - \xi)\right]^{2^n}$$

follows. □

problem 26

Theorem. *suppose f is differentiable on $[a, b]$ with $f(a) = 0$, and such that $\frac{f'}{f}$ is bounded on $[a, b]$, then $f = 0$ on this range.*

Proof. By MVT
$$|f(x)| \leq (x-a) \sup_{x_0 \in [a,x]} |f'(x_0)| \leq M(x-a) \sup_{x_0 \in [a,x]} |f(x_0)|$$
for some contant M. Now suppose there $\sup_{x_0 \in [a,x]} |f(x_0)| > 0$, and pick the x for which it is maximum, then $M(x_m - a) \geq 1$ but we can simply pick $x_m - a < \frac{1}{M}$ and this would be false. Hence f is 0 on $[a, a + \frac{1}{M})$. Now repeating this until we get to b is atmost countable, and covers the whole interval to have 0. □

problem 27

Theorem. *Let a solution to $y' = \phi(x,y), y(a) = c$ on $[a,b]$ be f. Suppose*
$$|\phi(x, y_1) - \phi(x, y_2)| \leq A|y_1 - y_2|$$
then there is a unique solution for y

Proof. Suppose y_1 and y_2 are solutions. let $z = y_1 - y_2$. Then
$$|z'| < A|z| \implies z = 0 \implies y_1 = y_2$$
Where problem 26 was used. □

problem 28

Theorem. *If $\boldsymbol{y}' = \boldsymbol{\phi}(x, \boldsymbol{y})$, and if*
$$|\boldsymbol{\phi}(x, \boldsymbol{y}_1) - \boldsymbol{\phi}(x, \boldsymbol{y}_2)| \leq A|\boldsymbol{y}_1 - \boldsymbol{y}_2|$$
then there is a unique solution for $\boldsymbol{y}$ given $\boldsymbol{y}(a) = \boldsymbol{c}$

Proof. Problem 26 did not use MVT for reals, but used the inequality that applies to vector valued functions too. Hence Same argument as 27. □

problem 29

Theorem. *Let*
$$y^{(k)} + g_k(x)y^{(k-1)} + \cdots + g_2(x)y' + g_1(x)y = f(x)$$
, and suppose $y(a) = c_1, y'(a) = c_2 \ldots y^{(k-1)}(a) = c_k$, then there is atmost one solution, given f and g are continuous.

Proof. take $\mathbf{y} = (y, y' \ldots, y^{(k-1)})$. Then $\mathbf{y}' = (y', y'' \ldots, y^{(k-1)}, y^{(k)})$ Now
$$|y_1^{(k)} - y_2^{(k)}| \leq \sum |g_n||y_1^{(n-1)} - y_2^{(n-1)}|$$
and hence clearly the $\mathbf{y}$ will satisfy 28 as g_n is bounded on $[a,b]$. □