problem 3

Theorem. *Suppose* $s_1 = \sqrt{2}$ *and* $s_{n+1} = \sqrt{2 + \sqrt{s_n}}$ *for* $n = 1, 2, 3 \ldots,$ *then* $\{s_n\}$ *converges*

Lemma. *For all natural* n, $1 < s_n < 2$

Proof. the base case $1 < s_1 < 2$ is clearly true. Suppose $1 < s_n < 2$, then

$$1 < \sqrt{2 + \sqrt{1}} < s_{n+1} < \sqrt{2 + \sqrt{2}} < 2.$$

Then by principle of induction, $1 < s_n < 2$ is true for all naturals. □

Now for the proof of the main theorem:

Theorem. *Take* $\epsilon > 0$, *and then* $4^N > \frac{1}{\epsilon}$, *then* $|s_n - s_m| < \epsilon$ *whenever* $n, m > N$.

Proof. WLOG $n \geq m > N$. Now we have

$$|s_n - s_m| = \frac{|s_n^2 - s_m^2|}{s_n + s_m} = \frac{|s_{n-1} - s_{m-1}|}{(s_n + s_m)(\sqrt{s_{n-1}} + \sqrt{s_{m-1}})} < \frac{|s_{n-1} - s_{m-1}|}{4}$$

Now doing this $m - 1$ times we have

$$|s_n - s_{m-1}| < \frac{|s_{n-m+1} - s_1|}{4^{m-1}} < \frac{2 - \sqrt{2}}{4^{m-1}} < \frac{1}{4^N} < \epsilon$$

□

Hence $\{s_n\}$ is cauchy, meaning it converges in $\mathbb{R}$

problem 5

Theorem. *Given the upper limits are all finite,*

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty}(a_n) + \limsup_{n\to\infty}(b_n)$$

Proof. Let $x^\star = \limsup_{n\to\infty} x_n$. Now by defination

$$x_1 > a^\star \implies \exists N_1 \forall n > N_1 : a_n < x_1$$

$$x_2 > b^\star \implies \exists N_2 \forall n > N_2 : b_n < x_2$$

now let $N = \max(N_1, N_2)$, and $x = x_1 + x_2$, then

$$x > a^\star + b^\star \implies \forall n \geq N : a_n + b_n < x$$

Suppose $(a + b)^\star > a^\star + b^\star$. Now all neighborhoods of $(a + b)*$ contains arbritarily large members of some subsequence. Take one that does not contain $a^\star + b^\star$, then there are arbritarily large member of the sequence in this ball that are bigger than $a^\star + b^\star$, which contradicts the inequality. Hence the theorem is true. □

problem 7

I am a loser and could not solve this, and looked up the sol.

problem 10

Theorem. *Suppose the coefficient of the power series $\sum a_n z^n$ are integers, infinitely many distinct from zero. Prove the radius of convergence is atmost 1.*

Proof. We know by the root test that $|z| \le \frac{1}{\alpha}$, where $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Now pick a subsequence $\{a'_n\}$, which removes all the zeroes. notice $|a'_n| \ge 1$. this means

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} \ge \limsup_{n\to\infty} \sqrt[n]{|a'_n|} \ge \limsup_{n\to\infty} \sqrt[n]{1} = 1$$

hence the radius of convergence is atmost 1. □

problem 11

Problem. *Suppose $a_n > 0$, and $s_n = a_1 + \cdots + a_n$, and that $\sum a_n$ diverges*

Theorem. *(a) $\sum \frac{a_n}{1+a_n}$ diverges.*

Proof. Suppose $a_n \to \infty$, then clearly $\frac{a_n}{1+a_n} \not\to 0$, and the sum diverges.
Suppose $a_n \not\to \infty$, then a_n is bounded. let this bound be $a_n \le a$, then $\frac{a_n}{1+a_n} \ge \frac{a_n}{1+a}$, and since $\sum \frac{a_n}{1+a}$ diverges, so does the given sum. □

Theorem. *(b) $\sum_{n=p+1}^{q} \frac{a_n}{s_n} \ge 1 - \frac{s_p}{s_q}$*

Proof. Since a_n is positive, s_n is increasing, hence

$$\sum_{n=p+1}^{q} \frac{a_n}{s_n} \ge \sum_{n=p+1}^{q} \frac{a_n}{s_q} = \frac{s_q - s_p}{s_q} = 1 - \frac{s_p}{s_q}$$

since $\frac{1}{s_q}$ tends to 0, This sequence is not cauchy, making it divergent. □

Theorem. *(c)*

$$\frac{a_n}{s_n^2} \le \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and hence $\sum \frac{a_n}{s_n^2}$ converges.

