ch 2

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ch 2

problem 2.1

Theorem. *Suppose G is a compact Abelian group, and π an irreducible complex representation, then it has dimension 1.*

Proof. The image of G inside $\mathrm{GL}(n, \mathbb{C})$ is a family of commuting matrices. Now if two matrices A, B commute then an eigenspace B_λ is preserved by A. This is because $Bv = \lambda v \implies ABv = \lambda Av \implies B(Av) = \lambda(Av)$. So any of the eigenspaces corrosponding to $\pi(g)$ will be invariant under G as they all commute, and since this is irreducible it would have to be the whole space. The only matrices who have a full eigenspace with only one eigenvalue are clearly multiples of the identity. These clearly fix any subspace and in particular any 1 dimensional subspace so n itself has to be 1. $\square$

Proof. (schur's) Since G is abelian, the maps $V \xrightarrow{\pi(g)} V$ are G equivariant, and hence by Schur's lemma corrosponds to λI for some λ. Conclusion follows as above. $\square$

problem 2.2

Theorem. *Suppose that G is a compact group and $G \xrightarrow{f} \mathbb{C}$ is a matrix coefficient of an representation π, then so is $\overline{f(g^{-1})}$.*

Proof. The space of matrix coefficients is generated by things of the form $f(g) = \langle \pi(g)w | v \rangle$ and so $f(g^{-1}) = \langle \pi(g)^{-1}w | v \rangle = \langle w | \pi(g)v \rangle = \overline{\langle \pi(g)v | w \rangle}$. $\square$

problem 2.3

For G a compact group, let $C(G)$ be the space of continous functions $G \longrightarrow \mathbb{C}$.Define the convolution $(f_1 \star f_2)(g) = \int_G f_1(gh^{-1}) f_2(h) \, \mathrm{d}h = \int_G f_1(h) f_2(h^{-1}g) \, \mathrm{d}h$

Theorem (I)**.** *$C(G)$ is a non unital ring under addition and convolution. M_π is a 2-sided ideal.*

Proof. The only non-trivial part to show this is a ring is to show $\star$ is associative.

$$\begin{aligned}(f_1 \star f_2) \star f_3 &= \int_{(ab)c=h} f_1(a) f_2(b) f_3(c) \, \mathrm{d}a \, \mathrm{d}b \\ &= \int_{a(bc)=h} f_1(a) f_2(b) f_3(c) \, \mathrm{d}a \, \mathrm{d}b \\ &= f_1 \star (f_2 \star f_3)\end{aligned}$$

Now for M_π, notice that for $f_1 = \langle \pi(g)v | w \rangle$,

$$\begin{aligned}f_1 \star f_2 &= \int_G \left\langle \pi(gh^{-1})v \middle| w \right\rangle f_2(h) \, \mathrm{d}h \\ &= \left\langle \pi(g) \int_G f_2(h) \pi(h)^{-1} v \, \mathrm{d}h \middle| w \right\rangle\end{aligned}$$

We were able to interchange the inner product and the integral due to the representation being finite dimensional. This is clearly a matrix coefficient. On the other hand take $f_2 = \langle \pi(g)v | w \rangle$ then

$$\begin{aligned}f_1 \star f_2 &= \int_G f_1(h) \left\langle \pi(h^{-1}g)v \middle| w \right\rangle \mathrm{d}h \\ &= \int_G f_1(h) \langle \pi(g)v | \pi(h)w \rangle \, \mathrm{d}h \\ &= \left\langle \pi(g)v \middle| \int_G \overline{f_1(h)} \pi(h) w \, \mathrm{d}h \right\rangle\end{aligned}$$

and so this is also a matrix coefficient. This shows M_π is a two sided ideal. $\square$

Corollary. *(schur's orthogonality) If π_1 and π_2 are two non-isomorphic irreducible representations of G then their matrix coefficients are orthogonal.*

Proof. First we prove something cannot be both of the form $\langle \pi_1(g)v_1 | w_1 \rangle$ and $\langle \pi_2(g)v_2 | w_2 \rangle$. define a map from $V_1 \xrightarrow{F} V_2$ by setting $F(\pi_1(g)v_1) = \pi_2(g)v_2$.This is well defined as if $\pi(h)v_1 = v_1$, then

$$\langle \pi_2(gh)v_2 | w_2 \rangle = \langle \pi_1(gh)v_1 | w_1 \rangle = \langle \pi_1(g)v_1 | w_1 \rangle = \langle \pi_2(g)v_2 | w_2 \rangle$$

and so $\langle \pi_2(g)(\pi_2(h)v_2 - v_2) | w_2 \rangle = 0$ for all g. Since this is an irrecubile representation, $\pi_2(g)(\pi_2(h)v_2 - v_2)$ is either always 0 or generates the whole space, non-degeneracy of the inner product implies $\pi_2(h)v_2 = v_2$. So this map is 0 map by schur's lemma and both inner coefficients were zero to begin with.

