problem 2

Problem. *Prove the set of all algebraic numbers is countable.*

Lemma. *Suppose A is a countable set, then $A[x]$, the set of all polynomials in A is also countable.*

Proof. Let $A_n[x]$ denote the subset of $A[x]$ such that all members have degree less than n.Then there is a bijection between $A_n[x]$ and A^n, as each of the n coefficients of the polynomials in $A_n[x]$ can take on values from A.Hence $A_n[x]$ is countable, and hence $A[x] = \bigcup_{n\in\mathbb{N}} A_n[x]$ is countable. $\square$

Corollary. $\mathbb{Z}[x]$ *is countable*

Let Ω be the set of all algebraic integers, and let $R(f)$ denote the roots of of polynomial f. notice $R(f)$ is a finite set, and that

$$\Omega = \bigcup_{f\in\mathbb{Z}[x]} R(f)$$

And since this is a countable union of atmost countable sets, Ω is countable$\square$

Corollary. *(problem 3) There exists real numbers that are not algebriac.*

Proof. If all real numbers were algebraic, then the identity map would be a bijection $\mathbb{R} \hookrightarrow\!\!\!\!\rightarrow \Omega$, but $\mathbb{R}$ is uncountable while Ω is countable, $\square$

problem 4

Problem. *Prove the set of irrationals is uncountable*

Proof. Let $\mathbb{Q}^c$ denote the set of irrationals. Suppose $\mathbb{Q}^c$ is countable, then since $\mathbb{Q}$ is countable, $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$ would be countable.But $\mathbb{R}$ is uncountable, hence $\mathbb{Q}^c$ must be uncountable. $\square$

problem 6

Problem. *let E' be the set of all limit points of E. Prove that E' is closed. Prove that E and $\bar{E} = E \cup E'$ have the same limit point. Do E and E' always have the same limit points?*

Theorem. *E' is closed.*

Proof. Let p be a limit point of E',then all neighborhoods of p intersects E' at some point $q \neq p$. But there exists a neighborhood of q contained entirely in the neighborhood of p, and since $q \in E'$ it contains a point of E. Hence the neighborhood of p contains an element of E and p is a limit point. Hence E' is closed. $\square$

Corollary. *E and $\bar{E}$ have the same limit points.*

Proof. If $q \in N_r(p)$, then $q \in N_{r'}(p)$ forall $r' \geq r$.This means that the limit points of $A \cup B$ must be limit point of either A or B, since if not then there would be a neighborhood of the limit point p not in A and another not in B and the smaller neighborhood would contain neither. So limit points of $\bar{E}$ are the limit points of E and E' which is precisely the limit points of E $\square$

take $E = \left\{\frac{1}{n} | n \in \mathbb{N}\right\}$, then $E' = \{0\}$ and $E'' = \varnothing$.

problem 9

Problem. *let E° denote the set of all interior points of a set E*

Theorem. *(a) E° is always open*

Proof. Let $p \in E^\circ$. Suppose p is not an interior point of E°, then every neighborhood of p has a point not in E°. There is a neighborhood of p contained in E, let q be a point in this neighborhood not in E°. But there is a neighborhood of q contained in the neighborhood of p and hence contained in E, meaning q is an interior point of E. This is a contradiction, and hence all points of E° must be a interior point of E° □

Theorem. *(b) E is open $\iff$ $E = E^\circ$.*

Proof. ($\leftarrow$) Suppose $E = E^\circ$, then $E \subset E^\circ$, and hence E is open.
($\rightarrow$) Suppose E is open, then by defination $E \subset E^\circ$. Since all interior points have neighborhoods contained in E, they themselves must be in E, i.e $E^\circ \subset E$, hence $E = E^\circ$ □

Theorem. *(c) If $G \subset E$ is an open set, then $G \subset E^\circ$*

Proof. Suppose G is open, then $G \subset G^\circ$. Now all the interior points of G are also interior points of E, so $G^\circ \subset E^\circ$, hence $G \subset E^\circ$. □

Theorem. *(d) The complement of E° is the closure of E^c.*

Proof. Suppose $x \notin E^\circ$, this is true iff x is not a interior point of E, i.e iff all neighborhoods of x contains a point not in E. This holds iff x is a limit point of E^c. This means that $x \in E^c$ and $x \in (E^c)'$, so $x \in \overline{E^c}$. Hence $(E^\circ)^c = \overline{E^c}$ □

