Understanding Analysis Solutions

Ulisse Mini

Preface

Huge thanks to the math discord for answering my questions! I don't know how I'd manage without them ♡

If you don't find your exercise here check linearalgebras.com or (god forbid) chegg.

I aim to complete abbott by new years. You can see my progress here.

Contents

Chapter 1

The Real Numbers

1.2 Some Preliminaries

Exercise 1.2.1

(a) Prove that $\sqrt{3}$ is irrational. Does a similar similar argument work to show $\sqrt{6}$ is irrational?

(b) Where does the proof break down if we try to prove $\sqrt{4}$ is irrational?

Solution

(a) Suppose for contradiction that p/q is a fraction in lowest terms, and that $(p/q)^2 = 3$. Then $p^2 = 3q^2$ implying p is a multiple of 3 since 3 is not a perfect square. Therefor we can write p as $3r$ for some r, substituting we get $(3r)^2 = 3q^2$ and $3r^2 = q^2$ implying q is also a multiple of 3 contradicting the assumption that p/q is in lowest terms.
For $\sqrt{6}$ the same argument applies, since 6 is not a perfect square.

(b) 4 is a perfect square, meaning $p^2 = 4q^2$ does not imply that p is a multiple of four as p could be 2.

Exercise 1.2.2

Show that there is no rational number satisfying $2^r = 3$

Solution

Letting $r = p/q$ we have $2^{p/q} = 3$ implying $2^p = 3^q$ which is impossible since 2 and 3 are coprime.

Exercise 1.2.3

Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all sets containing an infinite number of elements, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.

(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$.

(d) $A \cap (B \cap C) = (A \cap B) \cap C$.

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution

(a) False, consider $A_1 = \{1, 2, \dots\}, A_2 = \{2, 3, \dots\}$, ... has $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

(b) True.

(c) False, $A = \emptyset$ gives $\emptyset = C$.

(d) True, intersection is associative.

(e) True, draw a diagram.

Exercise 1.2.4

Produce an infinite collection of sets $A_1, A_2, A_3, \dots$ with the property that every A_i has an infinite number of elements, $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{\infty} A_i = \mathbf{N}$

Solution

This question is asking us to partition $\mathbf{N}$ into an infinite collection of sets. This is equivalent to asking us to unroll $\mathbf{N}$ into a square, which we can do along the diagonal

$$\begin{array}{cccccc} 1 & 3 & 6 & 10 & 15 & \cdots \\ 2 & 5 & 9 & 14 & \cdots & \\ 4 & 8 & 13 & \cdots & & \\ 7 & 12 & \cdots & & & \\ 11 & \dots & & & & \\ \vdots & & & & & \end{array}$$

Exercise 1.2.5 (De Morgan's Laws)

Let A and B be subsets of $\mathbf{R}$.

(a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$

(b) Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$

(c) Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

Solution

(a) If $x \in (A \cap B)^c$ then $x \notin A \cap B$ so $x \notin A$ or $x \notin B$ implying $x \in A^c$ or $x \in B^c$ which is the same as $x \in A^c \cup B^c$.

(b) Let $x \in A^c \cup B^c$ implying $x \in A^c$ or $x \in B^c$ meaning $x \notin A$ or $x \notin B$ implying $x \notin A \cap B$ which is the same as $x \in (A \cap B)^c$.

(c) First let $x \in (A \cup B)^c$ implying $x \notin A \cup B$ meaning $x \notin A$ and $x \notin B$ which is the same as $x \in A^c$ and $x \in B^c$ which is just $x \in A^c \cap B^c$. Second let $x \in A^c \cap B^c$ implying $x \in A^c$ and $x \in B^c$ implying $x \notin A$ and $x \notin B$ meaning $x \notin A \cup B$ which is just $x \in (A \cup B)^c$.

Exercise 1.2.6

(a) Verify the triangle inequality in the special case where a and b have the same sign.

(b) Find an efficient proof for all the cases at once by first demonstrating $(a+b)^2 \leq (|a|+|b|)^2$

(c) Prove $|a-b| \leq |a-c| + |c-d| + |d-b|$ for all a, b, c, and d.

