Understanding Analysis Solutions

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Contents

Chapter 1

The Real Numbers

1.2 Some Preliminaries

Exercise 1.2.1

(a) Prove that $\sqrt{3}$ is irrational. Does a similar similar argument work to show $\sqrt{6}$ is irrational?

(b) Where does the proof break down if we try to prove $\sqrt{4}$ is irrational?

Solution

(a) Suppose for contradiction that p/q is a fraction in lowest terms, and that $(p/q)^2 = 3$. Then $p^2 = 3q^2$ implying p is a multiple of 3 since 3 is not a perfect square. Therefor we can write p as $3r$ for some r, substituting we get $(3r)^2 = 3q^2$ and $3r^2 = q^2$ implying q is also a multiple of 3 contradicting the assumption that p/q is in lowest terms.
For $\sqrt{6}$ the same argument applies, since 6 is not a perfect square.

(b) 4 is a perfect square, meaning $p^2 = 4q^2$ does not imply that p is a multiple of four as p could be 2.

Exercise 1.2.2
Show that there is no rational number satisfying $2^r = 3$

Solution
Letting $r = p/q$ we have $2^{p/q} = 3$ implying $2^p = 3^q$ which is impossible since 2 and 3 are coprime.

Exercise 1.2.3
Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all sets containing an infinite number of elements, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.

(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$.

(d) $A \cap (B \cap C) = (A \cap B) \cap C$.

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution

(a) False, consider $A_1 = \{1, 2, \ldots\}, A_2 = \{2, 3, \ldots\}$, ... has $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

(b) True.

(c) False, $A = \emptyset$ gives $\emptyset = C$.

(d) True, intersection is associative.

(e) True, draw a diagram.

Exercise 1.2.4
Produce an infinite collection of sets $A_1, A_2, A_3, \ldots$ with the property that every A_i has an infinite number of elements, $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{\infty} A_i = \mathbf{N}$

Solution
This question is asking us to partition $\mathbf{N}$ into an infinite collection of sets. This is equivalent to asking us to unroll $\mathbf{N}$ into a square, which we can do along the diagonal

$$\begin{array}{llllll} 1 & 3 & 6 & 10 & 15 & \cdots \\ 2 & 5 & 9 & 14 & \cdots & \\ 4 & 8 & 13 & \cdots & & \\ 7 & 12 & \cdots & & & \\ 11 & \cdots & & & & \\ \vdots & & & & & \end{array}$$

Exercise 1.2.5 (De Morgan's Laws)
Let A and B be subsets of $\mathbf{R}$.

(a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$

(b) Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$

(c) Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

Solution
(a) **TODO**

(b) **TODO**

(c) **TODO**

Exercise 1.2.6
(a) Verify the triangle inequality in the special case where a and b have the same sign.

(b) Find an efficient proof for all the cases at once by first demonstrating $(a+b)^2 \leq (|a|+|b|)^2$

(c) Prove $|a-b| \leq |a-c| + |c-d| + |d-b|$ for all a, b, c, and d.

(d) Prove $||a| - |b|| \leq |a-b|$. (The unremarkable identity $a = a - b + b$ may be useful.)

Solution
(a) We have equality $|a+b| = |a| + |b|$ meaning $|a+b| \leq |a| + |b|$ also holds.

(b) $(a+b)^2 \leq (|a|+|b|)^2$ reduces to $2ab \leq 2|a||b|$ which is obviously true. and since squaring preserves inequality this implies $|a+b| \leq |a|+|b|$. Showing that squaring preserves inequality is left as an exercise to the reader.

(c) I would like to do this using the triangle inequality, I notice that $(a-c)+(c-d)+(d-b) = a-b$. Meaning I can use the triangle inequality for multiple terms

$$|a-b| = |(a-c)+(c-d)+(d-b)| \leq |a-c| + |c-d| + |d-b|$$

The general triangle inequality is proved by repeated application of the two variable inequality

$$|(a+b)+c| \leq |a+b| + |c| \leq |a| + |b| + |c|$$

(d) I would like to cancel the subtraction inside $||a| - |b||$ since then the inside will be positive, and the outer absolute value will vanish. **TODO**