Proof.

$$\frac{a_n}{s_n^2} = \frac{s_n - s_{n-1}}{s_n^2} = \frac{1}{s_n} - \frac{s_{n-1}}{s_n^2} \le \frac{1}{s_n} - \frac{1}{s_{n-1}}$$

The RHS converges as $\frac{1}{s_n}$ converges and the series telescopes. □

Theorem. $\sum \frac{a_n}{1+a_n n^2}$ *converges*

Proof. $\frac{a_n}{1+a_n n^2} \le \frac{1}{n^2}$ □

problem 12

Problem. *Suppose $a_n > 0$ and $\sum a_n$ converges. let $r_n = \sum_{m\geq n} a_m$.*

Theorem. *(a) $\sum_{k=m}^{n} \frac{a_k}{r_k} > 1 - \frac{r_n}{r_m}$ and hence $\sum \frac{a_k}{r_k}$ diverges*

Proof. We have $r_m > r_n$ for $m < n$, hence

$$\sum_{k=m}^{n} \frac{a_k}{r_k} > \sum_{k=m}^{n} \frac{a_k}{r_m} = \frac{r_m - r_{n+1}}{r_m} > 1 - \frac{r_n}{r_m}$$

we have $r_n \to 0$, hence this sequence isnt cauchy. □

Theorem.

$$\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$$

and hence $\sum \frac{a_n}{\sqrt{r_n}}$ converges

Proof. since $r_n > r_{n+1}$,

$$\frac{a_n}{\sqrt{r_n}} < \frac{2a_n}{\sqrt{r_n} + \sqrt{r_{n+1}}} = \frac{2a_n}{r_n - r_{n+1}}(\sqrt{r_n} - \sqrt{r_{n+1}}) = 2(\sqrt{r_n} - \sqrt{r_{n+1}})$$

convergence follows trivially as $r_n \to 0$ □

Problem 14

Problem. *Let $\{s_n\}$ be a complex sequence, and define its arithmetic mean*

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1}$$

Theorem. *Suppose $s_n \to s$, then $\sigma_n \to s$.*

Proof. Since s_n is convergent, it is bounded. let $b = \sup |s_n - s|$. Now fix $\epsilon > 0$, since $s_n \to s$, there exists N such that for all $n > N$:$|s_n - s| < \frac{\epsilon}{2}$.Now take $k > N + 2b\frac{N+1}{\epsilon}$, then

$$\left| s - \sum_{n\leq k} \frac{s_n}{k+1} \right| < \sum_{n\leq k} \frac{|s_n - s|}{k+1} = \frac{1}{k+1}\left(\sum_{n\leq N} |s_n - s| + \sum_{N<n\leq k} |s_n - s|\right) < \frac{(N+1)b + (N-n)\frac{\epsilon}{2}}{k+1} < \epsilon$$

□

Theorem. *$\sigma_n \to \sigma$ does not imply $s_n \to \sigma$*

Proof. Take $\{s_n\} = 1, -1, 1, -1 \ldots$, then $\sigma_n \to 0$ but s_n diverges. □

Theorem. *There exists a positive sequence $\{s_n\}$ with $\limsup_{n\to\infty} s_n = \infty$ but $\sigma_n \to 0$*

Proof. consider the sequence :$s_{2n} = -s_{2n+1} = \sqrt{2n+1}$, then the upper limit of s_n is clearly ∞ while $\sigma \to 0$ □

Theorem. *Let $a_n = s_n - s_{n-1}$, then*

$$s_n - \sigma_n = \frac{1}{n+1}\sum_{k=1}^{n} ka_k$$

and then the condition of $na_n \to 0$ in addition to convergence of σ_n implies convergence of s_n

