Understanding Analysis Solutions

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Preface

Huge thanks to the math discord for answering my questions! I don't know how I'd manage without them ♡

If you don't find your exercise here check linearalgebras.com or (god forbid) chegg.

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Chapter 1

The Real Numbers

1.2 Some Preliminaries

Exercise 1.2.1

(a) Prove that $\sqrt{3}$ is irrational. Does a similar similar argument work to show $\sqrt{6}$ is irrational?

(b) Where does the proof break down if we try to prove $\sqrt{4}$ is irrational?

Solution

(a) Suppose for contradiction that p/q is a fraction in lowest terms, and that $(p/q)^2 = 3$. Then $p^2 = 3q^2$ implying p is a multiple of 3 since 3 is not a perfect square. Therefore we can write p as $3r$ for some r, substituting we get $(3r)^2 = 3q^2$ and $3r^2 = q^2$ implying q is also a multiple of 3 contradicting the assumption that p/q is in lowest terms.
For $\sqrt{6}$ the same argument applies, since 6 is not a perfect square.

(b) 4 is a perfect square, meaning $p^2 = 4q^2$ does not imply that p is a multiple of four as p could be 2.

Exercise 1.2.2
Show that there is no rational number satisfying $2^r = 3$

Solution
If $r = 0$ clearly $2^r = 1 \neq 3$, if $r \neq 0$ set $p/q = r$ to get $2^p = 3^q$ which is impossible since 2 and 3 share no factors.

Exercise 1.2.3
Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all sets containing an infinite number of elements, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.

(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$.

(d) $A \cap (B \cap C) = (A \cap B) \cap C$.

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution

(a) False, consider $A_1 = \{1, 2, \dots\}, A_2 = \{2, 3, \dots\}$, ... has $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

(b) True, because we eventually reach $A_j = \{x\}$ and get stuck

(c) False, $A = \emptyset$ gives $\emptyset = C$.

(d) True, intersection is associative.

(e) True, draw a diagram.

Exercise 1.2.4

Produce an infinite collection of sets $A_1, A_2, A_3, \dots$ with the property that every A_i has an infinite number of elements, $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{\infty} A_i = \mathbf{N}$

Solution

This question is asking us to partition $\mathbf{N}$ into an infinite collection of sets. This is equivalent to asking us to unroll $\mathbf{N}$ into a square, which we can do along the diagonal

$$\begin{array}{cccccc}
1 & 3 & 6 & 10 & 15 & \cdots \\
2 & 5 & 9 & 14 & \cdots & \\
4 & 8 & 13 & \cdots & & \\
7 & 12 & \cdots & & & \\
11 & \dots & & & & \\
\vdots & & & & &
\end{array}$$

Exercise 1.2.5 (De Morgan's Laws)

Let A and B be subsets of $\mathbf{R}$.

(a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$

(b) Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$

(c) Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

Solution

(a) If $x \in (A \cap B)^c$ then $x \notin A \cap B$ so $x \notin A$ or $x \notin B$ implying $x \in A^c$ or $x \in B^c$ which is the same as $x \in A^c \cup B^c$.

(b) Let $x \in A^c \cup B^c$ implying $x \in A^c$ or $x \in B^c$ meaning $x \notin A$ or $x \notin B$ implying $x \notin A \cap B$ which is the same as $x \in (A \cap B)^c$.

(c) First let $x \in (A \cup B)^c$ implying $x \notin A \cup B$ meaning $x \notin A$ and $x \notin B$ which is the same as $x \in A^c$ and $x \in B^c$ which is just $x \in A^c \cap B^c$. Second let $x \in A^c \cap B^c$ implying $x \in A^c$ and $x \in B^c$ implying $x \notin A$ and $x \notin B$ meaning $x \notin A \cup B$ which is just $x \in (A \cup B)^c$.

Exercise 1.2.6

(a) Verify the triangle inequality in the special case where a and b have the same sign.

(b) Find an efficient proof for all the cases at once by first demonstrating $(a+b)^2 \leq (|a|+|b|)^2$

(c) Prove $|a-b| \leq |a-c| + |c-d| + |d-b|$ for all a, b, c, and d.

(d) Prove $||a|-|b|| \leq |a-b|$. (The unremarkable identity $a = a - b + b$ may be useful.)

Solution

(a) We have equality $|a+b| = |a| + |b|$ meaning $|a+b| \leq |a| + |b|$ also holds.

(b) $(a+b)^2 \leq (|a|+|b|)^2$ reduces to $2ab \leq 2|a||b|$ which is true as the left side can be negative but the right side can't. and since squaring preserves inequality this implies $|a+b| \leq |a| + |b|$.

(c) I would like to do this using the triangle inequality, I notice that $(a-c)+(c-d)+(d-b) = a-b$. Meaning I can use the triangle inequality for multiple terms

$$|a-b| = |(a-c)+(c-d)+(d-b)| \leq |a-c| + |c-d| + |d-b|$$

The general triangle inequality is proved by repeated application of the two variable inequality

$$|(a+b)+c| \leq |a+b| + |c| \leq |a| + |b| + |c|$$

(d) Since $||a|-|b|| = ||b|-|a||$ we can assume $|a| > |b|$ without loss of generality. Then

$$||a|-|b|| = |a| - |b| = |(a-b)+b| - |b| \leq |a-b| + |b| - |b| = |a-b|$$

Exercise 1.2.7

Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(A) = \{f(x) : x \in A\}$.

(a) Let $f(x) = x^2$. If $A = [0, 2]$ (the closed interval $\{x \in \mathbf{R} : 0 \leq x \leq 2\}$) and $B = [1, 4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?

(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

(c) Show that, for an arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbf{R}$

(d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary function g

Solution

(a) $f(A) = [0, 4]$, $f(B) = [1, 16]$, $f(A \cap B) = [1, 4] = f(A) \cap f(B)$ and $f(A \cup B) = [0, 16] = f(A) \cup f(B)$

(b) $A = \{-1\}$, $B = \{1\}$ thus $f(A \cap B) = \emptyset$ but $f(A) \cap f(B) = \{1\}$

(c) Suppose $y \in g(A \cap B)$, then $\exists x \in A \cap B$ such that $g(x) = y$. But if $x \in A \cap B$ then $x \in A$ and $x \in B$, meaning $y \in g(A)$ and $y \in g(B)$ implying $y \in g(A) \cap g(B)$ and thus $g(A \cap B) \subseteq g(A) \cap g(B)$.

Notice why it is possible to have $x \in g(A) \cap g(B)$ but $x \notin g(A \cap B)$, this happens when something in $A \setminus B$ and something in $B \setminus A$ map to the same thing. If g is 1-1 this does not happen.

(d) I conjecture that $g(A \cup B) = g(A) \cup g(B)$. To prove this we show inclusion both ways, First suppose $y \in g(A \cup B)$. then either $y \in g(A)$ or $y \in g(B)$, implying $y \in g(A) \cup g(B)$. Now suppose $y \in g(A) \cup g(B)$ meaning either $y \in g(A)$ or $y \in g(B)$ which is the same as $y \in g(A \cup B)$ as above.

Exercise 1.2.8

Here are two important definitions related to a function $f : A \to B$. The function f is *one-to-one* $(1 - 1)$ if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is *onto* if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$ Give an example of each or state that the request is impossible:

(a) $f : \mathbf{N} \to \mathbf{N}$ that is $1 - 1$ but not onto.

(b) $f : \mathbf{N} \to \mathbf{N}$ that is onto but not $1 - 1$.

(c) $f : \mathbf{N} \to \mathbf{Z}$ that is $1 - 1$ and onto.

Solution

(a) Let $f(n) = n + 1$ does not have a solution to $f(a) = 1$

(b) Let $f(1) = 1$ and $f(n) = n - 1$ for $n > 1$

(c) Let $f(n) = n/2$ for even n, and $f(n) = -(n + 1)/2$ for odd n.

Exercise 1.2.9

Given a function $f : D \to \mathbf{R}$ and a subset $B \subseteq \mathbf{R}$, let $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B; that is, $f^{-1}(B) = \{x \in D : f(x) \in B\}$. This set is called the *preimage* of B.

(a) Let $f(x) = x^2$. If A is the closed interval $[0, 4]$ and B is the closed interval $[-1, 1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case? Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?

(b) The good behavior of preimages demonstrated in (a) is completely general. Show that for an arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ for all sets $A, B \subseteq \mathbf{R}$

Solution

(a) $f^{-1}(A) = [-2, 2]$, $f^{-1}(B) = [-1, 1]$, $f^{-1}(A \cap B) = [-1, 1] = f^{-1}(A) \cap f^{-1}(B)$, $f^{-1}(A \cup B) = [-2, 2] = f^{-1}(A) \cup f^{-1}(B)$

(b) First let $x \in g^{-1}(A \cap B)$ meaning $g(x) \in A \cap B$ implying $g(x) \in A$ and $g(x) \in B$ which is the same as $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$ meaning $x \in g^{-1}(A) \cap g^{-1}(B)$.

Second let $x \in g^{-1}(A) \cap g^{-1}(B)$, this is the same as $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$ which is the same as $g(x) \in A$ and $g(x) \in B$ implying $g(x) \in A \cap B$ and thus $x \in g^{-1}(A \cap B)$. Thus $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$.

Seeing $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ is obvious. see 1.2.7 (d).

Exercise 1.2.10

Decide which of the following are true statements. Provide a short justification for those that are valid and a counterexample for those that are not:

(a) Two real numbers satisfy $a < b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

(b) Two real numbers satisfy $a < b$ if $a < b + \epsilon$ for every $\epsilon > 0$.

(c) Two real numbers satisfy $a \leq b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

Solution

(a) False, if $a = b$ then $a < b + \epsilon$ for all $\epsilon > 0$ but $a \not< b$

(b) False, consider $a = b$ as above

(c) True. First suppose $a < b + \epsilon$ for all $\epsilon > 0$, We want to show this implies $a \leq b$. We either have $a \leq b$ or $a > b$, but $a > b$ is impossible since the gap implies there exists an ϵ small enough such that $a > b + \epsilon$.

Second suppose $a \leq b$, obviously $a < b + \epsilon$ for all $\epsilon > 0$.

Exercise 1.2.11

Form the logical negation of each claim. One trivial way to do this is to simply add "It is not the case that..." in front of each assertion. To make this interesting, fashion the negation into a positive statement that avoids using the word "not" altogether. In each case, make an intuitive guess as to whether the claim or its negation is the true statement.

(a) For all real numbers satisfying $a < b$, there exists an $n \in \mathbf{N}$ such that $a + 1/n < b$

(b) There exists a real number $x > 0$ such that $x < 1/n$ for all $n \in \mathbf{N}$.

(c) Between every two distinct real numbers there is a rational number.

Solution

(a) There exist real numbers satisfying $a < b$ where $a + 1/n \geq b$ for all $n \in \mathbf{N}$ (false).

(b) For every real number $x > 0$ there exists an $n \in \mathbf{N}$ such that $x < 1/n$ (true).

(c) There exist two real numbers $a < b$ such that if $r < b$ then $r < a$ for all $r \in \mathbf{Q}$ (false).

Exercise 1.2.12
Let $y_1 = 6$, and for each $n \in \mathbf{N}$ define $y_{n+1} = (2y_n - 6)/3$

(a) Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in \mathbf{N}$.

(b) Use another induction argument to show the sequence $(y_1, y_2, y_3, \ldots)$ is decreasing.

Solution

(a) Suppose for induction that $y_n > -6$, our base case clearly satisfies $y_1 > -6$. then

$$\begin{aligned} y_{n+1} = (2y_n - 6)/3 &\implies y_n = (3y_{n+1} + 6)/2 > -6 \\ &\implies y_{n+1} > (2 \cdot (-6) - 6)/3 = -6 \end{aligned}$$

Thus $y_{n+1} > -6$

(b) Suppose $y_{n+1} < y_n$, the base case $2 < 6$ works. Now

$$\begin{aligned} y_{n+1} < y_n &\implies 2y_{n+1} < 2y_n \\ &\implies 2y_{n+1} - 6 < 2y_n - 6 \\ &\implies (2y_{n+1} - 6)/3 < (2y_n - 6)/3 \\ &\implies y_{n+2} < y_{n+1} \end{aligned}$$

Thus (y_n) is decreasing.

Exercise 1.2.13
For this exercise, assume Exercise 1.2.5 has been successfully completed.

(a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

for any finite $n \in \mathbf{N}$

(b) It is tempting to appeal to induction to conclude

$$\left(\bigcup_{i=1}^{\infty} A_i \right)^c = \bigcap_{i=1}^{\infty} A_i^c$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of $n \in \mathbf{N}$, but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets $B_1, B_2, B_3, \ldots$ where $\bigcap_{i=1}^{n} B_i \neq \emptyset$ is true for every $n \in \mathbf{N}$, but $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$ fails.

(c) Nevertheless, the infinite version of De Morgan's Law stated in (b) is a valid statement. Provide a proof that does not use induction.

Solution

(a) 1.2.5 Is our base case, Assume $(A_1 \cup \cdots \cup A_n)^c = A_1^c \cap \cdots \cap A_n^c$. We want to show the $n+1$ case. Using associativity we have

$$\begin{aligned}((A_1 \cup \cdots \cup A_n) \cup A_{n+1})^c &= (A_1 \cup \cdots \cup A_n)^c \cap A_{n+1}^c \\ &= (A_1^c \cap \cdots \cap A_n^c) \cap A_{n+1}^c \\ &= A_1^c \cap \cdots \cap A_n^c \cap A_{n+1}^c\end{aligned}$$

(b) $B_1 = \{1, 2, \dots\}, B_2 = \{2, 3, \dots\}, \dots$

(c) First suppose $x \in (\bigcap_{i=1}^\infty A_i)^c$, then $x \notin \bigcap_{i=1}^\infty A_i$ meaning $x \notin A_i$ for some i, which is the same as $x \in A_i^c$ for some i, meaning $x \in \bigcup_{i=1}^\infty A_i^c$. This shows

$$\left(\bigcap_{i=1}^\infty A_i\right) \subseteq \bigcup_{i=1}^\infty A_i^c$$

Now suppose $x \in \bigcup_{i=1}^\infty A_i^c$ meaning $x \notin A_i$ for some i, which is the same as $x \notin \bigcap_{i=1}^\infty A_i$ implying $x \notin (\bigcap_{i=1}^\infty A_i)^c$. This shows inclusion the other way and completes the proof.

1.3 The Axiom of Completeness

Exercise 1.3.1

(a) Write a formal definition in the style of Definition 1.3.2 for the *infimum* or *greatest lower bound* of a set.

(b) Now, state and prove a version of Lemma 1.3.8 for greatest lower bounds.

Solution

(a) We have $i = \inf A$ if and only if

  (i) Lower bound, $a \geq i$ for all $a \in A$

  (ii) Greatest lower bound, If b is a lower bound on A then $b \leq i$

(b) Suppose i is a lower bound for A, it is the greatest lower bound if and only if for all $\epsilon > 0$, there exists an $a \in A$ such that $i + \epsilon > a$.

First suppose $i = \inf A$, then for all $\epsilon > 0$, $i + \epsilon$ cannot be a lower bound on A because (ii) implies all lower bounds b obey $b \leq i$, therefore there must be some $a \in A$ such that $i + \epsilon > a$.

Second suppose for all $\epsilon > 0$ there exists an $a \in A$ such that $i + \epsilon > a$. In other words $i + \epsilon$ is not a lower bound for all ϵ, which is the same as saying every lower bound b must have $b \leq i$ implying (ii).

Exercise 1.3.2

Give an example of each of the following, or state that the request is impossible.

(a) A set B with $\inf B \geq \sup B$.

(b) A finite set that contains its infimum but not its supremum.

(c) A bounded subset of $\mathbf{Q}$ that contains its supremum but not its infimum.

Solution

(a) Let $B = \{0\}$ we have $\inf B = 0$ and $\sup B = 0$ thus $\inf B \geq \sup B$.

(b) Impossible, finite sets must contain their infimum and supremum.

(c) Let $B = \{r \in \mathbf{Q} \mid 1 < r \leq 2\}$ we have $\inf B = 1 \notin B$ and $\sup B = 2 \in B$.

Exercise 1.3.3

(a) Let A be nonempty and bounded below, and define $B = \{b \in \mathbf{R} : b \text{ is a lower bound for } A\}$. Show that $\sup B = \inf A$.

(b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

Solution

(a) By definition $\sup B$ is the greatest lower bound for A, meaning it equals $\inf A$.

(b) (a) Proves the greatest lower bound exists using the least upper bound.

Exercise 1.3.4
Let $A_1, A_2, A_3, \ldots$ be a collection of nonempty sets, each of which is bounded above.

(a) Find a formula for $\sup(A_1 \cup A_2)$. Extend this to $\sup(\bigcup_{k=1}^{n} A_k)$.

(b) Consider $\sup(\bigcup_{k=1}^{\infty} A_k)$. Does the formula in (a) extend to the infinite case?

Solution
(a) $\sup(\bigcup_{k=1}^{n} A_k) = \sup\{\sup A_k \mid k = 1, \ldots, n\}$

(b) In general no, since $\bigcup_{k=1}^{\infty} A_k$ may be unbounded, for example with $A_n = [n, n+1]$.

Exercise 1.3.5
As in Example 1.3.7, let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $c \in \mathbf{R}$. This time define the set $cA = \{ca : a \in A\}$.

(a) If $c \geq 0$, show that $\sup(cA) = c \sup A$.

(b) Postulate a similar type of statement for $\sup(cA)$ for the case $c < 0$.

Solution
(a) Let $s = c \sup A$. Suppose $ca > s$, then $a > \sup A$ which is impossible, meaning s is an upper bound on cA. Now suppose s' is an upper bound on cA and $s' < s$. Then $s'/c < s/c$ and $s'/c < \sup A$ meaning s'/c cannot bound A, so there exists $a \in A$ such that $s'/c > a$ meaning $s' > ca$ thus s' cannot be an upper bound on cA, and so $s = c \sup A$ is the least upper bound.

(b) $\sup(cA) = c \inf(A)$ for $c < 0$

Exercise 1.3.6
Given sets A and B, define $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Follow these steps to prove that if A and B are nonempty and bounded above then $\sup(A + B) = \sup A + \sup B$

(a) Let $s = \sup A$ and $t = \sup B$. Show $s + t$ is an upper bound for $A + B$.

(b) Now let u be an arbitrary upper bound for $A + B$, and temporarily fix $a \in A$. Show $t \leq u - a$.

(c) Finally, show $\sup(A + B) = s + t$.

(d) Construct another proof of this same fact using Lemma 1.3.8.

Solution
(a) We have $a \leq s$ and $b \leq t$, adding the equations gives $a + b \leq s + t$.

(b) $t \leq u - a$ should be true since $u - a$ is an upper bound on b, meaning it is greater then or equal to the least upper bonud t. Formally $a + b \leq u$ implies $b \leq u - a$ and since t is the least upper bound on b we have $t \leq u - a$.

(c) From (a) we know $s+t$ is an upper bound, so we must only show it is the least upper bound.

Let $u = \sup(A+B)$, from (a) we have $t \leq u-a$ and $s \leq u-b$ adding and rearranging gives $a+b \leq 2u-s-t$. since $2u-s-t$ is an upper bound on $A+B$ it is less then the least upper bound, so $u \leq 2u-s-t$ implying $s+t \leq u$. and since u is the least upper bound $s+t$ must equal u.

Stepping back, the key to this proof is that $a+b \leq s, \forall a, b$ implying $\sup(A+B) \leq s$ can be used to transition from all $a+b$ to a single value $\sup(A+B)$, avoiding the ϵ-hackery I would otherwise use.

(d) Showing $s+t-\epsilon$ is not an upper bound for any $\epsilon > 0$ proves it is the least upper bound by Lemma 1.3.8. Rearranging gives $(s-\epsilon/2)+(t-\epsilon/2)$ we know there exists $a > (s-\epsilon/2)$ and $b > (t-\epsilon/2)$ therefore $a+b > s+t-\epsilon$ meaning $s+t$ cannot be made smaller, and thus is the least upper bound.

Exercise 1.3.7
Prove that if a is an upper bound for A, and if a is also an element of A, then it must be that $a = \sup A$.

Solution
a is the least upper bound since any smaller bound $a' < a$ would not bound a.

Exercise 1.3.8
Compute, without proofs, the suprema and infima (if they exist) of the following sets:

(a) $\{m/n : m, n \in \mathbf{N} \text{ with } m < n\}$.

(b) $\{(-1)^m/n : m, n \in \mathbf{N}\}$.

(c) $\{n/(3n+1) : n \in \mathbf{N}\}$

(d) $\{m/(m+n) : m, n \in \mathbf{N}\}$

Solution
(a) $\sup = 1$, $\inf = 0$

(b) $\sup = 1$, $\inf = -1$

(c) $\sup = 1/3$, $\inf = 1/4$

(d) $\sup = 1$, $\inf = 0$

Exercise 1.3.9
(a) If $\sup A < \sup B$, show that there exists an element $b \in B$ that is an upper bound for A.

(b) Give an example to show that this is not always the case if we only assume $\sup A \leq \sup B$

Solution

(a) By Lemma 1.3.8 we know there exists a b such that $(\sup B) - \epsilon < b$ for any $\epsilon > 0$, We set ϵ to be small enough that $\sup A < (\sup B) - \epsilon$ meaning $\sup A < b$ for some b, and thus b is an upper bound on A.

(b) $A = \{x \mid x \leq 1\}$, $B = \{x \mid x < 1\}$ no $b \in B$ is an upper bound since $1 \in A$ and $1 > b$.

Exercise 1.3.10 (Cut Property)
The Cut Property of the real numbers is the following:

If A and B are nonempty, disjoint sets with $A \cup B = \mathbf{R}$ and $a < b$ for all $a \in A$ and $b \in B$, then there exists $c \in \mathbf{R}$ such that $x \leq c$ whenever $x \in A$ and $x \geq c$ whenever $x \in B$.

(a) Use the Axiom of Completeness to prove the Cut Property.

(b) Show that the implication goes the other way; that is, assume $\mathbf{R}$ possesses the Cut Property and let E be a nonempty set that is bounded above. Prove $\sup E$ exists.

(c) The punchline of parts (a) and (b) is that the Cut Property could be used in place of the Axiom of Completeness as the fundamental axiom that distinguishes the real numbers from the rational numbers. To drive this point home, give a concrete example showing that the Cut Property is not a valid statement when $\mathbf{R}$ is replaced by $\mathbf{Q}$.

Solution

(a) If $c = \sup A = \inf B$ then $a \leq c \leq b$ is obvious. So we must only prove $\sup A = \inf B$. If $\sup A < \inf B$ then consider $c = \frac{\sup A + \inf B}{2}$. $c > \sup A$ and therefore $c \notin A$; similarly $c < \inf B$ and therefore $c \notin B$, implying $A \cup B \neq \mathbf{R}$. If $\sup A > \inf B$ then we can find a such that $a > b$ by subtracting $\epsilon > 0$ and using the least upper/lower bound facts, similarly to Lemma 1.3.8. Thus $\sup A$ must equal $\inf B$ since we have shown both alternatives are impossible.

(b) Let $B = \{x \mid e < x, \forall e \in E\}$ and let $A = B^c$. Clearly $a < b$ so the cut property applies. We have $a \leq c \leq b$ and must show the two conditions for $c = \sup E$

  (i) Since $E \subseteq A$, $a \leq c$ implies $e \leq c$ thus c is an upper bound.

  (ii) $c \leq b$ implies c is the smallest upper bound.

  Note: Using (a) here would be wrong, it assumes the axiom of completeness so we would be making a circular argument.

(c) $A = \{r \in \mathbf{Q} \mid r^2 < 2 \text{ or } r < 0\}$, $B = A^c$ does not satisfy the cut property in $\mathbf{Q}$ since $\sqrt{2} \notin \mathbf{Q}$.

  (Alternatively, $A = \{r \in \mathbf{Q} \mid r^3 < 2\}$ can be used to avoid needing to single out negative numbers in defining A, but requires a proof that $\sqrt[3]{2}$ is irrational - left to the reader.)

Exercise 1.3.11
Decide if the following statements about suprema and infima are true or false. Give a short proof for those that are true. For any that are false, supply an example where the claim in question does not appear to hold.

(a) If A and B are nonempty, bounded, and satisfy $A \subseteq B$, then $\sup A \leq \sup B$.

(b) If $\sup A < \inf B$ for sets A and B, then there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.

(c) If there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

Solution

(a) True. We know $a \leq \sup A$ and $a \leq \sup B$ since $A \subseteq B$. since $\sup A$ is the least upper bound on A we have $\sup A \leq \sup B$.

(b) True. Let $c = (\sup A + \inf B)/2$, $c > \sup A$ implies $a < c$ and $c < \inf B$ implies $c < b$ giving $a < c < b$ as desired.

(c) False. consider $A = \{x \mid x < 1\}$, $B = \{x \mid x > 1\}$, $a < 1 < b$ but $\sup A \not< \inf B$ since $1 \not< 1$.

1.4 Consequences of Completeness

Exercise 1.4.1
Recall that $\mathbf{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbf{Q}$, then ab and $a + b$ are elements of $\mathbf{Q}$ as well.

(b) Show that if $a \in \mathbf{Q}$ and $t \in \mathbf{I}$, then $a + t \in \mathbf{I}$ and $at \in \mathbf{I}$ as long as $a \neq 0$.

(c) Part (a) can be summarized by saying that $\mathbf{Q}$ is closed under addition and multiplication. Is $\mathbf{I}$ closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s + t$ and st?

Solution

(a) Trivial.

(b) Suppose $a + t \in \mathbf{Q}$, then by (a) $(a + t) - a = t \in \mathbf{Q}$ contradicting $t \in \mathbf{I}$.

(c) $\mathbf{I}$ is not closed under addition or multiplication. consider $(1 - \sqrt{2}) \in \mathbf{I}$ by (b), and $\sqrt{2} \in \mathbf{I}$. the sum $(1 - \sqrt{2}) + \sqrt{2} = 1 \in \mathbf{Q} \notin \mathbf{I}$. Also $\sqrt{2} \cdot \sqrt{2} = 2 \in \mathbf{Q} \notin \mathbf{I}$.

Exercise 1.4.2
Let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $s \in \mathbf{R}$ have the property that for all $n \in \mathbf{N}, s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upper bound for A. Show $s = \sup A$.

Solution
This is basically a rephrasing of Lemma 1.3.8 using the archimedean property. The most straightforward approach is to argue by contradiction:

(i) If $s < \sup A$ then there exists an n such that $s + 1/n < \sup A$ contradicting $\sup A$ being the least upper bound.

(ii) If $s > \sup A$ then there exists an n such that $s - 1/n > \sup A$ where $s - 1/n$ is not an upper bound, contradicting $\sup A$ being an upper bound.

Thus $s = \sup A$ is the only remaining possibility.

Exercise 1.4.3
Prove that $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closed for the conclusion of the theorem to hold.

Solution
Suppose $x \in \bigcap_{n=1}^{\infty}(0, 1/n)$, then we have $0 < x < 1/n$ for all n, which is impossible by the archimedean property, In other words we can always set n large enough that x lies outside the interval.

Exercise 1.4.4
Let $a < b$ be real numbers and consider the set $T = \mathbf{Q} \cap [a, b]$. Show $\sup T = b$

Solution
We must show the two conditions for a least upper bound

(i) Clearly $t \leq b$ for all $t \in T$

(ii) Let $a < b' < b$. b' Cannot be an upper bound for T since the density theorem tells us we can find $r \in \mathbf{Q} \cap [a, b]$ such that $b' < r < b$.

Exercise 1.4.5
Using Exercise 1.4.1, supply a proof that $\mathbf{I}$ is dense in $\mathbf{R}$ by considering the real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$. In other words show for every two real numbers $a < b$ there exists an irrational number t with $a < t < b$.

Solution
The density theorem lets us find a rational number r with $a - \sqrt{2} < r < b - \sqrt{2}$, adding $\sqrt{2}$ to both sides gives $a < r + \sqrt{2} < b$. From 1.4.1 we know $r + \sqrt{2}$ is irrational, so setting $t = r + \sqrt{2}$ gives $a < t < b$ as desired.

Exercise 1.4.6
Recall that a set B is dense in $\mathbf{R}$ if an element of B can be found between any two real numbers $a < b$. Which of the following sets are dense in $\mathbf{R}$? Take $p \in \mathbf{Z}$ and $q \in \mathbf{N}$ in every case.

(a) The set of all rational numbers p/q with $q \leq 10$.

(b) The set of all rational numbers p/q with q a power of 2 .

(c) The set of all rational numbers p/q with $10|p| \geq q$.

Solution
(a) Not dense since we cannot make $|p|/q$ smaller then $1/10$.

(b) Dense.

(c) Not dense since we cannot make $|p|/q$ smaller then $1/10$.

Exercise 1.4.7
Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ leads to a contradiction of the fact that $\alpha = \sup T$

Solution
Recall $T = \{t \in \mathbf{R} \mid t^2 < 2\}$ and $\alpha = \sup T$. suppose $\alpha^2 > 2$, we will show there exists an $n \in \mathbf{N}$ such that $(\alpha - 1/n)^2 > 2$ contradicting the assumption that α is the least upper bound.

We expand $(\alpha - 1/n)^2$ to find n such that $(\alpha^2 - 1/n) > 2$

$$2 < (\alpha - 1/n)^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} < \alpha^2 + \frac{1 - 2\alpha}{n}$$

Then

$$2 < \alpha^2 + \frac{1 - 2\alpha}{n} \implies n(2 - \alpha^2) < 1 - 2\alpha$$

Since $2-\alpha^2<0$ dividing reverses the inequality gives us

$$n > \frac{1-2\alpha}{2-\alpha^2}$$

This contradicts $\alpha^2 > 2$ since we have shown n can be picked such that $(\alpha^2 - 1/n) > 2$ meaning α is not the least upper bound.

Exercise 1.4.8
Give an example of each or state that the request is impossible. When a request is impossible, provide a compelling argument for why this is the case.

(a) Two sets A and B with $A \cap B = \emptyset, \sup A = \sup B, \sup A \notin A$ and $\sup B \notin B$.

(b) A sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.

(c) A sequence of nested unbounded closed intervals $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} L_n = \emptyset$. (An unbounded closed interval has the form $[a,\infty) = \{x \in R : x \geq a\}$.)

(d) A sequence of closed bounded (not necessarily nested) intervals $I_1, I_2, I_3, \ldots$ with the property that $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for all $N \in \mathbf{N}$, but $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

Solution

(a) $A = \mathbf{Q} \cap (0,1)$, $B = \mathbf{I} \cap (0,1)$. $A \cap B = \emptyset$, $\sup A = \sup B = 1$ and $1 \notin A$, $1 \notin B$.

(b) Defining $J_i = (a_i, b_i)$, $A = \{a_n : n \in \mathbf{N}\}$, $B = \{b_n : n \in \mathbf{N}\}$, $\bigcap_{n=1}^{\infty} J_n$ will at least contain $(\sup A, \inf B)$. Thus, a necessary condition to meet the request is $\sup A = \inf B$.

   $J_i = (-1/n, 1/n)$ satisfies this condition ($\sup A = \inf B = 0$) and by inspection, $\bigcap_{n=1}^{\infty} J_n = \{0\}$, which meets the request.

(c) $L_n = [n, \infty)$ has $\bigcap_{n=1}^{\infty} L_n = \emptyset$

(d) Impossible. Let $J_n = \bigcap_{k=1}^{n} I_k$ and observe the following

   (i) Since $\bigcap_{n=1}^{N} I_n \neq \emptyset$ we have $J_n \neq \emptyset$.

   (ii) J_n being the intersection of closed intervals makes it a closed interval.

   (iii) $J_{n+1} \subseteq J_n$ since $I_{n+1} \cap J_n \subseteq J_n$

   (iv) $\bigcap_{n=1}^{\infty} J_n = \bigcap_{n=1}^{\infty} (\bigcap_{k=1}^{n} I_k) = \bigcap_{n=1}^{\infty} I_n$

   By (i), (ii) and (iii) the Nested Interval Property tells us $\bigcap_{n=1}^{\infty} J_n \neq \emptyset$. Therefore by (iv) $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

1.5 Cardinality

Exercise 1.5.1
Finish the following proof for Theorem 1.5.7. Assume B is a countable set. Thus, there exists $f : \mathbf{N} \to B$, which is $1-1$ and onto. Let $A \subseteq B$ be an infinite subset of B. We must show that A is countable.

Let $n_1 = \min\{n \in \mathbf{N} : f(n) \in A\}$. As a start to a definition of $g : \mathbf{N} \to A$ set $g(1) = f(n_1)$. Show how to inductively continue this process to produce a 1-1 function g from $\mathbf{N}$ onto A.

Solution
Let $n_k = \min\{n \in \mathbf{N} \mid f(n) \in A, n \notin \{n_1, n_2, \dots, n_{k-1}\}\}$ and $g(k) = f(n_k)$. since $g : \mathbf{N} \to A$ is 1-1 and onto, A is countable.