(d) Prove $||a| - |b|| \leq |a-b|$. (The unremarkable identity $a = a - b + b$ may be useful.)

Solution

(a) We have equality $|a+b| = |a| + |b|$ meaning $|a+b| \leq |a| + |b|$ also holds.

(b) $(a+b)^2 \leq (|a|+|b|)^2$ reduces to $2ab \leq 2|a||b|$ which is obviously true. and since squaring preserves inequality this implies $|a+b| \leq |a| + |b|$.

(c) I would like to do this using the triangle inequality, I notice that $(a-c)+(c-d)+(d-b) = a-b$. Meaning I can use the triangle inequality for multiple terms

$$|a-b| = |(a-c)+(c-d)+(d-b)| \leq |a-c| + |c-d| + |d-b|$$

The general triangle inequality is proved by repeated application of the two variable inequality

$$|(a+b)+c| \leq |a+b| + |c| \leq |a| + |b| + |c|$$

(d) I would like to cancel the subtraction inside $||a| - |b||$ since then the inside will be positive, and the outer absolute value will vanish. Using the suggestion let $a = (a-b)+b$

$$||a| - |b|| = ||(a-b)+b| - |b|| \overset{!}{\leq} ||a-b| + |b| - |b|| = |a-b|$$

However this is incorrect by itself, since $|a| \leq |c|$ does not imply $||a| - |b|| \leq ||c| - |b||$ (draw a picture, or use the counterexample $a = 0, c = 1, b = 2$).

We can salvage this argument though, notice if $|a| \geq |b|$ then $|a| \leq |c|$ does imply $||a| - |b|| \leq ||c| - |b||$. And since we can swap a and b without changing anything, we can say without loss of generality assume $|a| \geq |b|$ and then apply the previous argument.

Exercise 1.2.7

Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(A) = \{f(x) : x \in A\}$.

(a) Let $f(x) = x^2$. If $A = [0, 2]$ (the closed interval $\{x \in \mathbf{R} : 0 \leq x \leq 2\}$) and $B = [1, 4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?

(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

(c) Show that, for an arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbf{R}$

(d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary function g

Solution

(a) $f(A) = [0, 4]$, $f(B) = [1, 16]$, $f(A \cap B) = [1, 4] = f(A) \cap f(B)$ and $f(A \cup B) = [0, 16] = f(A) \cup f(B)$

(b) $A = \{-1\}$, $B = \{1\}$ thus $f(A \cap B) = \emptyset$ but $f(A) \cap f(B) = \{1\}$

(c) Suppose $y \in g(A \cap B)$, then $\exists x \in A \cap B$ such that $g(x) = y$. But if $x \in A \cap B$ then $x \in A$ and $x \in B$, meaning $y \in g(A)$ and $y \in g(B)$ implying $y \in g(A) \cap g(B)$ and thus $g(A \cap B) \subseteq g(A) \cap g(B)$.

Notice why it is possible to have $x \in g(A) \cap g(B)$ but $x \notin g(A \cap B)$, this happens when something in $A \setminus B$ and something in $B \setminus A$ map to the same thing. If g is 1-1 this does not happen.

(d) I conjecture that $g(A \cup B) = g(A) \cup g(B)$. To prove this we show inclusion both ways, First suppose $y \in g(A \cup B)$. then either $y \in g(A)$ or $y \in g(B)$, implying $y \in g(A) \cup g(B)$. Now suppose $y \in g(A) \cup g(B)$ meaning either $y \in g(A)$ or $y \in g(B)$ which is the same as $y \in g(A \cup B)$ as above.

Exercise 1.2.8

Here are two important definitions related to a function $f : A \to B$. The function f is *one-to-one* $(1 - 1)$ if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is *onto* if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$ Give an example of each or state that the request is impossible:

(a) $f : \mathbf{N} \to \mathbf{N}$ that is $1 - 1$ but not onto.

(b) $f : \mathbf{N} \to \mathbf{N}$ that is onto but not $1 - 1$.

(c) $f : \mathbf{N} \to \mathbf{Z}$ that is $1 - 1$ and onto.