Now if f_1 and f_2 are primitive matrix coefficients for π_1 and π_2 respectively, then $f_1 \star f_2$ is a primitive coefficient for both by the previous theorem. As discussed above, this means that $f_1 \star f_2 = 0$. The $g = 1$ case of this gives the orthogonality of the coefficients. $\square$

problem 2.4

Theorem. *If we make M_π a $G \times G$ module by $(g, h)f(x) = f(g^{-1}xh)$ then M_π is the same as $\hat{\pi} \otimes \pi$ as $G \times G$ modules.*

Proof. Send $L \otimes v$ to $L(\pi(g)v)$. This is clearly a $G \times G$ equivariant map. This is by definition of a matrix coefficient surjective. Injectivity also trivially follows. $\square$

problem 2.5

Theorem. *Let G be compact, then g, h are conjugate $\iff$ $\chi(g) = \chi(h)$ for all irreducible characters.*

($\rightarrow$) Suppose $g = xhx^{-1}$, since trace is invariant under conjugation so are the characters.

($\leftarrow$) Suppose $\chi(g) = \chi(h)$ for all irreducible characters. Take a class function that sends the conjugacy class of x to 1 and everything else to 0. Since the characters form an orthonormal basis for L^2 class functions (theorem 2.6. combined with peter-weyl),this class function must have the same values for h and g and hence they are in the same class.

problem 2.6

Theorem. *Let G be compact. The space of invariant bilinear forms $V \times W \xrightarrow{B} \mathbb{C}$ with V, W being two irreducible G modules is atmost dimension 1, achieving that only when these are contragradient.*

Proof. Notice that the bilinear form gives rise to a morphism $V \xrightarrow{B(v,-)} W^\star$, and if $W^\star$ were irreducible by schurs lemma this would either be an isomorphism (in which case the hom space would be dimension 1) or the map would be 0. We prove that for any finite representations (doesnt need to be compact) irreducibility is preserved by taking contragradients. Suppose a representation σ were reducible, then $\sigma = \oplus \pi_i$ for irreducible representations π_i and hence $\hat{\sigma} = \oplus \hat{\pi}_i$ is also reducible. Now since $\pi \cong \hat{\hat{\pi}}$, $\hat{\pi}$ cannot be reducible if π is irreducible, proving the result. $\square$

ch 3

problem 3.1

Theorem. *Suppose that T is a bounded operator on Hilbert space H, and suppose that for each $\epsilon > 0$ there is a compact operator T_ϵ such that $|T - T_\epsilon| < \epsilon$. Then T is compact.*

Proof. Take a bounded sequence $\{x_i\}$ and wlog assume $|x_i| \leq 1$, then we want to show that $\{Tx_i\}$ has a convergent subsequence. We shall construct subsequences $S_n = \{Tx_{ni}\}$ such that $|Tx_{ni} - Tx_{nj}| \leq \frac{1}{n}$. Suppose S_{n-1} is already constructed.We know that we can pick some x_{ni} from S_{n-1} such that $\{T_{1/3n}x_{ni}\}$ converges, and by starting the sequence late enough we can assume $|T_{1/3n}(x_{ni} - x_{nj})| \leq \frac{1}{3n}$, so that

$$|T(x_{ni}-x_{nj})| \leq |(T-T_{1/3n})(x_{ni}-x_{nj})|+|T_{1/3n}(x_{ni}-x_{nj})| \leq \frac{1}{3n}|x_{ni}-x_{nj}|+\frac{1}{3n} \leq \frac{1}{n}$$

This completes the construction. Now $\{Tx_{ii}\}$ is the desired subsequence. □

problem 3.2

Theorem. *(Hilbert-Schmidt operators) Suppose X is a locally compact Haussdorf space with a positive Borel measure μ, and that $L^2(X)$ has a countable basis. Let $K \in L^2(X \times X)$, define an operator on $L^2(X)$ by*

$$Tf(x) = \int_X K(x,y)f(y)\,\mathrm{d}\mu(y)$$

Then T is a compact operator.