Theorem. *(e) E and $\bar{E}$ do not always have the same interiors.*

Proof. let $E = \mathbb{Q}$, then $\bar{E} = \mathbb{R}$, and clearly they have different interiors. □

Theorem. *(f) E and E° do not always have the same closures.*

Proof. take $E = \mathbb{Q}$, then $E^\circ = \varnothing$, again clearly different closures. □

problem 12

Problem. *Let $K = \left\{\frac{1}{n} | n \in \mathbb{N}\right\}$, prove directly that K is compact.*

Proof. take any cover $\{V_\alpha\}$ of K. 0 is contained in one of these open sets, call it V_0. 0 is an interior point of V_0, so some neighborhood $N_r(0)$ is contained entirely in this set. so V_0 covers everything from 0 to r. now the remaining members of K are with $\frac{1}{n} > r \implies \frac{1}{r} > n$. This set is clearly finite,so we can pick a singular V_n for each member of this set. $\{V_0\} \cup \{V_n\}$ is hence a finte subcover. □

problem 13

Problem. *Construct a subset of $\mathbb{R}$ that has countable limit points*

Proof. Take

$$A = \bigcup_{k \in \mathbb{Z}} \left\{k + \frac{1}{n} | n \in \mathbb{N}\right\}$$

The limit points of this are all of $\mathbb{Z}$. Infact this set is also closed since $\mathbb{Z} \subset A$ □

problem 14

Problem. *Construct a cover of* $(0,1)$ *with no finite subcovers*

Proof. Take the cover $V = \left\{ \left(\frac{x}{2}, x \right] | x \in (0,1) \right\}$, this clearly covers $(0,1)$ as $x \in \left(\frac{x}{2}, x \right]$. Suppose there is a finite subcover, then there would be an interval with smallest x in this finite subcover. But then all points less than $\frac{x}{2}$ would not be in this subcover, forming a contradiction. □

problem 17

Problem. *Let* E *be the set of all numbers in* $[0,1]$ *with only* $4s$ *and* $7s$ *in its decimal expansion*

Theorem. E *is uncountable.*

Proof. Define a map $f : E \hookrightarrow\!\!\!\!\to [0,1]$ as the composition of two maps: first by replacing 4s with 0s and 7s with 1s in E, then by converting the corrosponding binary number to decimal. Clearly both of these are bijective, hence the overall map is bijective, making E uncountable. □

Theorem. E *is not dense in* $[0,1]$

Proof. 0 is clearly not a limit point, as $N_{.1}(0)$ contains no point of E. and since $0 \notin E$, this means $0 \notin \overline{E} \implies \overline{E} \neq [0,1]$ □

Theorem. E *is compact.*

Proof. Suppose p is not in E, then there is a number that is not a 4 nor a 7 in its decimal expansion, let this be in the nth decimal place. Then $N_{10^{-n-2}}(p)$ has all elements with nth decimal place not being 4 or 7, meaining p is not a limit point. Hence E is closed. Since E is a closed and bounded subset of $\mathbb{R}$, by the Heine-Borel theorem it is compact. □

Theorem. E *is perfect*

Proof. let $p \in E$. let $r > 10^{-n}$, then $N_r(p)$ contains a point of E, obtained by changing the $(n+1)th$ decimal place from either 4 to 7 or 7 to 4. Hence p is a limit point of E. Since E is closed, E is perfect. □

problem 19

Theorem. *(a) If* A *and* B *are disjoint closed sets in some metric space* X*, they are seperate.*

Proof. since they are closed, $A = \overline{A}$ and $B = \overline{B}$ and then the result follows by defination. □

Theorem. *(b) If* A *and* B *are disjoint open sets in some metric space* X*, they are seperate.*

Proof. Suppose there is a limit point of B in A, call this p. There is a neighborhood of p contained in A since A open, and since p is a limit point:a point in B is contained in this neighborhood. Hence a point of B is contained in A, contradicting disjointness. This situation is symmetric, hence they are seperate. □

Theorem. *(c) fix* $p \in X, \delta > 0$*. let* $A = \{q \in X | d(p,q) < \delta\}$ *and* $B = \{q \in X | d(p,q) > \delta\}$*, then* A *and* B *are separate.*

Proof. A is just an open ball, and since B^c is the closed ball of the same radius, B is open. since A and B are disjoint open sets, they are seperate. □

Theorem. *(d) Every connected metric space with atleast* 2 *points is uncountable.*