Solution

(a) Let $f(n) = n + 1$ does not have a solution to $f(a) = 1$

(b) Let $f(1) = 1$ and $f(n) = n - 1$ for $n > 1$

(c) Let $f(n) = n/2$ for even n, and $f(n) = -(n + 1)/2$ for odd n.

Exercise 1.2.9

Given a function $f : D \to \mathbf{R}$ and a subset $B \subseteq \mathbf{R}$, let $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B; that is, $f^{-1}(B) = \{x \in D : f(x) \in B\}$. This set is called the *preimage* of B.

(a) Let $f(x) = x^2$. If A is the closed interval $[0, 4]$ and B is the closed interval $[-1, 1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case? Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?

(b) The good behavior of preimages demonstrated in (a) is completely general. Show that for an arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ for all sets $A, B \subseteq \mathbf{R}$

Solution

(a) $f^{-1}(A) = [-2, 2]$, $f^{-1}(B) = [-1, 1]$, $f^{-1}(A \cap B) = [-1, 1] = f^{-1}(A) \cap f^{-1}(B)$, $f^{-1}(A \cup B) = [-2, 2] = f^{-1}(A) \cup f^{-1}(B)$

(b) First let $x \in g^{-1}(A \cap B)$ meaning $g(x) \in A \cap B$ implying $g(x) \in A$ and $g(x) \in B$ which is the same as $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$ meaning $x \in g^{-1}(A) \cap g^{-1}(B)$.

Second let $x \in g^{-1}(A) \cap g^{-1}(B)$, this is the same as $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$ which is the same as $g(x) \in A$ and $g(x) \in B$ implying $g(x) \in A \cap B$ and thus $x \in g^{-1}(A \cap B)$.

Exercise 1.2.10

Decide which of the following are true statements. Provide a short justification for those that are valid and a counterexample for those that are not:

(a) Two real numbers satisfy $a < b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

(b) Two real numbers satisfy $a < b$ if $a < b + \epsilon$ for every $\epsilon > 0$.

(c) Two real numbers satisfy $a \leq b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

Solution

(a) False, if $a = b$ then $a < b + \epsilon$ forall $\epsilon > 0$ but $a \not< b$

(b) False, see above

(c) True, suppose $a < b + \epsilon$ forall $\epsilon > 0$, We want to show this implies $a \leq b$. We either have $a \leq b$ or $a > b$, but $a > b$ is impossible since the gap implies there exists an ϵ small enough such that $a > b + \epsilon$. Now suppose $a \leq b$, obviously $a < b + \epsilon$ forall $\epsilon > 0$.

Exercise 1.2.11

Form the logical negation of each claim. One trivial way to do this is to simply add "It is not the case that..." in front of each assertion. To make this interesting, fashion the negation into a positive statement that avoids using the word "not" altogether. In each case, make an intuitive guess as to whether the claim or its negation is the true statement.

(a) For all real numbers satisfying $a < b$, there exists an $n \in \mathbf{N}$ such that $a + 1/n < b$

(b) There exists a real number $x > 0$ such that $x < 1/n$ for all $n \in \mathbf{N}$.

(c) Between every two distinct real numbers there is a rational number.

Solution

(a) There exist real numbers satisfying $a < b$ where $a + 1/n \geq b$ forall $n \in \mathbf{N}$ (false).

(b) For every real number $x > 0$ there exists an $n \in \mathbf{N}$ such that $x < 1/n$ (true).

(c) There exist two real numbers $a < b$ such that if $r < b$ then $r < a$ forall $r \in \mathbf{Q}$ (false).

Exercise 1.2.12
Let $y_1 = 6$, and for each $n \in \mathbf{N}$ define $y_{n+1} = (2y_n - 6)/3$

(a) Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in \mathbf{N}$.

(b) Use another induction argument to show the sequence $(y_1, y_2, y_3, \dots)$ is decreasing.

Solution

(a) Suppose $y_n > -6$, then $y_{n+1} = (2y_n - 6)/3$ implying $y_n = (3y_{n+1} + 6)/2 > -6$ implying $y_{n+1} > -6$ by basic algebra.

