Understanding Analysis Solutions

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Preface

Huge thanks to the math discord for answering my questions! I don't know how I'd manage without them ♡

If you don't find your exercise here check linearalgebras.com or (god forbid) chegg.

I aim to complete abbott by new years. You can see my progress here.

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Chapter 1

The Real Numbers

1.2 Some Preliminaries

Exercise 1.2.1

(a) Prove that $\sqrt{3}$ is irrational. Does a similar similar argument work to show $\sqrt{6}$ is irrational?

(b) Where does the proof break down if we try to prove $\sqrt{4}$ is irrational?

Solution

(a) Suppose for contradiction that p/q is a fraction in lowest terms, and that $(p/q)^2 = 3$. Then $p^2 = 3q^2$ implying p is a multiple of 3 since 3 is not a perfect square. Therefor we can write p as $3r$ for some r, substituting we get $(3r)^2 = 3q^2$ and $3r^2 = q^2$ implying q is also a multiple of 3 contradicting the assumption that p/q is in lowest terms.
For $\sqrt{6}$ the same argument applies, since 6 is not a perfect square.

(b) 4 is a perfect square, meaning $p^2 = 4q^2$ does not imply that p is a multiple of four as p could be 2.

Exercise 1.2.2

Show that there is no rational number satisfying $2^r = 3$

Solution

Letting $r = p/q$ we have $2^{p/q} = 3$ implying $2^p = 3^q$ which is impossible since 2 and 3 are coprime.

Exercise 1.2.3

Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all sets containing an infinite number of elements, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.

(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$.

(d) $A \cap (B \cap C) = (A \cap B) \cap C$.

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution

(a) False, consider $A_1 = \{1, 2, \dots\}, A_2 = \{2, 3, \dots\}$, ... has $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

(b) True.

(c) False, $A = \emptyset$ gives $\emptyset = C$.

(d) True, intersection is associative.

(e) True, draw a diagram.

Exercise 1.2.4

Produce an infinite collection of sets $A_1, A_2, A_3, \dots$ with the property that every A_i has an infinite number of elements, $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{\infty} A_i = \mathbf{N}$

Solution

This question is asking us to partition $\mathbf{N}$ into an infinite collection of sets. This is equivalent to asking us to unroll $\mathbf{N}$ into a square, which we can do along the diagonal

$$\begin{array}{cccccc} 1 & 3 & 6 & 10 & 15 & \cdots \\ 2 & 5 & 9 & 14 & \cdots & \\ 4 & 8 & 13 & \cdots & & \\ 7 & 12 & \cdots & & & \\ 11 & \dots & & & & \\ \vdots & & & & & \end{array}$$

Exercise 1.2.5 (De Morgan's Laws)

Let A and B be subsets of $\mathbf{R}$.

(a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$

(b) Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$

(c) Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

Solution

(a) If $x \in (A \cap B)^c$ then $x \notin A \cap B$ so $x \notin A$ or $x \notin B$ implying $x \in A^c$ or $x \in B^c$ which is the same as $x \in A^c \cup B^c$.

(b) Let $x \in A^c \cup B^c$ implying $x \in A^c$ or $x \in B^c$ meaning $x \notin A$ or $x \notin B$ implying $x \notin A \cap B$ which is the same as $x \in (A \cap B)^c$.

(c) First let $x \in (A \cup B)^c$ implying $x \notin A \cup B$ meaning $x \notin A$ and $x \notin B$ which is the same as $x \in A^c$ and $x \in B^c$ which is just $x \in A^c \cap B^c$. Second let $x \in A^c \cap B^c$ implying $x \in A^c$ and $x \in B^c$ implying $x \notin A$ and $x \notin B$ meaning $x \notin A \cup B$ which is just $x \in (A \cup B)^c$.

Exercise 1.2.6

(a) Verify the triangle inequality in the special case where a and b have the same sign.

(b) Find an efficient proof for all the cases at once by first demonstrating $(a+b)^2 \leq (|a|+|b|)^2$

(c) Prove $|a-b| \leq |a-c|+|c-d|+|d-b|$ for all a, b, c, and d.

(d) Prove $||a|-|b|| \leq |a-b|$. (The unremarkable identity $a = a - b + b$ may be useful.)

Solution

(a) We have equality $|a+b| = |a|+|b|$ meaning $|a+b| \leq |a|+|b|$ also holds.

(b) $(a+b)^2 \leq (|a|+|b|)^2$ reduces to $2ab \leq 2|a||b|$ which is obviously true. and since squaring preserves inequality this implies $|a+b| \leq |a|+|b|$.

(c) I would like to do this using the triangle inequality, I notice that $(a-c)+(c-d)+(d-b) = a-b$. Meaning I can use the triangle inequality for multiple terms

$$|a-b| = |(a-c)+(c-d)+(d-b)| \leq |a-c|+|c-d|+|d-b|$$

The general triangle inequality is proved by repeated application of the two variable inequality

$$|(a+b)+c| \leq |a+b|+|c| \leq |a|+|b|+|c|$$

(d) I would like to cancel the subtraction inside $||a|-|b||$ since then the inside will be positive, and the outer absolute value will vanish. Using the suggestion let $a = (a-b)+b$

$$||a|-|b|| = ||(a-b)+b|-|b|| \overset{!}{\leq} ||a-b|+|b|-|b|| = |a-b|$$

However this is incorrect by itself, since $|a| \leq |c|$ does not imply $||a|-|b|| \leq ||c|-|b||$ (draw a picture, or use the counterexample $a = 0, c = 1, b = 2$).

We can salvage this argument though, notice if $|a| \geq |b|$ then $|a| \leq |c|$ does imply $||a|-|b|| \leq ||c|-|b||$. And since we can swap a and b without changing anything, we can say without loss of generality assume $|a| \geq |b|$ and then apply the previous argument.

Exercise 1.2.7

Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(A) = \{f(x) : x \in A\}$.

(a) Let $f(x) = x^2$. If $A = [0,2]$ (the closed interval $\{x \in \mathbf{R} : 0 \leq x \leq 2\}$) and $B = [1,4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?

(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

(c) Show that, for an arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbf{R}$

(d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary function g

Solution

(a) $f(A) = [0, 4]$, $f(B) = [1, 16]$, $f(A \cap B) = [1, 4] = f(A) \cap f(B)$ and $f(A \cup B) = [0, 16] = f(A) \cup f(B)$

(b) $A = \{-1\}$, $B = \{1\}$ thus $f(A \cap B) = \emptyset$ but $f(A) \cap f(B) = \{1\}$

(c) Suppose $y \in g(A \cap B)$, then $\exists x \in A \cap B$ such that $g(x) = y$. But if $x \in A \cap B$ then $x \in A$ and $x \in B$, meaning $y \in g(A)$ and $y \in g(B)$ implying $y \in g(A) \cap g(B)$ and thus $g(A \cap B) \subseteq g(A) \cap g(B)$.

Notice why it is possible to have $x \in g(A) \cap g(B)$ but $x \notin g(A \cap B)$, this happens when something in $A \setminus B$ and something in $B \setminus A$ map to the same thing. If g is 1-1 this does not happen.

(d) I conjecture that $g(A \cup B) = g(A) \cup g(B)$. To prove this we show inclusion both ways, First suppose $y \in g(A \cup B)$. then either $y \in g(A)$ or $y \in g(B)$, implying $y \in g(A) \cup g(B)$. Now suppose $y \in g(A) \cup g(B)$ meaning either $y \in g(A)$ or $y \in g(B)$ which is the same as $y \in g(A \cup B)$ as above.

Exercise 1.2.8

Here are two important definitions related to a function $f : A \to B$. The function f is *one-to-one* $(1 - 1)$ if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is *onto* if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$ Give an example of each or state that the request is impossible:

(a) $f : \mathbf{N} \to \mathbf{N}$ that is $1 - 1$ but not onto.

(b) $f : \mathbf{N} \to \mathbf{N}$ that is onto but not $1 - 1$.

(c) $f : \mathbf{N} \to \mathbf{Z}$ that is $1 - 1$ and onto.

Solution

(a) Let $f(n) = n + 1$ does not have a solution to $f(a) = 1$

(b) Let $f(1) = 1$ and $f(n) = n - 1$ for $n > 1$

(c) Let $f(n) = n/2$ for even n, and $f(n) = -(n + 1)/2$ for odd n.

Exercise 1.2.9

Given a function $f : D \to \mathbf{R}$ and a subset $B \subseteq \mathbf{R}$, let $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B; that is, $f^{-1}(B) = \{x \in D : f(x) \in B\}$. This set is called the *preimage* of B.

(a) Let $f(x) = x^2$. If A is the closed interval $[0, 4]$ and B is the closed interval $[-1, 1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case? Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?

(b) The good behavior of preimages demonstrated in (a) is completely general. Show that for an arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ for all sets $A, B \subseteq \mathbf{R}$

Solution

(a) $f^{-1}(A) = [-2, 2]$, $f^{-1}(B) = [-1, 1]$, $f^{-1}(A \cap B) = [-1, 1] = f^{-1}(A) \cap f^{-1}(B)$, $f^{-1}(A \cup B) = [-2, 2] = f^{-1}(A) \cup f^{-1}(B)$

(b) First let $x \in g^{-1}(A \cap B)$ meaning $g(x) \in A \cap B$ implying $g(x) \in A$ and $g(x) \in B$ which is the same as $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$ meaning $x \in g^{-1}(A) \cap g^{-1}(B)$.

Second let $x \in g^{-1}(A) \cap g^{-1}(B)$, this is the same as $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$ which is the same as $g(x) \in A$ and $g(x) \in B$ implying $g(x) \in A \cap B$ and thus $x \in g^{-1}(A \cap B)$.

Exercise 1.2.10

Decide which of the following are true statements. Provide a short justification for those that are valid and a counterexample for those that are not:

(a) Two real numbers satisfy $a < b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

(b) Two real numbers satisfy $a < b$ if $a < b + \epsilon$ for every $\epsilon > 0$.

(c) Two real numbers satisfy $a \le b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

Solution

(a) False, if $a = b$ then $a < b + \epsilon$ forall $\epsilon > 0$ but $a \not< b$

(b) False, see above

(c) True, suppose $a < b + \epsilon$ forall $\epsilon > 0$, We want to show this implies $a \le b$. We either have $a \le b$ or $a > b$, but $a > b$ is impossible since the gap implies there exists an ϵ small enough such that $a > b + \epsilon$. Now suppose $a \le b$, obviously $a < b + \epsilon$ forall $\epsilon > 0$.

Exercise 1.2.11

Form the logical negation of each claim. One trivial way to do this is to simply add "It is not the case that..." in front of each assertion. To make this interesting, fashion the negation into a positive statement that avoids using the word "not" altogether. In each case, make an intuitive guess as to whether the claim or its negation is the true statement.

(a) For all real numbers satisfying $a < b$, there exists an $n \in \mathbf{N}$ such that $a + 1/n < b$

(b) There exists a real number $x > 0$ such that $x < 1/n$ for all $n \in \mathbf{N}$.

(c) Between every two distinct real numbers there is a rational number.

Solution

(a) There exist real numbers satisfying $a < b$ where $a + 1/n \ge b$ forall $n \in \mathbf{N}$ (false).

(b) For every real number $x > 0$ there exists an $n \in \mathbf{N}$ such that $x < 1/n$ (true).

(c) There exist two real numbers $a < b$ such that if $r < b$ then $r < a$ forall $r \in \mathbf{Q}$ (false).

Exercise 1.2.12
Let $y_1 = 6$, and for each $n \in \mathbf{N}$ define $y_{n+1} = (2y_n - 6)/3$

(a) Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in \mathbf{N}$.

(b) Use another induction argument to show the sequence $(y_1, y_2, y_3, \dots)$ is decreasing.

Solution

(a) Suppose $y_n > -6$, then $y_{n+1} = (2y_n - 6)/3$ implying $y_n = (3y_{n+1} + 6)/2 > -6$ implying $y_{n+1} > -6$ by basic algebra.

(b) Suppose $y_{n+1} < y_n$ this implies $2y_{n+1} < 2y_n$ implying $2y_{n+1} - 6 < 2y_n - 6$ and finally $(2y_{n+1} - 6)/3 < (2y_n - 6)/3$ which shows $y_{n+2} < y_{n+1}$.

Exercise 1.2.13
For this exercise, assume Exercise 1.2.5 has been successfully completed.

(a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \dots \cup A_n)^c = A_1^c \cap A_2^c \cap \dots \cap A_n^c$$

for any finite $n \in \mathbf{N}$

(b) It is tempting to appeal to induction to conclude

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of $n \in \mathbf{N}$, but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets $B_1, B_2, B_3, \dots$ where $\bigcap_{i=1}^{n} B_i \neq \emptyset$ is true for every $n \in \mathbf{N}$, but $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$ fails.

(c) Nevertheless, the infinite version of De Morgan's Law stated in (b) is a valid statement. Provide a proof that does not use induction.

Solution

(a) 1.2.5 Is our base case, Assume $(A_1 \cup \dots \cup A_n)^c = A_1^c \cap \dots \cap A_n^c$. We want to show the $n+1$ case. Using associativity we have

$$\begin{aligned}((A_1 \cup \dots \cup A_n) \cup A_{n+1})^c &= (A_1 \cup \dots \cup A_n)^c \cap A_{n+1}^c \\ &= (A_1^c \cap \dots \cap A_n^c) \cap A_{n+1}^c \\ &= A_1^c \cap \dots \cap A_n^c \cap A_{n+1}^c\end{aligned}$$

(b) $B_1 = \{1, 2, \dots\}, B_2 = \{2, 3, \dots\}, \dots$

(c) First suppose $x \in \left(\bigcap_{i=1}^{\infty} A_i\right)^c$, then $x \notin \bigcap_{i=1}^{\infty} A_i$ meaning $x \notin A_i$ for some i, which is the same as $x \in A_i^c$ for some i, meaning $x \in \bigcup_{i=1}^{\infty} A_i^c$. This shows

$$\left(\bigcap_{i=1}^{\infty} A_i\right) \subseteq \bigcup_{i=1}^{\infty} A_i^c$$

Now suppose $x \in \bigcup_{i=1}^{\infty} A_i^c$ meaning $x \notin A_i$ for some i, which is the same as $x \notin \bigcap_{i=1}^{\infty} A_i$ implying $x \notin \left(\bigcap_{i=1}^{\infty} A_i\right)^c$. This shows inclusion the other way and completes the proof.

1.3 The Axiom of Completeness

Exercise 1.3.1

(a) Write a formal definition in the style of Definition 1.3.2 for the *infimum* or *greatest lower bound* of a set.

(b) Now, state and prove a version of Lemma 1.3.8 for greatest lower bounds.

Solution

(a) We have $i = \inf A$ if and only if

  (i) Lower bound, $a \geq i$ forall $a \in A$

  (ii) Greatest lower bound, If b is a lower bound on A then $b \leq i$

(b) Theorem: Suppose i is a lower bound for A, it is the greatest lower bound if and only if forall $\epsilon > 0$, there exists an $a \in A$ such that $i + \epsilon < a$.

Proof:
$(\implies)$ Suppose $i = \inf A$, then any $i + \epsilon$ cannot be a lower bound since i is defined as the greatest lower bound, and $i + \epsilon > i$.
$(\impliedby)$ Suppose $j > i$ is a lower bound on A, then set $\epsilon = j - i$ and we have that $i + \epsilon = j$ is not a lower bound since there exists $a \in A$ such that $j > a$. Therefor i is the greatest lower bound.

Exercise 1.3.2

Give an example of each of the following, or state that the request is impossible.

(a) A set B with $\inf B \geq \sup B$.

(b) A finite set that contains its infimum but not its supremum.

(c) A bounded subset of $\mathbf{Q}$ that contains its supremum but not its infimum.

Solution

(a) Let $B = \{0\}$ we have $\inf B = 0$ and $\sup B = 0$ thus $\inf B \leq \sup B$.

(b) Impossible, finite sets must contain their infimum and supremum.

(c) Let $B = \{r \in \mathbf{Q} \mid 1 < r \leq 2\}$ we have $\inf B = 1 \notin B$ and $\sup B = 2 \in B$.