Proof. Suppose a metric space X with atleast 2 points is atmost countable, then the set of all possible values for d is also atmost countable. Since $\mathbb{R}$ is uncountable, we can pick a δ which cannot be a value for d in this space. For some $p \in X$ take $A = \{q \in X | d(p,q) < \delta\}$ then $A^c = \{q \in X | d(p,q) > \delta\}$ which we saw was seperate and hence $X = A \cup A^c$ is not connected. □

problem 21

Problem. *let A and B be seperated subsets of some $\mathbb{R}^k$, suppose $a \in A$ and $b \in B$ and define*

$$p(t) = (1-t)a + tb, t \in \mathbb{R}$$

Let $A_0 = p^{-1}(A)$ and $B_0 = p^{-1}(B)$

Theorem. *(a) A_0 and B_0 are separated subsets of $\mathbb{R}$*

Proof. A_0 and B_0 are clearly disjoint. Now suppose a limit point t' of A_0 is in B_0. Take $p(t') = a + t'(b-a) \in B$. Consider $N_r(p(t'))$,a subset of this is $a + N_{r'}(t')(b-a)$ with $r' = \dfrac{r}{|a-b|}$. The t' neighborhood contains a point of A_0, and hence the $p(t')$ neighborhood contains a point of A, contradicting the seperatedness of A and B. Hence A_0 and B_0 are separated. □

Theorem. *(b) There exist $t_0 \in (0,1)$ such that $p(t_0) \notin A \cup B$*

Proof. $A_0 \cap (0,1)$ has a supremum less than 1, as $1 \in B_0$ cannot be a limit point of A_0. Let S_A denote this supremum. Let I_B be the smallest number in B_0 that is greater than S_A. Now since I_B has a neighborhood that is disjoint from A, there exists a t_0 such that $I_B > t_0 > S_A$.Now clearly $t_0 \notin A_0 \cup B_0$, and the problem follows. □

Theorem. *(c) Every convex subset of $\mathbb{R}^k$ is connected.*

Proof. Suppose E is a convex subset of $\mathbb{R}^k$ with $E = A \cup B$ where A and B are separated. Now pick $a \in A$ and $b \in B$, then by convexity all $(1-t)a + tb \in E$, but from (b) we know this is not true. Hence all convex subsets of $\mathbb{R}^k$ is connected. □

problem 23

Theorem. *Every seperable metric space has a countable base.*

Proof. Suppose X is seperable, and let X' be a countable dense subset of X. Then

$$V = \bigcup_{r \in \mathbb{Q}} \{N_r(x') | x' \in X'\}$$

is a countable union of countable sets, making it countable. Consider an open set $G \subset X$. let $x \in G$, then there is a neighborhood of x of some radius $2r$ contained in G. Take the neighborhood of x with radius r, this contains an element $x' \in X'$, and hence $x \in N_r(x') \subset N_{2r}(x) \subset G$ notice we could have picked a rational radius by simply decreasing the radius. Hence V is a basis for X, and all seperable metric spaces have countable basis. □

problem 24

Theorem. *Let X be a metric space where all infinite subsets have a limit point, then X is seperable.*

Proof. Fix $\delta > 0$, take $x_1 \in X$, and then for $x_1 \dots x_n \in X$, pick x_{n+1} such that $d(x_{n+1}, x_i) \geq \delta$ with $i = 1 \dots n$.Keep doing this until this is impossible. The set $s_\delta = \{x_i\}$ has no limit points, as any neighborhood of radius δ can contain atmost 1 point in the set and limit points must have neighborhoods that intersects the set infinitely. This set is hence finite. Every element of X is contained in δ neighborhoods of the x_i, as if a point was not contained in the neighborhoods then it could be added to the x_i. . Now take the set of all the centers of the balls, and take the union over $\delta = \dfrac{1}{n}, n \in \mathbb{N}$, this is a countable union of finite sets, making it countable. Now all neighborhoods of $x \in X$ contains one of these centers, making this dense. □

problem 25

Theorem. *Every compact metric space X has a countable basis, and is hence seperable.*

Proof. Take $\mathcal{N}_\delta$ as a cover of X made up of open balls around each of its elements with radius δ, then it has a finite subcover. Take the union of all such finite subcovers with $\delta \in \mathbb{Q}$, this is a countable union of finite sets making it countable. Using the same argument as 23, this is a basis for X. All neighborhoods of $x \in X$ contains one of the ball's center like in 24, and hence is seperable. $\square$