(b) Suppose $y_{n+1} < y_n$ this implies $2y_{n+1} < 2y_n$ implying $2y_{n+1} - 6 < 2y_n - 6$ and finally $(2y_{n+1} - 6)/3 < (2y_n - 6)/3$ which shows $y_{n+2} < y_{n+1}$.

Exercise 1.2.13
For this exercise, assume Exercise 1.2.5 has been successfully completed.

(a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

for any finite $n \in \mathbf{N}$

(b) It is tempting to appeal to induction to conclude

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of $n \in \mathbf{N}$, but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets $B_1, B_2, B_3, \dots$ where $\bigcap_{i=1}^{n} B_i \neq \emptyset$ is true for every $n \in \mathbf{N}$, but $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$ fails.

(c) Nevertheless, the infinite version of De Morgan's Law stated in (b) is a valid statement. Provide a proof that does not use induction.

Solution

(a) 1.2.5 Is our base case, Assume $(A_1 \cup \cdots \cup A_n)^c = A_1^c \cap \cdots \cap A_n^c$. We want to show the $n + 1$ case. Using associativity we have

$$\begin{aligned}((A_1 \cup \cdots \cup A_n) \cup A_{n+1})^c &= (A_1 \cup \cdots \cup A_n)^c \cap A_{n+1}^c \\ &= (A_1^c \cap \cdots \cap A_n^c) \cap A_{n+1}^c \\ &= A_1^c \cap \cdots \cap A_n^c \cap A_{n+1}^c\end{aligned}$$

(b) $B_1 = \{1, 2, \dots\}, B_2 = \{2, 3, \dots\}, \dots$

(c) First suppose $x \in \left(\bigcap_{i=1}^{\infty} A_i\right)^c$, then $x \notin \bigcap_{i=1}^{\infty} A_i$ meaning $x \notin A_i$ for some i, which is the same as $x \in A_i^c$ for some i, meaning $x \in \bigcup_{i=1}^{\infty} A_i^c$. This shows

$$\left(\bigcap_{i=1}^{\infty} A_i\right) \subseteq \bigcup_{i=1}^{\infty} A_i^c$$

Now suppose $x \in \bigcup_{i=1}^{\infty} A_i^c$ meaning $x \notin A_i$ for some i, which is the same as $x \notin \bigcap_{i=1}^{\infty} A_i$ implying $x \notin \left(\bigcap_{i=1}^{\infty} A_i\right)^c$. This shows inclusion the other way and completes the proof.

1.3 The Axiom of Completeness

Exercise 1.3.1

(a) Write a formal definition in the style of Definition 1.3.2 for the *infimum* or *greatest lower bound* of a set.

(b) Now, state and prove a version of Lemma 1.3.8 for greatest lower bounds.

Solution

(a) We have $i = \inf A$ if and only if

  (i) Lower bound, $a \geq i$ forall $a \in A$

  (ii) Greatest lower bound, If b is a lower bound on A then $b \leq i$

(b) Theorem: Suppose i is a lower bound for A, it is the greatest lower bound if and only if forall $\epsilon > 0$, there exists an $a \in A$ such that $i + \epsilon < a$.

Proof:
($\implies$) Suppose $i = \inf A$, then any $i + \epsilon$ cannot be a lower bound since i is defined as the greatest lower bound, and $i + \epsilon > i$.
($\impliedby$) Suppose $j > i$ is a lower bound on A, then set $\epsilon = j - i$ and we have that $i + \epsilon = j$ is not a lower bound since there exists $a \in A$ such that $j > a$. Therefor i is the greatest lower bound.

Exercise 1.3.2

Give an example of each of the following, or state that the request is impossible.

(a) A set B with $\inf B \geq \sup B$.

(b) A finite set that contains its infimum but not its supremum.

(c) A bounded subset of $\mathbf{Q}$ that contains its supremum but not its infimum.

Solution

(a) Let $B = \{0\}$ we have $\inf B = 0$ and $\sup B = 0$ thus $\inf B \leq \sup B$.

(b) Impossible, finite sets must contain their infimum and supremum.

(c) Let $B = \{r \in \mathbf{Q} \mid 1 < r \leq 2\}$ we have $\inf B = 1 \notin B$ and $\sup B = 2 \in B$.