Exercise 1.3.3

(a) Let A be nonempty and bounded below, and define $B = \{b \in \mathbf{R} : b$ is a lower bound for $A\}$. Show that $\sup B = \inf A$.

(b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

Solution

(a) By definition $\sup B$ is the greatest lower bound for A, meaning it equals $\inf A$.

(b) (a) Proves the greatest lower bound exists using the least upper bound.

Exercise 1.3.4
Let $A_1, A_2, A_3, \dots$ be a collection of nonempty sets, each of which is bounded above.

(a) Find a formula for $\sup(A_1 \cup A_2)$. Extend this to $\sup\left(\bigcup_{k=1}^{n} A_k\right)$.

(b) Consider $\sup\left(\bigcup_{k=1}^{\infty} A_k\right)$. Does the formula in (a) extend to the infinite case?

Solution
(a) $\sup\left(\bigcup_{k=1}^{n} A_k\right) = \sup\{\sup A_k \mid k = 1, \dots, n\}$

(b) Yes. Let $S = \{\sup A_k \mid k = 1, \dots, \}$ and $s = \sup S$. s is obviously an upper bound for $\bigcup_{k=1}^{\infty} A_k$. to see it is the least upper bound suppose $s' < s$, then by definition there exists a k such that $\sup A_k > s'$ implying s' is not an upper bound for A_k. Therefor s is the least upper bound.

Exercise 1.3.5
As in Example 1.3.7, let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $c \in \mathbf{R}$. This time define the set $cA = \{ca : a \in A\}$.

(a) If $c \geq 0$, show that $\sup(cA) = c \sup A$.

(b) Postulate a similar type of statement for $\sup(cA)$ for the case $c < 0$.

Solution
(a) Let $s = c \sup A$. Suppose $ca > s$, then $a > \sup A$ which is impossible, meaning s is an upper bound on cA. Now suppose s' is an upper bound on cA and $s' < s$. Then $s'/c < s/c$ and $s'/c < \sup A$ meaning s'/c cannot bound A, so there exists $a \in A$ such that $s'/c > a$ meaning $s' > ca$ thus s' cannot be an upper bound on cA, and so $s = c \sup A$ is the least upper bound.

(b) $\sup(cA) = c \inf(A)$ for $c < 0$

Exercise 1.3.6
Given sets A and B, define $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Follow these steps to prove that if A and B are nonempty and bounded above then $\sup(A + B) = \sup A + \sup B$

(a) Let $s = \sup A$ and $t = \sup B$. Show $s + t$ is an upper bound for $A + B$.

(b) Now let u be an arbitrary upper bound for $A + B$, and temporarily fix $a \in A$. Show $t \leq u - a$.

(c) Finally, show $\sup(A + B) = s + t$.

(d) Construct another proof of this same fact using Lemma 1.3.8.

Solution
(a) We have $a \leq s$ and $b \leq t$, adding the equations gives $a + b \leq s + t$.

(b) $t \leq u - a$ should be true since $u - a$ is an upper bound on b, meaning it is greater then or equal to the least upper bonud t. Formally $a + b \leq u$ implies $b \leq u - a$ and since t is the least upper bound on b we have $t \leq u - a$.

(c) From (a) we know $s + t$ is an upper bound, so we must only show it is the least upper bound.

Let $u = \sup(A + B)$, from (a) we have $t \leq u - a$ and $s \leq u - b$ adding and rearranging gives $a + b \leq 2u - s - t$. since $2u - s - t$ is an upper bound on $A + B$ it is less then the least upper bound, so $u \leq 2u - s - t$ implying $s + t \leq u$. and since u is the least upper bound $s + t$ must equal u.

Stepping back, the key to this proof is that $a + b \leq s, \forall a, b$ implying $\sup(A + B) \leq s$ can be used to transition from all $a + b$ to a single value $\sup(A + B)$, avoiding the ϵ-hackery I would otherwise use.

(d) Showing $s + t - \epsilon$ is not an upper bound for any $\epsilon > 0$ proves it is the least upper bound by Lemma 1.3.8. Rearranging gives $(s - \epsilon/2) + (t - \epsilon/2)$ we know there exists $a > (s - \epsilon/2)$ and $b > (t - \epsilon/2)$ therefor $a + b > s + t - \epsilon$ meaning $s + t$ cannot be made smaller, and thus is the least upper bound.

Exercise 1.3.7
Prove that if a is an upper bound for A, and if a is also an element of A, then it must be that $a = \sup A$.

Solution
a is the least upper bound since any smaller bound $a' < a$ would not bound a.

Exercise 1.3.8
Compute, without proofs, the suprema and infima (if they exist) of the following sets:

(a) $\{m/n : m, n \in \mathbf{N} \text{ with } m < n\}$.

(b) $\{(-1)^m/n : m, n \in \mathbf{N}\}$.

(c) $\{n/(3n + 1) : n \in \mathbf{N}\}$

(d) $\{m/(m + n) : m, n \in \mathbf{N}\}$

Solution
(a) $\sup = 1$, $\inf = 0$

(b) $\sup = 1$, $\inf = -1$

(c) $\sup = 1/3$, $\inf = 1/4$

(d) $\sup = 1$, $\inf = 0$

Exercise 1.3.9
(a) If $\sup A < \sup B$, show that there exists an element $b \in B$ that is an upper bound for A.

(b) Give an example to show that this is not always the case if we only assume $\sup A \leq \sup B$

Solution

(a) By Lemma 1.3.8 we know there exists a b such that $(\sup B) - \epsilon < b$ for any $\epsilon > 0$, We set ϵ to be small enough that $\sup A < (\sup B) - \epsilon$ meaning $\sup A < b$ for some b, and thus b is an upper bound on A.

(b) $A = \{x \mid x \leq 1\}$, $B = \{x \mid x < 1\}$ no $b \in B$ is an upper bound since $1 \in A$ and $1 > b$.

Exercise 1.3.10 (Cut Property)

The Cut Property of the real numbers is the following:

If A and B are nonempty, disjoint sets with $A \cup B = \mathbf{R}$ and $a < b$ for all $a \in A$ and $b \in B$, then there exists $c \in \mathbf{R}$ such that $x \leq c$ whenever $x \in A$ and $x \geq c$ whenever $x \in B$.

(a) Use the Axiom of Completeness to prove the Cut Property.

(b) Show that the implication goes the other way; that is, assume $\mathbf{R}$ possesses the Cut Property and let E be a nonempty set that is bounded above. Prove $\sup E$ exists.

(c) The punchline of parts (a) and (b) is that the Cut Property could be used in place of the Axiom of Completeness as the fundamental axiom that distinguishes the real numbers from the rational numbers. To drive this point home, give a concrete example showing that the Cut Property is not a valid statement when $\mathbf{R}$ is replaced by $\mathbf{Q}$.

Solution

(a) If $c = \sup A = \inf B$ then $a \leq c \leq b$ is obvious. So we must only prove $\sup A = \inf B$. If $\sup A < \inf B$ then we can find c between A and B implying $A \cup B \neq \mathbf{R}$. If $\sup A > \inf B$ then we can find a such that $a > b$ by subtracting $\epsilon > 0$ and using the least upper/lower bound facts, similarly to Lemma 1.3.8. Thus $\sup A$ must equal $\inf B$ since we have shown both alternatives are impossible.

(b) Let $B = \{x \mid e < x, \forall e \in E\}$ and let $A = B^c$. Clearly $a < b$ so the cut property applies. We have $a \leq c \leq b$ and must show the two conditions for $c = \sup E$

(i) Since $E \subseteq A$, $a \leq c$ implies $e \leq c$ thus c is an upper bound.

(ii) $c \leq b$ implies c is the smallest upper bound.

Note: Using (a) here would be wrong, it assumes the axiom of completeness so we would be making a circular argument.

(c) $A = \{r \in \mathbf{Q} \mid r^2 < 2\}$, $B = A^c$ does not satisfy the cut property in $\mathbf{Q}$ since $\sqrt{2} \notin \mathbf{Q}$

Exercise 1.3.11

Decide if the following statements about suprema and infima are true or false. Give a short proof for those that are true. For any that are false, supply an example where the claim in question does not appear to hold.

(a) If A and B are nonempty, bounded, and satisfy $A \subseteq B$, then $\sup A \leq \sup B$.

(b) If $\sup A < \inf B$ for sets A and B, then there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.

(c) If there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

Solution

(a) True. We know $a \leq \sup A$ and $a \leq \sup B$ since $A \subseteq B$. since $\sup A$ is the least upper bound on A we have $\sup A \leq \sup B$.

(b) True. Let $c = (\sup A + \inf B)/2$, $c > \sup A$ implies $a < c$ and $c < \inf B$ implies $c < b$ giving $a < c < b$ as desired.

(c) False. consider $A = \{x \mid x < 1\}$, $B = \{x \mid x > 1\}$, $a < 1 < b$ but $\sup A \not< \inf B$ since $1 \not< 1$.

1.4 Consequences of Completeness

Exercise 1.4.1
Recall that $\mathbf{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbf{Q}$, then ab and $a+b$ are elements of $\mathbf{Q}$ as well.

(b) Show that if $a \in \mathbf{Q}$ and $t \in \mathbf{I}$, then $a+t \in \mathbf{I}$ and $at \in \mathbf{I}$ as long as $a \neq 0$.

(c) Part (a) can be summarized by saying that $\mathbf{Q}$ is closed under addition and multiplication. Is $\mathbf{I}$ closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s+t$ and st?

Solution

(a) Trivial.

(b) Suppose $a+t \in \mathbf{Q}$, then by (a) $(a+t)-a = t \in \mathbf{Q}$ contradicting $t \in \mathbf{I}$.

(c) $\mathbf{I}$ is not closed under addition or multiplication. consider $(1-\sqrt{2}) \in \mathbf{I}$ by (b), and $\sqrt{2} \in \mathbf{I}$. the sum $(1-\sqrt{2}) + \sqrt{2} = 1 \in \mathbf{Q} \notin \mathbf{I}$. Also $\sqrt{2} \cdot \sqrt{2} = 2 \in \mathbf{Q} \notin \mathbf{I}$.

Exercise 1.4.2
Let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $s \in \mathbf{R}$ have the property that for all $n \in \mathbf{N}, s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upper bound for A. Show $s = \sup A$.

Solution
This is basically a rephrasing of Lemma 1.3.8 using the archimedean property. The most straightforward approach is to argue by contradiction:

(i) If $s < \sup A$ then there exists an n such that $s + 1/n < \sup A$ contradicting $\sup A$ being the least upper bound.

(ii) If $s > \sup A$ then there exists an n such that $s - 1/n > \sup A$ where $s - 1/n$ is not an upper bound, contradicting $\sup A$ being an upper bound.

Thus $s = \sup A$ is the only remaining possibility.

Exercise 1.4.3
Prove that $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closed for the conclusion of the theorem to hold.

Solution
Suppose $x \in \bigcap_{n=1}^{\infty}(0, 1/n)$, then we have $0 < x < 1/n$ forall n, which is impossible by the archimedean property, In other words we can always set n large enough that x lies outside the interval.

Exercise 1.4.4
Let $a < b$ be real numbers and consider the set $T = \mathbf{Q} \cap [a, b]$. Show $\sup T = b$

Solution
We must show the two conditions for a least upper bound

(i) Clearly $t \leq b$ forall $t \in T$

(ii) Let $a < b' < b$. b' Cannot be an upper bound for T since the density theorem tells us we can find $r \in \mathbf{Q} \cap [a, b]$ such that $b' < r < b$.

Exercise 1.4.5
Using Exercise 1.4.1, supply a proof that $\mathbf{I}$ is dense in $\mathbf{R}$ by considering the real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$. In other words show for every two real numbers $a < b$ there exists an irrational number t with $a < t < b$.

Solution
The density theorem lets us find a rational number r with $a - \sqrt{2} < r < b - \sqrt{2}$, adding $\sqrt{2}$ to both sides gives $a < r + \sqrt{2} < b$. From 1.4.1 we know $r + \sqrt{2}$ is irrational, so setting $t = r + \sqrt{2}$ gives $a < t < b$ as desired.

Exercise 1.4.6
Recall that a set B is dense in $\mathbf{R}$ if an element of B can be found between any two real numbers $a < b$. Which of the following sets are dense in $\mathbf{R}$? Take $p \in \mathbf{Z}$ and $q \in \mathbf{N}$ in every case.

(a) The set of all rational numbers p/q with $q \leq 10$.

(b) The set of all rational numbers p/q with q a power of 2 .

(c) The set of all rational numbers p/q with $10|p| \geq q$.

Solution
(a) Dense.

(b) Dense.

(c) Not dense since we cannot make $|p|/q$ smaller then $1/10$.

Exercise 1.4.7
Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ leads to a contradiction of the fact that $\alpha = \sup T$

Solution
Recall $T = \{t \in \mathbf{R} \mid t^2 < 2\}$ and $\alpha = \sup T$. suppose $\alpha^2 > 2$, we will show there exists an $n \in \mathbf{N}$ such that $(\alpha - 1/n)^2 > 2$ contradicting the assumption that α is the least upper bound.

We expand $(\alpha - 1/n)^2$ to find n such that $(\alpha^2 - 1/n) > 2$

$$2 < (\alpha - 1/n)^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} < \alpha^2 + \frac{1 - 2\alpha}{n}$$

Then

$$2 < \alpha^2 + \frac{1 - 2\alpha}{n} \implies n(2 - \alpha^2) < 1 - 2\alpha$$

Since $2-\alpha^2<0$ dividing reverses the inequality gives us

$$n > \frac{1-2\alpha}{2-\alpha^2}$$

This contradicts $\alpha^2 > 2$ since we have shown n can be picked such that $(\alpha^2 - 1/n) > 2$ meaning α is not the least upper bound.

Exercise 1.4.8

Give an example of each or state that the request is impossible. When a request is impossible, provide a compelling argument for why this is the case.

(a) Two sets A and B with $A \cap B = \emptyset, \sup A = \sup B, \sup A \notin A$ and $\sup B \notin B$.

(b) A sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.

(c) A sequence of nested unbounded closed intervals $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} L_n = \emptyset$. (An unbounded closed interval has the form $[a, \infty) = \{x \in R : x \geq a\}$.)

(d) A sequence of closed bounded (not necessarily nested) intervals $I_1, I_2, I_3, \ldots$ with the property that $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for all $N \in \mathbf{N}$, but $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

Solution

(a) $A = \mathbf{Q} \cap (0,1)$, $B = \mathbf{I} \cap (0,1)$. $A \cap B = \emptyset$, $\sup A = \sup B = 1$ and $1 \notin A$, $1 \notin B$.

(b) Impossible. $\bigcap_{n=1}^{\infty} J_n$ is the same as asking what happens to J_n as n goes to ∞. since every J_n is nonempty, $\bigcap_{n=1}^{\infty} = J_\infty$ must have an uncountably infinite number of elements.

(c) $L_n = [n, \infty)$ has $\bigcap_{n=1}^{\infty} L_n = \emptyset$

(d) Impossible. Let $J_n = \bigcap_{k=1}^{n} I_k$ and observe the following

  (i) Since $\bigcap_{n=1}^{N} I_n \neq \emptyset$ we have $J_n \neq \emptyset$.

  (ii) J_n being the intersection of closed intervals makes it a closed interval.

  (iii) $J_{n+1} \subseteq J_n$ since $I_{n+1} \cap J_n \subseteq J_n$

  (iv) $\bigcap_{n=1}^{\infty} J_n = \bigcap_{n=1}^{\infty} (\bigcap_{k=1}^{n} I_k) = \bigcap_{n=1}^{\infty} I_n$

  By (i), (ii) and (iii) the Nested Interval Property tells us $\bigcap_{n=1}^{\infty} J_n \neq \emptyset$. Therefor by (iv) $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

1.5 Cardinality

Exercise 1.5.1
Finish the following proof for Theorem 1.5.7. Assume B is a countable set. Thus, there exists $f : \mathbf{N} \to B$, which is $1-1$ and onto. Let $A \subseteq B$ be an infinite subset of B. We must show that A is countable.

Let $n_1 = \min\{n \in \mathbf{N} : f(n) \in A\}$. As a start to a definition of $g : \mathbf{N} \to A$ set $g(1) = f(n_1)$. Show how to inductively continue this process to produce a 1-1 function g from $\mathbf{N}$ onto A.

