problem 1

Theorem. *Suppose α is increasing on $[a,b]$ and that $f(x) = \begin{cases} 1 & x = x_0 \\ 0 & x \neq x_0 \end{cases}$ and α is continuous on x_0. Then $\int f\, d\alpha = 0$.*

Proof. we know that there is some δ such that $|x_0 - y| < \delta \implies |\alpha(x_0) - \alpha(y)| < \epsilon$. take a partition where the interval containing x_0 is $[x_0 - \delta, x_0 + \delta]$, the L on this would be 0 and the $U = \alpha(x_0 + \delta) - \alpha(x_0 + \delta) < 2\epsilon$. Hence $U - L < 2\epsilon$, and hence the this is riemann integrable. Since L is always 0, the integral is also hence 0. $\square$

problem 2

Theorem. *Suppose $f \geq 0$ and is continuous on $[a,b]$. Suppose $\int_a^b f\, dx$ is 0, show that so is f.*

Proof. Suppose $f(n) = m \neq 0$ for some n. since f is continous on n, we know for some δ, $|n - x| < \delta \implies |f(n) - f(x)| < \epsilon$, or that in $(n - \delta, n + \delta)$, $f(x)$ has a lower bound of $m - \epsilon$. Hence the integral is atleast $2\delta(m - \epsilon)$, and since $m \neq 0$, we can pick ϵ such that this is non 0, contradiction. $\square$

problem 4

Theorem. *let $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$, then $f \notin \mathfrak{R}$*

Proof. Note that since the rationals are dense in $\mathbb{R}$, in each interval $f(x)$ attains a max of 1 and a min of 0.Hence $U = b - a$ and $L = 0$ for all partitions. This is hence not Riemann integrable. $\square$

problem 5

Theorem. *Suppose f is a bounded function on $[a,b]$. $f^{2n} \in \mathfrak{R} \not\Rightarrow f \in \mathfrak{R}$ but $f^{2n+1} \in \mathfrak{R} \implies f \in \mathfrak{R}$, with $n \in \mathbb{N}$*

Proof. For the even case, let $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \notin \mathbb{Q} \end{cases}$, then this is clearly not integrable on $[a,b]$ while $f(x)^{2n} = 1$ is always integrable on $[a,b]$.

For the odd case, since $\psi(y) = \sqrt[2n+1]{y}$ is continuous on $\mathbb{R}$, $\psi(f^{2n+1}) = f$ is Riemann integrable. $\square$

problem 7

Problem. *Let f be defined on $[a,b]$ with f riemann integrable over all $[c,1]$ with $c > 0$. Define*

$$\int_0^1 f\, dx = \lim_{c \to 0} \int_c^1 f\, dx$$

Theorem (a)**.** *If f is riemann integrable on $[0,1]$, then this defination coincides with the usual defination.*

Proof. fix $\epsilon > 0$. Then we have for some Partion P:

$$\left| \int_0^1 f\, dx - \int_c^1 f\, dx \right| = \left| \int_0^c f\, dx \right| \leq \left| \int_0^c f\, dx - \sum f(x_i)\Delta x_i \right| + \left| \sum f(x_i)\Delta x_i \right| < \epsilon + c \sup_{x \in [0,c]} f(c)$$

As c can be arbritarily small, and as f is bounded, we are done. $\square$

problem 8

Theorem. *Suppose $f(x) \geq 0$ and is decreasing monotonically on $[1, \infty)$. then $\int_1^\infty f(x)\,dx$ converges $\iff$ $\sum_{n\geq 1} f(n)$ converges.*

Proof. ($\leftarrow$) Suppose the sum converges, then for all $\epsilon > 0$ there is some N such that $q \geq p > N$ means $\sum_p^q f(n) < \epsilon$. Notice the LHS is a upper sum for $\int_p^q f(x)\,dx$, hence $\int_p^q f(x)\,dx < \epsilon$, showing the integral converges as it is cauchy.

($\rightarrow$) Suppose the integral converges, then for all $\epsilon > 0$ there is some N such that $q \geq p > N$ means $\int_p^q f(x)\,dx < \epsilon$. A Lower sum of this is $\sum_p^q f(n+1)$, meaning $\sum_p^q f(n+1) < \epsilon$. Hence the sum converges as it is cauchy. $\square$

problem 9

Theorem. *Suppose all the limits exists, and F, G are both differentiable functions whose derivatives are riemann integrable. Then*

$$\int_a^\infty F(x)G'(x)\,dx = \lim_{b\to\infty} F(b)G(b) - F(a)G(a) - \int_a^\infty G(x)F'(x)\,dx$$

Proof. Take the limit of IBP. $\square$

problem 10

Suppose $p, q \in \mathbb{R}^+$ with

$$\frac{1}{p} + \frac{1}{q} = 1$$

Theorem (a). *Show*

$$\frac{u^p}{p} + \frac{v^q}{q} \geq uv$$

And equality holds iff $u^q = v^p$.

Proof. Call $f(x)$ concave if for each $0 < \lambda < 1$, $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$. Replicating problem 14 from chapter 5, this is true iff $f'(x)$ is monotonuously decreasing.