Exercise 1.3.3

(a) Let A be nonempty and bounded below, and define $B = \{b \in \mathbf{R} : b \text{ is a lower bound for } A\}$. Show that $\sup B = \inf A$.

(b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

Solution

(a) By definition $\sup B$ is the greatest lower bound for A, meaning it equals $\inf A$.

(b) (a) Proves the greatest lower bound exists using the least upper bound.

Exercise 1.3.4
Let $A_1, A_2, A_3, \ldots$ be a collection of nonempty sets, each of which is bounded above.

(a) Find a formula for $\sup(A_1 \cup A_2)$. Extend this to $\sup\left(\bigcup_{k=1}^n A_k\right)$.

(b) Consider $\sup\left(\bigcup_{k=1}^\infty A_k\right)$. Does the formula in (a) extend to the infinite case?

Solution
(a) $\sup\left(\bigcup_{k=1}^n A_k\right) = \sup\{\sup A_k \mid k = 1, \ldots, n\}$

(b) Yes. Let $S = \{\sup A_k \mid k = 1, \ldots,\}$ and $s = \sup S$. s is obviously an upper bound for $\bigcup_{k=1}^\infty A_k$. to see it is the least upper bound suppose $s' < s$, then by definition there exists a k such that $\sup A_k > s'$ implying s' is not an upper bound for A_k. Therefor s is the least upper bound.

Exercise 1.3.5
As in Example 1.3.7, let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $c \in \mathbf{R}$. This time define the set $cA = \{ca : a \in A\}$.

(a) If $c \geq 0$, show that $\sup(cA) = c \sup A$.

(b) Postulate a similar type of statement for $\sup(cA)$ for the case $c < 0$.

Solution
(a) Let $s = c \sup A$. Suppose $ca > s$, then $a > \sup A$ which is impossible, meaning s is an upper bound on cA. Now suppose s' is an upper bound on cA and $s' < s$. Then $s'/c < s/c$ and $s'/c < \sup A$ meaning s'/c cannot bound A, so there exists $a \in A$ such that $s'/c > a$ meaning $s' > ca$ thus s' cannot be an upper bound on cA, and so $s = c \sup A$ is the least upper bound.

(b) $\sup(cA) = c \inf(A)$ for $c < 0$

Exercise 1.3.6
Given sets A and B, define $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Follow these steps to prove that if A and B are nonempty and bounded above then $\sup(A + B) = \sup A + \sup B$

(a) Let $s = \sup A$ and $t = \sup B$. Show $s + t$ is an upper bound for $A + B$.

(b) Now let u be an arbitrary upper bound for $A + B$, and temporarily fix $a \in A$. Show $t \leq u - a$.

(c) Finally, show $\sup(A + B) = s + t$.

(d) Construct another proof of this same fact using Lemma 1.3.8.

Solution
(a) We have $a \leq s$ and $b \leq t$, adding the equations gives $a + b \leq s + t$.

(b) $t \leq u - a$ should be true since $u - a$ is an upper bound on b, meaning it is greater then or equal to the least upper bonud t. Formally $a + b \leq u$ implies $b \leq u - a$ and since t is the least upper bound on b we have $t \leq u - a$.

(c) From (a) we know $s + t$ is an upper bound, so we must only show it is the least upper bound.

Let $u = \sup(A + B)$, from (a) we have $t \leq u - a$ and $s \leq u - b$ adding and rearranging gives $a + b \leq 2u - s - t$. since $2u - s - t$ is an upper bound on $A + B$ it is less then the least upper bound, so $u \leq 2u - s - t$ implying $s + t \leq u$. and since u is the least upper bound $s + t$ must equal u.

Stepping back, the key to this proof is that $a + b \leq s, \forall a, b$ implying $\sup(A + B) \leq s$ can be used to transition from all $a + b$ to a single value $\sup(A + B)$, avoiding the ϵ-hackery I would otherwise use.