Solution
Let $n_k = \min\{n \in \mathbf{N} \mid f(n) \in A, n \notin \{n_1, n_2, \ldots, n_{k-1}\}\}$ and $g(k) = f(n_k)$. since $g : \mathbf{N} \to A$ is 1-1 and onto, A is countable.

Exercise 1.5.2
Review the proof of Theorem 1.5.6, part (ii) showing that $\mathbf{R}$ is uncountable, and then find the flaw in the following erroneous proof that $\mathbf{Q}$ is uncountable:

Assume, for contradiction, that $\mathbf{Q}$ is countable. Thus we can write $\mathbf{Q} = \{r_1, r_2, r_3, \ldots\}$ and, as before, construct a nested sequence of closed intervals with $r_n \notin I_n$. Our construction implies $\bigcap_{n=1}^{\infty} I_n = \emptyset$ while NIP implies $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. This contradiction implies Q must therefore be uncountable.

Solution
The nested interval property is not true for $\mathbf{Q}$. Consider I_n being rational bounds for $\sqrt{2}$ with n decimal places, then $\bigcap_{n=1}^{\infty} I_n = \emptyset$ since $\sqrt{2} \notin \mathbf{Q}$.

Exercise 1.5.3

(a) Prove if $A_1, \ldots, A_m$ are countable sets then $A_1 \cup \cdots \cup A_m$ is countable.

(b) Explain why induction *cannot* be used to prove that if each A_n is countable, then $\bigcup_{n=1}^{\infty} A_n$ is countable.

(c) Show how arranging $\mathbf{N}$ into the two-dimensional array

$$\begin{array}{llllll} 1 & 3 & 6 & 10 & 15 & \cdots \\ 2 & 5 & 9 & 14 & \cdots & \\ 4 & 8 & 13 & \cdots & & \\ 7 & 12 & \cdots & & & \\ 11 & \cdots & & & & \\ \vdots & & & & & \end{array}$$

leads to a proof for the infinite case.

Solution

(a) Let B, C be disjoint countable sets. We use the same trick as with the integers and list them as

$$B \cup C = \{b_1, c_1, b_2, c_2, \ldots\}$$

Meaning $B \cup C$ is countable, and $A_1 \cup A_2$ is also countable since we can let $B = A_1$ and $C = A_2 \setminus A_1$.

Now induction: suppose $A_1 \cup \cdots \cup A_n$ is countable, $(A_1 \cup \cdots \cup A_n) \cup A_{n+1}$ is the union of two countable sets which by above is countable.

(b) Induction shows something for each $n \in \mathbf{N}$, it does not apply in the infinite case.

(c) Rearranging $\mathbf{N}$ as in (c) gives us disjoint sets C_n such that $\bigcup_{n=1}^{\infty} C_n = \mathbf{N}$. Let B_n be disjoint, constructed as $B_1 = A_1, B_2 = A_1 \setminus B_1, \ldots$ we want to do something like

$$f(\mathbf{N}) = f\left(\bigcup_{n=1}^{\infty} C_n\right) = \bigcup_{n=1}^{\infty} f_n(C_n) = \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$$

Let $f_n : C_n \to B_n$ be bijective since B_n is countable, define $f : \mathbf{N} \to \bigcup_{n=1}^{\infty} B_n$ as

$$f(n) = \begin{cases} f_1(n) & \text{if } n \in C_1 \\ f_2(n) & \text{if } n \in C_2 \\ \vdots \end{cases}$$

(i) Since each C_n is disjoint and each f_n is 1-1, $f(n_1) = f(n_2)$ implies $n_1 = n_2$ meaning f is 1-1.

(ii) Since any $b \in \bigcup_{n=1}^{\infty} B_n$ has $b \in B_n$ for some n, we know $b = f_n(x)$ has a solution since f_n is onto. Letting $x = f_n^{-1}(b)$ we have $f(x) = f_n(x) = b$ since $f_n^{-1}(b) \in C_n$ meaning f is onto.

By (i) and (ii) f is bijective and so $\bigcup_{n=1}^{\infty} B_n$ is countable. And since

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$$

We have that $\bigcup_{n=1}^{\infty} A_n$ is countable, completing the proof.

Exercise 1.5.4

(a) Show $(a, b) \sim \mathbf{R}$ for any interval (a, b).

(b) Show that an unbounded interval like $(a, \infty) = \{x : x > a\}$ has the same cardinality as $\mathbf{R}$ as well.

(c) Using open intervals makes it more convenient to produce the required 1-1, onto functions, but it is not really necessary. Show that $[0, 1) \sim (0, 1)$ by exhibiting a 1-1 onto function between the two sets.

Solution

(a) We will start by finding $f : (-1, 1) \to \mathbf{R}$ and then transform it to (a, b). Example 1.5.4 gives a suitable f

$$f(x) = \frac{x}{x^2 - 1}$$

The book says to use calculus to show f is bijective, first we will examine the derivative

$$f'(x) = \frac{x^2 - 1 - 2x^2}{(x^2-1)^2} = -\frac{x^2+1}{(x^2-1)^2}$$

The denominator and numerator are positive, so $f'(x) < 0$ forall $x \in (0,1)$. This means no two inputs will be mapped to the same output, meaning f is one to one (a rigorous proof is beyond our current ability)

To show that f is onto, we examine the limits

$$\lim_{x \to 1^-} \frac{x}{x^2-1} = -\infty$$
$$\lim_{x \to -1^+} \frac{x}{x^2-1} = +\infty$$

Then use the intermediate value theorem to conclude f is onto.

Now we shift f to the interval (a,b)

$$g(x) = f\left(\frac{2x-1}{b-a} - a\right)$$

Proving $g(x)$ is also bijective is a straightforward application of the chain rule.

(b) We want a bijective $h(x)$ such that $h(x) : (a,\infty) \to (-1,1)$ because then we could compose them to get a new bijective function $f(h(x)) : (a,\infty) \to \mathbf{R}$.

Let
$$h(x) = \frac{2}{x-a+1} - 1$$
We have $h : (a,\infty) \to (1,-1)$ since $h(a) = 1$ and $\lim_{x\to\infty} h(x) = 1$.

Meaning that $f(h(x)) : (a,\infty) \to \mathbf{R}$ is our bijective map.

(c) With countable sets adding a single element doesn't change cardinality since we can just shift by one to get a bijective map. we'll use a similar technique here to essentially outrun our problems. Define $f : [0,1) \to (0,1)$ as

$$f(x) = \begin{cases} 1/2 & \text{if } x = 0 \\ 1/4 & \text{if } x = 1/2 \\ 1/8 & \text{if } x = 1/4 \\ \vdots & \\ x & \text{otherwise} \end{cases}$$

Now we prove f is bijective by showing $y = f(x)$ has exactly one solution for all $y \in (0,1)$.

If $y = 1/2^n$ then the only solution is $y = f(1/2^{n-1})$ (or $x = 0$ in the special case $n = 1$), If $y \neq 1/2^n$ then the only solution is $y = f(y)$.

Exercise 1.5.5

(a) Why is $A \sim A$ for every set A?

(b) Given sets A and B, explain why $A \sim B$ is equivalent to asserting $B \sim A$.

(c) For three sets A, B, and C, show that $A \sim B$ and $B \sim C$ implies $A \sim C$. These three properties are what is meant by saying that $\sim$ is an *equivalence relation*.

Solution

(a) The identity function $f(x) = x$ is a bijection

(b) If $f : A \to B$ is bijective then $f^{-1} : B \to A$ is bijective.

(c) Let $f : A \to B$ and $g : B \to C$, since $g \circ f : A \to C$ is bijective we have $A \sim C$.

Exercise 1.5.6

(a) Give an example of a countable collection of disjoint open intervals.

(b) Give an example of an uncountable collection of disjoint open intervals, or argue that no such collection exists.

Solution

(a) $I_1 = (0, 1)$, $I_2 = (1, 2)$ and in general $I_n = (n - 1, n)$

(b) Let A denote this set. Intuitively no such collection should exist since each I_n has nonzero length.

The key here is to try and show $A \sim \mathbf{Q}$ instead of directly showing $A \sim \mathbf{N}$.

For any nonempty interval I_n the density theorem tells us there exists an $r \in \mathbf{Q}$ such that $r \in I_n$. Assigning each $I \in A$ a rational number $r \in I$ proves $I \subseteq \mathbf{Q}$ and thus I is countable.

Exercise 1.5.7

Consider the open interval $(0, 1)$, and let S be the set of points in the open unit square; that is, $S = \{(x, y) : 0 < x, y < 1\}$.

(a) Find a 1-1 function that maps $(0, 1)$ into, but not necessarily onto, S. (This is easy.)

(b) Use the fact that every real number has a decimal expansion to produce a $1-1$ function that maps S into $(0, 1)$. Discuss whether the formulated function is onto. (Keep in mind that any terminating decimal expansion such as .235 represents the same real number as .234999 . . . .)

The Schröder-Bernstein Theorem discussed in Exercise 1.5.11 can now be applied to conclude that $(0, 1) \sim S$.

Solution

(a) We scale and shift up into the square. $f(x) = \frac{1}{2}x + \frac{1}{3}$

(b) Let $g : S \to (0,1)$ be a function that interleaves decimals in the representation without trailing nines. $g(0.32, 0.45) = 0.3425$ and $g(0.1\bar{9}, 0.2) = g(0.2, 0.2) = 0.22$ etc.

Every real number can be written with two digit representations, one with trailing 9's and one without. However $g(x, y) = 0.d_1d_2 \ldots \bar{9}$ is impossible since it would imply $x = 0.d_1 \ldots \bar{9}$ and $y = 0.d_2 \ldots \bar{9}$ but the definition of g forbids this. Therefor $g(s)$ is unique, and so g is 1-1.

Is g onto? No since $g(x, y) = 0.1$ has no solutions, since we would want $x = 0.1$ and $y = 0$ but $0 \notin (0,1)$.

Exercise 1.5.8
Let B be a set of positive real numbers with the property that adding together any finite subset of elements from B always gives a sum of 2 or less. Show B must be finite or countable.

Solution
Notice $B \cap (a, 2)$ is finite forall $a > 0$, since if it was infinite we could make a set with sum greater then two. And since B is the countable union of finite sets $\bigcup_{n=1}^{\infty} B \cap (1/n, 2)$, B must be countable or finite.

Exercise 1.5.9
A real number $x \in \mathbf{R}$ is called algebraic if there exist integers $a_0, a_1, a_2, \ldots, a_n \in \mathbf{Z}$, not all zero, such that

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

Said another way, a real number is algebraic if it is the root of a polynomial with integer coefficients. Real numbers that are not algebraic are called *transcendental* numbers. Reread the last paragraph of Section 1.1. The final question posed here is closely related to the question of whether or not transcendental numbers exist.

(a) Show that $\sqrt{2}$, $\sqrt[3]{2}$, and $\sqrt{3} + \sqrt{2}$ are algebraic.

(b) Fix $n \in \mathbf{N}$, and let A_n be the algebraic numbers obtained as roots of polynomials with integer coefficients that have degree n. Using the fact that every polynomial has a finite number of roots, show that A_n is countable.

(c) Now, argue that the set of all algebraic numbers is countable. What may we conclude about the set of transcendental numbers?

Solution
(a) $x^2 - 2 = 0$, $x^3 - 2 = 0$ are obvious. Now consider $\sqrt{3} + \sqrt{2}$. The key is setting $x = \sqrt{3} + \sqrt{2}$ then using algebra on x to concoct an integer, and thus find the polynomial with x as a root.

We have $x^2 = 5 + 2\sqrt{6}$ meaning $x^2 - 5 = 2\sqrt{6}$ and thus $(x^2-5)^2 = 24$ so $(x^2-5)^2 - 24 = 0$ is a polynomial with $\sqrt{3} + \sqrt{2}$ as a root.

(b) Basically $A_n \sim \mathbf{Z}^n \sim \mathbf{N}^n \sim \mathbf{N}$.

(i) $A_n \sim \mathbf{Z}^n$ since integer polynomials of degree n are identical to an ordered list of n integers.

(ii) $\mathbf{Z}^n \sim \mathbf{N}^n$ since $f : \mathbf{N}^n \to \mathbf{Z}^n$ is just the piecewise application of $g : \mathbf{N} \to \mathbf{Z}$.

(iii) $\mathbf{N}^n \sim \mathbf{N}$ since it is the intersection of finite sets $\bigcup_{n=2}^{\infty} \{(a,b) : a+b=n\}$.

In general if V is countable then $V^n = (v_1, \ldots, v_n)$ is also countable.

(c) By 1.5.3 the set of all algebraic numbers $\bigcup_{n=1}^{\infty} A_n$ is countable.

Exercise 1.5.10

(a) Let $C \subseteq [0,1]$ be uncountable. Show that there exists $a \in (0,1)$ such that $C \cap [a,1]$ is uncountable.

(b) Now let A be the set of all $a \in (0,1)$ such that $C \cap [a,1]$ is uncountable, and set $\alpha = \sup A$. Is $C \cap [\alpha, 1]$ an uncountable set?

(c) Does the statement in (a) remain true if "uncountable" is replaced by "infinite"?

Solution

(a) Suppose a does not exist, then $C \cap [a,1]$ is countable for all $a \in (0,1)$ meaning

$$\bigcup_{n=1}^{\infty} C \cap [1/n, 1] = C \cap [0,1]$$

Is countable (by 1.5.3), contradicting our assumption that $C \cap [0,1]$ is uncountable.

(b) If $\alpha = 1$ then $C \cap [\alpha, 1]$ is finite. Now if $\alpha < 1$ we have $C \cap [\alpha+\epsilon, 1]$ countable for $\epsilon > 0$ (otherwise the set would be in A, and hence α would not be an upper bound). Take

$$\bigcup_{n=1}^{\infty} C \cap [\alpha + 1/n, 1] = C \cap [\alpha, 1]$$

Which is countable by 1.5.3.

(c) No, consider the set $C = \{1/n : n \in \mathbf{N}\}$ it has $C \cap [\alpha, 1]$ finite for every α, but $C \cap [0,1]$ is infinite.

Exercise 1.5.11 (Schröder-Bernstein Theorem)

Assume there exists a 1-1 function $f : X \to Y$ and another 1-1 function $g : Y \to X$. Follow the steps to show that there exists a 1-1, onto function $h : X \to Y$ and hence $X \sim Y$. The strategy is to partition X and Y into components

$$X = A \cup A' \quad \text{and} \quad Y = B \cup B'$$

with $A \cap A' = \emptyset$ and $B \cap B' = \emptyset$, in such a way that f maps A onto B, and g maps B' onto A'.

(a) Explain how achieving this would lead to a proof that $X \sim Y$.

(b) Set $A_1 = X \setminus g(Y) = \{x \in X : x \notin g(Y)\}$ (what happens if $A_1 = \emptyset$?) and inductively define a sequence of sets by letting $A_{n+1} = g(f(A_n))$. Show that $\{A_n : n \in \mathbf{N}\}$ is a pairwise disjoint collection of subsets of X, while $\{f(A_n) : n \in \mathbf{N}\}$ is a similar collection in Y.

(c) Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} f(A_n)$. Show that f maps A onto B.

(d) Let $A' = X \setminus A$ and $B' = Y \setminus B$. Show g maps B' onto A'.

Solution

(a) **TODO**

(b) **TODO**

(c) **TODO**

(d) **TODO**

1.6 Cantor's theorem

Exercise 1.6.1
Show that $(0,1)$ is uncountable if and only if $\mathbf{R}$ is uncountable.

Solution
Exercise 1.5.4 tells us $(0,1)$ has the same cardinality as $\mathbf{R}$.

TODO Prove without using exercise 1.5.4 (probably what was intended)

Exercise 1.6.2
Let $f : \mathbf{N} \to \mathbf{R}$ be a way to list every real number (hence show $\mathbf{R}$ is countable).

Define a new number x with digits $b_1 b_2 \ldots$ given by

$$b_n = \begin{cases} 2 & \text{if } a_{nn} \neq 2 \\ 3 & \text{if } a_{nn} = 2 \end{cases}$$

(a) Explain why the real number $x = .b_1 b_2 b_3 b_4 \ldots$ cannot be $f(1)$.