Exercise 1.5.2
Review the proof of Theorem 1.5.6, part (ii) showing that $\mathbf{R}$ is uncountable, and then find the flaw in the following erroneous proof that $\mathbf{Q}$ is uncountable:

Assume, for contradiction, that $\mathbf{Q}$ is countable. Thus we can write $\mathbf{Q} = \{r_1, r_2, r_3, \dots\}$ and, as before, construct a nested sequence of closed intervals with $r_n \notin I_n$. Our construction implies $\bigcap_{n=1}^{\infty} I_n = \emptyset$ while NIP implies $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. This contradiction implies Q must therefore be uncountable.

Solution
The nested interval property is not true for $\mathbf{Q}$. Consider I_n being rational bounds for $\sqrt{2}$ with n decimal places, then $\bigcap_{n=1}^{\infty} I_n = \emptyset$ since $\sqrt{2} \notin \mathbf{Q}$.

Exercise 1.5.3

(a) Prove if $A_1, \dots, A_m$ are countable sets then $A_1 \cup \cdots \cup A_m$ is countable.

(b) Explain why induction *cannot* be used to prove that if each A_n is countable, then $\bigcup_{n=1}^{\infty} A_n$ is countable.

(c) Show how arranging $\mathbf{N}$ into the two-dimensional array

$$\begin{array}{llllll} 1 & 3 & 6 & 10 & 15 & \cdots \\ 2 & 5 & 9 & 14 & \cdots & \\ 4 & 8 & 13 & \cdots & & \\ 7 & 12 & \cdots & & & \\ 11 & \cdots & & & & \\ \vdots & & & & & \end{array}$$

leads to a proof for the infinite case.

Solution

(a) Let B, C be disjoint countable sets. We use the same trick as with the integers and list them as

$$B \cup C = \{b_1, c_1, b_2, c_2, \dots\}$$

Meaning $B \cup C$ is countable, and $A_1 \cup A_2$ is also countable since we can let $B = A_1$ and $C = A_2 \setminus A_1$.

Now induction: suppose $A_1 \cup \cdots \cup A_n$ is countable, $(A_1 \cup \cdots \cup A_n) \cup A_{n+1}$ is the union of two countable sets which by above is countable.

(b) Induction shows something for each $n \in \mathbf{N}$, it does not apply in the infinite case.

(c) Rearranging $\mathbf{N}$ as in (c) gives us disjoint sets C_n such that $\bigcup_{n=1}^{\infty} C_n = \mathbf{N}$. Let B_n be disjoint, constructed as $B_1 = A_1, B_2 = A_1 \setminus B_1, \ldots$ we want to do something like

$$f(\mathbf{N}) = f\left(\bigcup_{n=1}^{\infty} C_n\right) = \bigcup_{n=1}^{\infty} f_n(C_n) = \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$$

Let $f_n : C_n \to B_n$ be bijective since B_n is countable, define $f : \mathbf{N} \to \bigcup_{n=1}^{\infty} B_n$ as

$$f(n) = \begin{cases} f_1(n) & \text{if } n \in C_1 \\ f_2(n) & \text{if } n \in C_2 \\ \vdots \end{cases}$$

(i) Since each C_n is disjoint and each f_n is 1-1, $f(n_1) = f(n_2)$ implies $n_1 = n_2$ meaning f is 1-1.

(ii) Since any $b \in \bigcup_{n=1}^{\infty} B_n$ has $b \in B_n$ for some n, we know $b = f_n(x)$ has a solution since f_n is onto. Letting $x = f_n^{-1}(b)$ we have $f(x) = f_n(x) = b$ since $f_n^{-1}(b) \in C_n$ meaning f is onto.

By (i) and (ii) f is bijective and so $\bigcup_{n=1}^{\infty} B_n$ is countable. And since

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$$

We have that $\bigcup_{n=1}^{\infty} A_n$ is countable, completing the proof.

Exercise 1.5.4

(a) Show $(a, b) \sim \mathbf{R}$ for any interval (a, b).

(b) Show that an unbounded interval like $(a, \infty) = \{x : x > a\}$ has the same cardinality as $\mathbf{R}$ as well.

(c) Using open intervals makes it more convenient to produce the required 1-1, onto functions, but it is not really necessary. Show that $[0, 1) \sim (0, 1)$ by exhibiting a 1-1 onto function between the two sets.

Solution

(a) We will start by finding $f : (-1, 1) \to \mathbf{R}$ and then transform it to (a, b). Example 1.5.4 gives a suitable f

$$f(x) = \frac{x}{x^2 - 1}$$

The book says to use calculus to show f is bijective, first we will examine the derivative

$$f'(x) = \frac{x^2 - 1 - 2x^2}{(x^2-1)^2} = -\frac{x^2+1}{(x^2-1)^2}$$

The denominator and numerator are positive, so $f'(x) < 0$ for all $x \in (0,1)$. This means no two inputs will be mapped to the same output, meaning f is one to one (a rigorous proof is beyond our current ability)

To show that f is onto, we examine the limits

$$\lim_{x \to 1^-} \frac{x}{x^2-1} = -\infty$$
$$\lim_{x \to -1^+} \frac{x}{x^2-1} = +\infty$$

Then use the intermediate value theorem to conclude f is onto.

Now we shift f to the interval (a,b)

$$g(x) = f\left(\frac{2x-1}{b-a} - a\right)$$

Proving $g(x)$ is also bijective is a straightforward application of the chain rule.

(b) We want a bijective $h(x)$ such that $h(x) : (a,\infty) \to (-1,1)$ because then we could compose them to get a new bijective function $f(h(x)) : (a,\infty) \to \mathbf{R}$.

Let

$$h(x) = \frac{2}{x-a+1} - 1$$

We have $h : (a,\infty) \to (1,-1)$ since $h(a) = 1$ and $\lim_{x\to\infty} h(x) = 1$.

Meaning that $f(h(x)) : (a,\infty) \to \mathbf{R}$ is our bijective map.

(c) With countable sets adding a single element doesn't change cardinality since we can just shift by one to get a bijective map. we'll use a similar technique here to essentially outrun our problems. Define $f : [0,1) \to (0,1)$ as

$$f(x) = \begin{cases} 1/2 & \text{if } x = 0 \\ 1/4 & \text{if } x = 1/2 \\ 1/8 & \text{if } x = 1/4 \\ \vdots & \\ x & \text{otherwise} \end{cases}$$

Now we prove f is bijective by showing $y = f(x)$ has exactly one solution for all $y \in (0,1)$.

If $y = 1/2^n$ then the only solution is $y = f(1/2^{n-1})$ (or $x = 0$ in the special case $n = 1$), If $y \neq 1/2^n$ then the only solution is $y = f(y)$.

Exercise 1.5.5

(a) Why is $A \sim A$ for every set A?

(b) Given sets A and B, explain why $A \sim B$ is equivalent to asserting $B \sim A$.

(c) For three sets A, B, and C, show that $A \sim B$ and $B \sim C$ implies $A \sim C$. These three properties are what is meant by saying that $\sim$ is an *equivalence relation.*

Solution

(a) The identity function $f(x) = x$ is a bijection

(b) If $f : A \to B$ is bijective then $f^{-1} : B \to A$ is bijective.

(c) Let $f : A \to B$ and $g : B \to C$, since $g \circ f : A \to C$ is bijective we have $A \sim C$.

Exercise 1.5.6

(a) Give an example of a countable collection of disjoint open intervals.

(b) Give an example of an uncountable collection of disjoint open intervals, or argue that no such collection exists.

Solution

(a) $I_1 = (0, 1)$, $I_2 = (1, 2)$ and in general $I_n = (n - 1, n)$

(b) Let A denote this set. Intuitively no such collection should exist since each I_n has nonzero length.

The key here is to try and show $A \sim \mathbf{Q}$ instead of directly showing $A \sim \mathbf{N}$.

For any nonempty interval I_n the density theorem tells us there exists an $r \in \mathbf{Q}$ such that $r \in I_n$. Assigning each $I \in A$ a rational number $r \in I$ proves $I \subseteq \mathbf{Q}$ and thus I is countable.

Exercise 1.5.7

Consider the open interval $(0, 1)$, and let S be the set of points in the open unit square; that is, $S = \{(x, y) : 0 < x, y < 1\}$.

(a) Find a 1-1 function that maps $(0, 1)$ into, but not necessarily onto, S. (This is easy.)

(b) Use the fact that every real number has a decimal expansion to produce a $1-1$ function that maps S into $(0, 1)$. Discuss whether the formulated function is onto. (Keep in mind that any terminating decimal expansion such as .235 represents the same real number as $.234999\ldots$.)

The Schröder-Bernstein Theorem discussed in Exercise 1.5.11 can now be applied to conclude that $(0, 1) \sim S$.

Solution

(a) We scale and shift up into the square. $f(x) = \frac{1}{2}x + \frac{1}{3}$

(b) Let $g : S \to (0,1)$ be a function that interleaves decimals in the representation without trailing nines, padding with zeros if necessary. $g(0.32, 0.45) = 0.3425$, $g(0.1\bar{9}, 0.2) = g(0.2, 0.2) = 0.22$, $g(0.1, 0.23) = 0.1203$, $g(0.1, 0.\bar{2}) = 0.12\overline{02}$, etc.

Every real number can be written with two digit representations, one with trailing 9's and one without. However $g(x, y) = 0.d_1d_2\ldots\bar{9}$ is impossible since it would imply $x = 0.d_1\ldots\bar{9}$ and $y = 0.d_2\ldots\bar{9}$ but the definition of g forbids this. therefore $g(s)$ is unique, and so g is 1-1.

Is g onto? No since $g(x, y) = 0.1$ has no solutions, since we would want $x = 0.1$ and $y = 0$ but $0 \notin (0, 1)$.

Exercise 1.5.8
Let B be a set of positive real numbers with the property that adding together any finite subset of elements from B always gives a sum of 2 or less. Show B must be finite or countable.

Solution
Notice $B \cap (a, 2)$ is finite for all $a > 0$, since if it was infinite we could make a set with sum greater then two. And since B is the countable union of finite sets $\bigcup_{n=1}^{\infty} B \cap (1/n, 2)$, B must be countable or finite.

Exercise 1.5.9
A real number $x \in \mathbf{R}$ is called algebraic if there exist integers $a_0, a_1, a_2, \ldots, a_n \in \mathbf{Z}$, not all zero, such that

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$$

Said another way, a real number is algebraic if it is the root of a polynomial with integer coefficients. Real numbers that are not algebraic are called *transcendental* numbers. Reread the last paragraph of Section 1.1. The final question posed here is closely related to the question of whether or not transcendental numbers exist.

(a) Show that $\sqrt{2}$, $\sqrt[3]{2}$, and $\sqrt{3} + \sqrt{2}$ are algebraic.

(b) Fix $n \in \mathbf{N}$, and let A_n be the algebraic numbers obtained as roots of polynomials with integer coefficients that have degree n. Using the fact that every polynomial has a finite number of roots, show that A_n is countable.

(c) Now, argue that the set of all algebraic numbers is countable. What may we conclude about the set of transcendental numbers?

Solution
(a) $x^2 - 2 = 0$, $x^3 - 2 = 0$ are obvious. Now consider $\sqrt{3} + \sqrt{2}$. The key is setting $x = \sqrt{3}+\sqrt{2}$ then using algebra on x to concoct an integer, and thus find the polynomial with x as a root.

We have $x^2 = 5+2\sqrt{6}$ meaning $x^2-5 = 2\sqrt{6}$ and thus $(x^2-5)^2 = 24$ so $(x^2-5)^2-24 = 0$ is a polynomial with $\sqrt{3} + \sqrt{2}$ as a root.

(b) Basically $A_n \sim \mathbf{Z}^n \sim \mathbf{N}^n \sim \mathbf{N}$.

(i) $A_n \sim \mathbf{Z}^n$ since integer polynomials of degree n are identical to an ordered list of n integers.

(ii) $\mathbf{Z}^n \sim \mathbf{N}^n$ since $f : \mathbf{N}^n \to \mathbf{Z}^n$ is just the piecewise application of $g : \mathbf{N} \to \mathbf{Z}$.

(iii) $\mathbf{N}^n \sim \mathbf{N}$ since it is the intersection of finite sets $\bigcup_{n=2}^{\infty} \{(a,b) : a+b=n\}$.

In general if V is countable then $V^n = (v_1, \dots, v_n)$ is also countable.

(c) By 1.5.3 the set of all algebraic numbers $\bigcup_{n=1}^{\infty} A_n$ is countable.

Exercise 1.5.10

(a) Let $C \subseteq [0,1]$ be uncountable. Show that there exists $a \in (0,1)$ such that $C \cap [a,1]$ is uncountable.

(b) Now let A be the set of all $a \in (0,1)$ such that $C \cap [a,1]$ is uncountable, and set $\alpha = \sup A$. Is $C \cap [\alpha, 1]$ an uncountable set?

(c) Does the statement in (a) remain true if "uncountable" is replaced by "infinite"?

Solution

(a) Suppose a does not exist, then $C \cap [a,1]$ is countable for all $a \in (0,1)$ meaning

$$\bigcup_{n=1}^{\infty} C \cap [1/n, 1] = C \cap [0,1]$$

Is countable (by 1.5.3), contradicting our assumption that $C \cap [0,1]$ is uncountable.

(b) If $\alpha = 1$ then $C \cap [\alpha, 1]$ is finite. Now if $\alpha < 1$ we have $C \cap [\alpha + \epsilon, 1]$ countable for $\epsilon > 0$ (otherwise the set would be in A, and hence α would not be an upper bound). Take

$$\bigcup_{n=1}^{\infty} C \cap [\alpha + 1/n, 1] = C \cap [\alpha, 1]$$

Which is countable by 1.5.3.

(c) No, consider the set $C = \{1/n : n \in \mathbf{N}\}$ it has $C \cap [\alpha, 1]$ finite for every α, but $C \cap [0,1]$ is infinite.

Exercise 1.5.11 (Schröder-Bernstein Theorem)

Assume there exists a 1-1 function $f : X \to Y$ and another 1-1 function $g : Y \to X$. Follow the steps to show that there exists a 1-1, onto function $h : X \to Y$ and hence $X \sim Y$. The strategy is to partition X and Y into components

$$X = A \cup A' \quad \text{and} \quad Y = B \cup B'$$

with $A \cap A' = \emptyset$ and $B \cap B' = \emptyset$, in such a way that f maps A onto B, and g maps B' onto A'.

(a) Explain how achieving this would lead to a proof that $X \sim Y$.

(b) Set $A_1 = X \setminus g(Y) = \{x \in X : x \notin g(Y)\}$ (what happens if $A_1 = \emptyset$?) and inductively define a sequence of sets by letting $A_{n+1} = g(f(A_n))$. Show that $\{A_n : n \in \mathbf{N}\}$ is a pairwise disjoint collection of subsets of X, while $\{f(A_n) : n \in \mathbf{N}\}$ is a similar collection in Y.

(c) Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} f(A_n)$. Show that f maps A onto B.

(d) Let $A' = X \setminus A$ and $B' = Y \setminus B$. Show g maps B' onto A'.

Solution

(a) $f : A \to B$ and $g : B' \to A'$ are bijective, therefore we can define

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g^{-1}(x) & \text{if } x \in A' \end{cases}$$

which is bijective.

(b) If $A_1 = \emptyset$ then $g : Y \to X$ is 1-1 and onto so we are done. So assume $A_1 \neq \emptyset$, to show $\{A_n\}$ is pairwise disjoint, first consider how $A_1 \cap A_k = \emptyset$ since $A_1 = X \setminus g(Y)$ and $g(f(A_1)) \subseteq g(Y)$. Define $h(x) = g(f(x))$

Since h is injective we have $h(A \cap B) = h(A) \cap h(B)$ for all A, B in X. (Proof left as an exercise to the reader.) Using this we can prove pairwise disjointness, let $j > k$ and use the iterated function notation $h^2 = h \circ h$ and note that h^k is injective.

$$A_{j+1} \cap A_{k+1} = h^k(A_{j-k}) \cap h^k(A_1) = h^k(A_{j-k} \cap A_1) = h^k(\emptyset) = \emptyset$$

And since f is injective $f(A_j) \cap f(A_k) = f(A_j \cap A_k) = f(\emptyset) = \emptyset$.

(c) $f(A) = B$ because $f\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} f(A_n)$ thus $f : A \to B$ is onto. (B was basically defined as the range of f)

(d) We show inclusion both ways by deriving contradictions. Key facts we use: (i) $A_1 \cap g(Y) = \emptyset$ (ii) $g(B) = \bigcup_{n=2}^{\infty} A_n = A \setminus A_1 = A \cap g(Y)$

(i) $g(B') \subseteq A'$. SFC that $g(b') \in A$. Because $A_1 \cap g(Y) = \emptyset$, $g(b) \notin A_1$ meaning $g(b') \in \bigcup_{n=2}^{\infty} A_n = g(B)$, meaning $\exists b \in B$ with $g(b') = g(b)$ and $b' \neq b$, contradicting g being 1-1.

(ii) $A' \subseteq g(B')$. SFC $\exists a' \in A'$ with $a' \notin g(B')$. Because $A' \subseteq g(Y)$ we have $a' \in g(B)$ (since $a' \notin g(B')$) and $g(B) \subseteq A$ contradicting $a' \in A'$ (we can't have $a' \in A'$ and $a' \in A$.)

1.6 Cantor's theorem

Exercise 1.6.1
Show that $(0,1)$ is uncountable if and only if $\mathbf{R}$ is uncountable.

Solution
In Exercise 1.5.4 (a) we found a bijection $f : (0,1) \to \mathbf{R}$. Now suppose $g : (0,1) \to \mathbf{N}$ is some map, we must show g is bijective if and only if $(g \circ f) : \mathbf{R} \to \mathbf{N}$ is bijective. This is clearly true as if g is bijective then $(g \circ f)$ is bijective (composition of bijective functions), and if $(g \circ f)$ is bijective then $(g \circ f) \circ f^{-1} = g$ is bijective.

Exercise 1.6.2
Let $f : \mathbf{N} \to \mathbf{R}$ be a way to list every real number (hence show $\mathbf{R}$ is countable).

Define a new number x with digits $b_1 b_2 \dots$ given by

$$b_n = \begin{cases} 2 & \text{if } a_{nn} \neq 2 \\ 3 & \text{if } a_{nn} = 2 \end{cases}$$

(a) Explain why the real number $x = .b_1 b_2 b_3 b_4 \dots$ cannot be $f(1)$.

(b) Now, explain why $x \neq f(2)$, and in general why $x \neq f(n)$ for any $n \in \mathbf{N}$.

(c) Point out the contradiction that arises from these observations and conclude that $(0,1)$ is uncountable.

Solution
(a) The first digit is different

(b) The nth digit is different

(c) Therefore x is not in the list, since the nth digit is different

Exercise 1.6.3
Supply rebuttals to the following complaints about the proof of Theorem 1.6.1.

(a) Every rational number has a decimal expansion, so we could apply this same argument to show that the set of rational numbers between 0 and 1 is uncountable. However, because we know that any subset of $\mathbf{Q}$ must be countable, the proof of Theorem 1.6.1 must be flawed.

(b) Some numbers have two different decimal representations. Specifically, any decimal expansion that terminates can also be written with repeating 9's. For instance, $1/2$ can be written as $.5$ or as $.4999\dots$ Doesn't this cause some problems?

Solution
(a) False, since the constructed number has an infinite number of decimals it is irrational.

(b) No, since if we have $9999\dots$ and change the nth digit $9992999 = 9993$ is still different.

Exercise 1.6.4
Let S be the set consisting of all sequences of 0 's and 1 's. Observe that S is not a particular sequence, but rather a large set whose elements are sequences; namely,

$$S = \{(a_1, a_2, a_3, \ldots) : a_n = 0 \text{ or } 1\}$$

As an example, the sequence $(1, 0, 1, 0, 1, 0, 1, 0, \ldots)$ is an element of S, as is the sequence $(1, 1, 1, 1, 1, 1, \ldots)$. Give a rigorous argument showing that S is uncountable.

Solution
We flip every bit in the diagonal just like with $\mathbf{R}$. Another way would be to show $S \sim \mathbf{R}$ by writing real numbers in base 2.

Exercise 1.6.5
(a) Let $A = \{a, b, c\}$. List the eight elements of $P(A)$. (Do not forget that $\emptyset$ is considered to be a subset of every set.)

(b) If A is finite with n elements, show that $P(A)$ has 2^n elements.

Solution
(a) $A = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

(b) There are n elements, we can include or exclude each element so there are 2^n subsets.

Exercise 1.6.6
(a) Using the particular set $A = \{a, b, c\}$, exhibit two different $1-1$ mappings from A into $P(A)$.

(b) Letting $C = \{1, 2, 3, 4\}$, produce an example of a $1-1$ map $g : C \to P(C)$.

(c) Explain why, in parts (a) and (b), it is impossible to construct mappings that are onto.

Solution
(a) $f(x) = \{x\}$, $f(x) = \{x, b\}$ for $x \neq b$ and $f(x) = \{a, b, c\}$ for $x = b$.

(b) $f(x) = \{x\}$.

(c) We can hit at most n elements in the power set out of the 2^n total elements.

Theorem (Cantor's Theorem)
Given any set A, there does not exist a function $f : A \to P(A)$ that is onto.

Proof
Suppose $f : A \to P(A)$ is onto. We want to use the self referential nature of $P(A)$ to find a contradiction. Define

$$B = \{a : a \notin f(a)\}$$

Since f is onto we must have $f(a) = B$ for some $a \in A$. Then either

(i) $a \in B$ implies $a \in f(a)$ which by the definition of B implies $a \notin B$, so $a \in B$ is impossible.

(ii) $a \notin B$ implies $a \notin f(a)$ since $f(a) = B$. but if $a \notin f(a)$ then $a \in B$ by the definition of B, contradicting $a \notin B$.

Therefore f cannot be onto, since we have found a $B \in P(A)$ where $f(a) = B$ is impossible.

Stepping back, the pearl of the argument is that if $B = f(a)$ then $B = \{a : a \notin B\}$ is undecidable/impossible.

Exercise 1.6.7
See the proof of Cantor's theorem above (the rest is a computation)

Exercise 1.6.8
See the proof of Cantor's theorem above

Exercise 1.6.9
Using the various tools and techniques developed in the last two sections (including the exercises from Section 1.5), give a compelling argument showing that $P(\mathbf{N}) \sim \mathbf{R}$.

Solution
Recall from Exercise 1.6.4 that if

$$S = \{(a_1, a_2, a_3, \ldots) : a_n = 0 \text{ or } 1\}$$

then $S \sim \mathbf{R}$. Define $f : P(\mathbf{N}) \to S$ as $f(x) = (a_1, a_2, \ldots)$ where $a_i = 1$ if $i \in x$ and $a_i = 0$ otherwise. f is thus a one-to-one, onto map between $P(\mathbf{N})$ and S, hence $P(\mathbf{N}) \sim S$. Since $\sim$ is an equivalence relation, $P(\mathbf{N}) \sim \mathbf{R}$.

Exercise 1.6.10
As a final exercise, answer each of the following by establishing a $1-1$ correspondence with a set of known cardinality.

(a) Is the set of all functions from $\{0, 1\}$ to $\mathbf{N}$ countable or uncountable?

(b) Is the set of all functions from $\mathbf{N}$ to $\{0, 1\}$ countable or uncountable?

(c) Given a set B, a subset $\mathcal{A}$ of $P(B)$ is called an antichain if no element of $\mathcal{A}$ is a subset of any other element of $\mathcal{A}$. Does $P(\mathbf{N})$ contain an uncountable antichain?

Solution

(a) The set of functions from $\{0, 1\}$ to $\mathbf{N}$ is the same as $\mathbf{N}^2$ which we found was countable in Exercise 1.5.9.

(b) This is the same as an infinite list of zeros and ones which we showed was uncountable in Exercise 1.6.4.

(c) Let $\mathcal{A}$ be an antichain of $P(\mathbf{N})$, let $\mathcal{A}_l$ be the sets in $\mathcal{A}$ of size l. For finite l, $\mathcal{A}_l$ is countable since $\mathcal{A}_l \subseteq \mathbf{N}^l$ is countable (shown in 1.5.9). Therefore the countable union $\bigcup_{l=0}^{\infty} \mathcal{A}_l = \mathcal{A}$ is countable. Thus, if $P(\mathbf{N})$ contains an uncountable antichain, "most" sets in the antichain must be infinite (in that there will be uncountably many sets in the antichain will be infinite, whereas only countably many sets in the antichain will be finite).

This observation inspires the following construction, using a variant of the set S from Exercise 1.6.4. Define the set

$$\mathcal{A} = \{\{10n + d(x, n) : n \in \mathbf{N}\} : x \in (0, 1)\}$$

where $d(x, n)$ is the n'th digit in the decimal expansion of x. To avoid the issue of some numbers having two equivalent decimal representations, always use the representation with repeating 9's. In this manner, each element of $\mathcal{A}$ encodes a particular real number, in a similar way that each element of S encodes a particular real number through its binary expansion.

Note that each element of $\mathcal{A}$ is infinite. Note also that since any two distinct real numbers will differ in at least one place in their decimal expansions, the corresponding elements in $\mathcal{A}$ will differ there as well, and hence $\mathcal{A}$ is an antichain.

Formally, let x_1, x_2 be two distinct real numbers, A_1, A_2 be the elements of $\mathcal{A}$ corresponding to x_1, x_2 respectively, and n be the first decimal position where x_1 and x_2 differ. Then $10n + d(x_1, n)$ will be in A_1 but not A_2, and $10n + d(x_2, n)$ will be in A_2 but not A_1. Thus, neither $A_1 \subseteq A_2$ nor $A_2 \subseteq A_1$. Since $(0, 1)$ is uncountable, $\mathcal{A}$ is an uncountable antichain in $P(\mathbf{N})$.

Chapter 2

Sequences and Series

2.2 The Limit of a Sequence

Exercise 2.2.1
What happens if we reverse the order of the quantifiers in Definition 2.2.3?

Definition: A sequence (x_n) *verconges* to x if *there exists* an $\epsilon > 0$ such that *for all* $N \in \mathbf{N}$ it is true that $n \geq N$ implies $|x_n - x| < \epsilon$

Give an example of a vercongent sequence. Is there an example of a vercongent sequence that is divergent? Can a sequence verconge to two different values? What exactly is being described in this strange definition?

Solution
Firstly, since we have *for all* $N \in \mathbf{N}$ we can remove N entirely and just say $n \in \mathbf{N}$. Our new definition is

Definition: A sequence (x_n) *verconges* to x if *there exists* an $\epsilon > 0$ such that *for all* $n \in \mathbf{N}$ we have $|x_n - x| < \epsilon$.

In other words, a series (x_n) *verconges* to x if $|x_n - x|$ is bounded. This is a silly definition though since if $|x_n - x|$ is bounded, then $|x_n - x'|$ is bounded for all $x' \in \mathbf{R}$, meaning if a sequence is vercongent it verconges to every $x' \in \mathbf{R}$.

Put another way, a sequence is vercongent *if and only if* it is bounded.

Exercise 2.2.2
Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

(a) $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$.

(b) $\lim \frac{2n^2}{n^3+3} = 0$.

(c) $\lim \frac{\sin\left(n^2\right)}{\sqrt[3]{n}} = 0$.

Solution
(a) We have

$$\left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| = \left|\frac{5(2n+1) - 2(5n+4)}{5(5n+4)}\right| = \left|\frac{-3}{5(5n+4)}\right| = \frac{3}{5(5n+4)} < \epsilon$$

We now find n such that the distance is less then ϵ

$$\frac{3}{5(5n+4)} < \frac{1}{n} < \epsilon \implies n > \frac{1}{\epsilon}$$

You could also solve for the smallest n, which would give you

$$\frac{3}{5(5n+4)} < \epsilon \implies 5n+4 > \frac{3}{5\epsilon} \implies n > \frac{3}{25\epsilon} - \frac{4}{5}$$

I prefer the first approach, the second is better if you were doing numerical analysis and wanted a precise convergence rate.

(b) We have

$$\left|\frac{2n^2}{n^3+3} - 0\right| = \frac{2n^2}{n^3+3} < \frac{2n^2}{n^3} = \frac{2}{n} < \epsilon \implies n > \frac{2}{\epsilon}$$

(c) We have

$$\frac{\sin(n^2)}{n^{1/3}} \leq \frac{1}{n^{1/3}} < \epsilon \implies n > \frac{1}{\epsilon^3}$$

Really slow convergence! if $\epsilon = 10^{-2}$ we would require $n > 10^6$

Exercise 2.2.3

Describe what we would have to demonstrate in order to disprove each of the following statements.

(a) At every college in the United States, there is a student who is at least seven feet tall.

(b) For all colleges in the United States, there exists a professor who gives every student a grade of either A or B.

(c) There exists a college in the United States where every student is at least six feet tall.

Solution

(a) Find a collage in the United States with no students over seven feet tall.

(b) Find a collage in the United States with no professors that only give grades of A or B.

(c) Show that all collages in the United States have at least one student under six feet tall.

Exercise 2.2.4

Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

(a) A sequence with an infinite number of ones that does not converge to one.

(b) A sequence with an infinite number of ones that converges to a limit not equal to one.

(c) A divergent sequence such that for every $n \in \mathbf{N}$ it is possible to find n consecutive ones somewhere in the sequence.

Solution

(a) $a_n = (-1)^n$

(b) Impossible, if $\lim a_n = a \neq 1$ then for any $n \geq N$ we can find a n with $a_n = 1$ meaning $\epsilon < |1 - a|$ is impossible.

(c) $a_n = (1, 2, 1, 1, 3, 1, 1, 1, \dots)$

Exercise 2.2.5

Let $[[x]]$ be the greatest integer less than or equal to x. For example, $[[\pi]] = 3$ and $[[3]] = 3$. For each sequence, find $\lim a_n$ and verify it with the definition of convergence.

(a) $a_n = [[5/n]]$,

(b) $a_n = [[(12 + 4n)/3n]]$.

Reflecting on these examples, comment on the statement following Definition 2.2.3 that "the smaller the ϵ-neighborhood, the larger N may have to be."

Solution

(a) For all $n > 5$ we have $[[5/n]] = 0$ meaning $\lim a_n = 0$.

(b) The inside clearly converges to $4/3$ from above, so $\lim a_n = 1$.

Some sequences eventually reach their limit, meaning N no longer has to increase.

Exercise 2.2.6

Theorem 2.2.7 (Uniqueness of Limits). *The limit of a sequence, when it exists, must be unique.*

Prove Theorem 2.2.7. To get started, assume $(a_n) \to a$ and also that $(a_n) \to b$. Now argue $a = b$

Solution

If $a \neq b$ then we can set ϵ small enough that having both $|a_n - a| < \epsilon$ and $|a_n - b| < \epsilon$ is impossible. Therefore $a = b$.

(Making this rigorous is trivial and left as an exercise to the reader)

Exercise 2.2.7

Here are two useful definitions:

(i) A sequence (a_n) is *eventually* in a set $A \subseteq \mathbf{R}$ if there exists an $N \in \mathbf{N}$ such that $a_n \in A$ for all $n \geq N$.

(ii) A sequence (a_n) is *frequently* in a set $A \subseteq \mathbf{R}$ if, for every $N \in \mathbf{N}$, there exists an $n \geq N$ such that $a_n \in A$.

  (a) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?

  (b) Which definition is stronger? Does frequently imply eventually or does eventually imply frequently?

  (c) Give an alternate rephrasing of Definition 2.2.3B using either frequently or eventually. Which is the term we want?

(d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2 . Is (x_n) necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?

Solution

(a) Frequently, but not eventually.

(b) Eventually is stronger, it implies frequently.

(c) $(x_n) \to x$ *if and only if* x_n is eventually in any ϵ-neighborhood around x.

(d) (x_n) is frequently in $(1.9, 2.1)$ but not necessarily eventually (consider $x_n = 2(-1)^n$).

Exercise 2.2.8

For some additional practice with nested quantifiers, consider the following invented definition:

Let's call a sequence (x_n) zero-heavy if there exists $M \in \mathbf{N}$ such that for all $N \in \mathbf{N}$ there exists n satisfying $N \leq n \leq N + M$ where $x_n = 0$

(a) Is the sequence $(0, 1, 0, 1, 0, 1, \ldots)$ zero heavy?

(b) If a sequence is zero-heavy does it necessarily contain an infinite number of zeros? If not, provide a counterexample.

(c) If a sequence contains an infinite number of zeros, is it necessarily zeroheavy? If not, provide a counterexample.

(d) Form the logical negation of the above definition. That is, complete the sentence: A sequence is not zero-heavy if ....

Solution

(a) Yes. Choose $M = 1$; since the sequence has a 0 in every two spaces, for all N either $x_n = 0$ or $x_{n+1} = 0$.

(b) Yes. If there were a finite number of zeros, with the last zero at position K, then choosing $N > K$ would lead to a contradiction.

(c) No, consider $(0, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, \ldots)$ where the gap between 0's grows indefinitely. For any value of M, for large enough N the gap between zeros will be greater than M. Then we simply choose N so that x_N is the first 1 in a streak of at least $M + 1$ 1's.

(d) A sequence is not zero-heavy if for all $M \in \mathbf{N}$, there exists some $N \in \mathbf{N}$ such that for all $n \in \mathbf{N}, N \leq n \leq N + M$, $x_n \neq 0$.