Proof.
$$\frac{1}{n+1}\sum_{k\leq n}k(s_k-s_{k-1})=\frac{1}{n+1}\left(\sum_{k\leq n}(ks_k-(k-1)s_{k-1})-\sum_{k\leq n}s_{k-1}\right)=s_n-\sigma_n$$
Now fix $\epsilon>0$, and let $b=\sup na_n$ which exists since na_n is bounded. Then there is a N such that for all $n>N$:$|na_n|<\frac{\epsilon}{2}$. Take $k>N+2b\frac{N+1}{\epsilon}$, then
$$\left|\sum_{n\leq k}\frac{na_n}{k+1}\right|\leq\sum_{n\leq k}\frac{|na_n|}{k+1}=\frac{1}{k+1}\left(\sum_{n\leq N}|na_n|+\sum_{N<n\leq k}|na_n|\right)<\frac{(N+1)b+(N-n)\frac{\epsilon}{2}}{k+1}<\epsilon.$$
Hence $s_n-\sigma_n\to 0 \implies s_n\to\sigma$, where σ is the limit of σ_n. □

problem 16(a)

Theorem. *With $x_1>\sqrt{\alpha}$, the sequence $2x_{n+1}=x_n+\alpha x_n^{-1}$ converges to $\sqrt{\alpha}$*

Proof. First by AM-GM, $x_n\geq\sqrt{\alpha}\to x_n\geq\frac{\alpha}{x_n}$. Hence $x_{n+1}\geq x_n$, hence it suffices to find the infinum of this sequence for the limit. As already noted, $\sqrt{\alpha}$ is a lower bound. Suppose some $x>\sqrt{\alpha}$ is the least upperbound, then there are x_n arbritarily close to x, but then $x_{n+1}=\frac{1}{2}\left(x_n+\alpha x_n^{-1}\right)\leq\frac{1}{2}(x_n+\sqrt{\alpha})$ and this would be greater than x, forming a contradiction. Hence $x_n\to\sqrt{\alpha}$. □

problem 17

Problem. *Fix $\alpha>1$ and $x_1>\sqrt{\alpha}$, and define*
$$x_{n+1}=\frac{\alpha+x_n}{1+x_n}=x_n+\frac{\alpha-x_n^2}{1+x_n}$$
Theorem. *(a) and (b) : $x_1>x_3>x_5>\dots$ and $x_2<x_4<x_6<\dots$*

Proof. First $x_{n+1}=x_n+\frac{\alpha-x_n^2}{1+x_n}$, hence clearly $x_{2n+1}>\sqrt{\alpha}$ and $x_{2n}<\sqrt{\alpha}$ by induction. Next
$$x_{n+2}=\frac{\alpha+1}{2}-\frac{1}{2}\frac{(\alpha-1)^2}{2x_n+\alpha+1}\implies y_{n+2}=\frac{(\sqrt{\alpha}-1)^2}{2}-\frac{(\alpha-1)^2}{2(2y_n+(\sqrt{\alpha}+1)^2)}\implies\frac{y_{n+2}}{y_n}=\frac{(\sqrt{\alpha}-1)^2}{2y_n+(\sqrt{\alpha}+1)^2}$$
Where $y_n=x_n-\sqrt{\alpha}$, clearly $x_n>1$, and hence $y_n>1-\sqrt{\alpha}>-4\sqrt{\alpha}$ meaning
$$\frac{y_{n+2}}{y_n}<\frac{(\sqrt{\alpha}-1)^2}{-4\sqrt{\alpha}+(\sqrt{\alpha}+1)^2}=1$$
Hence $x_1>x_3>x_5\dots$ and $x_2<x_4<x_6\dots$ □

Theorem. *(c) $x_n\to\sqrt{\alpha}$*

Proof. Duh □

problem 20

Theorem. *Suppose $\{p_n\}$ is a Cauchy sequence in some metric space X, and some subsequence $\{p_{n_k}\}$ converges to $p\in X$, then $p_n\to p$.*

Proof. Fix $\epsilon>0$, now there is some N such that for all $n,m>N$, $d(p_n,p_m)<\frac{\epsilon}{2}$. There is also some K such that $k>K$ means $d(p_{n_k},p)<\frac{\epsilon}{2}$. Now take $M=\max(N,K)$, and then $d(p_m,p)\leq d(p_m,p_{n_k})+d(p_{n_k},p)<\epsilon$ for all $m>M$, notice the fact that $n_k\geq k$ was used. Hence p_m converges to p. □

problem 21

Theorem. *Suppose $\{E_n\}$ is a sequence of non-empty,closed and bounded subsets of complete metric space X with $E_n \supset E_{n+1}$, and with $\lim_{n\to\infty} diam(E_n) = 0$, then there is one and only one element x in each of the E_n.*