(b) Now, explain why $x \neq f(2)$, and in general why $x \neq f(n)$ for any $n \in \mathbf{N}$.

(c) Point out the contradiction that arises from these observations and conclude that $(0,1)$ is uncountable.

Solution

(a) The first digit is different

(b) The nth digit is different

(c) Therefor x is not in the list, since the nth digit is different

Exercise 1.6.3
Supply rebuttals to the following complaints about the proof of Theorem 1.6.1.

(a) Every rational number has a decimal expansion, so we could apply this same argument to show that the set of rational numbers between 0 and 1 is uncountable. However, because we know that any subset of $\mathbf{Q}$ must be countable, the proof of Theorem 1.6.1 must be flawed.

(b) Some numbers have two different decimal representations. Specifically, any decimal expansion that terminates can also be written with repeating 9's. For instance, $1/2$ can be written as .5 or as $.4999\ldots$ Doesn't this cause some problems?

Solution

(a) False, since the constructed number has an infinite number of decimals it is irrational.

(b) No, since if we have $9999\ldots$ and change the nth digit $9992999 = 9993$ is still different.

Exercise 1.6.4
Let S be the set consisting of all sequences of 0 's and 1 's. Observe that S is not a particular sequence, but rather a large set whose elements are sequences; namely,

$$S = \{(a_1, a_2, a_3, \ldots) : a_n = 0 \text{ or } 1\}$$

As an example, the sequence $(1, 0, 1, 0, 1, 0, 1, 0, \ldots)$ is an element of S, as is the sequence $(1, 1, 1, 1, 1, 1, \ldots)$. Give a rigorous argument showing that S is uncountable.

Solution
We flip every bit in the diagonal just like with $\mathbf{R}$. Another way would be to show $S \sim \mathbf{R}$ by writing real numbers in base 2.

Exercise 1.6.5

(a) Let $A = \{a, b, c\}$. List the eight elements of $P(A)$. (Do not forget that $\emptyset$ is considered to be a subset of every set.)

(b) If A is finite with n elements, show that $P(A)$ has 2^n elements.

Solution

(a) $A = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

(b) There are n elements, we can include or exclude each element so there are 2^n subsets.

Exercise 1.6.6

(a) Using the particular set $A = \{a, b, c\}$, exhibit two different $1-1$ mappings from A into $P(A)$.

(b) Letting $C = \{1, 2, 3, 4\}$, produce an example of a $1 - 1$ map $g : C \to P(C)$.

(c) Explain why, in parts (a) and (b), it is impossible to construct mappings that are onto.

Solution

(a) $f(x) = \{x\}$, $f(x) = \{x, b\}$ for $x \neq b$ and $f(x) = \{a, b, c\}$ for $x = b$.

(b) $f(x) = \{x\}$.

(c) We can hit at most n elements in the power set out of the 2^n total elements.

Theorem (Cantor's Theorem)
Given any set A, there does not exist a function $f : A \to P(A)$ that is onto.

Proof
Suppose $f : A \to P(A)$ is onto. We want to use the self referential nature of $P(A)$ to find a contradiction. Define

$$B = \{a : a \notin f(a)\}$$

Since f is onto we must have $f(a) = B$ for some $a \in A$. Then either

(i) $a \in B$ implies $a \in f(a)$ which by the definition of B implies $a \notin B$, so $a \in B$ is impossible.

(ii) $a \notin B$ implies $a \notin f(a)$ since $f(a) = B$. but if $a \notin f(a)$ then $a \in B$ by the definition of B, contradicting $a \notin B$.

Therefor f cannot be onto, since we have found a $B \in P(A)$ where $f(a) = B$ is impossible.

Stepping back, the pearl of the argument is that if $B = f(a)$ then $B = \{a : a \notin B\}$ is undecidable/impossible.

Exercise 1.6.7
See the proof of Cantor's theorem above (the rest is a computation)

Exercise 1.6.8
See the proof of Cantor's theorem above

Exercise 1.6.9
Using the various tools and techniques developed in the last two sections (including the exercises from Section 1.5), give a compelling argument showing that $P(\mathbf{N}) \sim \mathbf{R}$.

Solution
I will show $P(\mathbf{N}) \sim [0, 1]$ then use 1.5.3 to conclude $P(\mathbf{N}) \sim \mathbf{R}$.

Let $A \subseteq \mathbf{N}$ and let a_n be the nth smallest element of A. We can write a_n via the digit representation as $a_n = d_1d_2d_3 \dots d_m$, concatinating the digits of every a_n in order gives a possibly infinite sequence of digits $d_1d_2d_3 \dots$

This process is clearly 1-1, however it is not onto as $\{1, 2\}$ and $\{12\}$ both give the same digits. Thus $P(\mathbf{N})$ is "greater then or equal" $\mathbf{R}$, if we show a 1-1 map $\mathbf{R} \to P(\mathbf{N})$ we can complete the proof using 1.5.11.

TODO Finish (or take a different approach)

Exercise 1.6.10
As a final exercise, answer each of the following by establishing a $1 - 1$ correspondence with a set of known cardinality.

(a) Is the set of all functions from $\{0, 1\}$ to $\mathbf{N}$ countable or uncountable?

(b) Is the set of all functions from $\mathbf{N}$ to $\{0, 1\}$ countable or uncountable?

(c) Given a set B, a subset $\mathcal{A}$ of $P(B)$ is called an antichain if no element of $\mathcal{A}$ is a subset of any other element of $\mathcal{A}$. Does $P(\mathbf{N})$ contain an uncountable antichain?

Solution
(a) The set of functions from $\{0, 1\}$ to $\mathbf{N}$ is the same as $\mathbf{N}^2$ which we found was countable in Exercise 1.5.9.

(b) This is the same as an infinite list of zeros and ones which we showed was uncountable in Exercise 1.6.4.

(c) **TODO**

Chapter 2

Sequences and Series

2.2 The Limit of a Sequence

Exercise 2.2.1
What happens if we reverse the order of the quantifiers in Definition 2.2.3?

Definition: A sequence (x_n) *verconges* to x if *there exists* an $\epsilon > 0$ such that *for all* $N \in \mathbf{N}$ it is true that $n \geq N$ implies $|x_n - x| < \epsilon$

Give an example of a vercongent sequence. Is there an example of a vercongent sequence that is divergent? Can a sequence verconge to two different values? What exactly is being described in this strange definition?

Solution
Firstly, since we have *for all* $N \in \mathbf{N}$ we can remove N entirely and just say $n \in \mathbf{N}$. Our new definition is

Definition: A sequence (x_n) *verconges* to x if *there exists* an $\epsilon > 0$ such that *for all* $n \in \mathbf{N}$ we have $|x_n - x| < \epsilon$.

In other words, a series (x_n) *verconges* to x if $|x_n - x|$ is bounded. This is a silly definition though since if $|x_n - x|$ is bounded, then $|x_n - x'|$ is bounded for all $x' \in \mathbf{R}$, meaning if a sequence is vercongent it verconges to every $x' \in \mathbf{R}$.

Put another way, a sequence is vercongent *if and only if* it is bounded.

Exercise 2.2.2
Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

(a) $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$.

(b) $\lim \frac{2n^2}{n^3+3} = 0$.

(c) $\lim \frac{\sin(n^2)}{\sqrt[3]{n}} = 0$.

Solution
(a) We have

$$\left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| = \left|\frac{5(2n+1) - 2(5n+4)}{5(5n+4)}\right| = \left|\frac{-3}{5(5n+4)}\right| = \frac{3}{5(5n+4)} < \epsilon$$

We now find n such that the distance is less then ϵ

$$\frac{3}{5(5n+4)} < \frac{1}{n} < \epsilon \implies n > \frac{1}{\epsilon}$$

You could also solve for the smallest n, which would give you

$$\frac{3}{5(5n+4)} < \epsilon \implies 5n+4 > \frac{3}{5\epsilon} \implies n > \frac{3}{25\epsilon} - \frac{4}{5}$$

I prefer the first approach, the second is better if you were doing numerical analysis and wanted a precise convergence rate.

(b) We have

$$\left|\frac{2n^2}{n^3+3} - 0\right| = \frac{2n^2}{n^3+3} < \frac{2n^2}{n^3} = \frac{2}{n} < \epsilon \implies n > \frac{2}{\epsilon}$$

(c) We have

$$\frac{\sin(n^2)}{n^{1/3}} \le \frac{1}{n^{1/3}} < \epsilon \implies n > \frac{1}{\epsilon^3}$$

Really slow convergence! if $\epsilon = 10^{-2}$ we would require $n > 10^6$

Exercise 2.2.3
Describe what we would have to demonstrate in order to disprove each of the following statements.

(a) At every college in the United States, there is a student who is at least seven feet tall.

(b) For all colleges in the United States, there exists a professor who gives every student a grade of either A or B.

(c) There exists a college in the United States where every student is at least six feet tall.

Solution
(a) Find a collage in the United States with no students over seven feet tall.

(b) Find a collage in the United States with a professor who has given a grade other then an A or B.

(c) Find a collage in the united States with at least one student under six feet tall.

Exercise 2.2.4
Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

(a) A sequence with an infinite number of ones that does not converge to one.

(b) A sequence with an infinite number of ones that converges to a limit not equal to one.

(c) A divergent sequence such that for every $n \in \mathbf{N}$ it is possible to find n consecutive ones somewhere in the sequence.

Solution

(a) $a_n = (-1)^n$

(b) Impossible, if $\lim a_n = a \neq 1$ then for any $n \geq N$ we can find a n with $a_n = 1$ meaning $\epsilon < |1 - a|$ is impossible.

(c) $a_n = (1, 2, 1, 1, 3, 1, 1, 1, \dots)$

Exercise 2.2.5

Let $[[x]]$ be the greatest integer less than or equal to x. For example, $[[\pi]] = 3$ and $[[3]] = 3$. For each sequence, find $\lim a_n$ and verify it with the definition of convergence.

(a) $a_n = [[5/n]]$,

(b) $a_n = [[(12 + 4n)/3n]]$.

Reflecting on these examples, comment on the statement following Definition 2.2.3 that "the smaller the ϵ-neighborhood, the larger N may have to be."

Solution

(a) For all $n > 5$ we have $[[5/n]] = 0$ meaning $\lim a_n = 0$.

(b) The inside clearly converges to $4/3$ from above, so $\lim a_n = 1$.

Some sequences eventually reach their limit, meaning N no longer has to increase.

Exercise 2.2.6

Theorem 2.2.7 (Uniqueness of Limits). *The limit of a sequence, when it exists, must be unique.*

Prove Theorem 2.2.7. To get started, assume $(a_n) \to a$ and also that $(a_n) \to b$. Now argue $a = b$

Solution

If $a \neq b$ then we can set ϵ small enough that having both $|a_n - a| < \epsilon$ and $|a_n - b| < \epsilon$ is impossible. Therefor $a = b$.

(Making this rigorous is trivial and left as an exercise to the reader)

Exercise 2.2.7

Here are two useful definitions:

(i) A sequence (a_n) is *eventually* in a set $A \subseteq \mathbf{R}$ if there exists an $N \in \mathbf{N}$ such that $a_n \in A$ for all $n \geq N$.

(ii) A sequence (a_n) is *frequently* in a set $A \subseteq \mathbf{R}$ if, for every $N \in \mathbf{N}$, there exists an $n \geq N$ such that $a_n \in A$.

  (a) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?

  (b) Which definition is stronger? Does frequently imply eventually or does eventually imply frequently?

  (c) Give an alternate rephrasing of Definition 2.2.3B using either frequently or eventually. Which is the term we want?

(d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2 . Is (x_n) necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?

Solution

(a) Frequently, but not eventually.

(b) Eventually is stronger, it implies frequently.

(c) $(x_n) \to x$ *if and only if* x_n is eventually in any ϵ-neighborhood around x.

(d) (x_n) is frequently in $(1.9, 2.1)$ but not necessarily eventually (consider $x_n = 2(-1)^n$).

Exercise 2.2.8

For some additional practice with nested quantifiers, consider the following invented definition:

Let's call a sequence (x_n) zero-heavy if there exists $M \in \mathbf{N}$ such that for all $N \in \mathbf{N}$ there exists n satisfying $N \leq n \leq N + M$ where $x_n = 0$

(a) Is the sequence $(0, 1, 0, 1, 0, 1, \ldots)$ zero heavy?

(b) If a sequence is zero-heavy does it necessarily contain an infinite number of zeros? If not, provide a counterexample.

(c) If a sequence contains an infinite number of zeros, is it necessarily zeroheavy? If not, provide a counterexample.

(d) Form the logical negation of the above definition. That is, complete the sentence: A sequence is not zero-heavy if ....

Solution

(a) No.

(b) Yes. as any finite number of zeros K would lead to a contradiction when $M > K$.

(c) No, consider $(0, 1, 0, \ldots)$ from (a).

(d) A sequence is not zero-heavy if there exists an $M \in \mathbf{N}$ such that for all $N \in \mathbf{N}$ there exists an $n \in \mathbf{N}$ such that $N \leq n \leq N + M$ but $x_n \neq 0$.

2.3 The Algebraic and Order Limit Theorems

Exercise 2.3.1
Let $x_n \geq 0$ for all $n \in \mathbf{N}$.

(a) If $(x_n) \to 0$, show that $\left(\sqrt{x_n}\right) \to 0$.

(b) If $(x_n) \to x$, show that $\left(\sqrt{x_n}\right) \to \sqrt{x}$.

Solution

(a) Setting $x_n < \epsilon^2$ implies $\sqrt{x_n} < \epsilon$ (for all $n \geq N$ of course)

(b) We want $|\sqrt{x_n} - \sqrt{x}| < \epsilon$ multiplying by $(\sqrt{x_n} + \sqrt{x})$ gives $|x_n - x| < (\sqrt{x_n} + \sqrt{x})\epsilon$ since x_n is convergent, it is bounded $|x_n| \leq M$ implying $\sqrt{|x_n|} \leq \sqrt{M}$, multiplying gives

$$|x_n - x| < \left(\sqrt{x_n} + \sqrt{x}\right)\epsilon \leq \left(\sqrt{M} + \sqrt{x}\right)\epsilon$$

Since $|x_n - x|$ can be made arbitrarily small we can make this true for some $n \geq N$. Now dividing by $\sqrt{M} + \sqrt{x}$ gives us

$$|\sqrt{x_n} - \sqrt{x}| \leq \frac{|x_n - x|}{\sqrt{M} + \sqrt{x}} < \epsilon$$

Therefor $|\sqrt{x_n} - \sqrt{x}| < \epsilon$ completing the proof.

Exercise 2.3.2
Using only Definition 2.2.3, prove that if $(x_n) \to 2$, then

(a) $\left(\frac{2x_n - 1}{3}\right) \to 1$;

(b) $(1/x_n) \to 1/2$.

(For this exercise the Algebraic Limit Theorem is off-limits, so to speak.)

Solution

(a) We have $\left|\frac{2}{3}x_n - \frac{4}{3}\right| = \frac{2}{3}|x_n - 2| < \epsilon$ which can always be done since $|x_n - 2|$ can be made arbitrarily small.

(b) Since x_n is bounded we have $|x_n| \leq M$

$$|1/x_n - 1/2| = \frac{|2 - x_n|}{|2x_n|} \leq \frac{|2 - x_n|}{|2M|} < \epsilon$$

Letting $|2 - x_n| < \epsilon/|2M|$ gives $|1/x_n - 1/2| < \epsilon$.

Exercise 2.3.3 (Squeeze Theorem)
Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbf{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

Solution
Let $y = \lim y_n$. By the order limit theorem we have $l \leq y \leq l$ implying $y = l$.

Exercise 2.3.4
Let $(a_n) \to 0$, and use the Algebraic Limit Theorem to compute each of the following limits (assuming the fractions are always defined):

(a) $\lim \left(\frac{1+2a_n}{1+3a_n-4a_n^2} \right)$

(b) $\lim \left(\frac{(a_n+2)^2-4}{a_n} \right)$

(c) $\lim \left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5} \right)$.