2.3 The Algebraic and Order Limit Theorems

Exercise 2.3.1
Let $x_n \geq 0$ for all $n \in \mathbf{N}$.

(a) If $(x_n) \to 0$, show that $\left(\sqrt{x_n}\right) \to 0$.

(b) If $(x_n) \to x$, show that $\left(\sqrt{x_n}\right) \to \sqrt{x}$.

Solution
(a) Setting $x_n < \epsilon^2$ implies $\sqrt{x_n} < \epsilon$ (for all $n \geq N$ of course)

(b) We want $|\sqrt{x_n} - \sqrt{x}| < \epsilon$ multiplying by $(\sqrt{x_n} + \sqrt{x})$ gives $|x_n - x| < (\sqrt{x_n} + \sqrt{x})\epsilon$ since x_n is convergent, it is bounded $|x_n| \leq M$ implying $\sqrt{|x_n|} \leq \sqrt{M}$, multiplying gives

$$|x_n - x| < \left(\sqrt{x_n} + \sqrt{x}\right)\epsilon \leq \left(\sqrt{M} + \sqrt{x}\right)\epsilon$$

Since $|x_n - x|$ can be made arbitrarily small we can make this true for some $n \geq N$. Now dividing by $\sqrt{M} + \sqrt{x}$ gives us

$$|\sqrt{x_n} - \sqrt{x}| \leq \frac{|x_n - x|}{\sqrt{M} + \sqrt{x}} < \epsilon$$

therefore $|\sqrt{x_n} - \sqrt{x}| < \epsilon$ completing the proof.

Exercise 2.3.2
Using only Definition 2.2.3, prove that if $(x_n) \to 2$, then

(a) $\left(\frac{2x_n - 1}{3}\right) \to 1$;

(b) $(1/x_n) \to 1/2$.

(For this exercise the Algebraic Limit Theorem is off-limits, so to speak.)

Solution
(a) We have $\left|\frac{2}{3}x_n - \frac{4}{3}\right| = \frac{2}{3}|x_n - 2| < \epsilon$ which can always be done since $|x_n - 2|$ can be made arbitrarily small.

(b) Since x_n is bounded we have $|x_n| \leq M$

$$|1/x_n - 1/2| = \frac{|2 - x_n|}{|2x_n|} \leq \frac{|2 - x_n|}{|2M|} < \epsilon$$

Letting $|2 - x_n| < \epsilon/|2M|$ gives $|1/x_n - 1/2| < \epsilon$.

Exercise 2.3.3 (Squeeze Theorem)
Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbf{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

Solution

Let $\epsilon > 0$, set N so that $|x_n - l| < \epsilon/4$ and $|z_n - l| < \epsilon/4$. Use the triangle inequality to see $|x_n - z_n| < |x_n - l| + |l - z_n| < \epsilon/2$. Note that since $x_n \le y_n \le z_n$, $|y_n - x_n| = y_n - x_n \le z_n - x_n = |z_n - x_n|$. Apply the triangle inequality again to get

$$|y_n - l| \le |y_n - x_n| + |x_n - l| \le |z_n - x_n| + |x_n - l| < \epsilon/2 + \epsilon/4 < \epsilon$$

Exercise 2.3.4

Let $(a_n) \to 0$, and use the Algebraic Limit Theorem to compute each of the following limits (assuming the fractions are always defined):

(a) $\lim \left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right)$

(b) $\lim \left(\frac{(a_n+2)^2-4}{a_n}\right)$

(c) $\lim \left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right)$.

Solution

(a) Apply the ALT

$$\begin{aligned}\lim\left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right) &= \frac{\lim(1+2a_n)}{\lim(1+3a_n-4a_n^2)} \\ &= \frac{1+2\lim(a_n)}{1+3\lim a_n - 4\lim a_n^2} \\ &= 1\end{aligned}$$

Showing $a_n^2 \to 0$ is easy so I've omitted it

(b) Apply the ALT

$$\begin{aligned}\lim\left(\frac{(a_n+2)^2-4}{a_n}\right) &= \lim\left(\frac{a_n^2+2a_n}{a_n}\right) \\ &= \lim(a_n+2) = 2 + \lim a_n = 2\end{aligned}$$

(c) Multiply the top and bottom by a_n then apply the ALT

$$\begin{aligned}\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right) &= \lim\left(\frac{2+3a_n}{1+5a_n}\right) \\ &= \frac{2+3\lim a_n}{1+5\lim a_n} \\ &= 2\end{aligned}$$

Exercise 2.3.5

Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, x_3, y_3, \dots, x_n, y_n, \dots)$. Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

Solution
Obviously if $\lim x_n = \lim y_n = l$ then $z_n \to l$. To show the other way suppose $(z_n) \to l$, then $|z_n - l| < \epsilon$ for all $n \geq N$ meaning $|y_n - l| < \epsilon$ and $|x_n - l| < \epsilon$ for $n \geq N$ aswell. Thus $\lim x_n = \lim y_n = l$.

Exercise 2.3.6
Consider the sequence given by $b_n = n - \sqrt{n^2 + 2n}$. Taking $(1/n) \to 0$ as given, and using both the Algebraic Limit Theorem and the result in Exercise 2.3.1, show $\lim b_n$ exists and find the value of the limit.

Solution
I'm going to find the value of the limit before proving it. We have

$$n - \sqrt{n^2 + 2n} = n - \sqrt{(n+1)^2 - 1}$$

For large n, $\sqrt{(n+1)^2 - 1} \approx n + 1$ so $\lim b_n = -1$.

Factoring out n we get $n\left(1 - \sqrt{1 + 2/n}\right)$. Tempting as it is to apply the ALT here to say $(b_n) \to 0$ it doesn't work since n diverges.

How about if I get rid of the radical, then use the ALT to go back to what we had before?

$$(n - \sqrt{n^2 + 2n})(n + \sqrt{n^2 + 2n}) = n^2 - (n^2 + 2n) = -2n$$

Then we have

$$b_n = n - \sqrt{n^2 + 2n} = \frac{-2n}{n + \sqrt{n^2 + 2n}} = \frac{-2}{1 + \sqrt{1 + 2/n}}$$

Now we can finally use the algebraic limit theorem!

$$\lim\left(\frac{-2}{1 + \sqrt{1 + 2/n}}\right) = \frac{-2}{1 + \sqrt{1 + \lim(2/n)}} = \frac{-2}{1 + \sqrt{1 + 0}} = -1$$

Stepping back the key to this technique is removing the radicals via a difference of squares, then dividing both sides by the growth rate n and applying the ALT.

Exercise 2.3.7
Give an example of each of the following, or state that such a request is impossible by referencing the proper theorem(s):

(a) sequences (x_n) and (y_n), which both diverge, but whose sum $(x_n + y_n)$ converges;

(b) sequences (x_n) and (y_n), where (x_n) converges, (y_n) diverges, and $(x_n + y_n)$ converges;

(c) a convergent sequence (b_n) with $b_n \neq 0$ for all n such that $(1/b_n)$ diverges;

(d) an unbounded sequence (a_n) and a convergent sequence (b_n) with $(a_n - b_n)$ bounded;

(e) two sequences (a_n) and (b_n), where (a_nb_n) and (a_n) converge but (b_n) does not.

Solution
(a) $(x_n) = n$ and $(y_n) = -n$ diverge but $x_n + y_n = 0$ converges

(b) Impossible, the algebraic limit theorem implies $\lim(x_n+y_n)-\lim(x_n) = \lim y_n$ therefore (y_n) must converge if (x_n) and (x_n+y_n) converge.

(c) $b_n = 1/n$ has $b_n \to 0$ and $1/b_n$ diverges. If $b_n \to b \neq 0$ then $1/b_n \to 1/b$, but since $b = 0$ ALT doesn't apply.

(d) Impossible, $|b_n|$ is convergent and therefore bounded (Theorem 2.3.2) so $|b_n| \leq M_1$, and $|a_n - b_n| \leq M_2$ is bounded, therefore

$$|a_n| \leq |a_n - b_n| + |b_n| \leq M_1 + M_2$$

must be bounded.

(e) $b_n = n$ and $a_n = 0$ works. However if $(a_n) \to a$, $a \neq 0$ and $(a_n b_n) \to p$ then the ALT would imply $(b_n) \to p/a$.

Exercise 2.3.8
Let $(x_n) \to x$ and let $p(x)$ be a polynomial.

(a) Show $p(x_n) \to p(x)$.

(b) Find an example of a function $f(x)$ and a convergent sequence $(x_n) \to x$ where the sequence $f(x_n)$ converges, but not to $f(x)$.

Solution
(a) Applying the algebraic limit theorem multiple times gives $(x_n^d) \to x^d$ meaning

$$\lim p(x_n) = \lim \left(a_d x_n^d + a_{d-1} x_n^{d-1} + \cdots + a_0\right) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_0 = p(x).$$

As a cute corollary, any continuous function f has $\lim f(x_n) = f(x)$ since polynomials can approximate continuous functions arbitrarily well by the Weierstrass approximation theorem.

(b) Let $(x_n) = 1/n$ and define f as

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{otherwise} \end{cases}$$

We have $f(1/n) = 1$ for all n, meaning $\lim f(1/n) = 1$ but $f(0) = 0$.

Exercise 2.3.9
(a) Let (a_n) be a bounded (not necessarily convergent) sequence, and assume $\lim b_n = 0$. Show that $\lim(a_n b_n) = 0$. Why are we not allowed to use the Algebraic Limit Theorem to prove this?

(b) Can we conclude anything about the convergence of $(a_n b_n)$ if we assume that (b_n) converges to some nonzero limit b ?

(c) Use (a) to prove Theorem 2.3.3, part (iii), for the case when $a = 0$.

Solution

(a) We can't use the ALT since a_n is not necessarily convergent. a_n being bounded gives $|a_n| \leq M$ for some M giving

$$|a_n b_n| \leq M|b_n| < \epsilon$$

Which can be accomplished by letting $|b_n| < \epsilon/M$ since $(b_n) \to 0$.

(b) No

(c) In (a) we showed $\lim(a_n b_n) = 0 = ab$ for $b = 0$ which proves part (iii) of the ALT.

Exercise 2.3.10

Consider the following list of conjectures. Provide a short proof for those that are true and a counterexample for any that are false.

(a) If $\lim(a_n - b_n) = 0$, then $\lim a_n = \lim b_n$.

(b) If $(b_n) \to b$, then $|b_n| \to |b|$.

(c) If $(a_n) \to a$ and $(b_n - a_n) \to 0$, then $(b_n) \to a$.

(d) If $(a_n) \to 0$ and $|b_n - b| \leq a_n$ for all $n \in \mathbf{N}$, then $(b_n) \to b$.

Solution

(a) False, consider $a_n = n$ and $b_n = -n$.

(b) True since if $|b_n - b| < \epsilon$ then $||b_n| - |b|| \leq |b_n - b| < \epsilon$ by Exercise 1.2.6 (d).

(c) True by ALT since $\lim(b_n - a_n) + \lim a_n = \lim b_n = a$.

(d) True, since $0 \leq |b_n - b| \leq a_n$ we have $a_n \geq 0$. Let $\epsilon > 0$ and pick N such that $a_n < \epsilon$ for all $n \geq N$. Therefor

$$|b_n - b| \leq a_n < \epsilon$$

Proving $(b_n) \to b$.

Exercise 2.3.11 (Cesaro Means)

(a) Show that if (x_n) is a convergent sequence, then the sequence given by the averages

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

also converges to the same limit.

(b) Give an example to show that it is possible for the sequence (y_n) of averages to converge even if (x_n) does not.

Solution

(a) Let $D = \sup\{|x_n - x| : n \in \mathbf{N}\}$ and let $0 < \epsilon < D$, we have

$$|y_n - x| = \left|\frac{x_1 + \cdots + x_n}{n} - x\right| \leq \left|\frac{|x_1 - x| + \cdots + |x_n - x|}{n}\right| \leq D$$

Let $|x_n - x| < \epsilon/2$ for $n > N_1$ giving

$$|y_n - x| \leq \left|\frac{|x_1 - x| + \cdots + |x_{N_1} - x| + \cdots + |x_n - x|}{n}\right| \leq \left|\frac{N_1 D + (n - N_1)\epsilon/2}{n}\right|$$

Let N_2 be large enough that for all $n > N_2$ (remember $0 < \epsilon < D$ so $(D - \epsilon/2) > 0$.)

$$0 < \frac{N_1(D - \epsilon/2)}{n} < \epsilon/2$$

Therefor

$$|y_n - x| \leq \left|\frac{N_1(D - \epsilon/2)}{n} + \epsilon/2\right| < \epsilon$$

Letting $N = \max\{N_1, N_2\}$ completes the proof as $|y_n - x| < \epsilon$ for all $n > N$.

(Note: I could have used any $\epsilon' < \epsilon$ instead of $\epsilon/2$, I just needed some room.)

(b) $x_n = (-1)^n$ diverges but $(y_n) \to 0$.

Exercise 2.3.12

A typical task in analysis is to decipher whether a property possessed by every term in a convergent sequence is necessarily inherited by the limit. Assume $(a_n) \to a$, and determine the validity of each claim. Try to produce a counterexample for any that are false.

(a) If every a_n is an upper bound for a set B, then a is also an upper bound for B.

(b) If every a_n is in the complement of the interval $(0, 1)$, then a is also in the complement of $(0, 1)$.

(c) If every a_n is rational, then a is rational.

Solution

(a) True, let $s = \sup B$, we know $s \leq a_n$ so by the order limit theorem $s \leq a$ meaning a is also an upper bound on B.

(b) True, since if $a \in (0, 1)$ then there would exist an ϵ-neighborhood inside $(0, 1)$ that a_n would have to fall in, contradicting the fact that $a_n \notin (0, 1)$.

(c) False, consider the sequence of rational approximations to $\sqrt{2}$

Exercise 2.3.13 (Iterated Limits)

Given a doubly indexed array a_{mn} where $m, n \in \mathbf{N}$, what should $\lim_{m,n\to\infty} a_{mn}$ represent?

(a) Let $a_{mn} = m/(m+n)$ and compute the iterated limits

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) \quad \text{and} \quad \lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right)$$

Define $\lim_{m,n\to\infty} a_{mn} = a$ to mean that for all $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that if both $m, n \geq N$, then $|a_{mn} - a| < \epsilon$

(b) Let $a_{mn} = 1/(m+n)$. Does $\lim_{m,n\to\infty} a_{mn}$ exist in this case? Do the two iterated limits exist? How do these three values compare? Answer these same questions for $a_{mn} = mn/\left(m^2+n^2\right)$

(c) Produce an example where $\lim_{m,n\to\infty} a_{mn}$ exists but where neither iterated limit can be computed.

(d) Assume $\lim_{m,n\to\infty} a_{mn} = a$, and assume that for each fixed $m \in \mathbf{N}$, $\lim_{n\to\infty}(a_{mn}) \to b_m$. Show $\lim_{m\to\infty} b_m = a$

(e) Prove that if $\lim_{m,n\to\infty} a_{mn}$ exists and the iterated limits both exist, then all three limits must be equal.

Solution

(a)

$$\lim_{n\to\infty}\left(\lim_{m\to\infty}\frac{m}{m+n}\right) = 1, \text{ and } \lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{m}{m+n}\right) = 0$$

(b) For $a_{mn} = 1/(m+n)$ all three limits are zero. For $a_{mn} = mn/(m^2+n^2)$ iterated limits are zero, and $\lim_{m,n\to\infty} a_{mn}$ does not exist since for $m, n \geq N$ setting $m = n$ gives

$$\frac{n^2}{n^2+n^2} = \frac{1}{2}$$

Which cannot be made smaller then $\epsilon = 1/2$.

The reason you would think to set $m = n$ is in trying to maximize $mn/(m^2+n^2)$ notice if $m > n$ then $mn > n^2$ so we are adding more to the numerator then the denominator, hence the ratio is increasing. And if $m < n$ then the ratio is decreasing. therefore the maximum point is at $m = n$.

(c) Intuitively, in order for $\lim_{m,n\to\infty} a_{mn}$ to exist, neither iterated limit can diverge to infinity - otherwise, a_{mn} can also diverge to infinity, by holding letting the index that causes divergence to grow while holding the other index fixed. Therefore, we must rely on each iterated limit diverging due to oscillation.

The key additional "ability" that $\lim_{m,n\to\infty}$ gives over an iterated limit is that both m and n can be forced to grow big at the same time, whereas with an iterated limit only one of them is forced to grow big.

Note that since iterated limits can only increase one of m and n, $\min\{m,n\}$ can't be increased indefinitely - but with $\lim_{m,n\to\infty}$, it can. Thus, the idea is to introduce oscillation in the sequence, then use $\min\{m,n\}$ to cause the oscillation to die out. Define

$$a_{mn} = \frac{(-1)^{m+n}}{\min\{m,n\}}$$

For a fixed m, once $n > m$, a_{mn} will oscillate between $1/m$ and $-1/m$, and thus $\lim_{n\to\infty} a_{mn}$ does not exist. Similar reasoning shows that for a fixed n, $\lim_{m\to\infty} a_{mn}$ does not exist either. But clearly $\lim_{m,n\to\infty} a_{mn} = 0$.

(d) Choose $\epsilon > 0$ and let $0 < \epsilon' < \epsilon$. We need to find N so that $|b_m - a| < \epsilon$ for all $m > N$. Set N such that $|a_{mn} - a| < \epsilon'$ when $n, m \geq N$. Then fix $m \geq N$, I will show $|b_m - a| < \epsilon$. apply the triangle inequality to get

$$|b_m - a| \leq |b_m - a_{mn}| + |a_{mn} - a| \quad \forall n \in \mathbf{N}$$

This inequality is true for all n, we will pick n to make it strict enough to complete the proof. Set $n \geq \max\{N, N_m\}$ where N_m (dependent on m) is big enough that $|b_m - a_{mn}| < \epsilon - \epsilon'$. We also have $|a_{mn} - a| < \epsilon'$ since $m \geq N$ and $n \geq N$. So finally

$$|b_m - a| \leq |b_m - a_{mn}| + |a_{mn} - a| < (\epsilon - \epsilon') + \epsilon' = \epsilon$$

And we are done. The key is that we can make $|b_m - a_{mn}|$ as small as we want *independent of* m, so we take the limit as $n \to \infty$ to show $|b_m - a| \leq |a_{mn} - a|$.

(e) Let $b_m = \lim_{n\to\infty}(a_{mn})$, $c_n = \lim_{m\to\infty}(a_{mn})$, and $a = \lim_{m,n\to\infty}(a_{mn})$. In (d) we showed $(b_m) \to a$; a similar argument shows $(c_n) \to a$. Thus all three limits are equal to a.

2.4 The Monotone Convergence Theorem and a First Look at Infinite Series

Exercise 2.4.1

(a) Prove that the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

converges.

(b) Now that we know $\lim x_n$ exists, explain why $\lim x_{n+1}$ must also exist and equal the same value.

(c) Take the limit of each side of the recursive equation in part (a) to explicitly compute $\lim x_n$.

Solution

(a) $x_2 = 1$ makes me conjecture x_n is monotonic. For induction suppose $x_n > x_{n+1}$ then we have

$$4 - x_n < 4 - x_{n+1} \implies \frac{1}{4 - x_n} > \frac{1}{4 - x_{n+1}} \implies x_{n+1} > x_{n+2}$$

Thus x_n is decreasing, to show x_n is bounded notice x_n cannot be negative since $x_n < 3$ means $x_{n+1} = 1/(4 - x_n) > 0$. therefore by the monotone convergence theorem (x_n) converges.

(b) Clearly skipping a single term does not change what the series converges to.

(c) Since $x = \lim(x_n) = \lim(x_{n+1})$ we must have

$$x = \frac{1}{4 - x} \iff x^2 - 4x + 1 = 0 \iff (x - 2)^2 = 3 \iff x = 2 \pm \sqrt{3}$$

$2 + \sqrt{3} > 3$ is impossible since $x_n < 3$ thus $x = 2 - \sqrt{3}$.

Exercise 2.4.2

(a) Consider the recursively defined sequence $y_1 = 1$

$$y_{n+1} = 3 - y_n$$

and set $y = \lim y_n$. Because (y_n) and (y_{n+1}) have the same limit, taking the limit across the recursive equation gives $y = 3 - y$. Solving for y, we conclude $\lim y_n = 3/2$ What is wrong with this argument?

(b) This time set $y_1 = 1$ and $y_{n+1} = 3 - \frac{1}{y_n}$. Can the strategy in (a) be applied to compute the limit of this sequence?

Solution

(a) The sequence $y_n = (1, 2, 1, 2, \dots)$ does not converge.

(b) Yes, y_n converges by the monotone convergence theorem since $0 < y_n < 3$ and y_n is increasing.

Exercise 2.4.3

(a) Show that

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \dots$$

converges and find the limit.

(b) Does the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots$$

converge? If so, find the limit.

Solution

(a) Let $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2+x_n}$ clearly $x_2 > x_1$. assuming $x_{n+1} > x_n$ gives

$$2 + x_{n+1} > 2 + x_n \iff \sqrt{2+x_{n+1}} > \sqrt{2+x_n} \iff x_{n+2} > x_{n+1}$$

Since x_n is monotonically increasing and bounded the monotone convergence theorem tells us $(x_n) \to x$. Equating both sides like in 2.4.1 gives

$$x = \sqrt{2+x} \iff x^2 - x - 2 = 0 \iff x = \frac{1}{2} \pm \frac{3}{2}$$

Since $x > 0$ we must have $x = 2$.

(b) We have $x_1 = 2^{1/2}$ and $x_{n+1} = (2x_n)^{1/2}$. We have

$$x_{n+1} = (2x_n)^{1/2} \geq x_n \iff 2x_n \geq x_n^2 \iff 2 \geq x_n$$

Since $x_1 = 2^{1/2} \leq 2$ induction implies x_n is increasing. Now to show x_n is bounded notice that $x_1 \leq 2$ and if $x_n \leq 2$ then

$$2x_n \leq 4 \implies (2x_n)^{1/2} \leq 2$$

Now the monotone convergence theorem tells us (x_n) converges. To find the limit use $\lim x_n = \lim x_{n+1} = x$ to get

$$x = (2x)^{1/2} \implies x^2 = 2x \implies x = \pm 2$$

Since $x_n \geq 0$ we have $x = 2$.

Exercise 2.4.4

(a) In Section 1.4 we used the Axiom of Completeness (AoC) to prove the Archimedean Property of **R** (Theorem 1.4.2). Show that the Monotone Convergence Theorem can also be used to prove the Archimedean Property without making any use of AoC.

(b) Use the Monotone Convergence Theorem to supply a proof for the Nested Interval Property (Theorem 1.4.1) that doesn't make use of AoC.

These two results suggest that we could have used the Monotone Convergence Theorem in place of AoC as our starting axiom for building a proper theory of the real numbers.

Solution

(a) MCT tells us $(1/n)$ converges, obviously it must converge to zero therefore we have $|1/n - 0| = 1/n < \epsilon$ for any ϵ, which is the Archimedean Property.

(b) We have $I_n = [a_n, b_n]$ with $a_n \leq b_n$ since $I_n \neq \emptyset$. Since $I_{n+1} \subseteq I_n$ we must have $b_{n+1} \leq b_n$ and $a_{n+1} \geq a_n$ the MCT tells us that $(a_n) \to a$ and $(b_n) \to b$. by the Order Limit Theorem we have $a \leq b$ since $a_n \leq b_n$, therefore $a \in I_n$ for all n meaning $a \in \bigcap_{n=1}^{\infty} I_n$ and thus $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Exercise 2.4.5 (Calculating Square Roots)
Let $x_1 = 2$, and define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

(a) Show that x_n^2 is always greater than or equal to 2 , and then use this to prove that $x_n - x_{n+1} \geq 0$. Conclude that $\lim x_n = \sqrt{2}$.

(b) Modify the sequence (x_n) so that it converges to $\sqrt{c}$.

Solution

(a) Clearly $x_1^2 \geq 2$, now procede by induction. if $x_n^2 \geq 2$ then we have

$$x_{n+1}^2 = \frac{1}{4}\left(\frac{x_n^2+2}{x_n}\right)^2 = \frac{1}{4}\left(\frac{(x_n^2+2)^2}{x_n^2}\right) \geq \frac{1}{4}\left(\frac{(x_n^2+2)^2}{2}\right)$$

Now since $x_n^2 \geq 2$ we have $(x_n^2+2)^2 \geq 16$ meaning

$$x_{n+1}^2 = \frac{1}{4}\left(\frac{(x_n^2+2)^2}{2}\right) \geq 2.$$

Now to show $x_n - x_{n+1} \geq 0$ we use $x_n \geq 0$

$$\begin{aligned} x_n - x_{n+1} &= x_n - \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \\ &= \frac{1}{2}x_n + \frac{1}{x_n} \geq 0 \end{aligned}$$

Now we know $(x_n) \to x$ converges by MCT, to show $x^2 = 2$ we equate $x_n = x_{n+1}$ (true in the limit since $|x_n - x_{n+1}|$ becomes arbitrarily small)

$$x = \frac{1}{2}\left(x + \frac{2}{x}\right) \iff x^2 = \frac{1}{2}x^2 + 1 \iff x^2 = 2$$

therefore $x = \pm\sqrt{2}$, and since every x_n is positive $x = \sqrt{2}$.

(b) Let

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right)$$

I won't go through the convergence analysis again, but the only fixed point is

$$x = \frac{1}{2}\left(x + \frac{c}{x}\right) \implies \frac{1}{2}x^2 = \frac{1}{2}c \implies x^2 = c$$

So if x_n converges, it must converge to $x^2 = c$.

Exercise 2.4.6 (Arithmetic-Geometric Mean)

(a) Explain why $\sqrt{xy} \leq (x+y)/2$ for any two positive real numbers x and y. (The geometric mean is always less than the arithmetic mean.)

(b) Now let $0 \leq x_1 \leq y_1$ and define

$$x_{n+1} = \sqrt{x_n y_n} \quad \text{and} \quad y_{n+1} = \frac{x_n + y_n}{2}$$

Show $\lim x_n$ and $\lim y_n$ both exist and are equal.

Solution

(a) We have

$$\sqrt{xy} \leq (x+y)/2 \iff 4xy \leq x^2 + 2xy + y^2 \iff 0 \leq (x-y)^2$$

(b) The only fixed point is $x_n = y_n$ so we only need to show both sequences converge.

The inequality $x_1 \leq y_1$ is always true since

$$\sqrt{x_n y_n} \leq \frac{x_n + y_n}{2} \implies x_{n+1} \leq y_{n+1}$$

Also $x_n \leq y_n$ implies $(x_n + y_n)/2 = y_{n+1} \leq y_n$, similarly $\sqrt{x_n y_n} = x_{n+1} \geq x_n$ meaning both sequences converge by the monotone convergence theorem.

Exercise 2.4.7 (Limit Superior)

Let (a_n) be a bounded sequence.

(a) Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges.

(b) The limit superior of (a_n), or $\limsup a_n$, is defined by

$$\limsup a_n = \lim y_n$$

where y_n is the sequence from part (a) of this exercise. Provide a reasonable definition for $\liminf a_n$ and briefly explain why it always exists for any bounded sequence.

(c) Prove that $\liminf a_n \leq \limsup a_n$ for every bounded sequence, and give an example of a sequence for which the inequality is strict.

(d) Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

Solution

(a) (y_n) is decreasing and converges by the monotone convergence theorem.

(b) Define $\liminf a_n = \lim z_n$ for $z_n = \inf\{a_n : k \geq n\}$. z_n converges since it is increasing and bounded.

(c) Obviously $\inf\{a_k : k \geq n\} \leq \sup\{a_n : k \geq n\}$ so by the Order Limit Theorem $\liminf a_n \leq \limsup a_n$.

(d) If $\liminf a_n = \limsup a_n$ then the squeeze theorem (Exercise 2.3.3) implies a_n converges to the same value, since $\inf\{a_{k\geq n}\} \leq a_n \leq \sup\{a_{k\geq n}\}$.

Exercise 2.4.8

For each series, find an explicit formula for the sequence of partial sums and determine if the series converges.

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

(c) $\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$

(In (c), $\log(x)$ refers to the natural logarithm function from calculus.)

Solution

(a) This is a geometric series, we can use the usual trick to derive s_n. Let $r = 1/2$ for convenience

$$\begin{aligned} s_n &= 1 + r + r^2 + \cdots + r^n \\ rs_n &= r + r^2 + \cdots + r^{n+1} \\ rs_n - s_n &= r^{n+1} - 1 \implies s_n = \frac{r^{n+1}-1}{r-1} \end{aligned}$$

This is the formula when n starts at zero, but the sum in question starts at one so we subtract the first term to correct this

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = -1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = -1 + \lim_{n\to\infty} \frac{(1/2)^{n+1} - 1}{1/2 - 1} = -1 + \frac{-1}{-1/2} = 1$$

(b) We can use partial fractions to get

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

Which gives us a telescoping series, most of the terms cancel and we get

$$s_n = 1 - \frac{1}{n+1}$$

Therefor

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$$

(c) Another telescoping series, since

$$\log\left(\frac{n+1}{n}\right) = \log(n+1) - \log(n)$$

therefore most of the terms cancel and we get

$$s_n = \log(n+1)$$

Which doesn't converge.

Exercise 2.4.9
Complete the proof of Theorem 2.4.6 by showing that if the series $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges, then so does $\sum_{n=1}^{\infty} b_n$. Example 2.4.5 may be a useful reference.

Solution
Let $s_n = b_1 + b_2 + \cdots + b_n$ and $t_k = b_1 + 2b_2 + \cdots + 2^k b_{2^k}$.

We want to show s_n is unbounded, first we find a series similar to t_k that is less then s_n, then rewrite it in terms of t_k.

Let $n = 2^k$ so things match up nicely. We get

$$\begin{aligned} s_n &= b_1 + b_2 + (b_3 + b_4) + \cdots + (b_{2^{k-1}} + \cdots + b_{2^k}) \\ &\leq b_1 + b_2 + (b_4 + b_4) + \cdots + 2^{k-1} b_{2^k} \end{aligned}$$

(Notice there are $2^k - 2^{k-1} = 2^{k-1}$ terms in the last term)

Now define t'_k to be our new series $b_1 + b_2 + 2b_4 + 4b_8 + \cdots + 2^{k-1} b_{2^k}$. This looks a lot like t_k, and in fact some algebra gives

$$t'_k = \frac{1}{2}\left(b_1 + 2b_2 + 4b_4 + \cdots + 2^k b_k\right) + \frac{1}{2} b_1 = \frac{1}{2} t_k + \frac{1}{2} b_1$$

therefore we are justified in writing

$$s_n \geq t'_k \geq \frac{1}{2} t_k$$

And since $t_k/2$ diverges and s_n is bigger, s_n must also diverge.

Summary: s_n converges iff t_k conv since $t_k \geq s_n \geq t_k/2$ for $n = 2^k$.

Exercise 2.4.10 (Infinite Products)
A close relative of infinite series is the infinite product

$$\prod_{n=1}^{\infty} b_n = b_1 b_2 b_3 \cdots$$

which is understood in terms of its sequence of partial products

$$p_m = \prod_{n=1}^{m} b_n = b_1 b_2 b_3 \cdots b_m$$

Consider the special class of infinite products of the form

$$\prod_{n=1}^{\infty} (1 + a_n) = (1 + a_1)(1 + a_2)(1 + a_3) \cdots, \quad \text{where } a_n \geq 0$$

(a) Find an explicit formula for the sequence of partial products in the case where $a_n = 1/n$ and decide whether the sequence converges. Write out the first few terms in the sequence of partial products in the case where $a_n = 1/n^2$ and make a conjecture about the convergence of this sequence.

(b) Show, in general, that the sequence of partial products converges if and only if $\sum_{n=1}^{\infty} a_n$ converges. (The inequality $1 + x \le 3^x$ for positive x will be useful in one direction.)

Solution

(a) This is a telescoping product, most of the terms cancel

$$p_m = \prod_{n=1}^{m}(1 + 1/n) = \prod_{n=1}^{m} \frac{n+1}{n} = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{2} \cdots \frac{m+1}{m} = m + 1$$

therefore (p_m) diverges.

In the cast $a_n = 1/n^2$ we get

$$\prod_{n=1}^{\infty}(1 + 1/n^2) = \prod_{n=1}^{\infty} \frac{1+n^2}{n^2} = \frac{2}{1} \cdot \frac{5}{4} \cdot \frac{10}{9} \cdots$$

The growth seems slower, I conjecture it converges now.

(b) Using the inequality suggested we have $1 + a_n \le 3^{a_n}$ letting $s_m = a_1 + \cdots + a_m$ we get

$$p_m = (1 + a_1) \cdots (1 + a_m) \le 3^{a_1} 3^{a_2} \cdots 3^{a_m} = 3^{s_m}$$

Now if s_m converges it is bounded by some M meaning p_m is bounded by 3^M. and because $a_n \ge 0$ the partial products p_m are increasing, so they converge by the MCT. This shows s_m converging implies p_m converges.

For the other direction suppose $p_m \to p$. Distributing inside the products gives $p_2 = a_1 + a_2 + 1 + a_1 a_2 > s_2$ and in general $p_m > s_m$ implying that if p_m is bounded then s_n is bounded aswell. This completes the proof.