(d) Showing $s + t - \epsilon$ is not an upper bound for any $\epsilon > 0$ proves it is the least upper bound by Lemma 1.3.8. Rearranging gives $(s - \epsilon/2) + (t - \epsilon/2)$ we know there exists $a > (s - \epsilon/2)$ and $b > (t - \epsilon/2)$ therefor $a + b > s + t - \epsilon$ meaning $s + t$ cannot be made smaller, and thus is the least upper bound.

Exercise 1.3.7
Prove that if a is an upper bound for A, and if a is also an element of A, then it must be that $a = \sup A$.

Solution
a is the least upper bound since any smaller bound $a' < a$ would not bound a.

Exercise 1.3.8
Compute, without proofs, the suprema and infima (if they exist) of the following sets:

(a) $\{m/n : m, n \in \mathbf{N} \text{ with } m < n\}$.

(b) $\{(-1)^m/n : m, n \in \mathbf{N}\}$.

(c) $\{n/(3n + 1) : n \in \mathbf{N}\}$

(d) $\{m/(m + n) : m, n \in \mathbf{N}\}$

Solution
(a) $\sup = 1$, $\inf = 0$

(b) $\sup = 1$, $\inf = -1$

(c) $\sup = 1/3$, $\inf = 1/4$

(d) $\sup = 1$, $\inf = 0$

Exercise 1.3.9
(a) If $\sup A < \sup B$, show that there exists an element $b \in B$ that is an upper bound for A.

(b) Give an example to show that this is not always the case if we only assume $\sup A \leq \sup B$

Solution

(a) By Lemma 1.3.8 we know there exists a b such that $(\sup B) - \epsilon < b$ for any $\epsilon > 0$, We set ϵ to be small enough that $\sup A < (\sup B) - \epsilon$ meaning $\sup A < b$ for some b, and thus b is an upper bound on A.

(b) $A = \{x \mid x \leq 1\}$, $B = \{x \mid x < 1\}$ no $b \in B$ is an upper bound since $1 \in A$ and $1 > b$.

Exercise 1.3.10 (Cut Property)
The Cut Property of the real numbers is the following:

If A and B are nonempty, disjoint sets with $A \cup B = \mathbf{R}$ and $a < b$ for all $a \in A$ and $b \in B$, then there exists $c \in \mathbf{R}$ such that $x \leq c$ whenever $x \in A$ and $x \geq c$ whenever $x \in B$.

(a) Use the Axiom of Completeness to prove the Cut Property.

(b) Show that the implication goes the other way; that is, assume $\mathbf{R}$ possesses the Cut Property and let E be a nonempty set that is bounded above. Prove $\sup E$ exists.

(c) The punchline of parts (a) and (b) is that the Cut Property could be used in place of the Axiom of Completeness as the fundamental axiom that distinguishes the real numbers from the rational numbers. To drive this point home, give a concrete example showing that the Cut Property is not a valid statement when $\mathbf{R}$ is replaced by $\mathbf{Q}$.

Solution

(a) If $c = \sup A = \inf B$ then $a \leq c \leq b$ is obvious. So we must only prove $\sup A = \inf B$. If $\sup A < \inf B$ then we can find c between A and B implying $A \cup B \neq \mathbf{R}$. If $\sup A > \inf B$ then we can find a such that $a > b$ by subtracting $\epsilon > 0$ and using the least upper/lower bound facts, similarly to Lemma 1.3.8. Thus $\sup A$ must equal $\inf B$ since we have shown both alternatives are impossible.

(b) Let $B = \{x \mid e < x, \forall e \in E\}$ and let $A = B^c$. Clearly $a < b$ so the cut property applies. We have $a \leq c \leq b$ and must show the two conditions for $c = \sup E$

  (i) Since $E \subseteq A$, $a \leq c$ implies $e \leq c$ thus c is an upper bound.

  (ii) $c \leq b$ implies c is the smallest upper bound.

  Note: Using (a) here would be wrong, it assumes the axiom of completeness so we would be making a circular argument.

(c) $A = \{r \in \mathbf{Q} \mid r^2 < 2\}$, $B = A^c$ does not satisfy the cut property in $\mathbf{Q}$ since $\sqrt{2} \notin \mathbf{Q}$

Exercise 1.3.11
Decide if the following statements about suprema and infima are true or false. Give a short proof for those that are true. For any that are false, supply an example where the claim in question does not appear to hold.