Solution
(a) Divide by a_n^2, then apply the ALT

$$\lim \left(\frac{1+2a_n}{1+3a_n-4a_n^2} \right) = \frac{\lim \left(\frac{1}{a_n^2} + \frac{2}{a_n} \right)}{\lim \left(\frac{1}{a_n^2} + \frac{3}{a_n} - 4 \right)} = \frac{0}{-4} = 0$$

(b)

$$\lim \left(\frac{(a_n+2)^2-4}{a_n} \right) = \lim \left(\frac{a_n^2+2a_n}{a_n} \right) = \lim (a_n+2) = \infty$$

(c) This one is a straightforward application of the algebraic limit theorem

$$\lim \left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5} \right) = 3/5$$

Exercise 2.3.5
Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, x_3, y_3, \ldots, x_n, y_n, \ldots)$. Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

Solution
Obviously if $\lim x_n = \lim y_n = l$ then $z_n \to l$. To show the other way suppose $(z_n) \to l$, then $|z_n - l| < \epsilon$ for all $n \geq N$ meaning $|y_n - l| < \epsilon$ and $|x_n - l| < \epsilon$ for $n \geq N$ aswell. Thus $\lim x_n = \lim y_n = l$.

Exercise 2.3.6
Consider the sequence given by $b_n = n - \sqrt{n^2+2n}$. Taking $(1/n) \to 0$ as given, and using both the Algebraic Limit Theorem and the result in Exercise 2.3.1, show $\lim b_n$ exists and find the value of the limit.

Solution
I'm going to find the value of the limit before proving it. We have

$$n - \sqrt{n^2+2n} = n - \sqrt{(n+1)^2 - 1}$$

For large n, $\sqrt{(n+1)^2-1} \approx n+1$ so $\lim b_n = -1$.

Factoring out n we get $n\left(1-\sqrt{1+2/n}\right)$. Tempting as it is to apply the ALT here to say $(b_n) \to 0$ it doesn't work since n diverges.

How about if I get rid of the radical, then use the ALT to go back to what we had before?

$$(n-\sqrt{n^2+2n})(n+\sqrt{n^2+2n}) = n^2-(n^2+2n) = -2n$$

Then we have

$$b_n = n-\sqrt{n^2+2n} = \frac{-2n}{n+\sqrt{n^2+2n}} = \frac{-2}{1+\sqrt{1+2/n}}$$

Now we can finally use the algebraic limit theorem!

$$\lim\left(\frac{-2}{1+\sqrt{1+2/n}}\right) = \frac{-2}{1+\sqrt{1+\lim(2/n)}} = \frac{-2}{1+\sqrt{1+0}} = -1$$

Stepping back the key to this technique is removing the radicals via a difference of squares, then dividing both sides by the growthrate n and applying the ALT.

Exercise 2.3.7

Give an example of each of the following, or state that such a request is impossible by referencing the proper theorem(s):

(a) sequences (x_n) and (y_n), which both diverge, but whose sum (x_n+y_n) converges;

(b) sequences (x_n) and (y_n), where (x_n) converges, (y_n) diverges, and (x_n+y_n) converges;

(c) a convergent sequence (b_n) with $b_n \neq 0$ for all n such that $(1/b_n)$ diverges;

(d) an unbounded sequence (a_n) and a convergent sequence (b_n) with (a_n-b_n) bounded;

(e) two sequences (a_n) and (b_n), where (a_nb_n) and (a_n) converge but (b_n) does not.

Solution

(a) $(x_n)=n$ and $(y_n)=-n$ diverge but $x_n+y_n=0$ converges

(b) Impossible, the algebraic limit theorem implies $\lim(x_n+y_n)-\lim(x_n) = \lim y_n$ therefor (y_n) must converge if (x_n) and (x_n+y_n) converge.

(c) Impossible, the algebraic limit theorem implies that if $(b_n) \to b$ then $(1/b_n) \to 1/b$.

(d) Impossible, letting $|b_n| \leq M$ we have $|a_n-b_n| \leq |a_n-M|$ being bounded, which is impossible since a_n is unbounded and shifting by a constant M cannot not change that.

(e) $b_n=n$ and $a_n=0$ works. However if $(a_n) \to a$, $a \neq 0$ and $(a_nb_n) \to p$ then the ALT would imply $(b_n) \to p/a$.

Exercise 2.3.8

Let $(x_n) \to x$ and let $p(x)$ be a polynomial.

(a) Show $p\left(x_n\right) \rightarrow p(x)$.

(b) Find an example of a function $f(x)$ and a convergent sequence $\left(x_n\right) \rightarrow x$ where the sequence $f\left(x_n\right)$ converges, but not to $f(x)$.

Solution

(a) Applying the algebraic limit theorem multiple times gives $\left(x_n^d\right) \rightarrow x^d$ meaning

$$\lim p(x_n) = \lim \left(a_d x_n^d + a_{d-1} x_n^{d-1} + \cdots + a_0\right) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_0 = p(x).$$

As a cute corollary, any continuous function f has $\lim f(x_n) = f(x)$ since polynomials can approximate continuous functions arbitrarily well by the Weierstrass approximation theorem.

(b) Let $(x_n) = 1/n$ and define f as

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{otherwise} \end{cases}$$

We have $f(1/n) = 1$ for all n, meaning $\lim f(1/n) = 1$ but $f(0) = 0$.

Exercise 2.3.9

(a) Let $\left(a_n\right)$ be a bounded (not necessarily convergent) sequence, and assume $\lim b_n = 0$. Show that $\lim \left(a_n b_n\right) = 0$. Why are we not allowed to use the Algebraic Limit Theorem to prove this?

(b) Can we conclude anything about the convergence of $\left(a_n b_n\right)$ if we assume that $\left(b_n\right)$ converges to some nonzero limit b ?

(c) Use (a) to prove Theorem 2.3.3, part (iii), for the case when $a = 0$.

Solution

(a) We can't use the ALT since a_n is not necessarily convergent. a_n being bounded gives $|a_n| \leq M$ for some M giving

$$|a_n b_n| \leq M|b_n| < \epsilon$$

Which can be accomplished by letting $|b_n| < \epsilon/M$ since $(b_n) \rightarrow 0$.

(b) No

(c) In (a) we showed $\lim(a_n b_n) = 0 = ab$ for $b = 0$ which proves part (iii) of the ALT.

Exercise 2.3.10

Consider the following list of conjectures. Provide a short proof for those that are true and a counterexample for any that are false.

(a) If $\lim \left(a_n - b_n\right) = 0$, then $\lim a_n = \lim b_n$.

(b) If $\left(b_n\right) \rightarrow b$, then $\left|b_n\right| \rightarrow |b|$.

(c) If $(a_n) \to a$ and $(b_n - a_n) \to 0$, then $(b_n) \to a$.

(d) If $(a_n) \to 0$ and $|b_n - b| \leq a_n$ for all $n \in \mathbf{N}$, then $(b_n) \to b$.

Solution

(a) False, consider $a_n = n$ and $b_n = -n$.

(b) True since if $|b_n - b| < \epsilon$ then $||b_n| - |b|| \leq |b_n - b| < \epsilon$ by Exercise 1.2.6 (d).

(c) True by ALT since $\lim(b_n - a_n) + \lim a_n = \lim b_n = a$.

(d) True, since $0 \leq |b_n - b| \leq a_n$ we have $a_n \geq 0$. Let $\epsilon > 0$ and pick N such that $a_n < \epsilon$ for all $n \geq N$. Therefor

$$|b_n - b| \leq a_n < \epsilon$$

Proving $(b_n) \to b$.

Exercise 2.3.11 (Cesaro Means)

(a) Show that if (x_n) is a convergent sequence, then the sequence given by the averages

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

also converges to the same limit.

(b) Give an example to show that it is possible for the sequence (y_n) of averages to converge even if (x_n) does not.

Solution

(a) Let $D = \sup\{|x_n - x| : n \in \mathbf{N}\}$ and let $0 < \epsilon < D$, we have

$$|y_n - x| = \left|\frac{x_1 + \cdots + x_n}{n} - x\right| \leq \left|\frac{|x_1 - x| + \cdots + |x_n - x|}{n}\right| \leq D$$

Let $|x_n - x| < \epsilon/2$ for $n > N_1$ giving

$$|y_n - x| \leq \left|\frac{|x_1 - x| + \cdots + |x_{N_1} - x| + \cdots + |x_n - x|}{n}\right| \leq \left|\frac{N_1 D + (n - N_1)\epsilon/2}{n}\right|$$

Let N_2 be large enough that for all $n > N_2$ (remember $0 < \epsilon < D$ so $(D - \epsilon/2) > 0$.)

$$0 < \frac{N_1(D - \epsilon/2)}{n} < \epsilon/2$$

Therefor

$$|y_n - x| \leq \left|\frac{N_1(D - \epsilon/2)}{n} + \epsilon/2\right| < \epsilon$$

Letting $N = \max\{N_1, N_2\}$ completes the proof as $|y_n - x| < \epsilon$ for all $n > N$.

(Note: I could have used any $\epsilon' < \epsilon$ instead of $\epsilon/2$, I just needed some room.)

(b) $x_n = (-1)^n$ diverges but $(y_n) \to 0$.

Exercise 2.3.12
A typical task in analysis is to decipher whether a property possessed by every term in a convergent sequence is necessarily inherited by the limit. Assume $(a_n) \to a$, and determine the validity of each claim. Try to produce a counterexample for any that are false.

(a) If every a_n is an upper bound for a set B, then a is also an upper bound for B.

(b) If every a_n is in the complement of the interval $(0, 1)$, then a is also in the complement of $(0, 1)$.

(c) If every a_n is rational, then a is rational.

Solution
(a) True, let $s = \sup B$, we know $s \leq a_n$ so by the order limit theorem $s \leq a$ meaning a is also an upper bound on B.

(b) True, since if $a \in (0, 1)$ then there would exist an ϵ-neighborhood inside $(0, 1)$ that a_n would have to fall in, contradicting the fact that $a_n \notin (0, 1)$.

(c) False, consider the sequence of rational approximations to $\sqrt{2}$

Exercise 2.3.13 (Iterated Limits)
Given a doubly indexed array a_{mn} where $m, n \in \mathbf{N}$, what should $\lim_{m,n\to\infty} a_{mn}$ represent?

(a) Let $a_{mn} = m/(m + n)$ and compute the iterated limits

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) \quad \text{and} \quad \lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right)$$

Define $\lim_{m,n\to\infty} a_{mn} = a$ to mean that for all $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that if both $m, n \geq N$, then $|a_{mn} - a| < \epsilon$

(b) Let $a_{mn} = 1/(m + n)$. Does $\lim_{m,n\to\infty} a_{mn}$ exist in this case? Do the two iterated limits exist? How do these three values compare? Answer these same questions for $a_{mn} = mn/\left(m^2 + n^2\right)$

(c) Produce an example where $\lim_{m,n\to\infty} a_{mn}$ exists but where neither iterated limit can be computed.

(d) Assume $\lim_{m,n\to\infty} a_{mn} = a$, and assume that for each fixed $m \in \mathbf{N}$, $\lim_{n\to\infty}(a_{mn}) \to b_m$. Show $\lim_{m\to\infty} b_m = a$

(e) Prove that if $\lim_{m,n\to\infty} a_{mn}$ exists and the iterated limits both exist, then all three limits must be equal.

Solution
(a)

$$\lim_{n\to\infty}\left(\lim_{m\to\infty}\frac{m}{m+n}\right) = 1, \text{ and } \lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{m}{m+n}\right) = 0$$

(b) For $a_{mn} = 1/(m+n)$ all three limits are zero. For $a_{mn} = mn/(m^2+n^2)$ iterated limits are zero, and $\lim_{m,n\to\infty} a_{mn}$ does not exist since for $m, n \geq N$ setting $m = n$ gives

$$\frac{n^2}{n^2+n^2} = \frac{1}{2}$$

Which cannot be made smaller then $\epsilon = 1/2$.

The reason you would think to set $m = n$ is in trying to maximize $mn/(m^2+n^2)$ notice if $m > n$ then $mn > n^2$ so we are adding more to the numerator then the denominator, hence the ratio is increasing. And if $m < n$ then the ratio is decreasing. Therefor the maximum point is at $m = n$.

(c) **TODO**

(d) We are given $|b_m - a_{mn}| < \epsilon$ for all $n > N$, By the triangle inequality

$$|b_m - a| \leq |b_m - a_{mn}| + |a_{mn} - a| < \epsilon/2 + \epsilon/2 = \epsilon$$

(e) Let $b_m = \lim_{n\to\infty}(a_{mn})$ and $a = \lim_{m,n\to\infty}(a_{mn})$. In (d) we showed $(b_m) \to a$, A similar argument shows $(c_n) \to a$. Thus all three limits are equal to a.

2.4 The Monotone Convergence Theorem and a First Look at Infinite Series

Exercise 2.4.1

(a) Prove that the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

converges.

(b) Now that we know $\lim x_n$ exists, explain why $\lim x_{n+1}$ must also exist and equal the same value.

(c) Take the limit of each side of the recursive equation in part (a) to explicitly compute $\lim x_n$.

Solution

(a) $x_2 = 1$ makes me conjecture x_n is monotonic. For induction suppose $x_n > x_{n+1}$ then we have

$$4 - x_n < 4 - x_{n+1} \implies \frac{1}{4 - x_n} > \frac{1}{4 - x_{n+1}} \implies x_{n+1} > x_{n+2}$$

Thus x_n is decreasing, to show x_n is bounded notice x_n cannot be negative since $x_n < 3$ means $x_{n+1} = 1/(4 - x_n) > 0$. Therefor by the monotone convergence theorem (x_n) converges.

(b) Clearly skipping a single term does not change what the series converges to.

(c) Since $x = \lim(x_n) = \lim(x_{n+1})$ we must have

$$x = \frac{1}{4 - x} \iff x^2 - 4x + 1 = 0 \iff (x - 2)^2 = 3 \iff x = 2 \pm \sqrt{3}$$

$2 + \sqrt{3} > 3$ is impossible since $x_n < 3$ thus $x = 2 - \sqrt{3}$.

Exercise 2.4.2

(a) Consider the recursively defined sequence $y_1 = 1$

$$y_{n+1} = 3 - y_n$$

and set $y = \lim y_n$. Because (y_n) and (y_{n+1}) have the same limit, taking the limit across the recursive equation gives $y = 3 - y$. Solving for y, we conclude $\lim y_n = 3/2$ What is wrong with this argument?

(b) This time set $y_1 = 1$ and $y_{n+1} = 3 - \frac{1}{y_n}$. Can the strategy in (a) be applied to compute the limit of this sequence?

Solution

(a) The series $y_n = (1, 2, 1, 2, \dots)$ does not converge.

(b) Yes, y_n converges by the monotone convergence theorem since $0 < y_n < 3$ and y_n is increasing.

Exercise 2.4.3

(a) Show that

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$$

converges and find the limit.

(b) Does the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$$

converge? If so, find the limit.

Solution

(a) Let $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2+x_n}$ clearly $x_2 > x_1$. assuming $x_{n+1} > x_n$ gives

$$2 + x_{n+1} > 2 + x_n \iff \sqrt{2+x_{n+1}} > \sqrt{2+x_n} \iff x_{n+2} > x_{n+1}$$

Since x_n is monotonically increasing and bounded the monotone convergence theorem tells us $(x_n) \to x$. Equating both sides like in 2.4.1 gives

$$x = \sqrt{2+x} \iff x^2 - x - 2 = 0 \iff x = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$$

Since $x > 0$ we must have $x = (1+\sqrt{5})/2$.

(b) We have $x_1 = 2^{1/2}$ and $x_{n+1} = (2x_n)^{1/2}$. We have

$$x_{n+1} = (2x_n)^{1/2} \geq x_n \iff 2x_n \geq x_n^2 \iff 2 \geq x_n$$

Since $x_1 = 2^{1/2} \leq 2$ induction implies x_n is increasing. Now to show x_n is bounded notice that $x_1 \leq 2$ and if $x_n \leq 2$ then

$$2x_n \leq 4 \implies (2x_n)^{1/2} \leq 2$$

Now the monotone convergence theorem tells us (x_n) converges. To find the limit use $\lim x_n = \lim x_{n+1} = x$ to get

$$x = (2x)^{1/2} \implies x^2 = 2x \implies x = \pm 2$$

Since $x_n \geq 0$ we have $x = 2$.