Summary: Convergence is if and only if because $s_m \le p_m \le 3^{s_m}$.

(By the way the inequality $1 + x \le 3^x$ can be derived from $\log(1 + x) \le x$ implying $1 + x \le e^x$, I assume abbott rounded up to 3.)

2.5 Subsequences and the Bolzano–Weierstrass Theorem

Exercise 2.5.1
Give an example of each of the following, or argue that such a request is impossible.

(a) A sequence that has a subsequence that is bounded but contains no subsequence that converges.

(b) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.

(c) A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, 1/5, \dots\}$.

(d) A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, 1/5, \dots\}$, and no subsequences converging to points outside of this set.

Solution

(a) Impossible, the Bolzano–Weierstrass theorem tells us a convergent subsequence of that subsequence exists, and that sub-sub sequence is also a subsequence of the original sequence.

(b) $(1 + 1/n) \to 1$ and $(1/n) \to 0$ so $(1/2, 1 + 1/2, 1/3, 1 + 1/3, \dots)$ has subsequences converging to 0 and 1.

(c) Copy the finitely many previous terms before proceeding to a new term

$$(1, 1/2, 1, 1/3, 1, 1/2, 1/4, 1, 1/2, 1/3, \dots)$$

The sequence contains infinitely many terms in $\{1, 1/2, 1/3, \dots\}$ hence subsequences exist converging to each of these values.

(d) Impossible, the sequence must converge to zero which is not in the set.

Proof: Let $\epsilon > 0$ be arbitrary, pick N large enough that $1/n < \epsilon/2$ for $n > N$. We can find a subsequence $(b_m) \to 1/n$ meaning $|b_m - 1/n| < \epsilon/2$ for some m. using the triangle inequality we get

$$|b_m - 0| \leq |b_n - 1/n| + |1/n - 0| < \epsilon/2 + \epsilon/2 = \epsilon$$

therefore we have found a number b_m in the sequence a_m with $|b_m| < \epsilon$. This process can be repeated for any ϵ therefore a sequence which converges to zero can be constructed.

Exercise 2.5.2
Decide whether the following propositions are true or false, providing a short justification for each conclusion.

(a) If every proper subsequence of (x_n) converges, then (x_n) converges as well.

(b) If (x_n) contains a divergent subsequence, then (x_n) diverges.

(c) If (x_n) is bounded and diverges, then there exist two subsequences of (x_n) that converge to different limits.

(d) If (x_n) is monotone and contains a convergent subsequence, then (x_n) converges.

Solution

(a) True, removing the first term gives us the proper subsequence $(x_2, x_3, \dots)$ which converges. This implies $(x_1, x_2, \dots)$ also converges to the same value, since discarding the first term doesn't change the limit behavior of a sequence.

(b) True, the divergent subsequence is unbounded, hence (x_n) is also unbounded and divergent.

(c) True, since x_n is bounded $\limsup x_n$ and $\liminf x_n$ both converge. And since x_n diverges Exercise 2.4.7 tells us $\limsup x_n \neq \liminf x_n$.

(d) True, The subsequence (x_{n_k}) converges meaning it is bounded $|x_{n_k}| \leq M$. Suppose (x_n) is increasing, then x_n is bounded since picking k so that $n_k > n$ we have $x_n \leq x_{n_k} \leq M$. A similar argument applies if x_n is decreasing, therefore x_n is monotonic bounded and so must converge.

Exercise 2.5.3

(a) Prove that if an infinite series converges, then the associative property holds. Assume $a_1 + a_2 + a_3 + a_4 + a_5 + \cdots$ converges to a limit L (i.e., the sequence of partial sums $(s_n) \to L$). Show that any regrouping of the terms

$$(a_1 + a_2 + \cdots + a_{n_1}) + (a_{n_1+1} + \cdots + a_{n_2}) + (a_{n_2+1} + \cdots + a_{n_3}) + \cdots$$

leads to a series that also converges to L.

(b) Compare this result to the example discussed at the end of Section 2.1 where infinite addition was shown not to be associative. Why doesn't our proof in (a) apply to this example?

Solution

(a) Let s_n be the original partial sums, and let s'_m be the regrouping. Since s'_m is a subsequence of s_n, $(s_n) \to s$ implies $(s'_m) \to s$.

(b) The subsequence $s'_m = (1-1) + \cdots = 0$ converging does not imply the parent sequence s_n converges. In fact BW tells us any bounded sequence of partial sums will have a convergent subsequence (regrouping in this case).

Exercise 2.5.4

The Bolzano-Weierstrass Theorem is extremely important, and so is the strategy employed in the proof. To gain some more experience with this technique, assume the Nested Interval Property is true and use it to provide a proof of the Axiom of Completeness. To prevent the argument from being circular, assume also that $(1/2^n) \to 0$. (Why precisely is this last assumption needed to avoid circularity?)

Solution
Let A be a bounded set, we're basically going to binary search for $\sup A$ and then use NIP to prove the limit exists.

Let M be an upper bound on A, and pick any $L \in A$ as our starting lower bound for $\sup A$ and define $I_1 = [L, M]$. Doing binary search gives $I_{n+1} \subseteq I_n$ with length proportional to $(1/2)^n$. Applying the Nested Interval Property gives

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

As the length $(1/2)^n$ goes to zero, there is a single $s \in \bigcap_{n=1}^{\infty} I_n$ which must be the least upper bound since $I_n = [L_n, M_n]$ gives $L_n \leq x \leq M_n$ for all n meaning $s = \sup A$ since

(i) $s \geq L_n$ implies s is an upper bound

(ii) $s \leq M_n$ implies s is the least upper bound

The assumption that $(1/2^n) \to 0$ is necessary because otherwise, we would need to invoke the Archimedian Property (Theorem 1.4.2), which is proved using the Axiom of Completeness.

Exercise 2.5.5
Assume (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbf{R}$. Show that (a_n) must converge to a.

Solution
$(a_2, a_3, \dots)$ Is a convergent subsequence, so obviously if $(a_2, a_3, \dots) \to a$ then $(a_n) \to a$ also.

Exercise 2.5.6
Use a similar strategy to the one in Example 2.5.3 to show $\lim b^{1/n}$ exists for all $b \geq 0$ and find the value of the limit. (The results in Exercise 2.3.1 may be assumed.)

Solution
To show $b^{1/n}$ is monotone bounded consider two cases (I won't prove each rigorously to avoid clutter, you can if you want)

(i) If $b > 1$ then $b^{1/n}$ is decreasing, and bounded $b^{1/n} > 1 \iff b > 1^n$ (raise both to n)

(ii) If $b < 1$ then $b^{1/n}$ is increasing, and bounded $b^{1/n} < 1 \iff b < 1^n$ (raise both to n)

therefore $b^{1/n}$ converges for each $b \geq 0$ by the monotone convergence theorem. to find the limit equate terms

$$\begin{aligned} b^{1/(n+1)} = b^{1/n} &\implies b = b^{\frac{n+1}{n}} = b^{1+1/n} && \text{Raise both to } (n+1)\text{th power} \\ &\implies 1 = b^{1/n} && \text{Divide by } b\text{, this assumes } b \neq 0 \end{aligned}$$

Thus $\lim b^{1/n} = 1$ if $b \neq 0$, if $b = 0$ then $\lim b^{1/n} = 0$.

Exercise 2.5.7
Extend the result proved in Example 2.5.3 to the case $|b| < 1$; that is, show $\lim(b^n) = 0$ if and only if $-1 < b < 1$.

Solution
If $|b| \geq 1$ then $\lim(b^n) \neq 0$ (diverges for $b \neq 1$).

Now for the other direction, if $|b| < 1$ we immediately get $|b^n| < 1$ thus b^n is bounded. Since it is decreasing the monotone convergence theorem implies it converges. To find the limit equating terms $b^{n+1} = b^n$ gives $b = 0$ or $b = 1$, since b is *strictly* decreasing we have $b = 0$.

Exercise 2.5.8
Another way to prove the Bolzano-Weierstrass Theorem is to show that every sequence contains a monotone subsequence. A useful device in this endeavor is the notion of a peak term. Given a sequence (x_n), a particular term x_m is a peak term if no later term in the sequence exceeds it; i.e., if $x_m \geq x_n$ for all $n \geq m$.

(a) Find examples of sequences with zero, one, and two peak terms. Find an example of a sequence with infinitely many peak terms that is not monotone.

(b) Show that every sequence contains a monotone subsequence and explain how this furnishes a new proof of the Bolzano-Weierstrass Theorem.

Solution
(a) $(1, 2, \dots)$ has zero peak terms, $(1, 0, 1/2, 2/3, 3/4, \dots)$ has a single peak term, $(2, 1, 1/2, 2/3, \dots)$ has two peak terms (a similar argument works for k peak terms) and $(1, 1/2, 1/3, \dots)$ has infinitely many peak terms. The sequence $(1, -1/2, 1/3, -1/4, \dots)$ has infinitely many peak terms, but is not monotone.

(b) Note that the (possibly finite) sequence of peak terms is monotonic decreasing. There are two possibilities - either there are infinitely many peak terms, or only finitely many. If there are infinitely many peak terms, simply take the subsequence of peak terms; if the parent sequence is bounded we have found a subsequence which converges (by the Monotone Convergence Theorem), hence proving BW in this case.

If there are finitely many peak terms, let the last peak term be at position k. Consider the terms after the last peak term. Since after this point there are no more peak terms, then for every term x_n there must be at least one term $x_m \geq x_n$ where $m > n > k$. Therefore we can define the monotone subsequence x' as $x'_1 = x_{k+1}$, x'_n as the first term after x'_{n-1} such that $x'_n > x'_{n-1}$. By MCT this subsequence converges, hence proving BW in this case as well.

Exercise 2.5.9
Let (a_n) be a bounded sequence, and define the set

$$S = \{x \in \mathbf{R} : x < a_n \text{ for infinitely many terms } a_n\}$$

Show that there exists a subsequence (a_{n_k}) converging to $s = \sup S$. (This is a direct proof of the Bolzano-Weierstrass Theorem using the Axiom of Completeness.)

Solution
For every $\epsilon > 0$ there exists an $x \in S$ with $x > s - \epsilon$ implying $|s - x| < \epsilon$. therefore we can get arbitrarily close to $s = \sup S$ so there is a subsequence converging to this value.

To make this more rigorous, pick $x_n \in S$ such that $|x_n - s| < 1/n$ then pick $N > 1/\epsilon$ to get $|x_n - s| < \epsilon$ for all $n > N$.

2.6 The Cauchy Criterion

Exercise 2.6.1
Prove every convergent sequence is a Cauchy sequence. (Theorem 2.6.2)

Solution
Suppose (x_n) is convergent, we must show that for $m, n > N$ we have $|x_n - x_m| < \epsilon$
Set $|x_n - x| < \epsilon/2$ for $n > N$.
We get $|x_n - x_m| \le |x_n - x| + |x - x_m| \le \epsilon/2 + \epsilon/2 = \epsilon$

Exercise 2.6.2
Give an example of each of the following, or argue that such a request is impossible.

(a) A Cauchy sequence that is not monotone.

(b) A Cauchy sequence with an unbounded subsequence.

(c) A divergent monotone sequence with a Cauchy subsequence.

(d) An unbounded sequence containing a subsequence that is Cauchy.

Solution
(a) $x_n = (-1)^n/n$ is Cauchy by Theorem 2.6.2.

(b) Impossible since all Cauchy sequences converge and are therefore bounded.

(c) Impossible, if a subsequence was Cauchy it would converge, implying the subsequence would be bounded and therefore the parent sequence would be bounded (because it is monotone) and thus would converge.

(d) $(2, 1/2, 3, 1/3, \dots)$ has subsequence $(1/2, 1/3, \dots)$ which is Cauchy.

Exercise 2.6.3
If (x_n) and (y_n) are Cauchy sequences, then one easy way to prove that $(x_n + y_n)$ is Cauchy is to use the Cauchy Criterion. By Theorem 2.6.4, (x_n) and (y_n) must be convergent, and the Algebraic Limit Theorem then implies $(x_n + y_n)$ is convergent and hence Cauchy.

(a) Give a direct argument that $(x_n + y_n)$ is a Cauchy sequence that does not use the Cauchy Criterion or the Algebraic Limit Theorem.

(b) Do the same for the product $(x_n y_n)$.

Solution
(a) We have $|(x_n + y_n) - (x_m + y_m)| \le |x_n - x_m| + |y_n - y_m| < \epsilon/2 + \epsilon/2 = \epsilon$

(b) Bound $|x_n| \le M_1$, and $|y_n| \le M_2$ then

$$\begin{aligned}
|x_n y_n - x_m y_m| &= |(x_n y_n - x_n y_m) + (x_n y_m - x_m y_m)| \\
&\le |x_n(y_n - y_m)| + |y_m(x_n - x_m)| \\
&\le M_1|y_n - y_m| + M_2|x_n - x_m| \\
&< \epsilon/2 + \epsilon/2 = \epsilon
\end{aligned}$$

After setting $|y_n - y_m| < \epsilon/(2M_1)$ and $|x_n - x_m| < \epsilon/(2M_2)$.

Exercise 2.6.4
Let (a_n) and (b_n) be Cauchy sequences. Decide whether each of the following sequences is a Cauchy sequence, justifying each conclusion.

(a) $c_n = |a_n - b_n|$

(b) $c_n = (-1)^n a_n$

(c) $c_n = [[a_n]]$, where $[[x]]$ refers to the greatest integer less than or equal to x.

Solution

(a) Yes. Note that by the Triangle Inequality,

$$|a_n - a_m| + |b_m - b_n| + |a_m - b_m| \geq |a_n - b_n| \Rightarrow |a_n - a_m| + |b_m - b_n| \geq |a_n - b_n| - |b_m - a_m|$$

and

$$|a_m - a_n| + |b_n - b_m| + |a_n - b_m| \geq |a_m - b_m| \Rightarrow |a_n - a_m| + |b_m - b_n| \geq |a_m - b_m| - |b_n - a_n|$$

therefore

$$|c_n - c_m| = \big||a_n - b_n| - |a_m - b_m|\big| \leq |a_n - a_m| + |b_m - b_n| < \epsilon/2 + \epsilon/2 = \epsilon$$

(b) No, if $a_n = 1$ then $(-1)^n a_n$ diverges, and thus is not Cauchy.

(c) No, if $a_n = 1 - (-1)^n/n$ then $[[a_n]]$ fluctuates between 0 and 1 and so cannot be Cauchy.

Exercise 2.6.5
Consider the following (invented) definition: A sequence (s_n) is pseudo-Cauchy if, for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then $|s_{n+1} - s_n| < \epsilon$

Decide which one of the following two propositions is actually true. Supply a proof for the valid statement and a counterexample for the other.

(i) Pseudo-Cauchy sequences are bounded.

(ii) If (x_n) and (y_n) are pseudo-Cauchy, then $(x_n + y_n)$ is pseudo-Cauchy as well.

Solution

(i) False, consider $s_n = \log n$. clearly $|s_{n+1} - s_n|$ can be made arbitrarily small but s_n is unbounded.

(ii) True, as $|(x_{n+1} + y_{n+1}) - (x_n + y_n)| \leq |x_{n+1} - x_n| + |y_{n+1} - y_n| < \epsilon/2 + \epsilon/2 = \epsilon$.

Exercise 2.6.6
Let's call a sequence (a_n) quasi-increasing if for all $\epsilon > 0$ there exists an N such that whenever $n > m \geq N$ it follows that $a_n > a_m - \epsilon$

(a) Give an example of a sequence that is quasi-increasing but not monotone or eventually monotone.

(b) Give an example of a quasi-increasing sequence that is divergent and not monotone or eventually monotone.

(c) Is there an analogue of the Monotone Convergence Theorem for quasiincreasing sequences? Give an example of a bounded, quasi-increasing sequence that doesn't converge, or prove that no such sequence exists.

Solution

Think of "quasi-increasing" as "eventually the n'th term will be almost smaller then all terms after it"

(a) $a_n = (-1)^n/n$ is quasi-increasing since $(-1)^m/m - (-1)^n/n \leq 1/m + 1/n < 2/N < \epsilon$ after picking some $N > 2/\epsilon$.

(b) $a_n = (2, 2 - 1/2, 3, 3 - 1/3, \dots)$ is quasi-increasing. Let $\epsilon > 0$ and set $N > 1/\epsilon$, for $n > m \geq N$ consider two cases We have $a_n > a_m$ as long as $a_n \neq m - 1/m$. If $a_n = m - 1/m$ then $a_n > a_m - \epsilon$ since $m - 1/m > m - \epsilon$ as $1/m < \epsilon$.

(c) Suppose (a_n) is quasi-increasing and bounded and let $\epsilon > 0$.

Let N_1 be large enough that $n > m \geq N_1$ implies $a_n > a_m - \epsilon$.

Since (a_n) is bounded we can set $s = \sup a_n$ applying Lemma 1.3.8 tells us there exists an $N > N_1$ such that $a_N > s - \epsilon$.

Now for all $n > N$ we have $a_n > a_N - \epsilon$, and since $a_N > s - \epsilon$ we have $a_n > s - 2\epsilon$.

This completes the proof as $s \geq a_n > s - 2\epsilon$ implies $|a_n - s| < 2\epsilon$ *for all* $n \geq N$, thus $s = \lim a_n$.

Exercise 2.6.7

Exercises 2.4.4 and 2.5.4 establish the equivalence of the Axiom of Completeness and the Monotone Convergence Theorem. They also show the Nested Interval Property is equivalent to these other two in the presence of the Archimedean Property.

(a) Assume the Bolzano-Weierstrass Theorem is true and use it to construct a proof of the Monotone Convergence Theorem without making any appeal to the Archimedean Property. This shows that BW, AoC, and MCT are all equivalent.

(b) Use the Cauchy Criterion to prove the Bolzano-Weierstrass Theorem, and find the point in the argument where the Archimedean Property is implicitly required. This establishes the final link in the equivalence of the five characterizations of completeness discussed at the end of Section 2.6.

(c) How do we know it is impossible to prove the Axiom of Completeness starting from the Archimedean Property?

Solution

(a) Suppose (x_n) is increasing and bounded, BW tells us there exists a convergent subsequence $(x_{n_k}) \to x$, We will show $(x_n) \to x$. First note $x_k \leq x_{n_k}$ implies $x_n \leq x$ by the Order Limit Theorem.

Pick K such that for $k \geq K$ we have $|x_{n_k} - x| < \epsilon$. Since (x_n) is increasing and $x_n \leq x$ every $n \geq n_K$ satisfies $|x_n - x| < \epsilon$ as well. Thus (x_n) converges, completing the proof.

(b) We're basically going to use the Cauchy criterion as a replacement for NIP in the proof of BW. Recall we had $I_{n+1} \subseteq I_n$ with $a_{n_k} \in I_k$, we will show a_{n_k} is Cauchy.

The length of I_k is $M(1/2)^{k-1}$ by construction, so clearly $|a_{n_k} - a_{n_j}| < M(1/2)^{N-1}$ for $k, j \geq N$, implying (a_{n_k}) converges by the Cauchy criterion.

We needed the Archimedean Property to conclude $M(1/2)^{N-1} \in \mathbf{Q}$ can be made smaller then any $\epsilon \in \mathbf{R}^+$.

(c) The Archimedean Property is true for $\mathbf{Q}$ meaning it cannot prove AoC which is only true for $\mathbf{R}$. (If we did, then we would have proved AoC for $\mathbf{Q}$ which is obviously false.)

2.7 Properties of Infinite Series

Exercise 2.7.1

Proving the Alternating Series Test (Theorem 2.7.7) amounts to showing that the sequence of partial sums

$$s_n = a_1 - a_2 + a_3 - \cdots \pm a_n$$

converges. (The opening example in Section 2.1 includes a typical illustration of (s_n).) Different characterizations of completeness lead to different proofs.

(a) Prove the Alternating Series Test by showing that (s_n) is a Cauchy sequence.

(b) Supply another proof for this result using the Nested Interval Property (Theorem 1.4.1).

(c) Consider the subsequences (s_{2n}) and (s_{2n+1}), and show how the Monotone Convergence Theorem leads to a third proof for the Alternating Series Test.

Solution

(a) Let $N \in \mathbf{N}$ be even and let $n \geq N$. because the series is alternating we have

$$s_N \leq s_n \leq s_{N+1}$$

Obviously $|s_{N+1} - s_N| = |a_N|$ can be made as small as we like by increasing N, setting N large enough to make $|a_N| < \epsilon/2$ gives

$$|s_m - s_n| \leq |s_m - s_N| + |s_N - s_n| < \epsilon/2 + \epsilon/2 = \epsilon$$

Which shows (s_n) is Cauchy, and hence converges by the Cauchy Criterion.

(b) Let I_1 be the interval $[a_1 - a_2, a_1]$ and in general $I_n = [a_n - a_{n+1}, a_n]$, since (a_n) is decreasing we have $I_{n+1} \subseteq I_n$. Applying the nested interval property gives

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

Let $x \in \bigcap_{n=1}^{\infty} I_n$, since $a_n \in I_n$ and $x \in I_n$ the distance $|a_n - x|$ must be less then the length $|I_n|$. and since the length goes to zero $|a_n - x|$ can be made less then any ϵ.

(c) If we can show $\lim s_{2n} = \lim s_{2n+1} = s$ that will imply $\lim s_n = s$ since each s_n is either in (s_{2n+1}) or in (s_{2n}) as n is must be even or odd.

We have $s_{2n+1} \leq a_1$ since

$$s_{2n+1} = a_1 - (a_2 - a_3) - \cdots - (a_{2n} - a_{2n+1}) \leq a_1$$

Thus $s_{2n+1} \to s$ by the Monotone Convergence Theorem, to show $(s_{2n}) \to s$ notice $s_{2n} = s_{2n+1} - a_{2n+1}$ with $(a_{2n+1}) \to 0$ meaning we can use the triangle inequality

$$|s_{2n} - s| \leq \underbrace{|s_{2n} - s_{2n+1}|}_{a_{2n+1}} + |s_{2n+1} - s| < \epsilon/2 + \epsilon/2 < \epsilon$$

Thus $(s_{2n}) \to s$ aswell finally implying $(s_n) \to s$.

Summary: Partition the alternating series into two subsequences of partial sums, then use MCT to show they both converge to the same limit.

Exercise 2.7.2
Decide whether each of the following series converges or diverges:

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$

(b) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

(c) $1 - \frac{3}{4} + \frac{4}{6} - \frac{5}{8} + \frac{6}{10} - \frac{7}{12} + \cdots$

(d) $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots$

(e) $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \frac{1}{7} - \frac{1}{8^2} + \cdots$

Solution

(a) Converges by a comparison test with $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

(b) Converges by a comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(c) Diverges since $(n+1)/2n = 1/2 + 1/2n$ never gets smaller then $1/2$.

(d) Grouping terms gives

$$\frac{1}{n} + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) \geq \frac{1}{n}$$

Which shows the subsequence (s_{3n}) diverges (via comparison test with the harmonic series) hence (s_n) also diverge.

(e) Intuitively this should diverge since it is a mixture of $1/n$ (divergent) and $1/n^2$ (convergent). To make this rigorous examine the subsequence (s_{2n})

$$s_{2n} = 1 - 1/2^2 + 1/3 - 1/4^2 + \cdots + 1/(2n)^2$$

Let $t_n = \sum_{k=1}^{n} \frac{1}{2k-1}$ and $v_n = \sum_{k=1}^{n} \frac{1}{(2k)^2}$ so $s_{2n} = t_n - v_n$.

A comparison test with the harmonic series (after some manipulation) shows that (t_n) diverges, and p-series tells us (v_n) converges. Therefore their difference $s_{2n} = t_n - v_n$ must diverge, which implies (s_n) diverges as desired.

Exercise 2.7.3

(a) Provide the details for the proof of the Comparison Test (Theorem 2.7.4) using the Cauchy Criterion for Series.

(b) Give another proof for the Comparison Test, this time using the Monotone Convergence Theorem.

Solution
Suppose $a_n, b_n \geq 0$, $a_n \leq b_n$ and define $s_n = a_1 + \cdots + a_n$, $t_n = b_1 + \cdots + b_n$.

(a) We have $|a_{m+1} + \cdots + a_n| \leq |b_{m+1} + \cdots + b_n| < \epsilon$ implying $\sum_{n=1}^{\infty} a_n$ converges by the Cauchy criterion. The other direction is analogous, if (s_n) diverges then (t_n) must also diverge since $s_n \leq t_n$.

(b) Since $(t_n) \to t$. This implies that s_n is bounded, and since $s_n \leq t_n$ implies $s_n \leq t$ by the order limit theorem, we can use the monotone convergence theorem to conclude (s_n) converges.

Exercise 2.7.4
Give an example of each or explain why the request is impossible referencing the proper theorem(s).

(a) Two series $\sum x_n$ and $\sum y_n$ that both diverge but where $\sum x_n y_n$ converges.

(b) A convergent series $\sum x_n$ and a bounded sequence (y_n) such that $\sum x_n y_n$ diverges.

(c) Two sequences (x_n) and (y_n) where $\sum x_n$ and $\sum (x_n + y_n)$ both converge but $\sum y_n$ diverges.

(d) A sequence (x_n) satisfying $0 \leq x_n \leq 1/n$ where $\sum (-1)^n x_n$ diverges.

Solution
(a) $x_n = 1/n$ and $y_n = 1/n$ have their respective series diverge, but $\sum x_n y_n = \sum 1/n^2$ converges since it is a p-series with $p > 1$.

(b) Let $x_n = (-1)^n/n$ and $y_n = (-1)^n$. $\sum x_n$ converges but $\sum x_n y_n = \sum 1/n$ diverges.

(c) Impossible as the algebraic limit theorem for series implies $\sum (x_n + y_n) - \sum x_n = \sum y_n$ converges.

(d) The sequence

$$x_n = \begin{cases} \frac{1}{n} \text{ if } n \text{ even} \\ 0 \text{ otherwise} \end{cases}$$

diverges for the same reason the harmonic series does.

Exercise 2.7.5
Prove the series $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$. (Corollary 2.4.7)

Solution
Eventually we have $1/n^p < 1/p^n$ for $p > 1$ (polynomial vs exponential) meaning we can use the comparison test to conclude $\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$.

Now suppose $p \leq 1$, since $1/n^p \leq 1/n$ a comparsion test with the harmonic series implies $p \leq 1$ diverges.

Exercise 2.7.6
Let's say that a series subverges if the sequence of partial sums contains a subsequence that converges. Consider this (invented) definition for a moment, and then decide which of the following statements are valid propositions about subvergent series:

(a) If (a_n) is bounded, then $\sum a_n$ subverges.

(b) All convergent series are subvergent.

(c) If $\sum |a_n|$ subverges, then $\sum a_n$ subverges as well.

(d) If $\sum a_n$ subverges, then (a_n) has a convergent subsequence.

Solution

(a) False, consider $a_n = 1$ then $s_n = n$ does not have a convergent subsequence.

(b) True, every subsequence converges to the same limit in fact.

(c) True, since $s_n = \sum_{k=1}^n |a_k|$ converges it is bounded $|s_n| \leq M$, and since $t_n = \sum_{k=1}^n a_k$ is smaller $t_n \leq s_n$ it is bounded $t_n \leq M$ which by BW implies there exists a convergent subsequence (t_{n_k}).

(d) False, $a_n = (1, -1, 2, -2, \dots)$ has no convergent subsequence but the sum $s_n = \sum_{k=1}^n a_k$ has the subsequence $(s_{2n}) \to 0$.

Exercise 2.7.7

(a) Show that if $a_n > 0$ and $\lim (na_n) = l$ with $l \neq 0$, then the series $\sum a_n$ diverges.

(b) Assume $a_n > 0$ and $\lim (n^2 a_n)$ exists. Show that $\sum a_n$ converges.

Solution

Note: This is kind of like a wierd way to do a comparison with $1/n$ and $1/n^2$.

(a) If $\lim(na_n) = l \neq 0$ then $na_n \in (l - \epsilon, l + \epsilon)$, setting $\epsilon = l/2$ gives $na_n \in (l/2, 3l/2)$ implying $a_n > (l/2)(1/n)$. But if $a_n > (l/2)(1/n)$ then $\sum a_n$ diverges as it is a multiple of the harmonic series. (note that $a_n > 0$ ensures $l \geq 0$.)

(b) Letting $l = \lim(n^2 a_n)$ we have $n^2 a_n \in (l - \epsilon, l + \epsilon)$ setting $\epsilon = l$ gives $n^2 a_n \in (0, 2l)$ implying $0 \leq a_n \leq 2l/n^2$ and so $\sum a_n$ converges by a comparsion test with $\sum 2l/n^2$.

Exercise 2.7.8

Consider each of the following propositions. Provide short proofs for those that are true and counterexamples for any that are not.

(a) If $\sum a_n$ converges absolutely, then $\sum a_n^2$ also converges absolutely.

(b) If $\sum a_n$ converges and (b_n) converges, then $\sum a_n b_n$ converges.

(c) If $\sum a_n$ converges conditionally, then $\sum n^2 a_n$ diverges.

Solution

(a) True since $(a_n) \to 0$ so eventually $a_n^2 \leq |a_n|$ meaning $\sum a_n^2$ converges by a comparsion test with $\sum |a_n|$.

(b) False, let $a_n = (-1)^n/\sqrt{n}$ and $b_n = (-1)^n/\sqrt{n}$. $\sum a_n$ converges by the alternating series test, but $\sum a_n b_n = \sum 1/n$ diverges.

(c) True, suppose $(n^2 a_n)$ converges, since $(n^2 a_n) \to 0$ we have $|n^2 a_n| < 1$ for $n > N$, implying $|a_n| < 1/n^2$. But if $|a_n| < 1/n^2$ then a comparsion test with $1/n^2$ implies a_n converges absolutely, contradicting the assumption that a_n converges conditionally. Therefore $\sum n^2 a_n$ must diverge.

Exercise 2.7.9 (Ratio Test)
Given a series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$, the Ratio Test states that if (a_n) satisfies

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = r < 1$$

then the series converges absolutely.

(a) Let r' satisfy $r < r' < 1$. Explain why there exists an N such that $n \geq N$ implies $|a_{n+1}| \leq |a_n| r'$.

(b) Why does $|a_N| \sum (r')^n$ converge?

(c) Now, show that $\sum |a_n|$ converges, and conclude that $\sum a_n$ converges.

Solution

(a) We are given

$$\left| \frac{a_{n+1}}{a_n} - r \right| < \epsilon$$

Since $1 > r' > r$ we can set $\epsilon = r' - r$ meaning the neighborhood

$$\frac{a_{n+1}}{a_n} \in (r - \epsilon, r + \epsilon) = (2r - r', r')$$

Is all less then r' meaning

$$\left| \frac{a_{n+1}}{a_n} \right| \leq r' \implies |a_{n+1}| \leq r' |a_n|$$

(b) Let N be large enough that for $n > N$ we have $|a_{n+1}| \leq |a_n| r'$. Applying this multiple times gives $|a_n| \leq (r')^{n-N} |a_N|$ which gives

$$|a_N| + |a_{N+1}| + \cdots + |a_n| \leq |a_N| + r'|a_N| + \cdots + (r')^{n-N} |a_N|$$

Factoring out $|a_N|$ and writing with sums gives

$$\sum_{k=N}^{n} |a_k| \leq |a_N| \sum_{k=0}^{n-1} (r')^k$$

Which converges as $n \to \infty$ since $|r'| < 1$ and $|a_N|$ is constant. Implying $\sum_{k=N}^{n} |a_k|$ converges and thus $\sum_{k=1}^{n} |a_k|$ also converges since we only omitted finitely many terms.

(c) See (b)

Exercise 2.7.10 (Infinite Products)
Review Exercise 2.4.10 about infinite products and then answer the following questions:

(a) Does $\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{9}{8} \cdot \frac{17}{16} \cdots$ converge?

(b) The infinite product $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{9}{10} \cdots$ certainly converges. (Why?) Does it converge to zero?

(c) In 1655, John Wallis famously derived the formula

$$\left(\frac{2 \cdot 2}{1 \cdot 3}\right)\left(\frac{4 \cdot 4}{3 \cdot 5}\right)\left(\frac{6 \cdot 6}{5 \cdot 7}\right)\left(\frac{8 \cdot 8}{7 \cdot 9}\right) \cdots = \frac{\pi}{2}$$

Show that the left side of this identity at least converges to something. (A complete proof of this result is taken up in Section 8.3.)

Solution

(a) Rewriting the terms as $a_n = (1 + 1/n)$ and using the result from 2.4.10 implies the product diverges since $\sum 1/n$ diverges.

(b) Converges by the monotone convergence theorem since the partial products are decreasing and greater then zero. To show that the product converges to zero, the key insight is to rewrite each term $a_n = (2n-1)/(2n) = 1/(2n/(2n-1)) = 1/b_n$, where $b_n = 2n/(2n-1)$. Then the partial products

$$p_n = \prod_{i=1}^{n} a_n = \frac{1}{\prod_{i=1}^{n} b_n}$$

But

$$\prod_{i=1}^{n} b_n = \prod_{i=1}^{n} \left(1 + \frac{1}{2n-1}\right)$$

and since $\sum 1/(2n-1)$ diverges by comparison against a multiple of the harmonic series, $\prod b_n$ diverges. Thus, to show there exists some N so that $n \geq N$ implies $p_n < \epsilon$, simply take N large enough so that $\prod_{i=1}^{n} b_n > 1/\epsilon$.