Proof. Choose $x_n \in E_n$, this is possible since each E_n is non empty. Now $\{x_n\}$ as a cauchy sequence, as if not, then there is some ϵ such that $d(x_n, x_m) > \epsilon$ for arbritarily large n, m but then this is less than the diameter of E_k for arbitrarily large k, contradicting the diameter converging to 0. Since X is complete, $x_n \to x$ for some x.

Now all neighborhoods of x contain infinitely many x_n, i.e points of E_n, meaning x is either limit point of E_n or in E_n for each E_n. Since E_n is closed, x is in each E_n, meaning the intersection contains x.

If the intersection had more than one point, the diameter would be more than 0, forming a contradiction. □

problem 24

Theorem (a). *Call cauchy sequences $\{p_n\} \sim \{q_n\}$ if $\lim_{n\to\infty} d(p_n, q_n) = 0$. This is an equivalence relationship.*

Proof. This is clearly reflexive, as $d(p_n, p_n) = 0$. This is clearly symmetric as d is symmetric.

Suppose $\{p_n\} \sim \{q_n\}$ and $\{q_n\} \sim \{r_n\}$, then $d(p_n, r_n) \le d(p_n, q_n) + d(q_n, r_n) \implies \lim d(p_n, r_n) \le 0$ and since d is non-negative, the limit is 0, meaning $\{p_n\} \sim \{r_n\}$ □

Theorem (b). *Let $X^\star$ be the set of equivalence relations. Define*

$$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n)$$

with $\{p_n\} \in P, \{q_n\} \in Q$. Then this is a well defined distance function in $X^\star$

Proof. Suppose $\{p'_n\} \sim \{p_n\}$, then

$$d(p_n, q_n) - d(p_n, p'_n) \le d(p'_n, q_n) \le d(p_n, q_n) + d(p_n, p'_n)$$

Since $d(p, p'_n)$ goes to 0,

$$\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(p'_n, q_n)$$

Clearly Δ satisfies the triangle inequality, is symmetric, and by defination is 0 iff $P = Q$. Hence this is a distance function. □

Theorem (c). *$X^\star$ is complete.*

Proof. Take a cauchy sequence $\{P_n\}$, Call $A_{m,n} = \Delta(P_m, P_n)$.Fix $\epsilon > 0$. We know there is some N such that $m, n > N$ implies $A_{m,n} < \dfrac{\epsilon}{2}$. We know that there is some$K$ such that $k > K$ means,

$$|d(p_k^m, p_k^n) - A_{m,n}| < \frac{\epsilon}{2} \implies d(p_k^m, p_k^n) < \frac{\epsilon}{2} + A_{m,n} < \epsilon$$

with $\{p_k^m\} \in P_m$ and same for n.Since $\mathbb{R}$ is complete, this means $\{p_k^m\}_m$ converges for arbritarily large k, which means the whole sequence convergences, as for Δ only arbritarily large terms matter. □

Theorem (d). *Let $\psi : X \hookrightarrow X^\star$ be the canonical map that sends $\{p\}$ with $p \in X$ to its equivalence class, p''. This map is isometric.*

Proof.

$$\Delta(p'', q'') = \lim_{n\to\infty} d(p, q) = d(p, q)$$

□

Theorem (e)**.** *$\psi(X)$ is dense in $X^\star$, and $\psi(X) = X^\star$ if X is complete.*

Proof. take a $P \in X^\star$, then take a cauchy sequence in it, $\{p_n\}$. Now consider the cauchy sequence $\{P''_n\}$, this converges to P, as $\Delta(P, P''_m) = \lim_{n\to\infty} d(p_n, p_m)$ and this gets arbitrarily close to zero for large m, n as p is cauchy. This means every neighborhood of P contains a $P''_n \in \psi(X)$, or that $\psi(X)$ is dense in X.

Trivially if X is complete, then all equivalence class has all sequences converge. This means $X^\star$ is bijective to the set of all possible convergences of sequences in X, which is just X. Result follows. □