Exercise 2.4.4

(a) In Section 1.4 we used the Axiom of Completeness (AoC) to prove the Archimedean Property of $\mathbf{R}$ (Theorem 1.4.2). Show that the Monotone Convergence Theorem can also be used to prove the Archimedean Property without making any use of AoC.

(b) Use the Monotone Convergence Theorem to supply a proof for the Nested Interval Property (Theorem 1.4.1) that doesn't make use of AoC.

These two results suggest that we could have used the Monotone Convergence Theorem in place of AoC as our starting axiom for building a proper theory of the real numbers.

Solution

(a) MCT tells us $(1/n)$ converges, obviously it must converge to zero therefor we have $|1/n - 0| = 1/n < \epsilon$ for any ϵ, which is the Archimedean Property.

(b) We have $I_n = [a_n, b_n]$ with $a_n \leq b_n$ since $I_n \neq \emptyset$. Since $I_{n+1} \subseteq I_n$ we must have $b_{n+1} \leq b_n$ and $a_{n+1} \geq a_n$ the MCT tells us that $(a_n) \to a$ and $(b_n) \to b$. by the Order Limit Theorem we have $a \leq b$ since $a_n \leq b_n$, therefor $a \in I_n$ for all n meaning $a \in \bigcap_{n=1}^{\infty} I_n$ and thus $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Exercise 2.4.5 (Calculating Square Roots)
Let $x_1 = 2$, and define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

(a) Show that x_n^2 is always greater than or equal to 2 , and then use this to prove that $x_n - x_{n+1} \geq 0$. Conclude that $\lim x_n = \sqrt{2}$.

(b) Modify the sequence (x_n) so that it converges to $\sqrt{c}$.

Solution

(a) Clearly $x_1^2 \geq 2$, now procede by induction. if $x_n^2 \geq 2$ then we have

$$x_{n+1}^2 = \frac{1}{4}\left(\frac{x_n^2+2}{x_n}\right)^2 = \frac{1}{4}\left(\frac{(x_n^2+2)^2}{x_n^2}\right) \geq \frac{1}{4}\left(\frac{(x_n^2+2)^2}{2}\right)$$

Now since $x_n^2 \geq 2$ we have $(x_n^2+2)^2 \geq 16$ meaning

$$x_{n+1}^2 = \frac{1}{4}\left(\frac{(x_n^2+2)^2}{2}\right) \geq 2.$$

To show $x_n - x_{n+1} \geq 0$ we examine $(x_n - x_{n+1})^2 \geq 0$ with the hope of using $x_n^2 \geq 2$.

$$\begin{aligned}(x_n - x_{n+1})^2 &= x_n^2 - 2x_n x_{n+1} + x_{n+1}^2 \\ &= x_n^2 - 2x_n\frac{1}{2}\left(x_n + \frac{1}{x_n}\right) + x_{n+1}^2 \\ &= x_n^2 - (x_n^2 + 2) + x_{n+1}^2 \\ &= x_{n+1}^2 - 2 \geq 0.\end{aligned}$$

Now we know $(x_n) \to x$ converges, to show $x^2 = 2$ observe that $\lim x_n = \lim x_{n+1}$ so

$$x = \frac{1}{2}\left(x + \frac{2}{x}\right) \iff x^2 = \frac{1}{2}x^2 + 1 \iff x^2 = 2$$

Therefor $x = \pm\sqrt{2}$, since every x_n is positive $x = \sqrt{2}$.

(b) Let

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right)$$

I won't go through the convergence analysis again, but the only fixed point is

$$x = \frac{1}{2}\left(x + \frac{c}{x}\right) \implies \frac{1}{2}x^2 = \frac{1}{2}c \implies x^2 = c$$

So if x_n converges, it must converge to $x^2 = c$.

Exercise 2.4.6 (Arithmetic-Geometric Mean)

(a) Explain why $\sqrt{xy} \leq (x+y)/2$ for any two positive real numbers x and y. (The geometric mean is always less than the arithmetic mean.)

(b) Now let $0 \leq x_1 \leq y_1$ and define

$$x_{n+1} = \sqrt{x_n y_n} \quad \text{and} \quad y_{n+1} = \frac{x_n + y_n}{2}$$

Show $\lim x_n$ and $\lim y_n$ both exist and are equal.

Solution

(a) We have

$$\sqrt{xy} \leq (x+y)/2 \iff 4xy \leq x^2 + 2xy + y^2 \iff 0 \leq (x-y)^2$$

(b) The only fixed point is $x_n = y_n$ so we only need to show both sequences converge.

The inequality $x_1 \leq y_1$ is always true since

$$\sqrt{x_n y_n} \leq \frac{x_n + y_n}{2} \implies x_{n+1} \leq y_{n+1}$$

Also $x_n \leq y_n$ implies $(x_n + y_n)/2 = y_{n+1} \leq y_n$, similarly $\sqrt{x_n y_n} = x_{n+1} \geq x_n$ meaning both sequences converge by the monotone convergence theorem.

Exercise 2.4.7 (Limit Superior)
Let (a_n) be a bounded sequence.

(a) Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges.

(b) The limit superior of (a_n), or $\limsup a_n$, is defined by

$$\limsup a_n = \lim y_n$$

where y_n is the sequence from part (a) of this exercise. Provide a reasonable definition for $\liminf a_n$ and briefly explain why it always exists for any bounded sequence.

(c) Prove that $\liminf a_n \leq \limsup a_n$ for every bounded sequence, and give an example of a sequence for which the inequality is strict.

(d) Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

Solution

(a) (y_n) is decreasing and converges by the monotone convergence theorem.

(b) Define $\liminf a_n = \lim z_n$ for $z_n = \inf\{a_n : k \geq n\}$. z_n converges since it is increasing and bounded.

(c) Obviously $\inf\{a_k : k \geq n\} \leq \sup\{a_n : k \geq n\}$ so by the Order Limit Theorem $\liminf a_n \leq \limsup a_n$.

(d) If $\liminf a_n = \limsup a_n$ then the squeeze theorem (Exercise 2.3.3) implies a_n converges to the same value, since $\inf\{a_{k\geq n}\} \leq a_n \leq \sup\{a_{k\geq n}\}$.

Exercise 2.4.8

For each series, find an explicit formula for the sequence of partial sums and determine if the series converges.

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

(c) $\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$

(In (c), $\log(x)$ refers to the natural logarithm function from calculus.)

Solution

(a) This is a geometric series, we can use the usual trick to derive s_n. Let $r = 1/2$ for convenience

$$\begin{aligned} s_n &= 1 + r + r^2 + \cdots + r^n \\ rs_n &= r + r^2 + \cdots + r^{n+1} \\ rs_n - s_n &= r^{n+1} - 1 \implies s_n = \frac{r^{n+1}-1}{r-1} \end{aligned}$$

So we have

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{n\to\infty} \frac{(1/2)^{n+1}-1}{1/2-1} = \frac{-1}{-1/2} = 2$$

(b) We can use partial fractions to get

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

Which gives us a telescoping series, most of the terms cancel and we get

$$s_n = 1 - \frac{1}{n+1}$$

Therefor

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$$

(c) Another telescoping series, since

$$\log\left(\frac{n+1}{n}\right) = \log(n+1) - \log(n)$$

Therefor most of the terms cancel and we get

$$s_n = \log(n+1)$$

Which doesn't converge.

Exercise 2.4.9
Complete the proof of Theorem 2.4.6 by showing that if the series $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges, then so does $\sum_{n=1}^{\infty} b_n$. Example 2.4.5 may be a useful reference.

Solution
Let $s_n = b_1 + b_2 + \cdots + b_n$ and $t_k = b_1 + 2b_2 + \cdots + 2^k b_{2^k}$.

We want to show s_n is unbounded, first we find a series similar to t_k that is less then s_n, then rewrite it in terms of t_k.

Let $n = 2^k$ so things match up nicely. We get

$$\begin{aligned} s_n &= b_1 + b_2 + (b_3 + b_4) + \cdots + (b_{2^{k-1}} + \cdots + b_{2^k}) \\ &\leq b_1 + b_2 + (b_4 + b_4) + \cdots + 2^{k-1} b_{2^k} \end{aligned}$$

(Notice there are $2^k - 2^{k-1} = 2^{k-1}$ terms in the last term)

Now define t_k' to be our new series $b_1 + b_2 + 2b_4 + 4b_8 + \cdots + 2^{k-1} b_{2^k}$. This looks a lot like t_k, and in fact some algebra gives

$$t_k' = \frac{1}{2}\left(b_1 + 2b_2 + 4b_4 + \cdots + 2^k b_k\right) + \frac{1}{2} b_1 = \frac{1}{2} t_k + \frac{1}{2} b_1$$

Therefor we are justified in writing

$$s_n \geq t_k' \geq \frac{1}{2} t_k$$

And since $t_k/2$ diverges and s_n is bigger, s_n must also diverge.

Summary: s_n converges iff t_k conv since $t_k \geq s_n \geq t_k/2$ for $n = 2^k$.

Exercise 2.4.10 (Infinite Products)
A close relative of infinite series is the infinite product

$$\prod_{n=1}^{\infty} b_n = b_1 b_2 b_3 \cdots$$

which is understood in terms of its sequence of partial products

$$p_m = \prod_{n=1}^{m} b_n = b_1 b_2 b_3 \cdots b_m$$

Consider the special class of infinite products of the form

$$\prod_{n=1}^{\infty} (1 + a_n) = (1 + a_1)(1 + a_2)(1 + a_3) \cdots, \qquad \text{where } a_n \geq 0$$

(a) Find an explicit formula for the sequence of partial products in the case where $a_n = 1/n$ and decide whether the sequence converges. Write out the first few terms in the sequence of partial products in the case where $a_n = 1/n^2$ and make a conjecture about the convergence of this sequence.

(b) Show, in general, that the sequence of partial products converges if and only if $\sum_{n=1}^{\infty} a_n$ converges. (The inequality $1 + x \leq 3^x$ for positive x will be useful in one direction.)

Solution

(a) This is a telescoping product, most of the terms cancel

$$p_m = \prod_{n=1}^{m}(1 + 1/n) = \prod_{n=1}^{m} \frac{n+1}{n} = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{2} \cdots \frac{m+1}{m} = m + 1$$

Therefor (p_m) diverges.

In the cast $a_n = 1/n^2$ we get

$$\prod_{n=1}^{\infty}(1 + 1/n^2) = \prod_{n=1}^{\infty} \frac{1+n^2}{n^2} = \frac{2}{1} \cdot \frac{5}{4} \cdot \frac{10}{9} \cdots$$

The growth seems slower, I conjecture it converges now.

(b) Using the inequality suggested we have $1 + a_n \leq 3^{a_n}$ letting $s_m = a_1 + \cdots + a_m$ we get

$$p_m = (1 + a_1) \cdots (1 + a_m) \leq 3^{a_1} 3^{a_2} \cdots 3^{a_m} = 3^{s_m}$$

Now if s_m converges it is bounded by some M meaning p_m is bounded by 3^M. and because $a_n \geq 0$ the partial products p_m are increasing, so they converge by the MCT. This shows s_m converging implies p_m converges.

For the other direction suppose $p_m \to p$. Distributing inside the products gives $p_2 = a_1 + a_2 + 1 + a_1 a_2 > s_2$ and in general $p_m > s_m$ implying that if p_m is bounded then s_n is bounded aswell. This completes the proof.

Summary: Convergence is if and only if because $s_m \leq p_m \leq 3^{s_m}$.

(By the way the inequality $1 + x \leq 3^x$ can be derived from $\log(1 + x) \leq x$ implying $1 + x \leq e^x$, I assume abbott rounded up to 3.)

2.5 Subsequences and the Bolzano–Weierstrass Theorem

Exercise 2.5.1
Give an example of each of the following, or argue that such a request is impossible.

(a) A sequence that has a subsequence that is bounded but contains no subsequence that converges.

(b) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.

(c) A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$.

(d) A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$, and no subsequences converging to points outside of this set.

Solution

(a) Impossible, the Bolzano–Weierstrass theorem tells us a convergent subsequence of that subsequence exists, and that sub-sub sequence is also a subsequence of the original sequence.

(b) $(1 + 1/n) \to 1$ and $(1/n) \to 0$ so $(1/2, 1 + 1/2, 1/3, 1 + 1/3, \ldots)$ has subsequences converging to 0 and 1.

(c) Copy the finitely many previous terms before proceeding to a new term

$$(1, 1/2, 1, 1/3, 1, 1/2, 1/4, 1, 1/2, 1/3, \ldots)$$

The sequence contains infinitely many terms in $\{1, 1/2, 1/3, \ldots\}$ hence subsequences exist converging to each of these values.

(d) Impossible, the sequence must converge to zero which is not in the set.

Proof: Let $\epsilon > 0$ be arbitrary, pick N large enough that $1/n < \epsilon/2$ for $n > N$. We can find a subsequence $(b_m) \to 1/n$ meaning $|b_m - 1/n| < \epsilon/2$ for some m. using the triangle inequality we get

$$|b_m - 0| \leq |b_n - 1/n| + |1/n - 0| < \epsilon/2 + \epsilon/2 = \epsilon$$

Therefor we have found a number b_m in the sequence a_m with $|b_m| < \epsilon$. This process can be repeated for any ϵ therefor a sequence which converges to zero can be constructed.

Exercise 2.5.2
Decide whether the following propositions are true or false, providing a short justification for each conclusion.

(a) If every proper subsequence of (x_n) converges, then (x_n) converges as well.

(b) If (x_n) contains a divergent subsequence, then (x_n) diverges.

(c) If (x_n) is bounded and diverges, then there exist two subsequences of (x_n) that converge to different limits.

(d) If (x_n) is monotone and contains a convergent subsequence, then (x_n) converges.

Solution

(a) True, removing the first term gives us the proper subsequence $(x_2, x_3, \dots)$ which converges, implying $(x_1, x_2, \dots)$ also converges.

(b) True, the divergent subsequence is unbounded, hence (x_n) is also unbounded and divergent.

(c) True, since x_n is bounded $\limsup x_n$ and $\liminf x_n$ both converge. And since x_n diverges Exercise 2.4.7 tells us $\limsup x_n \neq \liminf x_n$.

(d) True, The subsequence (x_{n_k}) converges meaning it is bounded $|x_{n_k}| \leq M$. Suppose (x_n) is increasing, then x_n is bounded since picking k so that $n_k > n$ we have $x_n \leq x_{n_k} \leq M$. A similar argument applies if x_n is decreasing, Therefor x_n is monotonic bounded and so must converge.

Exercise 2.5.3

(a) Prove that if an infinite series converges, then the associative property holds. Assume $a_1 + a_2 + a_3 + a_4 + a_5 + \cdots$ converges to a limit L (i.e., the sequence of partial sums $(s_n) \to L$). Show that any regrouping of the terms

$$(a_1 + a_2 + \cdots + a_{n_1}) + (a_{n_1+1} + \cdots + a_{n_2}) + (a_{n_2+1} + \cdots + a_{n_3}) + \cdots$$

leads to a series that also converges to L.

(b) Compare this result to the example discussed at the end of Section 2.1 where infinite addition was shown not to be associative. Why doesn't our proof in (a) apply to this example?

Solution

(a) Let s_n be the original partial sums, and let s'_m be the regrouping. Since s'_m is a subsequence of s_n, $(s_n) \to s$ implies $(s'_m) \to s$.

(b) The subsequence $s'_m = (1-1) + \cdots = 0$ converging does not imply the parent sequence s_n converges. In fact BW tells us any bounded sequence of partial sums will have a convergent subsequence (regrouping in this case).

Exercise 2.5.4

The Bolzano-Weierstrass Theorem is extremely important, and so is the strategy employed in the proof. To gain some more experience with this technique, assume the Nested Interval Property is true and use it to provide a proof of the Axiom of Completeness. To prevent the argument from being circular, assume also that $(1/2^n) \to 0$. (Why precisely is this last assumption needed to avoid circularity?)

Solution
Let A be a bounded set, we're basically going to binary search for $\sup A$ and then use NIP to prove the limit exists.