(c) In compontent form we have

$$a_n = \frac{(2n)^2}{(2n-1)(2n+1)} = \frac{(2n)^2}{(2n)^2 - 1} = 1 + \frac{(2n)^2 - ((2n)^2 - 1)}{(2n)^2 - 1} = 1 + \frac{1}{(2n)^2 - 1}$$

And since $\sum 1/((2n)^2 - 1)$ converges by a comparison test with $1/n^2$, exercise 2.4.10 implies

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{(2n)^2 - 1}\right)$$

also converges.

Exercise 2.7.11
Find examples of two series $\sum a_n$ and $\sum b_n$ both of which diverge but for which $\sum \min\{a_n, b_n\}$ converges. To make it more challenging, produce examples where (a_n) and (b_n) are strictly positive and decreasing.

Solution
Let $m_n = \min\{a_n, b_n\}$. Clearly (m_n) must take an infinite amount of (a_n) and (b_n) terms, as otherwise removing the finite terms would imply one of $\sum a_n$ or $\sum b_n$ converged.

The key insight is that as long as $a_n, b_n > 0$, we can simply repeat terms in one sequence (while letting m_n be governed by the other sequence) for as long as we want - say, until we have enough terms to e.g. sum to 1. Then we can switch the roles of the sequences. To start with the construction, take some converging series with all terms positive - say, $m_n = 1/2^n$. Then, define the first few terms of m_n, a_n, and b_n as:

n	1	2	3	$[4, 4+8=12)$	$[12, 12+2^{11})$	$[12+2^{11}, 12+2^{11}+2^{12+2^{11}})$
m_n	$1/2$	$1/4$	$1/8$	$1/2^n$	$1/2^n$	$1/2^n$
$\min\{a_n, b_n\}$	a_n	b_n	b_n	a_n	b_n	a_n
a_n	$1/2$	$1/2$	$1/2$	$1/2^n$	$1/2^{11}$	$1/2^n$
b_n	1	$1/4$	$1/8$	$1/8$	$1/2^n$	$1/2^{12+2^{11}}$

Specifically, $\min\{a_n, b_n\}$ will alternate between following a_n and b_n. Every "block" of the sequence (examples: each of the last three columns) that isn't being followed by $\min\{a_n, b_n\}$ sums to 1. Since each block is finite, $\min\{a_n, b_n\}$ will alternate between a_n and b_n infinitely, and thus both $\sum a_n$ and $\sum b_n$ will diverge.

For the sake of completeness, a_n and b_n are defined more formally below. Let $k_1 = 1$, $k_n = k_{n-1} + 2^{k_{n-1}-1}$. Then

$$a_n = \begin{cases} 1/2^n & \text{if } k_{2p-1} \le n < k_{2p} \\ 1/2^{k_{2p}-1} & \text{if } k_{2p} \le n < k_{2p+1} \end{cases} \text{ and } b_n = \begin{cases} 1/2^n & \text{if } k_{2p} \le n < k_{2p+1} \\ 1/2^{k_{2p-1}-1} & \text{if } k_{2p-1} \le n < k_{2p} \end{cases}$$

Exercise 2.7.12 (Summation-by-parts)
Let (x_n) and (y_n) be sequences, let $s_n = x_1 + x_2 + \cdots + x_n$ and set $s_0 = 0$. Use the observation that $x_j = s_j - s_{j-1}$ to verify the formula

$$\sum_{j=m}^{n} x_j y_j = s_n y_{n+1} - s_{m-1} y_m + \sum_{j=m}^{n} s_j (y_j - y_{j+1})$$

Solution
Since $x_j = s_j - s_{j-1}$ we can rewrite the sum as

$$\sum_{j=m}^{n} x_j y_j = \sum_{j=m}^{n} y_j (s_j - s_{j-1}) = s_n y_{n+1} - s_m y_{m-1} + \sum_{j=m}^{n} s_j (y_j - y_{j+1})$$

In the last part we combine each $s_j y_j$ term with the $-s_j y_{j+1}$ term which is next in the sum, then we add some correction terms for the start and ending points.

Note the symmetry here, we can turn a sum $\sum y_j (s_j - s_{j-1})$ into a sum $\sum s_j (y_j - y_{j+1})$ (at the cost of some correction terms). This is a useful pattern to keep in mind.

Exercise 2.7.13 (Abel's Test)
Abel's Test for convergence states that if the series $\sum_{k=1}^{\infty} x_k$ converges, and if (y_k) is a sequence satisfying

$$y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0$$

then the series $\sum_{k=1}^{\infty} x_k y_k$ converges.

(a) Use Exercise 2.7.12 to show that

$$\sum_{k=1}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=1}^{n} s_k \left(y_k - y_{k+1}\right)$$

where $s_n = x_1 + x_2 + \cdots + x_n$.

(b) Use the Comparison Test to argue that $\sum_{k=1}^{\infty} s_k \left(y_k - y_{k+1}\right)$ converges absolutely, and show how this leads directly to a proof of Abel's Test.

Solution

(a) Exercise 2.7.12 combined with $s_0 = 0$ gives

$$\sum_{k=1}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=1}^{n} s_k(y_k - y_{k+1})$$

as desired.

(b) $s_n y_{n+1}$ clearly converges since y_{n+1} is "eventually constant", so we must only show the right hand side converges.

We will show absolute convergence, note $y_k - y_{k+1} \geq 0$ and so

$$\sum_{k=1}^{n} |s_k|(y_k - y_{k+1}) \geq 0$$

Bounding $|s_n| \leq M$ gives

$$\sum_{k=1}^{n} |s_k|(y_k - y_{k+1}) \leq M \sum_{k=1}^{n} (y_k - y_{k+1})$$

Since $\sum_{k=1}^{n} (y_k - y_{k+1}) = y_1 - y_{n+1}$ is telescoping we can write

$$\sum_{k=1}^{n} |s_k|(y_k - y_{k+1}) \leq M(y_1 - y_{n+1}) \leq M y_1$$

Implying $\sum_{k=1}^{\infty} |s_k|(y_k - y_{k+1})$ converges since it is bounded and increasing. And since the series converges absolutely so does the original $\sum_{k=1}^{\infty} s_k(y_k - y_{k+1})$.

Summary: Bound $|s_k| \leq M$ and use the fact that $\sum (y_k - y_{k+1})$ is telescoping.

Exercise 2.7.14 (Dirichlet's Test)
Dirichlet's Test for convergence states that if the partial sums of $\sum_{k=1}^{\infty} x_k$ are bounded (but not necessarily convergent), and if (y_k) is a sequence satisfying $y_1 \geq y_2 \geq y_3 \geq \cdots \geq 0$ with $\lim y_k = 0$, then the series $\sum_{k=1}^{\infty} x_k y_k$ converges.

(a) Point out how the hypothesis of Dirichlet's Test differs from that of Abel's Test in Exercise 2.7.13, but show that essentially the same strategy can be used to provide a proof.

(b) Show how the Alternating Series Test (Theorem 2.7.7) can be derived as a special case of Dirichlet's Test.

Solution

(a) Abel's test gets it's convergence from $\sum x_n$ converging, while Dirichlet's test gets its convergence from $(y_n) \to 0$. Expanding on that, the proof that the $\sum_{k=1}^{n} s_k(y_k - y_{k-1})$ term converges is the same in Abel and Dirichlet, but the proof that $s_n y_{n+1}$ differs depending on if we get our convergence from $(y_n) \to 0$ and (s_n) bounded, or $(s_n) \to s$ and (y_n) bounded.

The proof is almost identical to Abel's test, bound $|s_n| < M$ and use the triangle inequality on the right hand side to get (note $y_k - y_{k+1} > 0$ because decreasing)

$$\sum_{k=1}^{n} |s_k|(y_k - y_{k+1}) \leq M \sum_{k=1}^{n} (y_k - y_{k+1}) = M(y_1 - y_{n+1}) \leq M y_1$$

Thus $\sum s_k(y_k - y_{k+1})$ is bounded and increasing, so it converges by MCT. To see the $s_n y_{n+1}$ term converges, simply note that $(y_n) \to 0$ and $|s_n| < M$.

Thus the original series converges, and furthermore

$$\sum_{k=1}^{\infty} x_k y_k = \sum_{k=1}^{\infty} s_k(y_k - y_{k+1})$$

Because $(s_n y_{n+1}) \to 0$.

(b) Let $a_n \geq 0$ with $a_1 \geq a_2 \geq \cdots \geq 0$ and $\lim a_n = 0$. The series $\sum (-1)^n$ is bounded, so Dirichlet's test implies $\sum (-1)^n a_n$ converges.

2.8 Double Summations and Products of Infinite Series

Exercise 2.8.1
Using the particular array a_{ij} from Section 2.1, compute $\lim_{n\to\infty} s_{nn}$. How does this value compare to the two iterated values for the sum already computed?

Solution
By inspection $s_{nn} = -2 + 1/2^{n-1}$, so $\lim_{n\to\infty} s_{nn} = -2$. This is the same as the result when fixing j and summing down each column, since each column series has finitely many non-zero elements.

Exercise 2.8.2
Show that if the iterated series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}|$$

converges (meaning that for each fixed $i \in \mathbf{N}$ the series $\sum_{j=1}^{\infty} |a_{ij}|$ converges to some real number b_i, and the series $\sum_{i=1}^{\infty} b_i$ converges as well), then the iterated series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$$

converges.

Solution
Since $\sum_{j=1}^{\infty} |a_{ij}|$ converges, $c_i = \sum_{j=1}^{\infty} a_{ij}$ converges as well; moreover $|c_i| \leq |b_i|$, so $\sum_{i=1}^{\infty} c_i$ converges by comparison with $\sum_{i=1}^{\infty} |b_i|$.

Exercise 2.8.3
Define

$$t_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|$$

(a) Prove that (t_{nn}) converges.

(b) Now, use the fact that (t_{nn}) is a Cauchy sequence to argue that (s_{nn}) converges.

Solution
(a) Note that t_{nn} is monotone increasing; moreover

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}| \geq \sum_{i=1}^{\infty}\sum_{j=1}^{n} |a_{ij}| \geq \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}| = t_{nn}$$

and therefore t_{nn} is bounded; by the Monotone Convergence Theorem t_{nn} converges.

(b) Since (t_{nn}) is a Cauchy sequence, for any $\epsilon > 0$ there exists N such that if $p > q > N$,

$$\begin{aligned}
\epsilon > |t_{pp} - t_{qq}| &= \left|\sum_{i=1}^{p}\sum_{j=1}^{p}|a_{ij}| - \sum_{i=1}^{q}\sum_{j=1}^{q}|a_{ij}|\right| \\
&= \left|\sum_{i=q+1}^{p}\sum_{j=1}^{p}|a_{ij}| + \sum_{i=1}^{q}\sum_{j=q+1}^{p}|a_{ij}| + \sum_{i=1}^{q}\sum_{j=1}^{q}|a_{ij}| - \sum_{i=1}^{q}\sum_{j=1}^{q}|a_{ij}|\right| \\
&= \left|\sum_{i=q+1}^{p}\sum_{j=1}^{p}|a_{ij}| + \sum_{i=1}^{q}\sum_{j=q+1}^{p}|a_{ij}|\right| \\
&\geq \left|\sum_{i=q+1}^{p}\sum_{j=1}^{p}a_{ij} + \sum_{i=1}^{q}\sum_{j=q+1}^{p}a_{ij}\right| \\
&= \left|\sum_{i=q+1}^{p}\sum_{j=1}^{p}a_{ij} + \sum_{i=1}^{q}\sum_{j=q+1}^{p}a_{ij} + \sum_{i=1}^{q}\sum_{j=1}^{q}a_{ij} - \sum_{i=1}^{q}\sum_{j=1}^{q}a_{ij}\right| \\
&= \left|\sum_{i=1}^{p}\sum_{j=1}^{p}a_{ij} - \sum_{i=1}^{q}\sum_{j=1}^{q}a_{ij}\right| \\
&= |s_{pp} - s_{qq}|
\end{aligned}$$

and therefore s_{nn} is also a Cauchy sequence and thus converges.

Exercise 2.8.4

(a) Let $\epsilon > 0$ be arbitrary and argue that there exists an $N_1 \in \mathbf{N}$ such that $m, n \geq N_1$ implies $B - \frac{\epsilon}{2} < t_{mn} \leq B$.

(b) Now, show that there exists an N such that

$$|s_{mn} - S| < \epsilon$$

for all $m, n \geq N$.

Solution

(a) $t_{mn} \leq B$ follows from the fact that B is an upper bound on $\{t_{mn} : m, n \in \mathbf{N}\}$. Lemma 1.3.8 indicates that there exists some p, q such that $t_{pq} > B - \epsilon/2$, and since $p_1 \leq p_2$ and $q_1 \leq q_2 \implies t_{p_1q_1} \leq t_{p_2q_2}$, we can choose $N_1 = \max\{p, q\}$.

(b) By the triangle inequality, $|s_{mn} - S| \leq |s_{mn} - s_{nn}| + |s_{nn} - S|$. Letting $n' = \min\{n, m\}$ and $m' = \max\{n, m\}$,

$$|s_{mn} - s_{nn}| = \left|\sum_{i=n'}^{m'}\sum_{j=1}^{n}a_{ij}\right| \leq \sum_{i=n'}^{m'}\sum_{j=1}^{n}|a_{ij}| = |t_{mn} - t_{nn}| \leq \frac{\epsilon}{2}$$

from Exercise 2.8.4a), as long as $m, n \geq N_1$. Since $S = \lim_{n\to\infty} s_{nn}$ there exists N_2 such that for $n \geq N_2$, $|s_{nn} - S| < \epsilon/2$; thus picking $N = \max\{N_1, N_2\}$ ensures

$$|s_{mn} - S| < \epsilon$$

for all $m, n \geq N$.

Exercise 2.8.5

(a) Show that for all $m \geq N$

$$|(r_1 + r_2 + \cdots + r_m) - S| \leq \epsilon$$

Conclude that the iterated sum $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$ converges to S.

(b) Finish the proof by showing that the other iterated sum, $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$, converges to S as well. Notice that the same argument can be used once it is established that, for each fixed column j, the sum $\sum_{i=1}^{\infty} a_{ij}$ converges to some real number c_j.

Solution

(a) For any given m, there must be some N_3 such that for $n > N_3$, $k \in \mathbf{N} \leq m$,

$$\left| r_k - \sum_{j=1}^{n} a_{kj} \right| < \frac{\epsilon}{2m}$$

Then there must exist some N such that when $m, n \geq N$,

$$\begin{aligned}
\left| \sum_{i=1}^{m} r_i - S \right| &\leq \left| \sum_{i=1}^{m} r_i - \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \right| + \left| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} - S \right| \\
&= \left| \sum_{i=1}^{m} \left(r_i - \sum_{j=1}^{n} a_{ij} \right) \right| + |s_{mn} - S| \\
&\leq \sum_{i=1}^{m} \left| r_k - \sum_{j=1}^{n} a_{kj} \right| + |s_{mn} - S| \\
&< \sum_{i=1}^{m} \left(\frac{\epsilon}{2m} \right) + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

and thus $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$ converges to S.

(b) $\sum_{i=1}^{\infty} |a_{ij}|$ converges for any fixed j by comparison with $\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} |a_{ik}|$ which converges by the hypothesis, and thus $\sum_{i=1}^{\infty} a_{ij}$ converges to some real number c_j. Then a similar argument to (a) can be used to show that there must be some Nsuch that when $n \geq N$,

$$\left| \sum_{j=1}^{n} c_j - S \right| \leq \epsilon$$

and thus $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$ converges to S.

Exercise 2.8.6

(a) Assuming the hypothesis - and hence the conclusion - of Theorem 2.8.1, show that $\sum_{k=2}^{\infty} d_k$ converges absolutely.

(b) Imitate the strategy in the proof of Theorem 2.8.1 to show that $\sum_{k=2}^{\infty} d_k$ converges to $S = \lim_{n\to\infty} s_{nn}$.

Solution

(a) Note that $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}$ contains all of the terms of $\sum_{k=2}^{\infty} d_k$, and thus by comparison to $\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|$, $\sum_{k=2}^{\infty} |d_k|$ must converge.

(b) What we need to show is that for all $\epsilon > 0$ there exists N such that for all $n > N$, $|S - \sum_{k=2}^{n} d_k| < \epsilon$. Note first that $\sum_{k=2}^{n} d_k$ contains all the elements of s_{pp} when $p \leq n/2$, and that s_{qq} contains all the elements of $\sum_{k=2}^{n} d_k$ as long as $q \geq n-1$.

Since $(s_{nn}) \to S$, for arbitrary $\epsilon_1 > 0$ we can choose n_1 large enough such that $|s_{n_1 n_1} - S| < \epsilon_1$. If we choose $N = 2n$ then whenever $n > N$, $\sum_{k=2}^{n} d_k$ will contain all terms in $s_{n_1} s_{n_1}$, and if we choose $m = n-1$ then s_{mm} will contain all terms in $\sum_{k=2}^{n} d_k$. Thus

$$\sum_{k=2}^{n} |d_k| - t_{n_1 n_1} \leq t_{mm} - t_{n_1 n_1}$$

where t_{nn} was defined near the start of the proof of Theorem 2.8.1. Moreover since t_{nn} converges (as proved in Exercise 2.8.3a) and is thus a Cauchy sequence, for arbitrary $\epsilon_2 > 0$, we can also choose n_1 large enough to ensure for any $m_1 > n_1$, $t_{m_1 m_1} - t_{n_1 n_1} < \epsilon_2$.

Putting it all together, choosing $\epsilon_1 = \epsilon_2 = \epsilon/2$, n_1 large enough to satisfy the two conditions discussed above, and $N = 2n_1$:

$$\begin{aligned}
\left| S - \sum_{k=2}^{n} d_k \right| &\leq |S - s_{n_1 n_1}| + \left| \sum_{k=2}^{n} d_k - s_{n_1 n_1} \right| \\
&< \epsilon_1 + \left| \sum_{k=2}^{n} |d_k| - t_{n_1 n_1} \right| \\
&\leq \epsilon_1 + t_{mm} - t_{n_1 n_1} < \epsilon_1 + \epsilon_2 \\
&= \epsilon
\end{aligned}$$

completing the proof.

Exercise 2.8.7

Assume that $\sum_{i=1}^{\infty} a_i$ converges absolutely to A, and $\sum_{j=1}^{\infty} b_j$ converges absolutely to B.

(a) Show that the iterated sum $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_i b_j|$ converges so that we may apply Theorem 2.8.1.

(b) Let $s_{nn} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j$, and prove that $\lim_{n\to\infty} s_{nn} = AB$. Conclude that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_i b_j = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_i b_j = \sum_{k=2}^{\infty} d_k = AB,$$

where, as before, $d_k = a_1 b_{k-1} + a_2 b_{k-2} + \cdots + a_{k-1} b_1$.

Solution

(a) Let $\sum_{i=1}^{\infty} |a_i|$ converge to A' and $\sum_{j=1}^{\infty} |b_j|$ converge to B'. By the Algebraic Limit Theorem for Series,

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_i b_j| = \sum_{i=1}^{\infty} \left(|a_i| \sum_{j=1}^{\infty} |b_j| \right) = \left(\sum_{i=1}^{\infty} |a_i| \right) \left(\sum_{j=1}^{\infty} |b_j| \right) = A'B'$$

(b) Theorem 2.8.1 shows that $\lim_{n\to\infty} s_{nn} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_i b_j$ which by the same manipulation as used in part (a), equals AB. By Theorem 2.8.1 the rest of the problem is solved.

Chapter 3

Basic Topology of R

3.2 Open and Closed Sets

Exercise 3.2.1

(a) Where in the proof of Theorem 3.2.3 part (ii) does the assumption that the collection of open sets be finite get used?

(b) Give an example of a countable collection of open sets $\{O_1, O_2, O_3, \ldots\}$ whose intersection $\bigcap_{n=1}^{\infty} O_n$ is closed, not empty and not all of $\mathbf{R}$.

Solution

(a) Taking $\min\{\epsilon_1, \ldots, \epsilon_N\}$ is only possible for finite sets.

(b) $O_n = (-1/n, 1 + 1/n)$ has $\bigcap_{n=1}^{\infty} O_n = [0, 1]$.

Exercise 3.2.2

Let

$$A = \left\{ (-1)^n + \frac{2}{n} : n = 1, 2, 3, \ldots \right\} \quad \text{and} \quad B = \{x \in \mathbf{Q} : 0 < x < 1\}$$

Answer the following questions for each set:

(a) What are the limit points?

(b) Is the set open? Closed?

(c) Does the set contain any isolated points?

(d) Find the closure of the set.

Solution

(a) The set of B's limit points is $[0, 1]$. The set of A's limit points is $\{1, -1\}$.

(b) B is not open since every $(a, b) \not\subseteq B$ and B is not closed since we can construct limits to irrational values outside B. A is closed since $\{1, -1\} \subseteq A$, but not open as it does not contain any irrationals meaning $(a, b) \not\subseteq A$ for all $a, b \in \mathbf{R}$.

(c) Every point of A except the limit points $\{1, -1\}$ is isolated, as if it were not isolated it would be a limit point. B has no isolated points since $B \setminus [0, 1] = \emptyset$, or in other words since B is dense in $[0, 1]$ every $b \in B \subseteq [0, 1]$ can be reached via a limit.

(d) $\overline{A} = A$ as A is already closed, and $\overline{B} = B \cup [0, 1] = [0, 1]$.

Exercise 3.2.3
Decide whether the following sets are open, closed, or neither. If a set is not open, find a point in the set for which there is no ϵ-neighborhood contained in the set. If a set is not closed, find a limit point that is not contained in the set.

(a) $\mathbf{Q}$.

(b) $\mathbf{N}$.

(c) $\{x \in \mathbf{R} : x \neq 0\}$.

(d) $\{1 + 1/4 + 1/9 + \cdots + 1/n^2 : n \in \mathbf{N}\}$

(e) $\{1 + 1/2 + 1/3 + \cdots + 1/n : n \in \mathbf{N}\}$

Solution
(a) Neither, not open as $(a, b) \subseteq \mathbf{Q}$ is impossible since $\mathbf{Q}$ contains no irrationals but (a, b) does. and not closed since every irrational can be reached as a limit of rationals ($\sqrt{2}$ is a simple example).

(b) Clearly not open, but ironically closed since it has no limit points.

(c) Open since every $x \in \{x \in \mathbf{R} : x \neq 0\}$ has an ϵ-neighborhood around it excluding zero. But closed since $(1/n) \to 0$.

(d) Neither, not closed, as the limit $\sum_k^n 1/n^2 = \pi^2/6$ is irrational but every term is rational. and not open as it does not contain any irrationals.

(e) Closed as it has no limit points, every sequence diverges. Not open because it contains no irrationals.

Exercise 3.2.4
Let A be nonempty and bounded above so that $s = \sup A$ exists.

(a) Show that $s \in \overline{A}$.

(b) Can an open set contain its supremum?

Solution
(a) Since every $s - \epsilon$ has an $a \in A$ with $a > s - \epsilon$ we can find $a \in V_\epsilon(s)$ for any $\epsilon > 0$, meaning s is a limit point of A and thus contained in $\overline{A}$.

(b) No, as $(s, s + \epsilon)$ contains no a's we have $V_\epsilon(s) \not\subseteq A$ for every $\epsilon > 0$.

Exercise 3.2.5
Prove that a set $F \subseteq \mathbf{R}$ is closed *if and only if* every Cauchy sequence contained in F has a limit that is also an element of F.

Solution
Let $F \subseteq \mathbf{R}$ be closed and suppose (x_n) is a Cauchy sequence in F, since Cauchy sequences converge $(x_n) \to x$ and finally since $x \in F$ since F contains its limit points.

Now suppose every Cauchy sequence (x_n) in F converges to a limit in F and let l be a limit point of F, as l is a limit point of F there exists a sequence (y_n) in F with $\lim(y_n) = l$. since (y_n) converges it must be Cauchy, and since every Cauchy sequence converges to a limit inside F we have $l \in F$.

Exercise 3.2.6
Decide whether the following statements are true or false. Provide counterexamples for those that are false, and supply proofs for those that are true.

(a) An open set that contains every rational number must necessarily be all of $\mathbf{R}$.

(b) The Nested Interval Property remains true if the term "closed interval" is replaced by "closed set."

(c) Every nonempty open set contains a rational number.

(d) Every bounded infinite closed set contains a rational number.

(e) The Cantor set is closed.

Solution
(a) False, $A = (-\infty, \sqrt{2}) \cup (\sqrt{2}, \infty)$ contains every rational number but not $\sqrt{2}$.

(b) False, $C_n = [n, \infty)$ is closed, has $C_{n+1} \subseteq C_n$ and $C_n \neq \emptyset$ but $\bigcap_{n=1}^{\infty} C_n = \emptyset$.

(c) True, let $x \in A$ since A is open we have $(a, b) \subseteq A$ with $x \in (a, b)$ the density theorem implies there exists an $r \in \mathbf{Q}$ with $r \in (a, b)$ and thus $r \in A$.

(d) False, $A = \{1/n + \sqrt{2} : n \in \mathbf{N}\} \cup \{\sqrt{2}\}$ is closed and contains no rational numbers.

(e) True, as it is the intersection of countably many closed intervals.

Exercise 3.2.7
Given $A \subseteq \mathbf{R}$, let L be the set of all limit points of A.

(a) Show that the set L is closed.

(b) Argue that if x is a limit point of $A \cup L$, then x is a limit point of A. Use this observation to furnish a proof for Theorem 3.2.12.

Solution
(a) Every $x_n \in L$ is $x_n = \lim_{m\to\infty} a_{mn}$ for $a_{mn} \in A$. Meaning if $\lim x_n = x$ then for $n > N$ and $m > M$ we have

$$|a_{mn} - x| \leq |a_{mn} - x_n| + |x_n - x| < \epsilon/2 + \epsilon/2 = \epsilon$$

and thus x is a limit point of A, so $x \in L$.

(b) Let $x_n \in A \cup L$ and $x = \lim x_n$. Since x_n is infinite there must be at least one subsequence $(x_{n_k}) \to x$ which is either all in A or all in L. If every $x_{n_k} \in L$ then we know $x \in L$ from (a), and if every $x_{n_k} \in A$ then $x \in L$ aswell.

Exercise 3.2.8
Assume A is an open set and B is a closed set. Determine if the following sets are definitely open, definitely closed, both, or neither.

(a) $\overline{A \cup B}$

(b) $A \backslash B = \{x \in A : x \notin B\}$

(c) $(A^c \cup B)^c$

(d) $(A \cap B) \cup (A^c \cap B)$

(e) $\overline{A}^c \cap \overline{A^c}$

Solution
For all of these keep in mind the only open and closed sets are $\mathbf{R}$ and $\emptyset$, and if A is open A^c is closed and vise versa.

(a) Closed, since the closure of a set is closed.

(b) Open since B being closed implies B^c is open and thus $A \cap B^c$ is open as it is an intersection of open sets.

(c) De Morgan's laws give $(A^c \cup B)^c = A \cap B^c$ which is the same as (b)

(d) Closed since $(A \cap B) \cup (A^c \cap B) = B$

(e) Neither in general. Note that $\overline{A}^c \neq \overline{A^c}$ consider how $A = \{1/n : n \in \mathbf{N}\}$ has $\overline{A^c} = \mathbf{R}$ but $\overline{A}^c \neq \mathbf{R}$.

Exercise 3.2.9 (De Morgan's Laws)
A proof for De Morgan's Laws in the case of two sets is outlined in Exercise 1.2.5. The general argument is similar.

(a) Given a collection of sets $\{E_\lambda : \lambda \in \Lambda\}$, show that

$$\left(\bigcup_{\lambda \in \Lambda} E_\lambda\right)^c = \bigcap_{\lambda \in \Lambda} E_\lambda^c \quad \text{and} \quad \left(\bigcap_{\lambda \in \Lambda} E_\lambda\right)^c = \bigcup_{\lambda \in \Lambda} E_\lambda^c$$

(b) Now, provide the details for the proof of Theorem 3.2.14.

Solution

(a) If $x \in \left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c$ then $x \notin \bigcup_{\lambda\in\Lambda} E_\lambda$ meaning $x \notin E_\lambda$ for all $\lambda \in \Lambda$ implying $x \in E_\lambda^c$ for all $\lambda \in \Lambda$ and so finally $x \in \bigcap_{\lambda\in\Lambda} E_\lambda^c$. This shows

$$\left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c \subseteq \bigcap_{\lambda\in\Lambda} E_\lambda^c$$

To show the reverse inclusion suppose $x \in \bigcap_{\lambda\in\Lambda} E_\lambda^c$ then $x \notin E_\lambda$ for all λ meaning $x \notin \bigcup_{\lambda\in\Lambda} E_\lambda$ and so the reverse inclusion

$$\bigcap_{\lambda\in\Lambda} E_\lambda^c \subseteq \left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c$$

Is true, completing the proof.

(b) Let $F = F_1 \cup F_2$, if $x_n \in F$ and $x = \lim x_n$. Let (x_{n_k}) be a subsequence of (x_n) fully contained in F_1 or F_2. the subsequence (x_{n_k}) must also converge to x, meaning x is in F_1 or F_2, the rest is by induction.

Now let $F = \bigcap_{\lambda\in\Lambda} F_\lambda$

$$F^c = \bigcup_{\lambda\in\Lambda} F_\lambda^c$$

Each F_λ^c is open by Theorem 3.2.13, thus Theorem 3.2.3 (ii) implies F^c is open, and so $(F^c)^c = F$ is closed.

Exercise 3.2.10

Only one of the following three descriptions can be realized. Provide an example that illustrates the viable description, and explain why the other two cannot exist.

(i) A countable set contained in $[0, 1]$ with no limit points.

(ii) A countable set contained in $[0, 1]$ with no isolated points.

(iii) A set with an uncountable number of isolated points.

Solution

(i) Cannot exist because taking any sequence (x_n) BW tells us there exists a convergent subsequence.

(ii) $\mathbf{Q} \cap [0, 1]$ is countable and has no isolated points.

(iii) Impossible, let $A \subseteq \mathbf{R}$ and let x be an isolated point of A. From the definition there exists a $\delta > 0$ with $V_\delta(x) \cap A = \{x\}$. in Exercise 1.5.6 we proved there cannot exist an uncountable collection of disjoint open intervals, meaning we cannot have an uncountable set of isolated points as we can map them to open sets in a 1-1 fashion.

Exercise 3.2.11

(a) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(b) Does this result about closures extend to infinite unions of sets?

Solution

(a) Recall that the set of limit points of a set is closed (Exercise 3.2.7). Let L be the set of limit points of $A \cup B$ and let L_a, L_b be the set of limit points for A and B respectively.

Let $x \in L$, thus there exists a sequence $x_n \in A \cup B$ with $x = \lim x_n$, since (x_n) is infinite there exists a subsequence (x_{n_k}) where every term is in A or B. Thus the limit $\lim(x_{n_k}) = x$ must be a limit point of A or B meaning $x \in L_a \cup L_b$. This shows $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

Now let $x \in L_a$ (L_b is the same). there exists a sequence $x_n \in A$ with $x = \lim x_n$, now since $x_n \in A \cup B$ as well, $x \in L$. Thus we have shown $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$ completing the proof.

(b) False, take $A_n = \{1/n\}$ as a counterexample

$$\overline{\bigcup_{n=1}^{\infty} A_n} = \{1/n : n \in \mathbf{N}\} \cup \{0\}, \text{ but } \bigcup_{n=1}^{\infty} \overline{A_n} = \{1/n : n \in \mathbf{N}\}$$

Exercise 3.2.12

Let A be an uncountable set and let B be the set of real numbers that divides A into two uncountable sets; that is, $s \in B$ if both $\{x : x \in A \text{ and } x < s\}$ and $\{x : x \in A \text{ and } x > s\}$ are uncountable. Show B is nonempty and open.

Solution

Our primary tool will be that countably infinite unions preserve countability (see Exercise 1.5.3).

Consider $B_1 = \{x \in \mathbf{R} : (-\infty, x) \cap A \text{ is uncountable}\}$. B_1 must be nonempty; otherwise, $A = \bigcup_{n=1}^{\infty}(-\infty, n) \cap A$ is a union of countable or finite sets, which by 1.5.3 means that A is countable (which it isn't). Note that if $x \in B_1$ and $y > x$, then $y \in B_1$. Moreover, $\exists \epsilon > 0$ so that $x - \epsilon \in B_1$; we can prove this by contradiction. If there is no such ϵ, then $(-\infty, x - 1/n) \cap A$ must be countable for all $n \in \mathbf{N}$; by 1.5.3,

$$\bigcup_{n=1}^{\infty}(-\infty, x - 1/n) \cap A = (-\infty, x) \cap A$$

is also countable, a contradiction. Therefore, B_1 is open. Now note that B_1 must be of the form $(-\infty, b_1)$, where $b_1 = \inf B_1$ (or $-\infty$ if $\inf B_1$ is undefined). Similarly, $B_2 = \{x \in \mathbf{R} : (x, \infty) \cap A \text{ is uncountable}\}$ is of the form (b_2, ∞).