(a) If A and B are nonempty, bounded, and satisfy $A \subseteq B$, then $\sup A \leq \sup B$.

(b) If $\sup A < \inf B$ for sets A and B, then there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.

(c) If there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

Solution

(a) True. We know $a \leq \sup A$ and $a \leq \sup B$ since $A \subseteq B$. since $\sup A$ is the least upper bound on A we have $\sup A \leq \sup B$.

(b) True. Let $c = (\sup A + \inf B)/2$, $c > \sup A$ implies $a < c$ and $c < \inf B$ implies $c < b$ giving $a < c < b$ as desired.

(c) False. consider $A = \{x \mid x < 1\}$, $B = \{x \mid x > 1\}$, $a < 1 < b$ but $\sup A \not< \inf B$ since $1 \not< 1$.

1.4 Consequences of Completeness

Exercise 1.4.1
Recall that $\mathbf{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbf{Q}$, then ab and $a + b$ are elements of $\mathbf{Q}$ as well.

(b) Show that if $a \in \mathbf{Q}$ and $t \in \mathbf{I}$, then $a + t \in \mathbf{I}$ and $at \in \mathbf{I}$ as long as $a \neq 0$.

(c) Part (a) can be summarized by saying that $\mathbf{Q}$ is closed under addition and multiplication. Is $\mathbf{I}$ closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s + t$ and st?

Solution

(a) Trivial.

(b) Suppose $a + t \in \mathbf{Q}$, then by (a) $(a + t) - a = t \in \mathbf{Q}$ contradicting $t \in \mathbf{I}$.

(c) $\mathbf{I}$ is not closed under addition or multiplication. consider $(1 - \sqrt{2}) \in \mathbf{I}$ by (b), and $\sqrt{2} \in \mathbf{I}$. the sum $(1 - \sqrt{2}) + \sqrt{2} = 1 \in \mathbf{Q} \notin \mathbf{I}$. Also $\sqrt{2} \cdot \sqrt{2} = 2 \in \mathbf{Q} \notin \mathbf{I}$.

Exercise 1.4.2
Let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $s \in \mathbf{R}$ have the property that for all $n \in \mathbf{N}, s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upper bound for A. Show $s = \sup A$.

Solution
This is basically a rephrasing of Lemma 1.3.8 using the archimedean property. The most straightforward approach is to argue by contradiction:

(i) If $s < \sup A$ then there exists an n such that $s + 1/n < \sup A$ contradicting $\sup A$ being the least upper bound.

(ii) If $s > \sup A$ then there exists an n such that $s - 1/n > \sup A$ where $s - 1/n$ is not an upper bound, contradicting $\sup A$ being an upper bound.

Thus $s = \sup A$ is the only remaining possibility.

Exercise 1.4.3
Prove that $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closed for the conclusion of the theorem to hold.

Solution
Suppose $x \in \bigcap_{n=1}^{\infty}(0, 1/n)$, then we have $0 < x < 1/n$ forall n, which is impossible by the archimedean property, In other words we can always set n large enough that x lies outside the interval.

Exercise 1.4.4
Let $a < b$ be real numbers and consider the set $T = \mathbf{Q} \cap [a, b]$. Show $\sup T = b$

Solution
We must show the two conditions for a least upper bound

(i) Clearly $t \leq b$ forall $t \in T$

(ii) Let $a < b' < b$. b' Cannot be an upper bound for T since the density theorem tells us we can find $r \in \mathbf{Q} \cap [a, b]$ such that $b' < r < b$.

Corollary 1 *Given any two real numbers $a < b$, there exists an irrational number t satisfying $a < t < b$*

Exercise 1.4.5
Using Exercise 1.4.1, supply a proof for Corollary 1.4.4 by considering the real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$.

Solution
By the density theorem we first find $a < p/q < b$, then we add $(\sqrt{2})/n$ with n small enough that $a < (p/q) + (\sqrt{2})/n < b$. By (a) we know $(p/q) + (\sqrt{2})/n \in \mathbf{I}$. Thus we have found $t = (p/q) + (\sqrt{2})/n$ with $a < t < b$ and $t \in \mathbf{I}$ which proves Corollary 1.4.4.