Proof. Choose an orthonormal basis ϕ_i of $L^2(X)$, then $K(x,y) = \sum \psi_i(x)\overline{\phi_i(y)}$. The Fourier coefficients $\psi_i(x)$ are given by $T\phi_i$. Define $K_N(x,y) = \sum_{i\leq N} \psi_i(x)\overline{\phi_i(y)}$ and T_N in a similar way to T. Now we are going to approximate T with the T_N, and then show these are compact and conclude using the last problem.

Now suppose we have a bounded sequence $\{f_k\}$ in $L^2(X)$, and suppose $f_k(y) = \sum a_{ki}\phi_i(y)$, then

$$T_n f_k = \int_X K_N(x,y)f_k(y)\,\mathrm{d}y = \sum_{i\leq N}\sum_j a_{ki}\psi_j(x)\int_X \phi_i(y)\overline{\phi_j(y)}\,\mathrm{d}y = \sum_{i\leq N} a_{ki}\psi_i(x)$$

And convergence of a subsequence of this really boils down to convergence of a subsequence the (a_{ki}) which by the local compactness of $\mathbb{C}$ follows. By Cauchy-Schwartz T is a bounded operator, and since the K_N converge to K, the T_N get arbitrarily close to T. So by using the last problem T itself is compact. □

ch 4

problem 4.1

Theorem. *If G is a totally disconnected group then the kernels of its finite representations are all open.*

Proof. We can clearly pick a small enough ball around the identity in $\mathrm{GL}(n, \mathbb{C})$ such that no subgroup is contained in it. Now the preimage of such an open set would be an open set in G such that the biggest subgroup it contains is $\ker(\pi)$. Now since totally disconnected groups have a neighborhood basis around the identity, the preimage has to contain an open subgroup, which must be contained in $\ker(\pi)$ as its the biggest subgroup in this. Now $\ker(\pi)$ is a union of the cosets of this open subgroup, and hence is open itself.

□

Corollary. $\mathrm{GL}(n, \mathbb{Z}_p)$ *has no faithful representations.*

Proof. $\{1\}$ is not an open subgroup. □

problem 4.2

Theorem. *Suppose G is a compact abelian group and $H \subset G$ a closed subgroup, then every character $H \xrightarrow{\chi} \mathbb{C}^\star$ can be extended to G.*

Proof. Consider the space of functions $V = \{f \in L^2(G) | f(hg) = \chi(h)f(g) \forall h \in H\}$.Now this is clearly a closed subspace and hence V is a Hilbert space. Now if V were non-empty, then we can take the action of G on V by left translations, and it is clearly invariant under this. So we get an unitary representation(since integral is invariant under translations, lets call thisρ) and hence by Peter-Weyl we get that this is a direct sum of irreducible finite representations. We know compact abelian groups only have dimension 1 irreducible representations by problem 2.1, so we get an irreducible representation $G \xrightarrow{\pi} \mathbb{C}$. Now notice $\rho(h)f(x) = f(hx) = \chi(h)f(x)$ so $\rho(h) = \chi(h)I$ and hence its restriction to a 1 dimensional subspace is $\chi(h)$. So these representations are the extensions of χ to G.

Now we shall prove that V is non-empty. If H had measure 0, then it would need to be discrete as open sets have positive measure. By compactness it would be finite then, and then we can simply take G/H and define a function by giving coset representatives value 1 and everything else value by the rule of V. This would clearly be square integrable then as its absolute value is 1 everywhere.

Now suppose H has positive measure. Notice that for a $\phi \in C(G)$, $f(g) = \int_H \phi(hg)\chi(h)^{-1} \, \mathrm{d}h$, we have

$$f(h'g) = \int_H \phi(hh'g)\chi(h)^{-1} \, \mathrm{d}h = \chi(h') \int_H \phi(hh'g)\chi(hh')^{-1} \, \mathrm{d}h = \chi(h')f(g)$$

whenver $h' \in H$. This is also L^2, because f is continuous, being the integral of two continuous functions. Now pick a neighborhood of the identity where the integral doesnt dissapear, and then a slighly larger one with the same property. Uryshons allows us to find a ϕ that is 1 on the (closure of) the smaller neighborhood and 0 outside the bigger neighborhood. We can make the things between the two neighborhoods arbritarily small and get a nonzero integral and f. □

ch 5

problem 5.1

Theorem.