Note that $B_1 \cup B_2 = \mathbf{R}$; therefore $b_1 > b_2$ and so $B = B_1 \cap B_2 \neq \emptyset$. Moreover since B_1 and B_2 are both open, so is B.

Exercise 3.2.13

Prove that the only sets that are both open and closed are $\mathbf{R}$ and the empty set $\emptyset$.

Solution
Let $A \neq \emptyset$ be open and closed, and suppose for contradiction that $A \neq \mathbf{R}$ and $r \notin A$. Note that every closed set must contain its supremum and infimum, but Exercise 3.2.4b shows that every open set cannot contain its supremum or its infimum; thus A must be unbounded.

$A \cap (-\infty, r)$ is open and closed since $A \cap (-\infty, r)$ is an intersection of open sets, and $A \cap (-\infty, r) = A \cap (-\infty, r]$ (since $r \notin A$) is an intersection of closed sets. Moreover, since A is unbounded below, $A \cap (-\infty, r) \neq \emptyset$.

Attempting to take $s = \sup A \cap (-\infty, r)$ gives a contradiction, since $s \in A \cap (-\infty, r)$ (because closed and bounded above) we can find $\epsilon > 0$ with $V_\epsilon(s) \subseteq A \cap (-\infty, r)$ (because open) which contradictions s being an upper bound of $A \cap (-\infty, r)$.

Therefore if $A \neq \emptyset$ we must have $A = \mathbf{R}$. The converse is simple, suppose $A \neq \mathbf{R}$ is open and closed, this happens iff A^c is open and closed, but since $A^c \neq \emptyset$ we have $A^c = \mathbf{R}$ implying $A = \emptyset$.

Exercise 3.2.14
A dual notion to the closure of a set is the *interior* of a set. The interior of E is denoted E° and is defined as

$$E^\circ = \{x \in E : \text{ there exists } V_\epsilon(x) \subseteq E\}$$

Results about closures and interiors possess a useful symmetry.

(a) Show that E is closed if and only if $\overline{E} = E$. Show that E is open if and only if $E^\circ = E$.

(b) Show that $\overline{E}^c = (E^c)^\circ$, and similarly that $(E^\circ)^c = \overline{E^c}$.

Solution
(a) (i) If $E = \overline{E}$ then E contains its limit points and so is closed. If E is closed then E contains its limit points so $\overline{E} = E$.

(ii) If $E^\circ = E$ then every $x \in E$ has $V_\epsilon(x) \subseteq E$ therefore E is open. If E is open then every $x \in E$ has $V_\epsilon(x) \subseteq E$ therefore $E = E^\circ$.

(b) $x \in \overline{E}^c$ iff $x \notin E$ and x is not a limit point of E, $x \in (E^c)^\circ$ iff $x \notin E$ and there exists $V_\epsilon(x) \subseteq E^c$. Notice "x is not a limit point of E" is equivilant to "there exists $V_\epsilon(x) \subseteq E^c$" therefore the sets are the same.

To show $(E^\circ)^c = \overline{E^c}$ let $D = E^c$ yielding $((D^c)^\circ)^c = \overline{D}$ taking the complement of both sides yields $(D^c)^\circ = \overline{D}^c$ which we showed earlier.

Exercise 3.2.15
A set A is called an F_σ set if it can be written as the countable union of closed sets. A set B is called a G_δ set if it can be written as the countable intersection of open sets.

(a) Show that a closed interval $[a, b]$ is a G_δ set.

(b) Show that the half-open interval $(a, b]$ is both a G_δ and an F_σ set.

(c) Show that $\mathbf{Q}$ is an F_σ set, and the set of irrationals $\mathbf{I}$ forms a G_δ set. (We will see in Section 3.5 that $\mathbf{Q}$ is not a G_δ set, nor is $\mathbf{I}$ an F_σ set.)

Solution

(a) $[a,b] = \bigcap_{n=1}^{\infty}(a-1/n, b+1/n)$

(b) $(a,b] = \bigcap_{n=1}^{\infty}(a, b+1/n) = \bigcup_{n=1}^{\infty}[a+1/n, b]$

(c) Let r_n be an enumeration of $\mathbf{Q}$ (possible since $\mathbf{Q}$ is countable), we have

$$\mathbf{Q} = \bigcup_{n=1}^{\infty}[r_n, r_n]$$

Applying De Morgan's laws combined with the complement of a closed set being open we get

$$\mathbf{Q}^c = \bigcap_{n=1}^{\infty}[r_n, r_n]^c$$

3.3 Compact Sets

Exercise 3.3.1
Show that if K is compact and nonempty, then $\sup K$ and $\inf K$ both exist and are elements of K.

Solution
Let $s = \sup K$, since s is the least upper bound for every $\epsilon > 0$ there exists an $x \in K$ with $s - \epsilon < x$. Picking $\epsilon_n = 1/n$ and x_n such that $s - \epsilon_n < x_n$ we get that $(x_n) \to s$ since $(\epsilon_n) \to 0$, and thus $s \in K$.

A similar argument applies to $\inf K$.

Exercise 3.3.2
Decide which of the following sets are compact. For those that are not compact, show how Definition 3.3.1 breaks down. In other words, give an example of a sequence contained in the given set that does not possess a subsequence converging to a limit in the set.

(a) $\mathbf{N}$.

(b) $\mathbf{Q} \cap [0,1]$.

(c) The Cantor set.

(d) $\{1 + 1/2^2 + 1/3^2 + \cdots + 1/n^2 : n \in N\}$.

(e) $\{1, 1/2, 2/3, 3/4, 4/5, \ldots\}$.

Solution

(a) Not compact, the sequence $x_n = n$ in $\mathbf{N}$ has no convergent subsequence in $\mathbf{N}$.

(b) Not compact, as we can construct a sequence $(x_n) \to 1/\sqrt{2} \notin \mathbf{Q} \cap [0,1]$ implying K is not closed, and thus cannot be compact.

(c) Compact, since the cantor set is bounded and closed since it is the infinite intersection of closed sets $\bigcap_{n=1}^{\infty} C_n$ where $C_1 = [0, 1/3] \cup [2/3, 1]$ etc where you keep removing the middle thirds of each interval.

(d) Not compact as every sequence (x_n) contained in the set converges to $\pi^2/6$ which is not in the set, meaning the set isn't closed and thus cannot be compact.

(e) Compact since it is bounded and closed, with every sequence in the set converging to one.

Exercise 3.3.3
Prove the converse of Theorem 3.3.4 by showing that if a set $K \subseteq \mathbf{R}$ is closed and bounded, then it is compact.

Solution
Let K be closed and bounded and let (x_n) be a sequence contained in K. BW tells us a convergent subsequence $(x_{n_k}) \to x$ exists since K is bounded, and since K is closed $x \in K$. Thus every sequence in K contains a subsequence convering to a limit in K, which is the definition of K being compact.

Exercise 3.3.4
Assume K is compact and F is closed. Decide if the following sets are definitely compact, definitely closed, both, or neither.

(a) $K \cap F$

(b) $\overline{F^c \cup K^c}$

(c) $K \backslash F = \{x \in K : x \notin F\}$

(d) $\overline{K \cap F^c}$

Solution
(a) Compact since $K \cap F$ is closed (finite intersection of closed sets) and bounded (since K is bounded)

(b) Closed but not Compact since K being bounded implies K^c is unbounded, meaning $\overline{F^c \cup K^c}$ is unbounded.

(c) $K \setminus F = K \cap F^c$ could be either, if $K = [0, 1], F^c = (0, 1)$ then $K \cap F^c$ is open, but if $K = [0, 1]$ and $F^c = (-1, 2)$ then $K \cap F^c = [0, 1]$ is compact.

(d) Compact since $K \cap F^c$ is bounded (since K is bounded) implies $\overline{K \cap F^c}$ is closed (closure of a set is closed) and bounded (if A is bounded then $\overline{A}$ is also bounded).

Exercise 3.3.5
Decide whether the following propositions are true or false. If the claim is valid, supply a short proof, and if the claim is false, provide a counterexample.

(a) The arbitrary intersection of compact sets is compact.

(b) The arbitrary union of compact sets is compact.

(c) Let A be arbitrary, and let K be compact. Then, the intersection $A \cap K$ is compact.

(d) If $F_1 \supseteq F_2 \supseteq F_3 \supseteq F_4 \supseteq \cdots$ is a nested sequence of nonempty closed sets, then the intersection $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Solution

(a) True, as it will be bounded and closed (since arbitrary intersections of closed sets are closed).

(b) False, $\bigcup_{n=1}^{\infty}[0, n]$ is unbounded and thus not compact.

(c) False, let $K = [0, 1]$ and $A = (0, 1)$. The intersection $K \cap A = (0, 1)$ is not compact.

(d) False as $\bigcap_{n=1}^{\infty}[n, \infty) = \emptyset$ (It is true for compact sets though)

Exercise 3.3.6

This exercise is meant to illustrate the point made in the opening paragraph to Section 3.3. Verify that the following three statements are true if every blank is filled in with the word "finite." Which are true if every blank is filled in with the word "compact"? Which are true if every blank is filled in with the word "closed"?

(a) Every _____ set has a maximum.

(b) If A and B are _____, then $A + B = \{a + b : a \in A, b \in B\}$ is also _____

(c) If $\{A_n : n \in \mathbf{N}\}$ is a collection of _____sets with the property that every finite subcollection has a nonempty intersection, then $\bigcap_{n=1}^{\infty} A_n$ is nonempty as well.

Solution

(a) Finite (by taking the maximum value), compact (by taking the supremum, which exists because of boundedness and is in the set because of closed-ness), but not closed ($\mathbf{R}$ is closed)

(b) Finite (by just enumerating through all possibilities of $a + b$). Compact - boundedness is obviously preserved. For closed-ness, note that if $c \in A + B$ then we can find a convergent sequence $(a_n + b_n) \to c$ in $A + B$. Now since (a_n) is bounded there must be a convergent subsequence which converges to some $a \in A$, and taking the corresponding elements in (b_n) we have a new bounded sequence from which we can get a convergent subsequence which converges to some $b \in B$. Taking the corresponding elements in $(a_n + b_n)$ shows that $c = a + b$. This argument fails if A and B are only closed, as boundedness no longer applies. E.g. for $A = \{n : n \in \mathbf{N}\}$, $B = \{-n + 1/n : n \in \mathbf{N}\}$ then the sequence $(1/n)$ is in $A + B$ which converges to 0, which is not in $A + B$.

(c) Finite - the only way every finite subcollection has a nonempty intersection is if there is at least one element all sets include. Compact, since letting $K_n = \bigcap_{k=1}^{n} A_k$ gives

$K_n \subseteq K_{n-1}$, we also have $K_n \neq \emptyset$ since every finite intersection is known to be nonempty. Applying the Nested Compact Set Property allows us to conclude

$$\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} K_n \neq \emptyset$$

Not closed, e.g. let $A_n = [n, \infty)$

Exercise 3.3.7
As some more evidence of the surprising nature of the Cantor set, follow these steps to show that the sum $C + C = \{x + y : x, y \in C\}$ is equal to the closed interval $[0, 2]$. (Keep in mind that C has zero length and contains no intervals.)

Because $C \subseteq [0, 1], C + C \subseteq [0, 2]$, so we only need to prove the reverse inclusion $[0, 2] \subseteq \{x + y : x, y \in C\}$. Thus, given $s \in [0, 2]$, we must find two elements $x, y \in C$ satisfying $x + y = s$

(a) Show that there exist $x_1, y_1 \in C_1$ for which $x_1 + y_1 = s$. Show in general that, for an arbitrary $n \in \mathbf{N}$, we can always find $x_n, y_n \in C_n$ for which $x_n + y_n = s$.

(b) Keeping in mind that the sequences (x_n) and (y_n) do not necessarily converge, show how they can nevertheless be used to produce the desired x and y in C satisfying $x + y = s$.

Solution

(a) Recall the definition for sets A and B, $A + B = \{x + y : x \in A, \ y \in B\}$. Note that $[a, b] + [c, d] = [a + c, b + d]$. Recall $C_1 = [0, 1/3] \cup [2/3, 1]$; we have

$$\begin{aligned} C_1 + C_1 &= ([0, 1/3] + [0, 1/3]) \cup ([2/3, 1] + [2/3, 1]) \cup ([0, 1/3] + [2/3, 1]) \\ &= [0, 2/3] \cup [4/3, 2] \cup [2/3, 4/3] \\ &= [0, 2] \end{aligned}$$

We prove the general case through induction; the base case has been demonstrated above. Define $A + r = \{x + r : x \in A\}$, $r \cdot A = \{rx : x \in A\}$ for $r \in \mathbf{R}$. Note that $[a, b] + c = [a + c, b + c]$ and $r \cdot (A + B) = r \cdot A + r \cdot B$.

The inductive hypothesis is that $C_n + C_n = [0, 2]$. We can complete the inductive step by noticing that if we scale up C_{n+1} a factor of 3, we simply get two copies of C_n, with one being offset 2 away; this ultimately makes it easy to express C_{n+1} in terms of C_n. Let $C'_{n+1} = C_n \cup (C_n + 2) = 3 \cdot C_{n+1}$ be this upscaled C_{n+1}. Then

$$\begin{aligned} C'_{n+1} + C'_{n+1} &= (C_n + C_n) \cup (C_n + (C_n + 2)) \cup ((C_n + 2) + (C_n + 2)) \\ &= [0, 2] \cup ([0, 2] + 2) \cup ([0, 2] + 4) \\ &= [0, 6] \end{aligned}$$

$$\frac{1}{3} \cdot (C'_{n+1} + C'_{n+1}) = C_{n+1} + C_{n+1} = \frac{1}{3} \cdot [0, 6] = [0, 2]$$

and the inductive step is complete.

(b) Since C is compact, there exists a subsequence $(x_{n_k}) \to x$ with $x \in C$. Now since $x_{n_k} + y_{n_k} = s$ for all k, we have $\lim y_{n_k} = \lim s - x_{n_k} = s - x$. Now since each $y_{n_k} \in C$ the limit $y = s - x \in C$ as well, thus we have found $x, y \in C$ with $x + y = s$.

Exercise 3.3.8

Let K and L be nonempty compact sets, and define

$$d = \inf\{|x - y| : x \in K \text{ and } y \in L\}$$

This turns out to be a reasonable definition for the distance between K and L.

(a) If K and L are disjoint, show $d > 0$ and that $d = |x_0 - y_0|$ for some $x_0 \in K$ and $y_0 \in L$.

(b) Show that it's possible to have $d = 0$ if we assume only that the disjoint sets K and L are closed.

Solution

(a) The set $|K - L| = \{|x - y| : x \in K, y \in L\}$ is compact since $K - L$ is compact by 3.3.6 (b) and $|\cdot|$ preserves compactness. Thus $d = \inf |K - L|$ has $d = |x_0 - y_0|$ for some $x_0 \in K$ and $y_0 \in L$.

(b) $K = \{n : n \in \mathbf{N}\}$ and $L = \{n + 1/n : n \in \mathbf{N}\}$ have $d = 0$, and both are closed since every limit diverges.

Exercise 3.3.9

Follow these steps to prove that being compact implies every open cover has a finite subcover.

Assume K is compact, and let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover for K. For contradiction, let's assume that no finite subcover exists. Let I_0 be a closed interval containing K.

(a) Show that there exists a nested sequence of closed intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ with the property that, for each n, $I_n \cap K$ cannot be finitely covered and $\lim |I_n| = 0$.

(b) Argue that there exists an $x \in K$ such that $x \in I_n$ for all n.

(c) Because $x \in K$, there must exist an open set O_{λ_0} from the original collection that contains x as an element. Explain how this leads to the desired contradiction.

Solution

(a) Bisect I_0 into two intervals, and let I_1 be the interval where $I_1 \cap K$ cannot be finitely covered. Repating in this fashion we have $\lim |I_n| = \lim |I_0|(1/2)^n = 0$.

(b) The nested compact set property with $K_n = I_n \cap K$ gives $x \in \bigcap_{n=1}^\infty K_n$ meaning $x \in K$ and $x \in I_n$ for all n.

(c) Since $x \in O_{\lambda_0}$ and $|I_n| \to 0$ with $x \in I_n$ for all n, there exists an N where $n > N$ implies $I_n \subseteq O_{\lambda_0}$ contradicting the assumption that $I_n \cap K$ cannot be finitely covered since $\{O_{\lambda_0}\}$ is a finite subcover for $I_n \cap K$.

Exercise 3.3.10
Here is an alternate proof to the one given in Exercise 3.3.9 for the final implication in the Heine-Borel Theorem.

Consider the special case where K is a closed interval. Let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover for $[a, b]$ and define S to be the set of all $x \in [a, b]$ such that $[a, x]$ has a finite subcover from $\{O_\lambda : \lambda \in \Lambda\}$.

(a) Argue that S is nonempty and bounded, and thus $s = \sup S$ exists.

(b) Now show $s = b$, which implies $[a, b]$ has a finite subcover.

(c) Finally, prove the theorem for an arbitrary closed and bounded set K.

Solution

(a) S is nonempty since $x = a$ has the finite subcover $\{O_{\lambda_0}\}$ for $a \in O_{\lambda_0}$. S is bounded since $x \leq b$ for all $x \in S$.

(b) Suppose for contradiction that $s < b$, letting $s \in O_{\lambda_0}$ implies $[a, s]$ is finitely coverable since we can take the finite cover of an $x \in O_{\lambda_0}$ with $x < s$. This is causes a contradiction however since there exist points $y > s$ with $y \in O_{\lambda_0}$ meaning $[a, y]$ is also finitely coverable. therefore the only option is $s = b$, since any $s < b$ doesn't work.

(c) (a) still works, for (b) we must also consider the case where y does not exist / there is a gap. Let $y = \inf\, [s, b] \cap K$ and suppose $y \neq s$. since $y \in [s, b] \cap K$ we know

$$[a, y] \cap K = ([a, s] \cap K) \cup ((s, y] \cap K) = [a, s] \cap K \cup \{y\}$$

therefore if $\{O_{\lambda_1}, \dots, O_{\lambda_n}\}$ covered $[a, s]$ then letting $y \in O_{\lambda_{n+1}}$ would give the finite cover $\{O_{\lambda_1}, \dots, O_{\lambda_{n+1}}\}$ contradicting the assumption that $s < b$, therefore $s = b$ is the only option, and so K can be finitely covered.

Exercise 3.3.11
Consider each of the sets listed in Exercise 3.3.2. For each one that is not compact, find an open cover for which there is no finite subcover.

Solution

(a) $\mathbf{N}$ and $\{V_1(n) : n \in \mathbf{N}\}$ has no finite subcover since each $V_1(n)$ covers exactly one $n \in \mathbf{N}$, meaning there are no subcovers at all!

(b) $\mathbf{Q} \cap [0, 1]$: Choose some $y \in \mathbf{Q}^c \cap [0, 1]$, for example $y = \sqrt{2}/2$. Consider the open cover $\{(-1, y)\} \cup \{(y + 1/n, 2) : n \in \mathbf{N}\}$. Since $\mathbf{Q}$ is dense in $\mathbf{R}$, for any finite subcover there must be some rational number $q \in (y, y + 1/n)$ where n is finite.

(c) The Cantor is compact

(d) $K = \{1 + 1/2^2 + 1/3^2 + \cdots + 1/n^2 : n \in \mathbf{N}\}$ and $\left\{V_{\frac{1}{2}|x-L|}(x) : x \in K\right\}$ for $L = \pi^2/6$ since any finite cover $\left\{V_{\frac{1}{2}|x_1-L|}(x_1), \dots, V_{\frac{1}{2}|x_n-L|}(x_n)\right\}$, letting$\epsilon = \min\{\frac{1}{2}|x_i - L|\}$ will make $V_\epsilon(L)$ not in the finite cover, meaning there exists an $x \in V_\epsilon(L)$ with $x \in K$ (since K gets arbitrarily close to L) but x not in the finite cover.

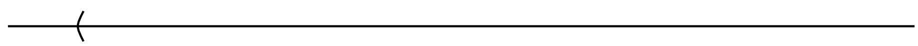
(e) $\{1, 1/2, 2/3, 3/4, 4/5, \ldots\}$ is compact

Exercise 3.3.12
Using the concept of open covers (and explicitly avoiding the Bolzano-Weierstrass Theorem), prove that every bounded infinite set has a limit point.

Solution
Let A be an infinite set bounded by M (i.e. $|x| < M$ for all $x \in A$), suppose for contradiction that A has no limit points, meaning there exists an $\epsilon > 0$ such that $V_\epsilon(x) \cap A = \{x\}$ for all $x \in A$.

This immediately implies there are only a finite number of sets in our cover, otherwise the union would be unbounded. Contradiction.

TODO Finish tikz picture and proof (its visually obivous)



Exercise 3.3.13
Let's call a set *clompact* if it has the property that every *closed* cover (i.e., a cover consisting of closed sets) admits a finite subcover. Describe all of the clompact subsets of $\mathbf{R}$.

Solution
K is clompact if and only if K is finite, since the closed cover $\{[x, x] : x \in K\} = K$ having a finite subcover implies $\{[x, x] : x \in K\}$ is finite (since it is the only subcover that works) therefore K is finite. If K is finite then it obviously permits a finite subcover.

3.4 Perfect Sets and Connected Sets

Exercise 3.4.1
If P is a perfect set and K is compact, is the intersection $P \cap K$ always compact? Always perfect?

Solution
Recall a perfect set is a closed set with no isolated points. Thus the intersection of a closed set P and a closed bounded set K gives a closed bounded (and thus compact) set $P \cap K$.

Now take $P = \mathbf{R}$, we get $P \cap K = K$ which is not nessesarily perfect.

Exercise 3.4.2
Does there exist a perfect set consisting of only rational numbers?

Solution
No, since any nonempty set $P \subseteq \mathbf{Q}$ is countable but, nonempty perfect sets are uncountable by Theorem 3.4.3

Exercise 3.4.3
Review the portion of the proof given in Example 3.4.2 and follow these steps to complete the argument.

(a) Because $x \in C_1$, argue that there exists an $x_1 \in C \cap C_1$ with $x_1 \neq x$ satisfying $|x - x_1| \leq 1/3$.

(b) Finish the proof by showing that for each $n \in \mathbf{N}$, there exists $x_n \in C \cap C_n$, different from x, satisfying $|x - x_n| \leq 1/3^n$.

Solution

(a) Noting that C_1 is the union of disjoint intervals of length $1/3$, and that C_2 divides each interval in C_1 into two, consider the intervals $[a, b] \subseteq C_1$ and $[c, d] \subseteq C_2$ that x is in. Then choose x_1 to be any other point $c \in C \cap ([a, b] \backslash [c, d])$ - i.e. it shares an interval with x in C_1 but is in a different interval in C_2; therefore it must be within $1/3$ of x but is different from x.

(b) Identical argument to part (a), replacing C_1 with C_n, C_2 with C_{n+1}, $1/3$ with $1/3^n$, and x_1 with x_n.

Exercise 3.4.4

Repeat the Cantor construction from Section 3.1 starting with the interval $[0, 1]$. This time, however, remove the open middle *fourth* from each component.

(a) Is the resulting set compact? Perfect?

(b) Using the algorithms from Section 3.1, compute the length and dimension of this Cantor-like set.

Solution

(a) The proofs that the Cantor set is compact and perfect can be copied and applied nearly word for word here. The Cantor-like set is obviously bounded, and it is closed because it is the intersection of countably many closed sets (see Exercise 3.2.6e); therefore it must be compact. Using the same strategy as Exercise 3.4.3, for any x in the Cantor-like set, we can find a sequence (x_n) where $x_n \neq x$ but $|x_n - x| \leq (3/8)^n$.

(b) The sum of the lengths of the removed segments is

$$\frac{1}{4} + \left(\frac{1}{4}\right)\left(\frac{3}{4}\right) + \cdots + \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{n-1} + \cdots = 1$$

and thus the Cantor-like set has zero length.

Magnifying the Cantor-like set by a factor of $8/3$ leaves us with two copies of the set, hence the dimension $d = \log 2 / \log(8/3) \approx 0.707$.

Exercise 3.4.5

Let A and B be nonempty subsets of $\mathbf{R}$. Show that if there exist disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$, then A and B are separated.

Solution

Disjoint open sets are separated, therefore so are their subsets.

Exercise 3.4.6
Prove that A set $E \subseteq \mathbf{R}$ is connected if and only if, for all nonempty disjoint sets A and B satisfying $E = A \cup B$, there always exists a convergent sequence $(x_n) \to x$ with (x_n) contained in one of A or B, and x an element of the other. (Theorem 3.4.6)

Solution
Both are obvious if you think about the definitions, here's some formal(ish) garbage though

Suppose $\overline{A} \cup B$ is nonempty and let x be an element in both, $x \in B$ implies $x \notin A$ therefore $x \in L$ (the set of limit points of A) meaning there must exist a sequence $(x_n) \to x$ contained in A.

Now suppose there exists an $(x_n) \to x$ in A with limit in B, then clearly $\overline{A} \cap B \subseteq \{x\}$ is nonempty.

Exercise 3.4.7
A set E is totally disconnected if, given any two distinct points $x, y \in E$, there exist separated sets A and B with $x \in A, y \in B$, and $E = A \cup B$.

(a) Show that $\mathbf{Q}$ is totally disconnected.

(b) Is the set of irrational numbers totally disconnected?

Solution
(a) Let $x, y \in \mathbf{Q}$, and let $z \in (x, y)$ with $z \in \mathbf{I}$. The sets $A = (-\infty, z) \cap \mathbf{Q}$ and $B = (z, \infty) \cap \mathbf{Q}$ are separated and have $A \cup B = \mathbf{Q}$.

(b) Now let $x, y \in \mathbf{I}$, and let $z \in (x, y)$ with $z \in \mathbf{Q}$. The sets $A = (-\infty, z) \cap \mathbf{I}$ and $B = (z, \infty) \cap \mathbf{I}$ are separated and have $A \cup B = \mathbf{I}$.

Exercise 3.4.8
Follow these steps to show that the Cantor set is totally disconnected in the sense described in Exercise 3.4.7. Let $C = \bigcap_{n=0}^{\infty} C_n$, as defined in Section 3.1.

(a) Given $x, y \in C$, with $x < y$, set $\epsilon = y - x$. For each $n = 0, 1, 2, \ldots$, the set C_n consists of a finite number of closed intervals. Explain why there must exist an N large enough so that it is impossible for x and y both to belong to the same closed interval of C_N.

(b) Show that C is totally disconnected.

Solution
(a) Since the length of every interval goes to zero, we set N large enough that the length of every interval is less then ϵ, meaning x and y cannot be in the same interval.

(b) Obvious

Exercise 3.4.9
Let $\{r_1, r_2, r_3, \ldots\}$ be an enumeration of the rational numbers, and for each $n \in \mathbf{N}$ set $\epsilon_n = 1/2^n$. Define $O = \bigcup_{n=1}^{\infty} V_{\epsilon_n}(r_n)$, and let $F = O^c$.

(a) Argue that F is a closed, nonempty set consisting only of irrational numbers.

(b) Does F contain any nonempty open intervals? Is F totally disconnected? (See Exercise 3.4.7 for the definition.)

(c) Is it possible to know whether F is perfect? If not, can we modify this construction to produce a nonempty perfect set of irrational numbers?

Solution

(a) O is a union of an arbitrary collection of open sets and therefore is open; therefore $O^c = F$ is closed. O contains all rational numbers, therefore F must consist only of irrational numbers. Intuitively, F must be nonempty, because the sum of the lengths of the intervals in O is 2 and can't cover the infinite real line - but the properties of interval lengths haven't been rigorously defined, and maybe something weird can happen (e.g. Banach-Tarski paradox), so it's best to prove it.

Let $O_n = \bigcup_{i=1}^{n} V_{\epsilon_i}(r_i)$, $F_n = O_n^c$, and $G_n = F_n \cap [1,3)$. Clearly $G_n \supseteq G_{n+1}$ and G_n is compact, so if we can show G_n is always nonempty, we can use the Nested Compact Set Property to show $F = \bigcap_{n=1}^{\infty} F_n \supseteq \bigcap_{n=1}^{\infty} G_n$ is nonempty.

Consider the set $M = \{m/2^{n-1} : m \in \mathbf{N}\} \cup [1,3)$, a set of evenly spaced numbers with $1/2^{n-1}$ between each number. Some arithmetic shows that there are 2^n elements in M. Since the length of O_i is $1/2^{i-1}$, there can only be at most 2^{n-i} elements of M in O_i, and therefore $2^n - 1$ elements of M in O. Since $G_n \supseteq M \backslash O$, G_n is nonempty, completing the proof.

(b) Since F contains no rational numbers, and any nonempty open interval will contain rational numbers (since $\mathbf{Q}$ is dense in $\mathbf{R}$), F cannot contain any open interval. The proof that F is totally disconnected is the same as that for $\mathbf{I}$ in Exercise 3.4.7b.

(c) It is possible for F to be not perfect. Our approach will be to create an isolated point in F, say, $\sqrt{2}$. To do so, we need to spilt the sequence of sets in O into three parts; the first will have their upper limits approach $\sqrt{2}$ from below, the second will have their lower limits approach $\sqrt{2}$ from above, and the third will be used to enumerate through the rest of the rational numbers.

First, we prove the following lemma - for any $L \in \mathbf{R}$ and a positive sequence $(\zeta_n) \to 0$, it is possible to construct a sequence $(x_n) \in \mathbf{Q}$ such that $[L, L+2\zeta_1) \backslash \bigcup_{i=1}^{\infty} V_{\zeta_i}(x_i) = \{L\}$. (We'll use this lemma for the sequences isolating $\sqrt{2}$.) To do so, let $V_{\zeta_n}(x_n) = (\alpha_n, \beta_n)$. Choose $L < \alpha_n < L + 2\zeta_{n+1}$.

We show by induction that $[L, L+2\zeta_1) \backslash \bigcup_{i=1}^{n} V_{\zeta_i}(x_i) \subseteq [L, L+2\zeta_{n+1})$. The base case $n = 1$ is trivial. For the inductive case: assume $[L, L+2\zeta_1) \backslash \bigcup_{i=1}^{n} V_{\zeta_i}(x_i) \subseteq [L, L+2\zeta_{n+1})$. Then

$$[L, L+\zeta_1) \backslash \bigcup_{i=1}^{n+1} V_{\zeta_i}(x_i) \subseteq [L, L+2\zeta_{n+1}) \backslash V_{\zeta_{n+1}}(x_{n+1}) = [L, L+2\zeta_{n+1}) \backslash (\alpha_{n+1}, \beta_{n+1})$$

Since $\alpha_{n+1} > L$, $\beta_{n+1} > L + 2\zeta_{n+1}$. Recall also that $\alpha_{n+1} < L + 2\zeta_{n+2}$, and therefore $[L, L+2\zeta_{n+1}) \backslash (\alpha_{n+1}, \beta_{n+1}) \subseteq [L, L+2\zeta_{n+2})$, completing the inductive step.

Now, for any $l \in [L, L+2\zeta_1) > L$, since $(\zeta_n) \to 0$ there must be some ζ_j so that $L + 2\zeta_j \leq l$ and $j > 1$, and therefore

$$\begin{aligned} & l \notin [L, L+2\zeta_j) \\ & \supseteq [L, L+2\zeta_1) \setminus \bigcup_{i=1}^{j-1} V_{\zeta_i}(x_i) \text{ (by induction above)} \\ & \supseteq [L, L+2\zeta_1) \setminus \bigcup_{i=1}^{\infty} V_{\zeta_i}(x_i) \\ & \implies l \notin [L, L+2\zeta_1) \setminus \bigcup_{i=1}^{\infty} V_{\zeta_i}(x_i) \end{aligned}$$

Also, since $\alpha_n > L \ \forall n$, $L \in [L, L+2\zeta_1) \setminus \bigcup_{i=1}^{\infty} V_{\zeta_i}(x_i)$. This completes the proof of the lemma.

Returning to the original problem of making F not perfect, we will construct a sequence (r_n) which isolates $\sqrt{2}$. Start with any enumeration of the rational numbers q_n. Our lemma above means we can assign r_{3n+1} and ϵ_{3n+1} for $n \geq 0$ to ensuring $(\sqrt{2}, \sqrt{2}+2\epsilon_1) \in O$ but $\sqrt{2}$ itself not in O. Similarly, a slight modification to the lemma lets us assign r_{3n+2} and ϵ_{3n+2} for $n \geq 0$ to ensuring $(\sqrt{2}-2\epsilon_2, \sqrt{2}) \in O$ while leaving $\sqrt{2}$ out. Finally, we assign r_{3n+3}, $n \geq 0$ to enumerating through the elements of q_n, skipping over any elements that will be present in r_{3n+1} and r_{3n+2}, and deferring any elements that would cause $\sqrt{2} \in V(q)$ until ϵ becomes small enough that this is no longer the case.