(I'm not sure what the exercise wanted me to do with $a - \sqrt{2}$ and $b - \sqrt{2}$. I'll figure out later)
TODO Make more rigorous

Exercise 1.4.6
Recall that a set B is dense in $\mathbf{R}$ if an element of B can be found between any two real numbers $a < b$. Which of the following sets are dense in $\mathbf{R}$? Take $p \in \mathbf{Z}$ and $q \in \mathbf{N}$ in every case.

(a) The set of all rational numbers p/q with $q \leq 10$.

(b) The set of all rational numbers p/q with q a power of 2 .

(c) The set of all rational numbers p/q with $10|p| \geq q$.

Solution

(a) Dense.

(b) Dense.

(c) Not dense since we cannot make $|p|/q$ smaller then $1/10$.

Exercise 1.4.7
Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ leads to a contradiction of the fact that $\alpha = \sup T$

Solution
Recall $T = \{t \in \mathbf{R} \mid t^2 < 2\}$ and $\alpha = \sup T$. suppose $\alpha^2 > 2$, we will show there exists an $n \in \mathbf{N}$ such that $(\alpha - 1/n)^2 > 2$ contradicting the assumption that α is the least upper bound.

We expand $(\alpha - 1/n)^2$ to find n such that $(\alpha^2 - 1/n) > 2$

$$2 < (\alpha - 1/n)^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} < \alpha^2 + \frac{1-2\alpha}{n}$$

Then

$$2 < \alpha^2 + \frac{1-2\alpha}{n} \implies n(2-\alpha^2) < 1 - 2\alpha$$

Since $2 - \alpha^2 < 0$ dividing reverses the inequality gives us

$$n > \frac{1-2\alpha}{2-\alpha^2}$$

This contradicts $\alpha^2 > 2$ since we have shown n can be picked such that $(\alpha^2 - 1/n) > 2$ meaning α is not the least upper bound.

Exercise 1.4.8

Give an example of each or state that the request is impossible. When a request is impossible, provide a compelling argument for why this is the case.

(a) Two sets A and B with $A \cap B = \emptyset, \sup A = \sup B, \sup A \notin A$ and $\sup B \notin B$.

(b) A sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.

(c) A sequence of nested unbounded closed intervals $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} L_n = \emptyset$. (An unbounded closed interval has the form $[a, \infty) = \{x \in R : x \geq a\}$.)

(d) A sequence of closed bounded (not necessarily nested) intervals $I_1, I_2, I_3, \ldots$ with the property that $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for all $N \in \mathbf{N}$, but $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

Solution

(a) $A = \mathbf{Q} \cap (0,1)$, $B = \mathbf{I} \cap (0,1)$. $A \cap B = \emptyset$, $\sup A = \sup B = 1$ and $1 \notin A$, $1 \notin B$.

(b) Impossible. $\bigcap_{n=1}^{\infty} J_n$ is the same as asking what happens to J_n as n goes to ∞. since every J_n is nonempty, $\bigcap_{n=1}^{\infty} = J_\infty$ must have an uncountably infinite number of elements.

(c) $L_n = [n, \infty)$ has $\bigcap_{n=1}^{\infty} L_n = \emptyset$

(d) Impossible. Let $J_n = \bigcap_{k=1}^{n} I_k$ and observe the following

   (i) Since $\bigcap_{n=1}^{N} I_n \neq \emptyset$ we have $J_n \neq \emptyset$.

   (ii) J_n being the intersection of closed intervals makes it a closed interval.

   (iii) $J_{n+1} \subseteq J_n$ since $I_{n+1} \cap J_n \subseteq J_n$

   (iv) $\bigcap_{n=1}^{\infty} J_n = \bigcap_{n=1}^{\infty} (\bigcap_{k=1}^{n} I_k) = \bigcap_{n=1}^{\infty} I_n$

   By (i), (ii) and (iii) the Nested Interval Property tells us $\bigcap_{n=1}^{\infty} J_n \neq \emptyset$. Therefor by (iv) $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.