Let M be an upper bound on A, and pick any $L \in A$ as our starting lower bound for $\sup A$ and define $I_1 = [L, M]$. Doing binary search gives $I_{n+1} \subseteq I_n$ with length proportional to $(1/2)^n$. Applying the Nested Interval Property gives

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

As the length $(1/2)^n$ goes to zero, there is a single $s \in \bigcap_{n=1}^{\infty} I_n$ which must be the least upper bound since $I_n = [L_n, M_n]$ gives $L_n \leq x \leq M_n$ for all n meaning $s = \sup A$ since

(i) $s \geq L_n$ implies s is an upper bound

(ii) $s \leq M_n$ implies s is the least upper bound

TODO Draw a diagram with tikz, Make this rigorous

Exercise 2.5.5
Assume (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbf{R}$. Show that (a_n) must converge to a.

Solution
$(a_2, a_3, \dots)$ Is a convergent subsequence, so obviously if $(a_2, a_3, \dots) \to a$ then $(a_n) \to a$ also.

Exercise 2.5.6
Use a similar strategy to the one in Example 2.5.3 to show $\lim b^{1/n}$ exists for all $b \geq 0$ and find the value of the limit. (The results in Exercise 2.3.1 may be assumed.)

Solution
Intuitively $\lim b^{1/n} = 1$

Two facts I'll take as granted (you can prove them if you wish)

(i) If $b < 1$ then $b^{1/n}$ is increasing

(ii) If $b > 1$ then $b^{1/n}$ is decreasing

Thus $b^{1/n}$ is monotonic, and bounded since

(i) If $b > 1$ then $b^{1/n} > 1$ since $b > 1^n$

(ii) If $b < 1$ then $b^{1/n} < 1$ since $b < 1^n$

Therefor $b^{1/n}$ converges by the monotone convergence theorem. to find the limit equate terms

$$b^{1/n+1} = b^{1/n} \implies b^1 = b^{\frac{n+1}{n}} = b^2 \implies b = 1$$

TODO Shorten this

Exercise 2.5.7
Extend the result proved in Example 2.5.3 to the case $|b| < 1$; that is, show $\lim(b^n) = 0$ if and only if $-1 < b < 1$.

Solution
If $|b| \geq 1$ then $\lim(b^n) \neq 0$ (diverges for $b \neq 1$).

Now for the other direction, if $|b| < 1$ we immediately get $|b^n| < 1$ thus b^n is bounded. Since it is decreasing the monotone convergence theorem implies it converges. To find the limit equating terms $b^{n+1} = b^n$ gives $b = 0$ or $b = 1$, since b is *strictly* decreasing we have $b = 0$.

Exercise 2.5.8
Another way to prove the Bolzano-Weierstrass Theorem is to show that every sequence contains a monotone subsequence. A useful device in this endeavor is the notion of a peak term. Given a sequence (x_n), a particular term x_m is a peak term if no later term in the sequence exceeds it; i.e., if $x_m \geq x_n$ for all $n \geq m$.

(a) Find examples of sequences with zero, one, and two peak terms. Find an example of a sequence with infinitely many peak terms that is not monotone.

(b) Show that every sequence contains a monotone subsequence and explain how this furnishes a new proof of the Bolzano-Weierstrass Theorem.

Solution
(a) $(1, 2, \dots)$ has zero peak terms, $(1, 0, 1/2, 2/3, 3/4, \dots)$ has a single peak term, $(2, 1, 1/2, 2/3, \dots)$ has two peak terms (a similar argument works for k peak terms) and $(1, 1/2, 1/3, \dots)$ has infinitely many peak terms. The sequence $(1, -1/2, 1/3, -1/4, \dots)$ has infinitely many peak terms, but is not monotone.

(b) The sequence of peak terms is monotonic decreasing, thus if the parent sequence is bounded we have found a subsequence which converges, hence proving BW. (If there aren't infinitely many peak terms, then take the sequence of valley terms)

Exercise 2.5.9
Let (a_n) be a bounded sequence, and define the set

$$S = \{x \in \mathbf{R} : x < a_n \text{ for infinitely many terms } a_n\}$$

Show that there exists a subsequence (a_{n_k}) converging to $s = \sup S$. (This is a direct proof of the Bolzano-Weierstrass Theorem using the Axiom of Completeness.)

Solution
For every $\epsilon > 0$ there exists an $x \in S$ with $x > s - \epsilon$ implying $|s - x| < \epsilon$. Therefor we can get arbitrarily close to $s = \sup S$ so there is a subsequence converging to this value.

To make this more rigorous, pick $x_n \in S$ such that $|x_n - s| < 1/n$ then pick $N > 1/\epsilon$ to get $|x_n - s| < \epsilon$ for all $n > N$.

2.6 The Cauchy Criterion

Exercise 2.6.1
Prove every convergent sequence is a Cauchy sequence. (Theorem 2.6.2)

Solution
Suppose (x_n) is convergent, we must show that for $m, n > N$ we have $|x_n - x_m| < \epsilon$
Set $|x_n - x| < \epsilon/2$ for $n > N$.
We get $|x_n - x_m| \leq |x_n - x| + |x - x_m| \leq \epsilon/2 + \epsilon/2 = \epsilon$

Exercise 2.6.2
Give an example of each of the following, or argue that such a request is impossible.

(a) A Cauchy sequence that is not monotone.

(b) A Cauchy sequence with an unbounded subsequence.

(c) A divergent monotone sequence with a Cauchy subsequence.

(d) An unbounded sequence containing a subsequence that is Cauchy.

Solution
(a) $x_n = (-1)^n/n$ is cauchy by Theorem 2.6.2.

(b) Impossible since all cauchy sequences converge.

(c) Impossible, If a subsequence was cauchy it would converge, implying the subsequence would be bounded and therefor the parent sequence would be bounded (because it is monotone) and thus would converge.

(d) $(2, 1/2, 3, 1/3, \dots)$ has subsequence $(1/2, 1/3, \dots)$ which is cauchy.

Exercise 2.6.3
If (x_n) and (y_n) are Cauchy sequences, then one easy way to prove that $(x_n + y_n)$ is Cauchy is to use the Cauchy Criterion. By Theorem 2.6.4, (x_n) and (y_n) must be convergent, and the Algebraic Limit Theorem then implies $(x_n + y_n)$ is convergent and hence Cauchy.

(a) Give a direct argument that $(x_n + y_n)$ is a Cauchy sequence that does not use the Cauchy Criterion or the Algebraic Limit Theorem.

(b) Do the same for the product $(x_n y_n)$.

Solution
(a) We have $|(x_n + y_n) - (x_m + y_m)| \leq |x_n - x_m| + |y_n - y_m| < \epsilon/2 + \epsilon/2 = \epsilon$

(b) Bound $|x_n| \leq M_1$, and $|y_n| \leq M_2$ then

$$\begin{aligned}
|x_n y_n - x_m y_m| &= |(x_n y_n - x_n y_m) + (x_n y_m - x_m y_m)| \\
&\leq |x_n(y_n - y_m)| + |y_m(x_n - x_m)| \\
&\leq M_1|y_n - y_m| + M_2|x_n - x_m| \\
&< \epsilon/2 + \epsilon/2 = \epsilon
\end{aligned}$$

After setting $|y_n - y_m| < \epsilon/(2M_1)$ and $|x_n - x_m| < \epsilon/(2M_2)$.

Exercise 2.6.4
Let (a_n) and (b_n) be Cauchy sequences. Decide whether each of the following sequences is a Cauchy sequence, justifying each conclusion.

(a) $c_n = |a_n - b_n|$

(b) $c_n = (-1)^n a_n$

(c) $c_n = [[a_n]]$, where $[[x]]$ refers to the greatest integer less than or equal to x.

Solution
(a) Yes, since $|(a_n - b_n) - (a_m - b_m)| \leq |a_n - a_m| + |b_m - b_n| < \epsilon/2 + \epsilon/2 = \epsilon$

(b) No, if $a_n = 1$ then $(-1)^n a_n$ diverges, and thus is not cauchy.

(c) No, if $a_n = 1-(-1)^n/n$ then $[[a_n]]$ fluctuates between 0 and 1 and so cannot be cauchy.

Exercise 2.6.5
Consider the following (invented) definition: A sequence (s_n) is pseudo-Cauchy if, for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then $|s_{n+1} - s_n| < \epsilon$

Decide which one of the following two propositions is actually true. Supply a proof for the valid statement and a counterexample for the other.

(i) Pseudo-Cauchy sequences are bounded.

(ii) If (x_n) and (y_n) are pseudo-Cauchy, then $(x_n + y_n)$ is pseudo-Cauchy as well.

Solution
(i) False, consider $s_n = \log n$. clearly $|s_{n+1} - s_n|$ can be made arbitrarily small but s_n is unbounded.

(ii) True, as $|(x_{n+1} + y_{n+1}) - (x_n + y_n)| \leq |x_{n+1} - x_n| + |y_{n+1} - y_n| < \epsilon/2 + \epsilon/2 = \epsilon$.

Exercise 2.6.6
Let's call a sequence (a_n) quasi-increasing if for all $\epsilon > 0$ there exists an N such that whenever $n > m \geq N$ it follows that $a_n > a_m - \epsilon$

(a) Give an example of a sequence that is quasi-increasing but not monotone or eventually monotone.

(b) Give an example of a quasi-increasing sequence that is divergent and not monotone or eventually monotone.

(c) Is there an analogue of the Monotone Convergence Theorem for quasiincreasing sequences? Give an example of a bounded, quasi-increasing sequence that doesn't converge, or prove that no such sequence exists.

Solution
(a) $a_n = (-1)^n/n$ is quasi-increasing since we can get $(-1)^m/m-(-1)^n/n \leq 1/m+1/n < \epsilon$ for large enough $n > m \geq N$.

(b) $a_n = (2, 1/2, 3, 1/3, \dots)$ is quasi-increasing since $n > m - \epsilon$ is clearly true as (n) is increasing. And $1/n > 1/m - \epsilon$ is true after picking N large enough that for $m \geq N$ we have $1/m < \epsilon$ and thus $1/n > 1/m - \epsilon$.

(c) In (b) I gave such an example, so there is no Monotone Convergence Theorem for quasiincreasing sequences (without modifying the definition that is.)

Exercise 2.6.7

Exercises 2.4.4 and 2.5.4 establish the equivalence of the Axiom of Completeness and the Monotone Convergence Theorem. They also show the Nested Interval Property is equivalent to these other two in the presence of the Archimedean Property.

(a) Assume the Bolzano-Weierstrass Theorem is true and use it to construct a proof of the Monotone Convergence Theorem without making any appeal to the Archimedean Property. This shows that BW, AoC, and MCT are all equivalent.

(b) Use the Cauchy Criterion to prove the Bolzano-Weierstrass Theorem, and find the point in the argument where the Archimedean Property is implicitly required. This establishes the final link in the equivalence of the five characterizations of completeness discussed at the end of Section 2.6.

(c) How do we know it is impossible to prove the Axiom of Completeness starting from the Archimedean Property?

Solution

(a) Suppose (x_n) is increasing and bounded, BW tells us there exists a convergent subsequence $(x_{n_k}) \to x$, We will show $(x_n) \to x$. First note $x_k \leq x_{n_k}$ implies $x_n \leq x$ by the Order Limit Theorem.

Pick K such that for $k \geq K$ we have $|x_{n_k} - x| < \epsilon$. Since (x_n) is increasing and $x_n \leq x$ every $n \geq n_K$ satisfies $|x_n - x| < \epsilon$ as well. Thus (x_n) converges, completing the proof.

(b) We're basically going to use the cauchy criterion as a replacement for NIP in the proof of BW. Recall we had $I_{n+1} \subseteq I_n$ with $a_{n_k} \in I_k$, we will show a_{n_k} is cauchy.

The length of I_k is $M(1/2)^{k-1}$ by construction, so clearly $|a_{n_k} - a_{n_j}| < M(1/2)^{N-1}$ for $k, j \geq N$, implying (a_{n_k}) converges by the cauchy criterion.

We needed the Archimedean Property to conclude $M(1/2)^{N-1} \in \mathbf{Q}$ can be made smaller then any $\epsilon \in \mathbf{R}^+$.

(c) The Archimedean Property is true for $\mathbf{Q}$ meaning it cannot prove AoC which is only true for $\mathbf{R}$. (If we did, then we would have proved AoC for $\mathbf{Q}$ which is obviously false.)

2.7 The Cauchy Criterion

Exercise 2.7.1
Proving the Alternating Series Test (Theorem 2.7.7) amounts to showing that the sequence of partial sums

$$s_n = a_1 - a_2 + a_3 - \cdots \pm a_n$$

converges. (The opening example in Section 2.1 includes a typical illustration of (s_n).) Different characterizations of completeness lead to different proofs.

(a) Prove the Alternating Series Test by showing that (s_n) is a Cauchy sequence.

(b) Supply another proof for this result using the Nested Interval Property (Theorem 1.4.1).

(c) Consider the subsequences (s_{2n}) and (s_{2n+1}), and show how the Monotone Convergence Theorem leads to a third proof for the Alternating Series Test.

Solution

(a) We would like to show $|a_{m+1} - a_{m+2} + \cdots \pm a_n|$ becomes arbitrarily small. **TODO**

(b) Let I_1 be the interval $[a_1 - a_2, a_1]$ and in general $I_n = [a_n - a_{n+1}, a_n]$ we have $I_{n+1} \subseteq I_n$ since (a_n) is decreasing. The nested interval property gives

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

Let $x \in \bigcap_{n=1}^{\infty} I_n$, since $a_n \in I_n$ and $x \in I_n$ the distance $|a_n - x|$ must be less then the length $|I_n|$. and since the length goes to zero $|a_n - x|$ can be made less then any ϵ.

(c) If we can show $\lim s_{2n} = \lim s_{2n+1} = s$ that will imply $\lim s_n = s$ since each s_n is either in (s_{2n+1}) or in (s_{2n}) as n is must be even or odd.

We have $s_{2n+1} \leq a_1$ since

$$s_{2n+1} = a_1 - (a_2 - a_3) - \cdots - (a_{2n} - a_{2n+1}) \leq a_1$$

Thus $s_{2n+1} \to s$ by the Monotone Convergence Theorem, to show $(s_{2n}) \to s$ notice $s_{2n} = s_{2n+1} - a_{2n+1}$ with $(a_{2n+1}) \to 0$ meaning we can use the triangle inequality

$$|s_{2n} - s| \leq \underbrace{|s_{2n} - s_{2n+1}|}_{a_{2n+1}} + |s_{2n+1} - s| < \epsilon/2 + \epsilon/2 < \epsilon$$

Thus $(s_{2n}) \to s$ aswell finally implying $(s_n) \to s$.

Summary: Partition the alternating series into two subsequences of partial sums, then use MCT to show they both converge to the same limit.

Exercise 2.7.2
Decide whether each of the following series converges or diverges:

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$

(b) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

(c) $1 - \frac{3}{4} + \frac{4}{6} - \frac{5}{8} + \frac{6}{10} - \frac{7}{12} + \cdots$

(d) $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots$

(e) $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \frac{1}{7} - \frac{1}{8^2} + \cdots$

Solution

(a) Converges by a comparison test with $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

(b) Converges by a comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(c) Diverges since $(n+1)/2n = 1/2 + 1/2n$ never gets smaller then $1/2$.

(d) The first thing I notice is the series cannot converge absolutely (as then it would be harmonic).

Grouping terms gives

$$\begin{aligned}\frac{1}{n} + \frac{1}{n+1} - \frac{1}{n+2} &= \frac{(n+1)(n+2) + n(n+2) - n(n+1)}{n(n+1)(n+2)} \\ &= \frac{(n^2+3n+2) + (n^2+2n) - (n^2+n)}{n(n+1)(n+2)} \\ &= \frac{n^2+4n+2}{n(n+1)(n+2)}\end{aligned}$$

Which diverges since

$$\frac{n^2+4n+2}{n(n+1)(n+2)} \geq \frac{n^2+4n+2}{n^3} \geq \frac{1}{n}$$

In other words, If we take terms three at a time (a subsequence of the partial sums) grows faster then the harmonic series implying the subsequence (and thus the parent sequence) of partial sums diverges.