In this manner, $\sqrt{2}$ has been surrounded, and $F = O^c$ will have $\sqrt{2}$ as an isolated point, and thus F is not perfect.

We can also construct (r_n) so that F is perfect. (Note: To simplify the notation a bit let $V_i = V_{\epsilon_i}(r_i)$.) Define $R_n = \bigcup_{i=1}^{n} V_i$, with $R_0 = \emptyset$. We will rely on the lemma that F is perfect if for all $i \in \mathbf{N}$, either $V_i \cap R_{i-1} = \emptyset$ or $V_i \subseteq R_{i-1}$. Informally, if V_i isn't redundant (in that it covers new numbers), then it is disjoint from all previous V_i. To prove this lemma, consider any element $x \in F$ and let $\epsilon > 0$ be arbitrary. Consider the interval $(x, x+\epsilon)$, and ignore any V_p if $V_p \subseteq R_{p-1}$. Since none of V_i overlap partially, this interval cannot be covered completely by some union of several V_i since any union would have gaps. Moreover, $(x, x+\epsilon)$ cannot be covered entirely by a single V_{ϵ_j}, since then V_j would be centered on $r_j = x + \epsilon_j$ which is irrational. Thus, there must be some other element $y \neq x$, $y \in F$ so that $y \in (x, x+\epsilon)$ and therefore x is not an isolated point and F must be perfect.

To construct an (r_n) which satisfies this condition, we start with an arbitrary (q_n). Define $R_n = \bigcup_{i=1}^{n} r_i$. For each element of (q_n), we add q_i to (r_n) only if either $V_{\epsilon_n}(q_i) \subseteq R_n$ or $V_{\epsilon_n}(q_i) \cap R_n = \emptyset$. Otherwise, we procrastinate on adding q_i by appending any rational number $s > \max(R_n) + 2\epsilon_n$. Clearly, for any q_i there will eventually be ϵ_n small enough that q_i can be added to (r_n) without violating our restrictions, and we don't need to worry about s's being added since they're far enough away from everything that they can't affect the restrictions.

3.5 Baire's Theorem

Exercise 3.5.1
Argue that a set A is a G_δ set if and only if its complement is an F_σ set.

Solution
If A is a G_δ set, then A^c is a F_σ set by De Morgan's laws. Likewise if A is an F_σ set then A^c must be a G_δ set.

Exercise 3.5.2
Replace each with the word finite or countable, depending on which is more appropriate.

(a) The ______ union of F_σ sets is an F_σ set.

(b) The ______ intersection of F_σ sets is an F_σ set.

(c) The ______ union of G_δ sets is a G_δ set.

(d) The ______ intersection of G_δ sets is a G_δ set.

Solution
(a) Countable, since two countable union can be written as a single countable union over the diagonal (see Exercise 1.2.4). Another way of seeing this is that we can form a bijection between $\mathbf{N}$ and $\mathbf{N}^2$, therefore a double infinite union can be written as a single infinite union.

(b) Finite

(c) Finite

(d) Countable, by the same logic as in (a) we can write two countable intersections as a single countable intersection.

Exercise 3.5.3
(a) Show that a closed interval $[a, b]$ is a G_δ set.

(b) Show that the half-open interval $(a, b]$ is both a G_δ and an F_σ set.

(c) Show that $\mathbf{Q}$ is an F_σ set, and the set of irrationals I forms a G_δ set.

Solution
This exercise has already appeared as Exercise 3.2.15.

Exercise 3.5.4
Let $\{G_1, G_2, G_3, \ldots\}$ be a countable collection of dense, open sets, we will prove that the intersection $\bigcap_{n=1}^{\infty} G_n$ is not empty.

Starting with $n = 1$, inductively construct a nested sequence of closed intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ satisfying $I_n \subseteq G_n$. Give special attention to the issue of the endpoints of each I_n. Show how this leads to a proof of the theorem.

Solution
Because G_1 is open there exists an open interval $(a_1, b_1) \subseteq G_1$, letting $[c_1, d_1]$ be a closed interval contained in (a_1, b_1) gives $I_1 \subseteq G_1$ as desired.

Now suppose $I_n \subseteq G_n$. because G_{n+1} is dense and $(c_n, d_n) \cap G_{n+1}$ is open there exists an interval $(a_{n+1}, b_{n+1}) \subseteq G_n \cap (c_n, d_n)$. Letting $[c_{n+1}, d_{n+1}] \subseteq (a_{n+1}, b_{n+1})$ gives us our new closed interval.

This gives us our collection of sets with $I_{n+1} \subseteq I_n$, $I_n \subseteq G_n$ and $I_n \neq \emptyset$ allowing us to apply the Nested Interval Property to conclude

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

and thus $\bigcap_{n=1}^{\infty} G_n \neq \emptyset$ since each $I_n \subseteq G_n$.

Exercise 3.5.5
Show that it is impossible to write

$$\mathbf{R} = \bigcup_{n=1}^{\infty} F_n$$

where for each $n \in \mathbf{N}, F_n$ is a closed set containing no nonempty open intervals.

Solution
This is just the complement of Exercise 3.5.4, If we had $\mathbf{R} = \bigcup_{n=1}^{\infty} F_n$ then we would also have

$$\emptyset = \bigcap_{n=1}^{\infty} G_n$$

for $G_n = F_n^c$. G_n is open as it is the complement of a closed set, and since F_n contains no nonempty open intervals G_n is dense. This contradicts $\bigcap_{n=1}^{\infty} G_n \neq \emptyset$ from 3.5.4.

To be totally rigorous we still have to justify F_n^c being dense. Let $a, b \in \mathbf{R}$ with $a < b$, since $(a, b) \not\subseteq F_n$ there exists a $c \in (a, b)$ with $c \in F_n^c$ and thus F_n^c is dense.

Exercise 3.5.6
Show how the previous exercise implies that the set $\mathbf{I}$ of irrationals cannot be an F_σ set, and $\mathbf{Q}$ cannot be a G_δ set.

Solution
Recall from 3.5.3 that $\mathbf{Q}$ is an F_σ set, suppose for contradiction that $\mathbf{I}$ were also an F_σ set. Then we could write

$$\mathbf{Q} = \bigcup_{n=1}^{\infty} F_n \quad \text{and} \quad \mathbf{I} = \bigcup_{n=1}^{\infty} F_n'$$

Each F_n and F_n' must contain no nonempty open intervals, since otherwise F_n would contain irrationals and vise versa. Combine the countable unions by setting $\tilde{F}_{2n} = F_n$ and $\tilde{F}_{2n-1} = F_n'$ to get

$$\mathbf{R} = \mathbf{Q} \cup \mathbf{I} = \bigcup_{n=1}^{\infty} \tilde{F}_n$$

But in 3.5.5 we showed

$$\mathbf{R} \neq \bigcup_{n=1}^{\infty} F_n$$

which gives our desired contradiction, hence $\mathbf{I}$ is not an F_σ set and $\mathbf{Q}$ is not a G_δ set (take complements).

Exercise 3.5.7
Using Exercise 3.5.6 and versions of the statements in Exercise 3.5.2, construct a set that is neither in F_σ nor in G_δ.

Solution
For a set $A \subseteq \mathbf{R}$ define $-A = \{-x : x \in A\}$. Note that if A is closed (open), then so is $-A$, and if A is a F_σ set (G_δ), so is $-A$.

Define $I^+ = \mathbf{I} \cap [0, \infty)$, $I^- = \mathbf{I} \cap (-\infty, 0]$, and Q^+ and Q^- be defined similarly for $\mathbf{Q}$. Consider the set $A = I^+ \cup Q^-$. If A is a F_σ set, then by Exercise 3.5.2 so is $A \cap [0, \infty) = I^+$, but so is $I^+ \cup -I^+ = I^+ \cup I^- = \mathbf{I}$, which is a contradiction. Similarly, A being a G_δ set implies $A \cap (-\infty, 0] = Q^-$ and $Q^- \cup -Q^- = \mathbf{Q}$ are both G_δ sets, a contradiction.

Exercise 3.5.8
Show that a set E is nowhere-dense in $\mathbf{R}$ if and only if the complement of $\overline{E}$ is dense in $\mathbf{R}$.

Solution
First suppose E is nowhere-dense, then $\overline{E}$ contains no nonempty open intervals meaning for every $a, b \in \mathbf{R}$ we have $(a, b) \not\subseteq \overline{E}$ meaning we can find a $c \in (a, b)$ with $c \notin \overline{E}$. But this is just saying $c \in \overline{E}^c$ which implies $\overline{E}^c$ is dense since for every $a, b \in \mathbf{R}$ we can find a $c \in \overline{E}^c$ with $a < c < b$.

Now suppose $\overline{E}^c$ is dense in $\mathbf{R}$, then then every interval (a, b) contains a point $c \in \overline{E}^c$, implying that $(a, b) \not\subseteq \overline{E}$ since $c \notin \overline{E}$ and $c \in (a, b)$. therefore $\overline{E}$ contains no nonempty open intervals and so E is nowhere-dense by defintion 3.5.3.

Exercise 3.5.9
Decide whether the following sets are dense in $\mathbf{R}$, nowhere-dense in $\mathbf{R}$, or somewhere in between.

(a) $A = \mathbf{Q} \cap [0, 5]$.

(b) $B = \{1/n : n \in \mathbf{N}\}$.

(c) the set of irrationals.

(d) the Cantor set.

Solution

(a) between, since A is dense in $[0, 5]$ but not in all of $\mathbf{R}$.

(b) nowhere-dense since $\overline{B} = B \cup \{0\}$ contains no nonempty open intervals

(c) dense since $\overline{\mathbf{I}} = \mathbf{R}$

(d) nowhere-dense since the Cantor set is closed, so $\overline{C} = C$, and C contains no intervals

Exercise 3.5.10 (Baire's Theorem)
Prove set of real numbers $\mathbf{R}$ cannot be written as the countable union of nowhere-dense sets.

To start, assume that $E_1, E_2, E_3, \ldots$ are each nowhere-dense and satisfy $\mathbf{R} = \bigcup_{n=1}^{\infty} E_n$ then find a contradiction to the results in this section.

Solution
By the definition of E_n being nowhere-dense, the closure $\overline{E_n}$ contains no nonempty open intervals meaning we can apply Exercise 3.5.5 to conclude that

$$\bigcup_{n=1}^{\infty} \overline{E_n} \neq \mathbf{R}$$

Since each $E_n \subseteq \overline{E_n}$ we have

$$\bigcup_{n=1}^{\infty} E_n \neq \mathbf{R}$$

as desired.

Chapter 4

Functional Limits and Continuity

4.2 Functional Limits

Exercise 4.2.1

(a) Supply the details for how Corollary 4.2.4 part (ii) follows from the Sequential Criterion for Functional Limits in Theorem 4.2.3 and the Algebraic Limit Theorem for sequences proved in Chapter 2.

(b) Now, write another proof of Corollary 4.2.4 part (ii) directly from Definition 4.2.1 without using the sequential criterion in Theorem 4.2.3.

(c) Repeat (a) and (b) for Corollary 4.2.4 part (iii).

Solution

(a) By the Sequential Criterion for Functional Limits, since $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = M$, all sequences $(x_n) \to c$ (where every $x_n \neq c$) have $f(x_n) \to L$ and $g(x_n) \to M$, which implies that $f(x_n) + g(x_n) \to L + M$ by the Algebraic Limit Theorem, which implies that $\lim_{x\to c}[f(x) + g(x)] = L + M$.

(b) Let $\epsilon > 0$, set δ_1 such that $0 < |x - c| < \delta_1$ implies $|f(x) - f(c)| < \epsilon/2$ and set δ_2 such that $0 < |x - c| < \delta_2$ implies $|g(x) - g(c)| < \epsilon/2$. Now let $\delta = \min\{\delta_1, \delta_2\}$ and use the triangle inequality to get

$$|f(x) + g(x) - (f(c) + g(c))| \leq |f(x) - f(c)| + |g(x) - g(c)| < \epsilon/2 + \epsilon/2 = \epsilon$$

for all $0 < |x - c| < \delta$.

(c) (a) is the same, if $f(x_n) \to L$ and $g(x_n) \to M$ then $(f(x_n)g(x_n)) \to LM$ by the sequential criterion for functional limits.

For (b) we add and subtract $f(c)g(x)$ then factor and use the triangle inequality (this is a common trick)

$$\begin{aligned}|f(x)g(x) - f(c)g(c)| &= |g(x)(f(x) - f(c)) - f(c)(g(c) - g(x))| \\ &\leq |g(x)||f(x) - f(c)| - |f(c)||g(c) - g(x)|\end{aligned}$$

Now we want a few things, (1) to bound $|g(x)|$ (2) to make $|f(x) - f(c)|$ small and (3) to make $|g(x) - g(c)|$ small. Whenever you want multiple things start thinking min/max!

In this case, set δ_1 so $|g(x) - g(c)| < 1$ giving the bound $|g(x)| < M + 1$. Set δ_2 so $|g(x) - g(c)| < \frac{\epsilon/2}{M+1}$ and set δ_3 so $|f(x) - f(c)| < \frac{\epsilon/2}{f(c)}$. Finally set $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ to get

$$|f(x)g(x) - f(c)g(c)| < \epsilon/2 + \epsilon/2 = \epsilon$$

as desired.

Exercise 4.2.2
For each stated limit, find the largest possible δ-neighborhood that is a proper response to the given ϵ challenge.

(a) $\lim_{x\to 3}(5x - 6) = 9$, where $\epsilon = 1$.

(b) $\lim_{x\to 4}\sqrt{x} = 2$, where $\epsilon = 1$.

(c) $\lim_{x\to\pi}[[x]] = 3$, where $\epsilon = 1$. (The function $[[x]]$ returns the greatest integer less than or equal to x.)

(d) $\lim_{x\to\pi}[[x]] = 3$, where $\epsilon = .01$.

Solution

(a) $|(5x - 6) - 9| = |5x - 15| = 5|x - 3| < 5\delta$ implies $\delta = 1/5$ for $\epsilon = 1$.

(b) Consider edge cases: We have $|\sqrt{9} - 2| = 1$ (x is 5 above) and $|\sqrt{1} - 2| = 1$ (x is 3 below) leading us to set $\delta = 3$. This δ must work since $\sqrt{x}$ is monotone.

(c) We must have $[[x]]] = 3$, since $|[[x]] - 3| = 1 \not< 1$. Therefor $\delta = \pi - 3$.

If the question was using $\leq$ instead of $<$ we would want $x \in (2, 5)$ as that is the largest neighborhood with $|[[x]] - 3| \leq 1$. Setting $\delta = \min\{|\pi - 2|, |\pi - 5|\} = \pi - 2$ achieves this maximum neighborhood.

(d) Since $[[x]]$ is an integer $\epsilon = .01$ is the same as saying $[[x]] = 3$. This happens precisely when $x \in (3, 4)$ hence we need $\delta = \min\{|\pi - 3|, |\pi - 4|\} = \pi - 3$.

Exercise 4.2.3
Review the definition of Thomae's function $t(x)$ from Section 4.1.

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \in \mathbf{Q}\backslash\{0\} \text{ is in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

(a) Construct three different sequences $(x_n), (y_n)$, and (z_n), each of which converges to 1 without using the number 1 as a term in the sequence.

(b) Now, compute $\lim t(x_n), \lim t(y_n)$, and $\lim t(z_n)$.

(c) Make an educated conjecture for $\lim_{x\to 1} t(x)$, and use Definition 4.2.1 B to verify the claim. (Given $\epsilon > 0$, consider the set of points $\{x \in \mathbf{R} : t(x) \geq \epsilon\}$ Argue that all the points in this set are isolated.)

Solution

(a) $x_n = (1+n)/n$, $y_n = 1 - 1/n^2$ and $z_n = 1 + 1/2^n$.

(b) $\lim t(x_n) = 0$ since the size of the denominator becomes arbitrarily large. Same for the others

(c) I claim $\lim_{x\to 1} t(x) = 0$. Let $\epsilon > 0$ be arbitrary; we must show there exists a δ where every $|x-1| < \delta$ has $t(x) < \epsilon$. For $x \notin \mathbf{Q}$ we have $t(x) = 0 < \epsilon$, and we can easily set δ small enough that $t(0) = 1$ is excluded. That leaves us with the case $x \in \mathbf{Q}$ in which case we can write $x - 1 = m/n$ in lowest terms.

To get $t(x) = 1/n < \epsilon$ we observe that $|m/n| < \delta$ implies $t(x) = 1/n \leq |m/n| < \delta$ so setting $\delta = \epsilon$ gives $t(x) < \epsilon$. To complete the proof set $\delta = \min\{\epsilon, 1\}$.

Exercise 4.2.4

Consider the reasonable but erroneous claim that

$$\lim_{x\to 10} 1/[[x]] = 1/10$$

(a) Find the largest δ that represents a proper response to the challenge of $\epsilon = 1/2$

(b) Find the largest δ that represents a proper response to $\epsilon = 1/50$.

(c) Find the largest ϵ challenge for which there is no suitable δ response possible.

Solution

(a) $\delta = 13$

(b) $\delta = 1$

(c) No matter how small δ is, $[[10 - \delta/2]] = 9$ can be obtained, meaning

$$|1/9 - 1/10| = 1/90$$

Is the largest ϵ with no suitable δ response.

Exercise 4.2.5

Use Definition 4.2.1 to supply a proper proof for the following limit statements.

(a) $\lim_{x\to 2}(3x + 4) = 10$

(b) $\lim_{x\to 0} x^3 = 0$

(c) $\lim_{x\to 2} (x^2 + x - 1) = 5.$

(d) $\lim_{x\to 3} 1/x = 1/3$

Solution
(Note that I use the largest δ choice that's easy to use)

(a) Since $|3x-6| = 3|x-2|$ setting $\delta = \epsilon/3$ gives $|3x-6| < \epsilon$ as desired.

(b) Since $|x^3| = |x|^3$ setting $\delta = \epsilon^{1/3}$ gives $|x|^3 < \epsilon$ as desired.

(c) Since $|x^2+x-6| = |x-2||x+3| < \delta(5+\delta)$ setting $\delta = \min\{1, \epsilon/6\}$ gives $\delta(5+\delta) < \delta(6) < \epsilon$ as desired. Another approach is to write $|x^2+x-6|$ in the "$(x-2)^n$ basis"

$$|x^2+x-6| = |(x-2)^2 + 5(x-2)| < \delta^2 + 5\delta = \delta(5+\delta)$$

(d) We have $|1/x - 1/3| = \frac{|3-x|}{3|x|}$ setting $\delta = \min\{1, 6\epsilon\}$ gives $1/3|x| < 1/6$ (because $|x| \in (2,4)$) and $|x-3| < 6\epsilon$ meaning

$$|1/x - 1/3| = \frac{|3-x|}{3|x|} < \frac{|3-x|}{6} < \epsilon$$

as desired.

Exercise 4.2.6
Decide if the following claims are true or false, and give short justifications for each conclusion.

(a) If a particular δ has been constructed as a suitable response to a particular ϵ challenge, then any smaller positive δ will also suffice.

(b) If $\lim_{x\to a} f(x) = L$ and a happens to be in the domain of f, then $L = f(a)$

(c) If $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a} 3[f(x)-2]^2 = 3(L-2)^2$

(d) If $\lim_{x\to a} f(x) = 0$, then $\lim_{x\to a} f(x)g(x) = 0$ for any function g (with domain equal to the domain of f.)

Solution

(a) Obviously, since if $\delta' < \delta$ then $|x-a| < \delta'$ implies $|x-a| < \delta$.

(b) False, consider $f(0) = 1$ and $f(x) = 0$ otherwise, the definition of a functional limit requires $|x-a| < \delta$ to imply $|f(x)-L| < \epsilon$ *for all x not equal to a* (This is the $0 < |x-a|$ part)

(c) True by the algebraic limit theorem for functional limits. (or composition of continuous functions, but that's unnecessary here)

(d) False, consider how $f(x) = x$ has $\lim_{x\to 0} f(x) = 0$ but $g(x) = 1/x$ has $\lim_{n\to 0} f(x)g(x) = 1$. (Fundementally this is because $1/x$ is not continuous at 0)

Exercise 4.2.7
Let $g : A \to \mathbf{R}$ and assume that f is a bounded function on A in the sense that there exists $M > 0$ satisfying $|f(x)| \leq M$ for all $x \in A$. Show that if $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c} g(x)f(x) = 0$ as well.

Solution
We have $|g(x)f(x)| \leq M|g(x)|$, set δ small enough that $|g(x)| < \epsilon/M$ to get

$$|g(x)f(x)| \leq M|g(x)| < M\frac{\epsilon}{M} = \epsilon$$

for all $|x-a| < \delta$.

Exercise 4.2.8
Compute each limit or state that it does not exist. Use the tools developed in this section to justify each conclusion.

(a) $\lim_{x\to 2} \frac{|x-2|}{x-2}$

(b) $\lim_{x\to 7/4} \frac{|x-2|}{x-2}$

(c) $\lim_{x\to 0}(-1)^{[[1/x]]}$

(d) $\lim_{x\to 0} \sqrt[3]{x}(-1)^{[[1/x]]}$

Solution

(a) Does not exist, the sequence $x_n = 2 + 1/n$ makes $|x-2|/(x-2)$ converge to 1, but $x_n = 2 - 1/n$ makes $|x-2|/x-2$ converge to -1. ($x \to |x|$ is not differentiable at zero for the same reason)

(b) -1. For $\delta < 1/4$ then $x < 2$ and we just have -1.

(c) Does not exist, let $(x_n) = 1/2n$ and $(y_n) = 1/(2n+1)$, then clearly $\lim x_n = \lim y_n = 0$ but $\lim f(x_n) = 1 \neq \lim f(y_n) = -1$

(d) $(-1)^{[[1/x]]}$ is bounded and $\lim_{x\to 0} \sqrt[3]{x} = 0$, so by Exercise 4.2.7 $\lim_{x\to 0} \sqrt[3]{x}(-1)^{[[1/x]]} = 0$. We can also show this directly, since $|\sqrt[3]{x}(-1)^{[[1/x]]}| = |\sqrt[3]{x}| < \epsilon$ when $\delta = \epsilon^3$.

Exercise 4.2.9 (Infinite Limits)
The statement $\lim_{x\to 0} 1/x^2 = \infty$ certainly makes intuitive sense. To construct a rigorous definition in the challenge response style of Definition 4.2.1 for an infinite limit statement of this form, we replace the (arbitrarily small) $\epsilon > 0$ challenge with an (arbitrarily large) $M > 0$ challenge:

Definition: $\lim_{x\to c} f(x) = \infty$ means that for all $M > 0$ we can find a $\delta > 0$ such that whenever $0 < |x-c| < \delta$, it follows that $f(x) > M$.

(a) Show $\lim_{x\to 0} 1/x^2 = \infty$ in the sense described in the previous definition.

(b) Now, construct a definition for the statement $\lim_{x\to\infty} f(x) = L$. Show $\lim_{x\to\infty} 1/x = 0$.

Solution

(a) For a given $M > 0$, if $0 < |x-0| = |x| < 1/\sqrt{M} = \delta$ then $1/|x|^2 = 1/x^2 < M$ as desired.

(b) $\lim_{x\to\infty} f(x) = L$ means that for all $\epsilon > 0$ we can find a N such that when $x > N$ it follows that $|f(x) - L| < \epsilon$. For a given $\epsilon > 0$, choosing $N = 1/\epsilon$ leaves us with $x > N \implies 1/N = \epsilon > 1/x$ hence $\lim_{x\to\infty} 1/x = 0$.

Exercise 4.2.10
Introductory calculus courses typically refer to the *right-hand limit* of a function as the limit obtained by "letting x approach a from the right-hand side."

(a) Give a proper definition in the style of Definition 4.2.1 for the right-hand and left-hand limit statements:

$$\lim_{x\to a^+} f(x) = L \text{ and } \lim_{x\to a^-} f(x) = M$$

(b) Prove that $\lim_{x\to a} f(x) = L$ if and only if both the right and left-hand limits equal L.

Solution

(a) Let $f : A \to \mathbf{R}$, and let c be a limit point of the domain A. We say that $\lim_{x\to c^+} f(x) = L$ provided that, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < x - c < \delta$ (and $x \in A$) it follows that $|f(x) - L| < \epsilon$. We say that $\lim_{x\to c^-} f(x) = L$ provided that, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < c - x < \delta$ (and $x \in A$) it follows that $|f(x) - L| < \epsilon$.

(b) ($\implies$) If $\lim_{x\to a} f(x) = L$ then for any $\epsilon > 0$, there exists a $\delta > 0$ so that $0 < |x-c| < \delta$ implies $|f(x) - L| < \epsilon$. Since both $0 < x - c < \delta$ and $0 < c - x < \delta$ will satisfy the requirement that $0 < |x - c| < \delta$, then $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$.

($\impliedby$) For a given $\epsilon > 0$, there exists $\delta_1, \delta_2 > 0$ so that either $0 < x - c < \delta_1$ or $0 > x - c > -\delta_2$ implies $|f(x) - L| < \epsilon$. If $0 < |x - c| < \delta = \min\{\delta_1, \delta_2\}$ then at least one of the preconditions is always true, so $\lim_{x\to a} f(x) = L$.

Exercise 4.2.11 (Squeeze Theorem)
Let f, g, and h satisfy $f(x) \leq g(x) \leq h(x)$ for all x in some common domain A. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} h(x) = L$ at some limit point c of A, show $\lim_{x\to c} g(x) = L$ as well.

Solution

$$\begin{aligned}
|g(x) - L| &\leq |g(x) - f(x)| + |f(x) - L| \\
&\leq |h(x) - f(x)| + |f(x) - L| \\
&\leq |h(x) - L| + |L - f(x)| + |f(x) - L| = |h(x) - L| + 2|f(x) - L|
\end{aligned}$$

For a given ϵ we can find $\delta > 0$ so that $|h(x) - L| < \epsilon/3$ and $|f(x) - L| < \epsilon/3$, hence $\lim_{x\to c} g(x) = L$.

4.3 Continuous Functions

Exercise 4.3.1
Let $g(x) = \sqrt[3]{x}$.

(a) Prove that g is continuous at $c = 0$.

(b) Prove that g is continuous at a point $c \neq 0$. (The identity $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ will be helpful.)

Solution

(a) Let $\epsilon > 0$ be arbitrary and set $\delta = \epsilon^3$. If $|x - 0| < \delta = \epsilon^3$ then taking the cube root of both sides gives $|x|^{1/3} < 1/\epsilon$ and since $(-x)^{1/3} = -(x^{1/3})$ we have $|x|^{1/3} = |x^{1/3}| < \epsilon$.

(b) We must make $|x^{1/3} - c^{1/3}| < \epsilon$ by making $|x - c|$ small. The identity given allows us to write

$$|x^{1/3} - c^{1/3}| = |x - c| \cdot |x^{2/3} + x^{1/3}c^{1/3} + c^{2/3}|$$

If we choose $\delta < c$ then $0 < |x| < 2|c|$. Keeping in mind that if $a > b > 0$ then $\sqrt[3]{a} > \sqrt[3]{b}$, we can now bound

$$\begin{aligned}|x^{2/3} + x^{1/3}c^{1/3} + c^{2/3}| &\leq |x^{2/3}| + |x^{1/3}c^{1/3}| + |c^{2/3}| \\ &\leq 2^{2/3}|c^{2/3}| + 2^{1/3}|c^{2/3}| + |c^{2/3}| \\ &= K\end{aligned}$$

where K is a constant. Then

$$|x^{1/3} - c^{1/3}| \leq |x - c| \cdot K$$

Setting $\delta = \frac{\epsilon}{K}$ gives $|x^{1/3} - c^{1/3}| \leq \epsilon$ completing the proof.

Exercise 4.3.2

To gain a deeper understanding of the relationship between ϵ and δ in the definition of continuity, let's explore some modest variations of Definition 4.3.1. In all of these, let f be a function defined on all of $\mathbf{R}$.

(a) Let's say f is *onetinuous* at c if for all $\epsilon > 0$ we can choose $\delta = 1$ and it follows that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$. Find an example of a function that is onetinuous on all of $\mathbf{R}$.

(b) Let's say f is *equaltinuous* at c if for all $\epsilon > 0$ we can choose $\delta = \epsilon$ and it follows that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$. Find an example of a function that is equaltinuous on $\mathbf{R}$ that is nowhere onetinuous, or explain why there is no such function.

(c) Let's say f is *lesstinuous* at c if for all $\epsilon > 0$ we can choose $0 < \delta < \epsilon$ and it follows that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$. Find an example of a function that is lesstinuous on $\mathbf{R}$ that is nowhere equaltinuous, or explain why there is no such function.

(d) Is every lesstinuous function continuous? Is every continuous function lesstinuous? Explain.

Solution

(a) The constant function $f(x) = k$ is onetinuous, in fact it is the only onetinuous function (Think about why)

(b) The line $f(x) = x$ is equaltinuous

(c) $f(x) = 2x$ is lesstinuous but nowhere-equaltinuous

(d) Every lesstinuous function is continuous, since the definition of lesstinuous is just continuous plus the requirement that $0 < \delta < \epsilon$.

And every continuous function is lesstinuous since if $\delta > 0$ works we can set $\delta' < \delta$ and $\delta' < \epsilon$ so that $|x - c| < \delta' < \delta$ still implies $|f(x) - f(c)| < \epsilon$

Exercise 4.3.3

(a) Supply a proof for Theorem 4.3.9 (Composition of continuous functions) using the $\epsilon-\delta$ characterization of continuity.

(b) Give another proof of this theorem using the sequential characterization of continuity (from Theorem 4.3.2 (iii)).

Solution

(a) Let f is continuous at c and g be continuous at $f(c)$. We will show $g \circ f$ is continuous at c. Let $\epsilon > 0$ be arbitrary, we want $|g(f(x)) - g(f(c))| < \epsilon$ for $|x - c| < \delta$. Pick $\alpha > 0$ so that $|y - f(c)| < \alpha$ implies $|g(y) - g(f(c))| < \epsilon$ (possible since g is continuous at $f(c)$) and pick $\delta > 0$ so that $|x - c| < \delta$ implies $|f(x) - f(c)| < \alpha$. Putting all of this together we have

$$|x - c| < \delta \implies |f(x) - f(c)| < \alpha \implies |g(f(x)) - g(f(c))| < \epsilon$$

(b) Let $(x_n) \to c$, we know $f(x_n)$ is a sequence converging to $f(c)$ since f is continuous at c, and since g is continuous at $f(c)$ any sequence $(y_n) \to f(c)$ has $g(y_n) \to g(f(c))$. Letting $y_n = f(x_n)$ gives $g(f(x_n)) \to g(f(c))$ as desired.

Exercise 4.3.4

Assume f and g are defined on all of $\mathbf{R}$ and that $\lim_{x\to p} f(x) = q$ and $\lim_{x\to q} g(x) = r$.

(a) Give an example to show that it may not be true that

$$\lim_{x\to p} g(f(x)) = r$$

(b) Show that the result in (a) does follow if we assume f and g are continuous.

(c) Does the result in (a) hold if we only assume f is continuous? How about if we only assume that g is continuous?

Solution

(a) Let $f(x) = q$ be constant and define $g(x)$ as

$$g(x) = \begin{cases} (r/q)x & \text{if } x \neq q \\ 0 & \text{if } x = q \end{cases}$$

We have $\lim_{x\to q} g(x) = r$ but $\lim_{x\to p} g(f(x)) = g(q) = 0$.

The problem is that functional limits allow jump discontinuities by requiring $y \neq q$ in $\lim_{y\to q} g(y)$ but $f(x)$ might not respect $f(x) \neq q$ as $x \to p$. Continuity fixes this by requiring $\lim_{y\to q} g(y) = g(q)$ so that $f(x) = q$ doesn't break anything.

Another fix would be requiring $f(x) \neq q$ for all $x \neq p$ - In other words that the error is always greater then zero $0 < |f(x) - q| < \epsilon$ similar to $0 < |x - p| < \delta$. This would allow chaining of functional limits, however it would make it impossible to take limits of "locally flat" functions.

(b) Theorem 4.3.9 (Proved in Exercise 4.3.3)

(c) Not if f is continuous (in our example f was continuous). Yes if g is continuous since it would get rid of the $f(x) = q$ problem.

Exercise 4.3.5
Show using Definition 4.3.1 that if c is an isolated point of $A \subseteq \mathbf{R}$, then $f : A \to \mathbf{R}$ is continuous at c

Solution
Since c is isolated, we can set δ small enough that the only $x \in A$ satisfying $|x - c| < \delta$ is $x = c$. Then clearly $|f(x) - f(c)| < \epsilon$ since $f(x) = f(c)$ for all $|x - c| < \delta$.

Exercise 4.3.6
Provide an example of each or explain why the request is impossible.

(a) Two functions f and g, neither of which is continuous at 0 but such that $f(x)g(x)$ and $f(x) + g(x)$ are continuous at 0

(b) A function $f(x)$ continuous at 0 and $g(x)$ not continuous at 0 such that $f(x) + g(x)$ is continuous at 0

(c) A function $f(x)$ continuous at 0 and $g(x)$ not continuous at 0 such that $f(x)g(x)$ is continuous at 0

(d) A function $f(x)$ not continuous at 0 such that $f(x) + \frac{1}{f(x)}$ is continuous at 0 .

