problem 1

Theorem. *There exist discontinous f on $\mathbb{R}$ such that*

$$\lim_{h\to 0}[f(x+h)-f(x-h)]=0$$

Proof. $f(x)=\dfrac{1}{x^2}$ $\square$

problem 2

Theorem. *For a continous map of metric spaces f, $f(\overline{E})\subset\overline{f(E)}$.*

Proof. Suppose $x\in E$, then trivially $f(x)\in\overline{f(E)}$.

Suppose $x\in E'$. Since f is continous, this means for all neighborhoods of $f(x)$, there is a neighborhood of x that is mapped into the neighborhood of $f(x)$. But since x is a limit point, there is some $p\in E$ in this neighborhood. It gets mapped to $f(p)\in f(E)$, meaning all neighborhoods of $f(x)$ contain a point of $f(E)$, meaning $f(x)\in\overline{f(E)}$ $\square$

problem 3

Theorem. *let $Z(f)$ be the set of zeros of continous map f of metric spaces, then $Z(f)$ is closed.*

Proof. By problem 2, $f(\overline{Z(f)})\subset\overline{f(Z(f))}=\{0\}$. This obviously forces $\overline{Z(f)}=Z(f)$. $\square$

problem 4

Theorem. *Let E be a dense subset of metric space X, and let f map X into metric space Y. Then $f(E)$ is dense in $f(X)$.*

Proof. First trivially $f(E)\subset f(X)$. Then by problem 2, $f(\overline{E})\subset\overline{f(E)}$ or $f(X)\subset\overline{f(E)}$, i.e $f(E)$ is dense in $f(X)$. $\square$

Theorem. *If $g(a)=f(a)$ for all $a\in E$, then $f(a)=g(a)$ forall $a\in X$.*

Proof. If $f=g$ in E, then $E\subset Z(f-g)$. But since $Z(f-g)$ is closed (problem 3), $X=\overline{E}\subset Z(f-g)$, hence $f=g$ in X. $\square$

problem 6

Theorem. *let the graph of a function $f:E\mapsto\mathbb{R}$, with $E\subset\mathbb{R}$ be the points $(x,f(x))$, then continous functions have continous graphs.*

Proof. Let $F:E\hookrightarrow E\times\mathbb{R}$ be the map defined as sending x to $(x,f(x))$. This is clearly injective.Fix $\epsilon>0$. Now since f is continous, we know there is a δ such that for all points $p\in E$, $d(x,p)<\delta$ means $d(f(x),f(p))<\dfrac{\sqrt{3}}{2}\epsilon$ with $x\in E$. Take $\delta'=\min(\delta,\epsilon/2)$, then $d(x,p)<\delta'$ still implies the same thing. Now this means

$$d(F(x),F(p))^2=d(f(x),f(p))^2+d(x,p)^2<\frac{3}{4}\epsilon^2+\delta'^2\leq\epsilon^2\implies d(F(x),F(p))<\epsilon$$

Hence F is a continous map. $\square$

Corollary. *If E is defined as above and is compact, then f is continous iff the graph of f is compact.*

Proof. $(\leftarrow)$ since F is continous, $F(E)$ is compact.

$(\rightarrow)$ since F is bijective with its image, $F^{-1}(F(E))=E$ must be compact. $\square$

problem 8

Theorem. *Suppose f is a real uniform map on bounded set $E \subset \mathbb{R}$, then $f(E)$ is bounded.*

Proof. Suppose E is finite, then so is $f(E)$, and trivially it has a finite diameter. Note each $f(x)$ must be finite since f is continuous.

Suppose E is infinite, and $f(E)$ unbounded. Take a subset $\tilde{E}$ with $\inf_{p,q \in \tilde{E}} d(f(p), f(q)) > 1$, notice this set can be chosen to be infinite. Now pick $\epsilon < 1$, then since f is uniformly continuous, there is some δ such that $d(p, q) < \delta \implies d(f(p), f(q)) < \epsilon$. The latter is never true in $\tilde{E}$, hence there must be some δ such that $d(p, q) < \delta$ is never true. But by Bolzano-Weierstrass If E is bounded we know $\tilde{E}$ has a limit point in $\mathbb{R}$, meaning there are points that are arbitrarily close to each other, forming a contradiction. $\square$

problem 11

Theorem. *Suppose $f : X \mapsto Y$ is a uniformly continuous mapping of metric spaces. then f maps cauchy sequences of X to cauchy sequences of Y.*

Proof. Let $\{x_n\}$ be cauchy in X. Fix $\epsilon > 0$. Since f is uniformly continous, we know there is a delta such that $d(x_n, x_m) < \delta \implies d(f(x_n), d(x_m)) < \epsilon$. Since $\{x_n\}$ is cauchy, we know there is some N such that $n, m \geq N$ implies $d(x_n, x_m) < \delta$, and hence that $d(f(x_n), d(x_m)) < \epsilon$. This proves the theorem. $\square$

problem 13

Theorem. *Let E be a dense subset of metric space X, and let f be a uniformly continuous map from E into some complete metric space Y, then f has a continous extention to X.*