Take $f(x) = \log(x)$. Now $f'(x) = \dfrac{1}{x} > 0$ and $f''(x) = -\dfrac{1}{x^2} < 0$, Hence f is concave. Now take $\lambda = \dfrac{1}{p}$ and hence

$$\log\left(\frac{x}{p} + \frac{y}{q}\right) \geq \frac{\log(x)}{p} + \frac{\log(y)}{q} \implies \frac{x}{p} + \frac{y}{q} \geq x^{1/p}y^{1/q}$$

Where the fact that log is strictly increasing was used. $x = y$ is clearly a equality point. $\square$

Proof. (alternate:Bernoulli). Take $g(x) = (1+x)^r$ with $r < 1$. then $g'(x) = r(1+x)^{r-1}$ and $g''(x) = r(r-1)(1+x)^{r-2} \leq 0$. Pick $x \geq -1$. By Taylors theorem we know there is some $a \in [0, x]$ or $[x, 0]$ (depending on the sign of x) such that

$$g(x) = 1 + rx + \frac{g''(a)}{2}x^2 \leq 1 + rx \implies (1+x)^r \leq 1 + rx$$

with the equality case being only when $x = 0$ ofcourse. Pick $r = \dfrac{1}{q}$ and $x = -1 + \dfrac{v^q}{u^p}$ and this gives our inequality, with equality case $x = 0 \implies v^q = u^p$ $\square$

Theorem (Hoelders inequality)**.** *let f and g be complex valued function in $\mathfrak{R}(\alpha)$. Then*

$$\left|\int_a^b fg\,d\alpha\right| \le \left\{\int_a^b |f|^p\,d\alpha\right\}^{1/p}\left\{\int_a^b |g|^q\,d\alpha\right\}^{1/q}$$

Proof. let

$$F = \frac{f}{\left\{\int_a^b |f|^p\,d\alpha\right\}^{1/p}}, G = \frac{g}{\left\{\int_a^b |g|^q\,d\alpha\right\}^{1/q}}$$

Then this means

$$\frac{|F|^p}{p} + \frac{|G|^q}{q} \ge |FG| \implies 1 = \int_a^b \frac{|F|^p}{p} + \frac{|G|^q}{q}\,d\alpha \ge \int_a^b |FG|\,d\alpha \ge \left|\int_b^a FG\,d\alpha\right|.$$

Notice this is precisely the statement of the theorem. □

This trivially works for improper integrals, as limits preserve order upto an equal sign.

Corollary. *Cauchy schwarz and Triangle inequalities.*

problem 13

Problem. *Let*

$$f(x) = \int_x^{x+1} \sin(t^2)\,dt$$

Theorem (a)**.** $|f(x)| < \dfrac{1}{x}$ *for $x > 0$.*

Proof. let $u = t^2$, then

$$4f(x) = \int_{x^2}^{(x+1)^2} 2\sin(u)u^{-1/2}\,du = -2(x+1)^{-1}\cos((x+1)^2) + 2x^{-1}\cos(x^2) + \int_{x^2}^{(x+1)^2} \cos(u)u^{-3/2}\,du$$

Notice the integral

$$\int_{x^2}^{(x+1)^2} \left|\cos(x)u^{-3/2}\right|\,du \le \int_{x^2}^{(x+1)^2} u^{-3/2}\,du = 2x^{-1} - 2(x+1)^{-1}$$

Hence

$$|4f(x)| \le 2(x+1)^{-1} + 2x^{-1} + \left(2x^{-1} - 2(x+1)^{-1}\right) = 4x^{-1}$$

proving the theorem. □

Theorem (b)**.**

$$2xf(x) = \cos(x^2) - \cos[(x+1)^2] + \mathcal{O}\left(\frac{1}{x}\right)$$

Proof. By (a), the $r(x)$ that remains to be compared to $\dfrac{1}{x}$ is

$$r(x) = \cos[(x+1)^2](x+1)^{-1} + 2^{-1}x\int_{x^2}^{(x+1)^2} \cos(u)u^{-3/2}\,du$$

Hence

$$|xr(x)| \le |\cos[(x+1)^2]x(x+1)^{-1}| + \left|2^{-1}x^2\int_{x^2}^{(x+1)^2} \cos(u)u^{-3/2}\,du\right| < 1 + x\left(1 - x(x+1)^{-1}\right) < 2$$

Hence $r(x) = \mathcal{O}\left(\dfrac{1}{x}\right)$ □

Theorem (d). *$\int_0^\infty \sin(t^2)\,dt$ converges.*

Proof. too lazy. Follows pretty easily from the last bound. □

problem 17

Problem. *Suppose α increases monotonically on $[a,b]$, g is continuous and $g(x) = G'(x)$. then*

$$\int_a^b \alpha g\,dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G\,d\alpha$$

Proof. For each interval, there is a $t_i \in [x_i, x_{i+1}]$ such that $g(t_i)\Delta x_i = G(x_{i+1}) - G(x_i)$. Hence using Summation by parts,

$$\sum \alpha(x_i)g(t_i)\Delta x_i = \sum \alpha(x_i)\left(G(x_{i+1}) - G(x_i)\right) = G(b)\alpha(b) - G(a)\alpha(a) - \sum G(x_i)\Delta\alpha(x_i)$$

Which is the statement of this theorem. □