TODO Find better proof

(e) Intuitively this should diverge since it is a mixture of $1/n$ (divergent) and $1/n^2$) (convergent). To make this rigorous examine the subsequence (s_{2n})

$$s_{2n} = 1 - 1/2^2 + 1/3 - 1/4^2 + \cdots + 1/(2n)^2$$

We can the rearrange terms in the partial sum to get

$$s_{2n} = \sum_{k=1}^{n} \frac{1}{2k-1} - \sum_{k=1}^{n} \frac{1}{(2k)^2} \equiv t_n - v_n$$

Now if $\lim(s_{2n})$ converges then $\lim(t_n - v_n)$ also converges. This is impossible however as (t_n) diverging and (v_n) converging would imply $\lim(t_n - v_n) + \lim(v_n) = \lim(t_n)$ converges by the ALT, but (t_n) diverges so we have a contradiction. Thus (s_{2n}) diverges and so the parent sequence (s_n) must also diverge.

Exercise 2.7.3

(a) Provide the details for the proof of the Comparison Test (Theorem 2.7.4) using the Cauchy Criterion for Series.

(b) Give another proof for the Comparison Test, this time using the Monotone Convergence Theorem.

Solution
Suppose $a_n, b_n \geq 0$, $a_n \leq b_n$ and define $s_n = a_1 + \cdots + a_n$, $t_n = b_1 + \cdots + b_n$.

(a) We have $|a_{m+1} + \cdots + a_n| \leq |b_{m+1} + \cdots + b_n| < \epsilon$ implying $\sum_{n=1}^{\infty} a_n$ converges by the cauchy criterion. The other direction is analogous, if (s_n) diverges then (t_n) must also diverge since $s_n \leq t_n$.

(b) Since $(t_n) \to t$. This implies that s_n is bounded, and since $s_n \leq t_n$ implies $s_n \leq t$ by the order limit theorem, we can use the monotone convergence theorem to conclude (s_n) converges.

Exercise 2.7.4
Give an example of each or explain why the request is impossible referencing the proper theorem(s).

(a) Two series $\sum x_n$ and $\sum y_n$ that both diverge but where $\sum x_n y_n$ converges.

(b) A convergent series $\sum x_n$ and a bounded sequence (y_n) such that $\sum x_n y_n$ diverges.

(c) Two sequences (x_n) and (y_n) where $\sum x_n$ and $\sum (x_n + y_n)$ both converge but $\sum y_n$ diverges.

(d) A sequence (x_n) satisfying $0 \leq x_n \leq 1/n$ where $\sum (-1)^n x_n$ diverges.

Solution

(a) $x_n = 1/n$ and $y_n = 1/n$ have their respective series diverge, but $\sum x_n y_n = \sum 1/n^2$ converges since it is a p-series with $p > 1$.

(b) Let $x_n = (-1)^n/n$ and $y_n = (-1)^n$. $\sum x_n$ converges but $\sum x_n y_n = \sum 1/n$ diverges.

(c) Impossible as the algebraic limit theorem for series implies $\sum (x_n + y_n) - \sum x_n = \sum y_n$ converges.

(d) Impossible as the alternating series test implies it converges.

Exercise 2.7.5
Prove the series $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$. (Corollary 2.4.7)

Solution
Eventually we have $1/n^p < 1/p^n$ for $p > 1$ (polynomial vs exponential) meaning we can use the comparison test to conclude $\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$.

Now suppose $p \leq 1$, since $1/n^p \leq 1/n$ a comparsion test with the harmonic series implies $p \leq 1$ diverges.

Exercise 2.7.6
Let's say that a series subverges if the sequence of partial sums contains a subsequence that converges. Consider this (invented) definition for a moment, and then decide which of the following statements are valid propositions about subvergent series:

(a) If (a_n) is bounded, then $\sum a_n$ subverges.

(b) All convergent series are subvergent.

(c) If $\sum |a_n|$ subverges, then $\sum a_n$ subverges as well.

(d) If $\sum a_n$ subverges, then (a_n) has a convergent subsequence.

Solution

(a) False, consider $a_n = 1$ then $s_n = n$ does not have a convergent subsequence.

(b) True, every subsequence converges to the same limit in fact.

(c) True, since $s_n = \sum_{k=1}^n |a_k|$ converges it is bounded $|s_n| \leq M$, and since $t_n = \sum_{k=1}^n a_k$ is smaller $t_n \leq s_n$ it is bounded $t_n \leq M$ which by BW implies there exists a convergent subsequence (t_{n_k}).

(d) False, $a_n = (1, -1, 2, -2, \dots)$ has no convergent subsequence but the sum $s_n = \sum_{k=1}^n a_k$ has the subsequence $(s_{2n}) \to 0$.

Exercise 2.7.7

(a) Show that if $a_n > 0$ and $\lim (na_n) = l$ with $l \neq 0$, then the series $\sum a_n$ diverges.

(b) Assume $a_n > 0$ and $\lim (n^2 a_n)$ exists. Show that $\sum a_n$ converges.

Solution
Note: This is kind of like a wierd way to do a comparison with $1/n$ and $1/n^2$.

(a) If $\lim(na_n) = l \neq 0$ then $na_n \in (l - \epsilon, l + \epsilon)$, setting $\epsilon = l/2$ gives $na_n \in (l/2, 3l/2)$ implying $a_n > (l/2)(1/n)$. But if $a_n > (l/2)(1/n)$ then $\sum a_n$ diverges as it is a multiple of the harmonic series. (note that $a_n > 0$ ensures $l \geq 0$.)

(b) Letting $l = \lim(n^2 a_n)$ we have $n^2 a_n \in (l - \epsilon, l + \epsilon)$ setting $\epsilon = l$ gives $n^2 a_n \in (0, 2l)$ implying $0 \leq a_n \leq 2l/n^2$ and so $\sum a_n$ converges by a comparsion test with $\sum 2l/n^2$.

Exercise 2.7.8
Consider each of the following propositions. Provide short proofs for those that are true and counterexamples for any that are not.

(a) If $\sum a_n$ converges absolutely, then $\sum a_n^2$ also converges absolutely.

(b) If $\sum a_n$ converges and (b_n) converges, then $\sum a_n b_n$ converges.

(c) If $\sum a_n$ converges conditionally, then $\sum n^2 a_n$ diverges.

Solution

(a) True since $(a_n) \to 0$ so eventually $a_n^2 \le |a_n|$ meaning $\sum a_n^2$ converges by a comparsion test with $\sum |a_n|$.

(b) False, let $a_n = (-1)^n/\sqrt{n}$ and $b_n = (-1)^n/\sqrt{n}$. $\sum a_n$ converges by the alternating series test, but $\sum a_n b_n = \sum 1/n$ diverges.

(c) True, suppose $(n^2 a_n)$ converges, since $(n^2 a_n) \to 0$ we have $|n^2 a_n| < 1$ for $n > N$, implying $|a_n| < 1/n^2$. But if $|a_n| < 1/n^2$ then a comparsion test with $1/n^2$ implies a_n converges absolutely, contradicting the assumption that a_n converges conditionally. Therefor $\sum n^2 a_n$ must diverge.

Exercise 2.7.9 (Ratio Test)

Given a series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$, the Ratio Test states that if (a_n) satisfies

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = r < 1$$

then the series converges absolutely.

(a) Let r' satisfy $r < r' < 1$. Explain why there exists an N such that $n \ge N$ implies $|a_{n+1}| \le |a_n| \, r'$.

(b) Why does $|a_N| \sum (r')^n$ converge?

(c) Now, show that $\sum |a_n|$ converges, and conclude that $\sum a_n$ converges.

Solution

(a) We are given

$$\left| \frac{a_{n+1}}{a_n} - r \right| < \epsilon$$

Since $1 > r' > r$ we can set $\epsilon = r' - r$ meaning the neighborhood

$$\frac{a_{n+1}}{a_n} \in (r - \epsilon, r + \epsilon) = (2r - r', r')$$

Is all less then r' meaning

$$\left| \frac{a_{n+1}}{a_n} \right| \le r' \implies |a_{n+1}| \le r' |a_n|$$

(b) Let N be large enough that for $n > N$ we have $|a_{n+1}| \le |a_n| r'$. Applying this multiple times gives $|a_n| \le (r')^{n-N} |a_N|$ which gives

$$|a_N| + |a_{N+1}| + \cdots + |a_n| \le |a_N| + r'|a_N| + \cdots + (r')^{n-N} |a_N|$$

Factoring out $|a_N|$ and writing with sums gives

$$\sum_{k=N}^{n} |a_k| \le |a_N| \sum_{k=0}^{n-1} (r')^k$$

Which converges as $n \to \infty$ since $|r'| < 1$ and $|a_N|$ is constant. Implying $\sum_{k=N}^{n} |a_k|$ converges and thus $\sum_{k=1}^{n} |a_k|$ also converges since we only omitted finitely many terms.

(c) See (b)

Exercise 2.7.10 (Infinite Products)
Review Exercise 2.4.10 about infinite products and then answer the following questions:

(a) Does $\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{9}{8} \cdot \frac{17}{16} \cdots$ converge?

(b) The infinite product $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{9}{10} \cdots$ certainly converges. (Why?) Does it converge to zero?

(c) In 1655, John Wallis famously derived the formula

$$\left(\frac{2 \cdot 2}{1 \cdot 3}\right)\left(\frac{4 \cdot 4}{3 \cdot 5}\right)\left(\frac{6 \cdot 6}{5 \cdot 7}\right)\left(\frac{8 \cdot 8}{7 \cdot 9}\right) \cdots = \frac{\pi}{2}$$

Show that the left side of this identity at least converges to something. (A complete proof of this result is taken up in Section 8.3.)

Solution

(a) Rewriting the terms as $a_n = (1 + 1/n)$ and using the result from 2.4.10 implies the product diverges since $\sum 1/n$ diverges.

(b) Converges by the monotone convergence theorem since the partial products are decreasin and greater then zero.

(c) In compontent form we have

$$a_n = \frac{(2n)^2}{(2n-1)(2n+1)} = \frac{(2n)^2}{(2n)^2 - 1} = 1 + \frac{(2n)^2 - ((2n)^2 - 1)}{(2n)^2 - 1} = 1 + \frac{1}{(2n)^2 - 1}$$

And since $\sum 1/((2n)^2 - 1)$ converges by a comparison test with $1/n^2$ 2.4.10 implies

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{(2n)^2 - 1}\right)$$

also converges.

Exercise 2.7.11
Find examples of two series $\sum a_n$ and $\sum b_n$ both of which diverge but for which $\sum \min\{a_n, b_n\}$ converges. To make it more challenging, produce examples where (a_n) and (b_n) are strictly positive and decreasing.

Solution
TODO

Exercise 2.7.12 (Summation-by-parts)
Let (x_n) and (y_n) be sequences, let $s_n = x_1 + x_2 + \cdots + x_n$ and set $s_0 = 0$. Use the observation that $x_j = s_j - s_{j-1}$ to verify the formula

$$\sum_{j=m}^{n} x_j y_j = s_n y_{n+1} - s_{m-1} y_m + \sum_{j=m}^{n} s_j (y_j - y_{j+1})$$

Solution
Since $x_j = s_j - s_{j-1}$ we can rewrite the sum as

$$
\begin{aligned}
&= \sum_{j=m}^{n} (s_j - s_{j-1}) y_j \\
&= \sum_{j=m}^{n} s_j y_j - \sum_{j=m}^{n} s_{j-1} y_j \\
&= (s_m y_m + \cdots + s_n y_n) - (s_{m-1} y_m + \cdots + s_{n-1} y_n) && \text{Factor out each } s_j \\
&= s_n(y_n) - s_{m-1}(y_m) + s_m(y_m - y_{m+1}) + \cdots + s_{n-1}(y_{n-1} - y_n) && \text{Add and subtract } s_n y_{n+1} \text{ for factoring} \\
&= s_n(y_{n+1}) - s_{m-1}(y_m) + s_m(y_m - y_{m+1}) + \cdots + s_n(y_n - y_{n+1}) && \text{Rewrite as a sum} \\
&= s_n(y_{n+1}) - s_{m-1}(y_m) + \sum_{j=m}^{n} s_j(y_j - y_{j+1}) && \text{Done :)}
\end{aligned}
$$

Exercise 2.7.13 (Abel's Test)
Abel's Test for convergence states that if the series $\sum_{k=1}^{\infty} x_k$ converges, and if (y_k) is a sequence satisfying

$$y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0$$

then the series $\sum_{k=1}^{\infty} x_k y_k$ converges.

(a) Use Exercise 2.7.12 to show that

$$\sum_{k=1}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=1}^{n} s_k \left(y_k - y_{k+1}\right)$$

where $s_n = x_1 + x_2 + \cdots + x_n$.

(b) Use the Comparison Test to argue that $\sum_{k=1}^{\infty} s_k \left(y_k - y_{k+1}\right)$ converges absolutely, and show how this leads directly to a proof of Abel's Test.

Solution
Intuitively, Abel's test is saying that multiplying by a positive decreasing sequence (y_n) is basically the same as multiplying by a constant in that it doesn't effect convergence.

(a) Exercise 2.7.12 combined with $s_0 = 0$ gives

$$\sum_{k=1}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=1}^{n} s_k (y_k - y_{k+1})$$

as desired.

(b) $s_n y_{n+1}$ clearly converges since y_{n+1} is "eventually constant", so we must only show the right hand side converges.

We will show absolute convergence, note $y_k - y_{k+1} \geq 0$ and so

$$\sum_{k=1}^{n} |s_k|(y_k - y_{k+1}) \geq 0$$

Bounding $|s_n| \leq M$ gives

$$\sum_{k=1}^{n} |s_k|(y_k - y_{k+1}) \leq M \sum_{k=1}^{n}(y_k - y_{k+1})$$

Since $\sum_{k=1}^{n}(y_k - y_{k+1}) = y_1 - y_{n+1}$ is telescoping we can write

$$\sum_{k=1}^{n} |s_k|(y_k - y_{k+1}) \leq M(y_1 - y_{n+1}) \leq My_1$$

Implying $\sum_{k=1}^{\infty} |s_k|(y_k - y_{k+1})$ converges since it is bounded and increasing. And since the series converges absolutely so does the original $\sum_{k=1}^{\infty} s_k(y_k - y_{k+1})$.

Summary: Bound $|s_k| \leq M$ and use the fact that $\sum(y_k - y_{k+1})$ is telescoping.

Exercise 2.7.14 (Dirichlet's Test)
Dirichlet's Test for convergence states that if the partial sums of $\sum_{k=1}^{\infty} x_k$ are bounded (but not necessarily convergent), and if (y_k) is a sequence satisfying $y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0$ with $\lim y_k = 0$, then the series $\sum_{k=1}^{\infty} x_k y_k$ converges.

(a) Point out how the hypothesis of Dirichlet's Test differs from that of Abel's Test in Exercise 2.7.13, but show that essentially the same strategy can be used to provide a proof.

(b) Show how the Alternating Series Test (Theorem 2.7.7) can be derived as a special case of Dirichlet's Test.

Solution

(a) Abel's test assumed $\sum x_n$ converged but did not require $\lim y_k = 0$. To prove it we use summation by parts to get

$$\sum_{k=1}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=1}^{n} s_k(y_k - y_{k+1})$$

Clearly $(s_n y_{n+1}) \to 0$ since $|s_n| \leq M$ and $(y_{n+1}) \to 0$. Now we show absolute convergence of the right hand side (note $|y_k - y_{k+1}| = (y_k - y_{k+1})$ as (y_n) is decreasing)

$$\sum_{k=1}^{n} |s_k|(y_k - y_{k+1}) \leq M \sum_{k=1}^{n}(y_k - y_{k+1}) = M(y_1 - y_{n+1}) \leq My_1$$

Thus $\sum s_k(y_k - y_{k+1})$ converges absolutely and so finally

$$s_n y_{n+1} + \sum_{k=1}^{n} s_k(y_k - y_{k+1}) = \sum_{k=1}^{n} x_k y_k$$

converges.

(b) Let $a_n \geq 0$ with $a_1 \geq a_2 \geq \cdots \geq 0$ and $\lim a_n = 0$. The series $\sum(-1)^n \leq 1$ is bounded, so Dirichlet's test implies $\sum(-1)^n a_n$ converges.

In fact any periodic b_n with $\sum b_n$ bounded has $\sum a_n b_n$ converging by Dirichlet.