Proof. Take $a \in X$. define $A_n := \overline{f(N_{E,1/n}(a))}$. A_n is a sequence of decreasing sets that are closed and bounded(since f is uniformly continuous), and non empty since E is dense in X . since Y is complete, $\bigcup A_n$ contains exactly one point. call this point $f'(a)$. Clearly f' is an extention of f, and continuous since every a is arbritarily close to some point in E, and the distance can hence be bounded this way. $\square$

problem 14

Theorem. *Every continuous $f : [0.1] \mapsto [0, 1]$ has a point in $[0, 1]$ such that $f(x) = x$*

Proof. Consider $g(x) = f(x) - x$, this is clearly continuous. If $g(0)$ or $g(1)$ is 0, we are done. If not, then clearly $g(0) > 0$ and $g(1) < 0$, and hence $g(0) > 0 > g(1)$. By IMV there is a $c \in (0, 1)$ such that $g(c) = 0$, and we are done. $\square$

problem 15

Theorem. *Suppose $f : \mathbb{R} \mapsto \mathbb{R}$ is an open continuous maping, then f is monotone.*

Proof. Suppose f is continuous but not monotone. This means theres an interval (a, b) where either the sup or inf of $f((a, b))$ isnt either of $f(a)$ or $f(b)$. Now since $f([a, b])$ is compact, this set contains its own sup or inf, and since one of these is not in the endpoints, $f((a, b))$ contains either its sup or inf. But no neighborhood around the sup or inf can be contained in $f((a, b))$, hence $f((a, b))$ is not open. This means f is not open. $\square$

problem 20

Problem. *let*

$$\rho_E(x) = \inf_{z \in E} d(x, z)$$

with $E \subset X$ and $x \in X$.

Theorem (a)**.** $\rho_E(x) = 0 \iff x \in \overline{E}$

Proof. ($\rightarrow$)Suppose $x \notin \overline{E}$, then there is some neighborhood of x that is disjoint from E, but $\rho_E(x)$ would be atleast this radius, meaning it is non-zero.

($\leftarrow$) Suppose $x \in \overline{E}$, then x is arbitritarily close to points of E or is a point of E. This means $\rho_E(x) = 0$. $\square$

Theorem (b)**.** *ρ_E is uniformly continuous on X.*

Proof.

$$\rho_E(x) = \inf d(x, z) \leq d(x, y) + \inf d(z, y) = d(x, y) + \rho_E(y)$$

Since this argument is symmetric, we can say $|\rho_E(x) - \rho_E(y)| \leq d(x, y)$. if we fix $\epsilon > 0$, then

$$d(x, y) < \epsilon \implies d_{\mathbb{R}}(\rho_E(x), \rho_E(y)) < \epsilon$$

Hence ρ_E is uniformly continuous. $\square$

problem 21

Theorem. *let K be a compact subset and F be closed subset of X. Show there is a $\delta > 0$ such that $d(p, q) > \delta$ whenever $p \in K, q \in F$.*

Proof. Suppose $d(p, q)$ is arbritarily close to 0. Now $\rho_F(K)$ is compact,since ρ_F is continuous. This means it is closed, and since 0 is a limit point, $0 \in \rho_F(K)$. But we know this is true iff there is a point $p \in K$ that is also $\in \overline{F} = F$. But we know these sets are disjoint, forming a contradiction. $\square$

problem 22

Theorem. *For non-empty,closed and empty sets A, B, let*

$$f(x) = \frac{\rho_A(x)}{\rho_B(x) + \rho_A(x)}$$

Then f is continuous on X with image contained in $[0, 1]$.

Proof. Clearly $\rho_B + \rho_A$ is continuous. Now since A, B are seperated, this is also always non zero. Hence f is also continuous.

$$f = \frac{1}{1 + \frac{\rho_B}{\rho_A}} \leq 1$$

$\square$

Theorem. $Z(f) = A, Z(f - 1) = B$

Proof. clearly $f(x) = 0 \iff \rho_A(x) = 0 \iff x \in A$. the other also follows like this. $\square$

Theorem. *Let $V = f^{-1}([0, .5))$ and $W = f^{-1}((.5, 1])$, then V, W are closed disjoint sets with $A \subset V$ and $B \subset W$. this is called normality of X.*

Proof. First $[0, .5)$ and $(.5, 1]$ are both open on $[0, 1]$, and since f is continuous their inverse images are also open. They are trivially open and trivially contain the respective closed sets of A, B. $\square$

problem 23

Theorem. *All convex functions are continous.*

Proof. Since f is convex, $f(y + \lambda(x - y)) - f(y) \leq \lambda(f(x) - f(y))$. Hence fixing $\epsilon > 0$ and y,

$$|p - y| < \left|\frac{x - y}{f(x) - f(y)}\right| \epsilon \implies |f(p) - f(y)| < \epsilon$$

$\square$