(e) A function $f(x)$ not continuous at 0 such that $[f(x)]^3$ is continuous at 0 .

Solution
(a) Let

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

And set $g(x) = -f(x)$. we have $f(x) + g(x) = 0$ which is continuous at zero, and we have $f(x)g(x) = -f(x)^2 = -1$ which is also continuous at zero.

(b) Impossible, since it would imply that $(f + g) - f = g$ is continuous at zero (sum of continuous functions is continuous).

(c) Let $f(x) = 0$, then $f(x)g(x) = 0$ is continuous at zero for any $g(x)$.

(d) Let

$$f(x) = \begin{cases} 2 & \text{if } x \geq 0 \\ 1/2 & \text{if } x < 0 \end{cases}$$

Then $f(x) + 1/f(x) = 2.5$ is continuous at zero.

(e) Impossible, if $[f(x)]^3$ was continuous at zero then $([f(x)]^3)^{1/3} = f(x)$ would also be continuous at zero since the composition of continuous functions is continuous

Exercise 4.3.7

(a) Referring to the proper theorems, give a formal argument that Dirichlet's function from Section 4.1 is nowhere-continuous on $\mathbf{R}$.

(b) Review the definition of Thomae's function in Section 4.1 and demonstrate that it fails to be continuous at every rational point.

(c) Use the characterization of continuity in Theorem 4.3.2 (iii) to show that Thomae's function is continuous at every irrational point in $\mathbf{R}$. (Given $\epsilon > 0$, consider the set of points $\{x \in \mathbf{R} : t(x) \geq \epsilon\}$.)

Solution

Recall Dirichlet's function is

$$g(x) = \begin{cases} 1 \text{ if } x \in \mathbf{Q} \\ 0 \text{ if } x \in \mathbf{I} \end{cases}$$

And Thomae's function is

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \text{ in lowest terms with } m, n \neq 0 \\ 0 & \text{if } x \in \mathbf{I} \end{cases}$$

(a) Let $a \in \mathbf{Q}$ and set $\epsilon = 1$. For any $\delta > 0$ there will exist points $x \in (a - \delta, a + \delta) \cap \mathbf{I}$ by the density of $\mathbf{I}$ in $\mathbf{R}$ with $|f(x) - f(a)| = |0 - 1| = 1$ not less then ϵ, therefore there does not exist a δ to match $\epsilon = 1$ and so f is discontinuous at a. Since a was arbitrary (the $a \in \mathbf{I}$ case is identical) g must be discontinuous at all of $\mathbf{R}$.

(b) By the same argument as in (a) for any $m/n \in \mathbf{Q}$ no matter how small δ is, we can find an irrational number within δ of m/n meaning ϵ cannot be made smaller then $|f(m/n) - f(x)| = 1/n$.

(c) Let $a \in \mathbf{I}$, we want to show $t(x) < \epsilon$ for $|x - a| < \delta$. I claim the set $\{x \in V_1(a) : t(x) \geq \epsilon\}$ is finite, this can be seen since the requirement that $t(x) \geq \epsilon$ is the same as $x = m/n$ and $1/n \geq \epsilon$. It is easy to see there are finitely many points like this (consider how there are finitely many n and finitely many m given n) thus we can say $\{x \in V_1(a) : t(x) \geq \epsilon\} = \{x_1, \ldots, x_n\}$ and set $\delta = \min\{|x_i - a| : i \in \{1, \ldots, n\}\}$ to ensure every $x \in V_\delta(a)$ has $t(x) < \epsilon$.

Exercise 4.3.8
Decide if the following claims are true or false, providing either a short proof or counterexample to justify each conclusion. Assume throughout that g is defined and continuous on all of $\mathbf{R}$.

(a) If $g(x) \geq 0$ for all $x < 1$, then $g(1) \geq 0$ as well.

(b) If $g(r) = 0$ for all $r \in \mathbf{Q}$, then $g(x) = 0$ for all $x \in \mathbf{R}$.

(c) If $g(x_0) > 0$ for a single point $x_0 \in \mathbf{R}$, then $g(x)$ is in fact strictly positive for uncountably many points.

Solution

(a) True, using the sequential definition for functional limits letting $(x_n) \to 1$ we have $g(x_n) \geq 0$ and $g(x_n) \to g(1)$ so by the Order Limit Theorem $g(1) \geq 0$

(b) True, since if there was some $x \in \mathbf{R}$ with $g(x) \neq 0$ then g would not be continuous at x because we could never make ϵ smaller then $|g(x) - g(r)| = |g(x)|$ as we can always find rational numbers satisfying $g(r) = 0$ inside any δ-neighborhood.

(c) True, let $\epsilon < g(x_0)$ and pick δ so that every $x \in V_\delta(x_0)$ satisfies $g(x) \in (g(x_0) - \epsilon, g(x_0) + \epsilon)$ and thus $g(x) > 0$ since $g(x_0) - \epsilon > 0$.

Exercise 4.3.9
Assume $h : \mathbf{R} \to \mathbf{R}$ is continuous on $\mathbf{R}$ and let $K = \{x : h(x) = 0\}$. Show that K is a closed set.

Solution
Let (x_n) be a convergent sequence in K and set $x = \lim x_n$. Since h is continuous the limit $\lim h(x_n) = h(x)$ and since $h(x_n) = 0$ for all n $\lim h(x_n) = 0$. therefore $h(x) = 0$ and $x \in K$.

Exercise 4.3.10
Observe that if a and b are real numbers, then

$$\max\{a, b\} = \frac{1}{2}[(a + b) + |a - b|]$$

(a) Show that if $f_1, f_2, \ldots, f_n$ are continuous functions, then

$$g(x) = \max\{f_1(x), f_2(x), \ldots, f_n(x)\}$$

is a continuous function.

(b) Let's explore whether the result in (a) extends to the infinite case. For each $n \in \mathbf{N}$, define f_n on $\mathbf{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } |x| \geq 1/n \\ n|x| & \text{if } |x| < 1/n \end{cases}$$

Now explicitly compute $h(x) = \sup\{f_1(x), f_2(x), f_3(x), \ldots\}$

Solution

(a) We will prove this by induction. The base case is

$$\max\{f_1(x), f_2(x)\} = \frac{1}{2}\left[(a+b) + |a-b|\right]$$

Which is obviously continuous. Now assume $\max\{f_1, \dots, f_{n-1}\}$ is continuous, letting $m(x) = \max\{f_1, \dots, f_{n-1}\}$ we can write

$$g(x) = \max\{f_1, \dots, f_{n-1}, f_n\} = \max\{f_n, m(x)\}$$

Now since $f_n(x)$ and $m(x)$ are continuous functions $\max\{f_n, m\}$ is continuous by the base case!

(b) We can reason by cases. if $x = 0$ then $f_n(0) = 0$ for all n so $h(0) = 0$. If $x \neq 0$ then $|x| > 1/n$ for all $n > N$ meaning we have $h(x) = \max\{f_1(x), \dots, f_N(x), 1\}$. Since $n|x| < 1$ for all $|x| < 1/n$ we have $h(x) = 1$ and so

$$h(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases}$$

Which is not continuous at $x = 0$, therefore (a) does not hold in the infinite case.

Exercise 4.3.11 (Contraction Mapping Theorem)

Let f be a function defined on all of $\mathbf{R}$, and assume there is a constant c such that $0 < c < 1$ and

$$|f(x) - f(y)| \leq c|x - y|$$

for all $x, y \in \mathbf{R}$.

(a) Show that f is continuous on $\mathbf{R}$.

(b) Pick some point $y_1 \in \mathbf{R}$ and construct the sequence

$$(y_1, f(y_1), f(f(y_1)), \dots).$$

In general, if $y_{n+1} = f(y_n)$, show that the resulting sequence (y_n) is a Cauchy sequence. Hence we may let $y = \lim y_n$.

(c) Prove that y is a fixed point of f (i.e., $f(y) = y$) and that it is unique in this regard.

(d) Finally, prove that if x is any arbitrary point in $\mathbf{R}$, then the sequence $(x, f(x), f(f(x)), \dots)$ converges to y defined in (b).

Solution

(a) Let $\delta = \epsilon/c$ to get $|f(x) - f(y)| \leq c|x - y| < \epsilon$ whenever $|x - y| < \delta$. (This is the general proof, we could make it shorter by letting $\delta = \epsilon$ since $0 < c < 1$)

(b) We want $|y_n - y_m| < \epsilon$. Since $0 < c < 1$ we have $|y_{n+1} - y_{m+1}| \le c|y_n - y_m|$, implying $|y_n - y_m| \le c^m |y_{n-m+1} - y_1|$. Thus if we can bound $|y_k - y_1| \le M$ for some constant M (which may depend on y_1) we will be done, by choosing m large enough so that $c^m \le \epsilon/M$.

In general, $|y_{a+1} - y_a| \le c|y_a - y_{a-1}|$, so

$$\begin{aligned}
|y_k - y_1| &\le |y_1 - y_2| + |y_2 - y_3| + \cdots + |y_{k-1} - y_k| \\
&\le |y_1 - y_2| + c|y_1 - y_2| + \cdots + c^{k-2}|y_1 - y_2| \\
&= |y_1 - y_2| \sum_{i=0}^{k-2} c^i \\
&< |y_1 - y_2| \sum_{i=0}^{\infty} c^i = \frac{|y_1 - y_2|}{1 - c}
\end{aligned}$$

which is bounded, hence proved.

(c) Since f is continuous, $f(\lim_{n\to\infty} y_n) = \lim_{n\to\infty} f(y_n)$ which is just the same as y shifted one element forward, so clearly $\lim f(y_n) = y$, showing that y is a fixed point.

Now, consider two similar sequences $(a_n) \to a$ and $(b_n) \to b$ where $a_1, b_1 \in \mathbf{R}$, $a_{n+1} = f(a_n)$, and $b_{n+1} = f(b_n)$. By the Algebraic Limit Theorem $b = a + \lim_{n\to\infty} b_n - a_n$. Now note

$$\lim_{n\to\infty} b_n - a_n \le \lim_{n\to\infty} |b_n - a_n| \le \lim_{n\to\infty} c^{n-1}|b_1 - a_1| = 0$$

Therefore $a = b$; this implies that regardless of our starting choice of y_1 we will end up at the same fixed point y. In particular, for a given fixed point z, if we start at $z_1 = z$ then clearly $\lim_{n\to\infty} z_n = z$ but also $\lim_{n\to\infty} z_n = y$ and therefore $z = y$ and y is a unique fixed point.

(d) See (c)

Exercise 4.3.12
Let $F \subseteq \mathbf{R}$ be a nonempty closed set and define $g(x) = \inf\{|x - a| : a \in F\}$. Show that g is continuous on all of $\mathbf{R}$ and $g(x) \neq 0$ for all $x \notin F$.

Solution
Let $x \in \mathbf{R}$ and let $a \in F$ be the element of F closest to x (must exist since F is closed), we have $0 \le g(y) \le |y - a|$ and $g(x) = |x - a|$. Similarly, let $b \in F$ be the element of F closest to y. For the rest of the argument to make sense, we need to pick a as our comparison point if $|x - a| \le |y - b|$ or b otherwise. Suppose, without loss of generality, that we pick a. Thus, not only $g(y) - g(x) \le |y - a| - |x - a|$ but also $|x - a| \le |y - b| \implies 0 \le g(y) - g(x)$. This allows us to write

$$|g(y) - g(x)| \le \big||y - a| - |x - a|\big|$$

Applying the bound from Exercise 1.2.6 (d) we get

$$\big||y - a| - |x - a|\big| \le |(y - a) - (x - a)| = |y - x| < \delta$$

Setting $\delta = \epsilon$ gives $|g(x) - g(y)| < \epsilon$ as desired. If we had to pick b, the argument is the same just replacing y with x and vice-versa.

To see $g(x) \neq 0$ for $x \notin F$ notice that F^c is open so there exists an $\alpha > 0$ so that $V_\alpha(x) \cap F = \emptyset$ meaning $g(x) \geq \alpha$ and so $g(x) \neq 0$.

Exercise 4.3.13

Let f be a function defined on all of $\mathbf{R}$ that satisfies the additive condition $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbf{R}$.

(a) Show that $f(0) = 0$ and that $f(-x) = -f(x)$ for all $x \in \mathbf{R}$.

(b) Let $k = f(1)$. Show that $f(n) = kn$ for all $n \in \mathbf{N}$, and then prove that $f(z) = kz$ for all $z \in \mathbf{Z}$. Now, prove that $f(r) = kr$ for any rational number r.

(c) Show that if f is continuous at $x = 0$, then f is continuous at every point in $\mathbf{R}$ and conclude that $f(x) = kx$ for all $x \in \mathbf{R}$. Thus, any additive function that is continuous at $x = 0$ must necessarily be a linear function through the origin.

Solution

(a) $f(0+0) = f(0) + f(0)$ implies $f(0) = 0$ and thus $f(x + (-x)) = 0 = f(x) + f(-x)$ meaning $f(-x) = -f(x)$.

(b) $f(n) = f(1) + \cdots + f(1) = k + \cdots + k = kn$. Now since $f(-n) = -f(n) = -kn$ we have $f(z) = kz$ for all $z \in \mathbf{Z}$. Finally let $r = p/q$ for $p \in \mathbf{Z}$ and $q \in \mathbf{N}$. notice that $f(qr) = k(qr)$ and since $f(qr) = f(r + \cdots + r) = f(r) + \cdots + f(r) = qf(r)$ we have $f(r) = k(qr)/q = kr$.

(c) Assume f is continuous at 0 and let $x \in \mathbf{R}$ be arbitrary. Let x_n be a sequence approaching x. since $(x - x_n) \to 0$ we have $f(x - x_n) \to 0$ because f is continuous at zero. Now since f is additive $f(x - x_n) = f(x) - f(x_n) \to 0$ implies $f(x_n) \to f(x)$ meaning f is continuous at $x \in \mathbf{R}$ by the sequential characterization of continuity.

Now to see that $f(x) = kx$ for $x \in \mathbf{I}$ simply take a limit of rationals approaching x.

Exercise 4.3.14

(a) Let F be a closed set. Construct a function $f : \mathbf{R} \to \mathbf{R}$ such that the set of points where f fails to be continuous is precisely F. (The concept of the interior of a set, discussed in Exercise 3.2.14, may be useful.)

(b) Now consider an open set O. Construct a function $g : \mathbf{R} \to \mathbf{R}$ whose set of discontinuous points is precisely O. (For this problem, the function in Exercise 4.3.12 may be useful.)

Solution

(a) Using the notation from Exercise 3.2.14, note that F^o, F^c, and $\overline{F^c} \backslash F^c$ are disjoint but their union is $\mathbf{R}$; moreover $F^o \cup \overline{F^c} \backslash F^c = F$. Let $d(x)$ denote Dirichlet's function (1 on rationals, 0 on irrationals), and consider

$$f(x) = \begin{cases} d(x) & x \in F^o \\ 2 & x \in \overline{F^c} \backslash F^c \\ 3 & x \in F^c \end{cases}$$

If $x \in F^o$ (which is open) then we can find $V_\epsilon(x) \subseteq F^o$ where there will be both irrational and rational numbers, indicating that f is discontinuous over F^o.

If $x \in \overline{F^c} \backslash F^c$, x must be a limit point of F^c, and therefore all $V_\epsilon(x)$ will intersect F^c at some point, and thus $f(y) = 3$ for some $y \in V_\epsilon(x)$, preventing f from being continuous in $\overline{F^c} \backslash F^c$.

If $x \in F^c$ (which is open) then we can find $V_\epsilon(x) \subseteq F^c$ which is a constant 3, and therefore f is continuous over F^c. Thus, f is discontinous only over F.

(b) Define
$$f(x) = d(x)\left(\inf\{|x-a| : a \in F^c\}\right)$$

$f(x) = 0$ for $x \in F^c$ and by choosing $\delta = \epsilon > 0$ we will have $\inf\{|y-a| : a \in F^c\} < \epsilon$ for $y \in V_\delta(x)$ (simply consider $a = x$) implying f is continuous over F^c.

Since F is open, for any given $x \in F$ we can find $\alpha > 0$ so that $\inf\{|y-a| : a \in F^c\} > \gamma > 0$ for all $y \in V_\alpha(x)$. (One way to do this is by choosing β so that $V_\beta(x) \subseteq F$, taking $\alpha = \beta/2$, noting that $\{a : \exists y \in V_\alpha(x) \text{ such that } |y-a| < \alpha\} = V_\beta(x)$, and concluding that $a \in F^c \implies a \notin V_\beta(x) \implies \forall y \in V_\alpha(x),\ |y-a| \geq \alpha$.) Then since for any $V_\delta(x)$, there must be points y_1, y_2 where $d(y_1) = 1,\ d(y_2) = 0$, it must be impossible to satisfy the definition of continuity for $\epsilon < \gamma$ (since in the δ-neighbourhood of x, $f(x)$ will jump by at least that amount between rational and irrational numbers), and therefore f is discontinuous for any $x \in F$.

4.4 Continuous Functions on Compact Sets

Exercise 4.4.1

(a) Show that $f(x) = x^3$ is continuous on all of $\mathbf{R}$.

(b) Argue, using Theorem 4.4.5, that f is not uniformly continuous on $\mathbf{R}$.

(c) Show that f is uniformly continuous on any bounded subset of $\mathbf{R}$.

Solution

(a) True since the product of continuous functions is continuous

(b) Take $x_n = n$ and $y_n = n + 1/n$ has $|x_n - y_n| \to 0$ but
$$|f(y_n) - f(x_n)| = \left|(n+1/n)^3 - n^3\right| = \left|3n^2 \cdot \frac{1}{n} + 3n \cdot \frac{1}{n^2} + \frac{1}{n^3}\right| \to \infty$$

Which shows x^3 is not uniformly continuous by Theorem 4.4.5

(c) Let A be a bounded subset of $\mathbf{R}$ with $A \subset (-M, M)$. Let $\epsilon > 0$ and note that
$$\left|\frac{x^3 - y^3}{x - y}\right| = \left|\frac{(x-y)(x^2+xy+y^2)}{(x-y)}\right| = \left|x^2 + xy + y^2\right|$$

Is clearly bounded on $(-M, M)$. Thus the Lipschitz condition allows us to conclude f is uniformly continuous on A.

Exercise 4.4.2

(a) Is $f(x) = 1/x$ uniformly continuous on $(0, 1)$?

(b) Is $g(x) = \sqrt{x^2 + 1}$ uniformly continuous on $(0, 1)$?

(c) Is $h(x) = x \sin(1/x)$ uniformly continuous on $(0, 1)$?

Solution

(a) No, intuitively because slope becomes unbounded as we approach zero. Rigorously consider $x_n = 2/n$ and $y_n = 1/n$ we have $|x_n - y_n| \to 0$ but $|1/x_n - 1/y_n| = |n/2 - n| = n/2$ is unbounded meaning f cannot be uniformly continuous by Theorem 4.4.5.

(b) Yes, since it's continuous on $[0, 1]$ Theorem 4.4.7 implies it is uniformly continuous on $[0, 1]$ and hence on any subset as well.

(c) Yes, since h is continuous over $[0, 1]$ implying it is also uniformly continuous over $[0, 1]$ by Theorem 4.4.7

Exercise 4.4.3

Show that $f(x) = 1/x^2$ is uniformly continuous on the set $[1, \infty)$ but not on the set $(0, 1]$.

Solution

By Lipschitz over $[1, \infty)$

$$\left|\frac{1/x^2 - 1/y^2}{x - y}\right| = \left|\frac{y^2 - x^2}{x^2y^2(x - y)}\right| = \left|\frac{(x - y)(x + y)}{x^2y^2(x - y)}\right| = \left|\frac{x + y}{x^2y^2}\right| = \left|\frac{1}{xy^2} + \frac{1}{x^2y}\right| \leq 2$$

For $(0, 1]$ consider $x_n = 1/n$ and $y_n = 1/2n$. we have $|x_n - y_n| \to 0$ but

$$|f(x_n) - f(y_n)| = |n^2 - 4n^2| = 3n^2$$

is unbounded, hence f is not uniformly continuous on $(0, 1]$ by Theorem 4.4.5.

Exercise 4.4.4

Decide whether each of the following statements is true or false, justifying each conclusion.

(a) If f is continuous on $[a, b]$ with $f(x) > 0$ for all $a \leq x \leq b$, then $1/f$ is bounded on $[a, b]$ (meaning $1/f$ has bounded range).

(b) If f is uniformly continuous on a bounded set A, then $f(A)$ is bounded.

(c) If f is defined on $\mathbf{R}$ and $f(K)$ is compact whenever K is compact, then f is continuous on $\mathbf{R}$.

Solution

(a) True, the Algebraic Limit Theorem implies $1/f$ is continuous (well defined since $f > 0$) and the Extreme Value Theorem implies $1/f$ attains a maximum and minimum and so is bounded.

(b) Let $\epsilon = 1$ and choose $\delta > 0$ so that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. Define the set $X = \{x_0, \ldots, x_n\}$ consisting of evenly spaced values x_i, ranging from $x_0 = \inf A$ to $x_n = \sup A$, with the spacing between each value less than $\delta/2$ (i.e. $\forall k,\ x_k - x_{k-1} < \delta/2$). Now define the set $P = \{p_0, \ldots, p_m\}$ where for each $x_i \in X$, we add one element $p_i \in A \cap V_{\delta/2}(x)$, if $A \cap V_{\delta/2}(x) \neq \emptyset$ (and do not add anything if $A \cap V_{\delta/2}(x) = \emptyset$).

Now, every element $a \in A$ is at most δ from an element $p \in P$ (i.e. $|a - x| < \delta$). To see this, for any $a \in A$, there must be some $x_i \in X$ so that $|a - x_i| < \delta/2$, and since $A \cap V_{\delta/2}(x) \neq \emptyset$ (it at least contains a), there must also be an element $p_i \in P$ so that $|x_i - p| < \delta/2$. By the Triangle Inequality, $|a - p| < \delta$ for some $p \in P$.

Noting that $P \subseteq A$ is finite, we can consider $M = \max(f(P))$. Let $a \in A$ be arbitrary, and identify the nearest $p \in P$. We have $|a - p| < \delta$ so $|f(y) - f(p)| < \epsilon$ and since $f(p) \leq M$, $f(a) < \epsilon + M$, completing the proof.

Alternative proof approach: $\bar{A}$ is closed, bounded, and thus compact. Extend the definition of f to cover limit points of A via a limit on f; these limits exist since f is uniformly continuous (**TODO** write explicit proof). The extended f is continuous, and thus preserves the compactness of $\bar{A}$; therefore $f(A)$ is bounded.

(c) Any function with finite range preserves compact sets, since all finite sets are compact. Meaning Dirichlet's function

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \in \mathbf{Q} \end{cases}$$

"preserves" compact sets, but is nowhere-continuous.

Exercise 4.4.5
Assume that g is defined on an open interval (a, c) and it is known to be uniformly continuous on $(a, b]$ and $[b, c)$, where $a < b < c$. Prove that g is uniformly continuous on (a, c).

Solution
Let $\epsilon > 0$ and choose δ_1 so that every $x, y \in (a, b]$ with $|x - y| < \delta_1$ has $|f(x) - f(y)| < \epsilon/2$, likewise choose δ_2 so that every $x, y \in [b, c)$ with $|x - y| < \delta_2$ has $|f(x) - f(y)| < \epsilon/2$. Finally let $\delta = \min\{\delta_1, \delta_2\}$. The final case is if $x \in (a, b]$ and $y \in [b, c]$ where we use the triangle inequality

$$|f(x) - f(y)| \leq |f(x) - f(b)| + |f(b) - f(y)| < \epsilon/2 + \epsilon/2 = \epsilon$$

Thus f is uniformly continuous on (a, c).

Exercise 4.4.6
Give an example of each of the following, or state that such a request is impossible. For any that are impossible, supply a short explanation for why this is the case.

(a) A continuous function $f : (0, 1) \to \mathbf{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence;

(b) A uniformly continuous function $f : (0, 1) \to \mathbf{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence;

(c) A continuous function $f : [0, \infty) \to \mathbf{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence;

Solution

(a) $f(x) = 1/x$ and $x_n = 1/n$ has $f(x_n)$ diverging, hence $f(x_n)$ is not Cauchy.

(b) Impossible since for all $\epsilon > 0$ we can find an N so that all $n \geq N$ has $|x_n - x_m| < \delta$ (since x_n is Cauchy) implying $|f(x_n) - f(x_m)| < \epsilon$ and thus $f(x_n)$ is Cauchy. (Uniform continuity is needed for the $\forall n \geq N$ part)

(c) Impossible since $[0, \infty)$ is closed $(x_n) \to x \in [0, \infty)$ implying $f(x_n) \to f(x)$ since f is continuous, thus $f(x_n)$ is a Cauchy sequence.

Exercise 4.4.7

Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Solution

We will show f is uniformly continuous on $[0, 1]$ and $[1, \infty)$ then combine them similar to Exercise 4.4.5.

(i) Since f is continuous over $[0, 1]$ Theorem 4.4.7 implies f is uniformly continuous on $[0, 1]$.

(ii) f is Lipschitz on $[1, \infty)$ since $\sqrt{x}$ is sublinear over $[1, \infty)$

$$\left|\frac{\sqrt{x} - \sqrt{y}}{x - y}\right| = \left|\frac{(\sqrt{x} - \sqrt{y})}{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})}\right| = \left|\frac{1}{\sqrt{x} + \sqrt{y}}\right| \leq 1$$

Let $\delta = \min\{\delta_1, \delta_2\}$ where δ_1 is for $[0, 1]$ and δ_2 is for $[1, \infty)$. If x, y are both in one of $[0, 1]$ or $[1, \infty)$ we have $|f(x) - f(y)| < \epsilon/2$ and are done. If $x \in [0, 1]$ and $y \in [1, \infty)$ then

$$|f(x) - f(y)| \leq |f(x) - f(1)| + |f(1) - f(y)| < \epsilon/2 + \epsilon/2 = \epsilon$$

Thus $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Exercise 4.4.8

Give an example of each of the following, or provide a short argument for why the request is impossible.

(a) A continuous function defined on $[0, 1]$ with range $(0, 1)$.

(b) A continuous function defined on $(0, 1)$ with range $[0, 1]$.

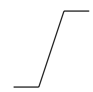
(c) A continuous function defined on $(0, 1]$ with range $(0, 1)$.

Solution

(a) Impossible as continuous functions preserve compact sets and $(0, 1)$ is not compact.

(b) Define

$$f(x) = \begin{cases} 3x - 1 & \text{if } x \in [1/3, 2/3] \\ 1 & \text{if } x \in (2/3, 1) \\ 0 & \text{if } x \in (0, 1/3) \end{cases}$$

f is continuous on $(0,1)$ and has range $[0,1]$. Here's a graph of f: 

(c) Consider $g(x) = \sin(1/x)(1-x)$ over $x \in (0,1]$; clearly $g(x)$ is continuous over this interval. The $1-x$ term bounds $g(x)$ to $(-1,1)$, while the $\sin(1/x)$ ensures that $g(x)$ will approach this bound arbitrarily close as $x \to 0$. Thus, the range of $g(x)$ is $(-1,1)$. We now just need to shape this to $(0,1)$ by defining $f(x) = (g(x)+1)/2$.

Exercise 4.4.9 (Lipschitz Functions)
A function $f : A \to \mathrm{R}$ is called Lipschitz if there exists a bound $M > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M$$

for all $x \neq y \in A$. Geometrically speaking, a function f is Lipschitz if there is a uniform bound on the magnitude of the slopes of lines drawn through any two points on the graph of f.

(a) Show that if $f : A \to \mathbf{R}$ is Lipschitz, then it is uniformly continuous on A.

(b) Is the converse statement true? Are all uniformly continuous functions necessarily Lipschitz?

Solution
(a) Choose $\epsilon > 0$ and set $\delta = \epsilon/M$ to get $|f(x) - f(y)| \leq M|x-y| < M\delta < \epsilon$.

(b) No, consider $f(x) = \sqrt{x}$ over $[0,1]$ which is uniformly continuous by Theorem 4.4.7 however at $x = 0$ we have

$$\left| \frac{f(x) - f(y)}{x - y} \right| = \left| \frac{-\sqrt{y}}{-y} \right| = \frac{1}{\sqrt{y}}$$

Which is unbounded for small y.

Exercise 4.4.10
Assume that f and g are uniformly continuous functions defined on a common domain A. Which of the following combinations are necessarily uniformly continuous on A:

$$f(x) + g(x), \quad f(x)g(x), \quad \frac{f(x)}{g(x)}, \quad f(g(x))?$$

(Assume that the quotient and the composition are properly defined and thus at least continuous.)

Solution

(i) $f(x) + g(x)$ is clearly uniformly continuous

(ii) $f(x) = g(x) = x$ are individually uniformly continuous over $\mathbf{R}$ but $f(x)g(x) = x^2$ is not.

Note this counterexample only works when A is unbounded. If A is bounded you can prove $f(x)g(x)$ must be uniformly continuous the same way you prove the product rule.

(iii) False, consider $f(x) = 1$ and $g(x) = x$ over $(0, 1)$. Both are uniformly continuous on $(0, 1)$ but $f/g = 1/x$ is not.

(iv) Want $|f(g(x)) - f(g(y))| < \epsilon$. Since f is uniformly continuous we can find an $\alpha > 0$ so that

$$|g(x) - g(y)| < \alpha \implies |f(g(x)) - f(g(y))| < \epsilon$$

Now since g is uniformly continuous we can find a $\delta > 0$ so that

$$|x - y| < \delta \implies |g(x) - g(y)| < \alpha$$

Combine the two to get

$$|x - y| < \delta \implies |f(g(x)) - f(g(y))| < \epsilon$$

as desired, which proves that $f(g(x))$ is uniformly continuous.

Exercise 4.4.11 (Topological Characterization of Continuity)

Let g be defined on all of $\mathbf{R}$. If B is a subset of $\mathbf{R}$, define the set $g^{-1}(B)$ by

$$g^{-1}(B) = \{x \in \mathbf{R} : g(x) \in B\}.$$

Show that g is continuous if and only if $g^{-1}(O)$ is open whenever $O \subseteq \mathbf{R}$ is an open set.

Solution

A fact we'll use is that $g(A) \subseteq B$ if and only if $A \subseteq g^{-1}(B)$. Which is true since

$$g(A) \subseteq B \implies A \subseteq g^{-1}(g(A)) \subseteq g^{-1}(B) \quad \text{and} \quad A \subseteq g^{-1}(B) \implies g(A) \subseteq B.$$

Fix $x \in \mathbf{R}$, we are given that $g^{-1}(V_\epsilon(x))$ is open, meaning there exists a neighborhood $V_\delta(x)$ with $V_\delta(x) \subseteq g^{-1}(V_\epsilon(x))$ implying $g(V_\delta(x)) \subseteq V_\epsilon(x)$ and thus g is continuous.

Now suppose g is continuous and let $O \subseteq \mathbf{R}$ be an open set. If $x \in g^{-1}(O)$ then $g(x) \in O$ and (since O is open) there exists a neighborhood $V_\epsilon(g(x)) \subseteq O$, now there exists $V_\delta(x)$ where $g(V_\delta(x)) \subseteq V_\epsilon(g(x)) \subseteq O$ so we have $V_\delta(x) \subseteq g^{-1}(O)$ and are done.

Exercise 4.4.12

Review Exercise 4.4.11, and then determine which of the following statements is true about a continuous function defined on $\mathbf{R}$:

(a) $f^{-1}(B)$ is finite whenever B is finite.

(b) $f^{-1}(K)$ is compact whenever K is compact.

(c) $f^{-1}(A)$ is bounded whenever A is bounded.

(d) $f^{-1}(F)$ is closed whenever F is closed.

Solution

(a) False, $f(x) = 0$ has $f^{-1}(\{0\}) = \mathbf{R}$

(b) False, $f(x) = 0$ has $\{0\}$ compact but $f^{-1}(\{0\}) = \mathbf{R}$ is not compact

(c) False, $f(x) = 0$ has $f^{-1}(\{0\}) = \mathbf{R}$

(d) True, let $(x_n) \to x$ be a convergent sequence in $f^{-1}(F)$, we know $f(x_n) \to f(x) \in F$ meaning $x \in f^{-1}(F)$ and so $f^{-1}(F)$ is closed.

Exercise 4.4.13 (Continuous Extension Theorem)

(a) Show that a uniformly continuous function preserves Cauchy sequences; that is, if $f : A \to \mathbf{R}$ is uniformly continuous and $(x_n) \subseteq A$ is a Cauchy sequence, then show $f(x_n)$ is a Cauchy sequence.

(b) Let g be a continuous function on the open interval (a, b). Prove that g is uniformly continuous on (a, b) if and only if it is possible to define values $g(a)$ and $g(b)$ at the endpoints so that the extended function g is continuous on $[a, b]$. (In the forward direction, first produce candidates for $g(a)$ and $g(b)$, and then show the extended g is continuous.)

Solution

(a) Let $\epsilon > 0$ and set N large enough that $n, m \geq N$ has $|x_n - x_m| < \delta$ implying $|f(x_n) - f(x_m)| < \epsilon$ by the uniformly continuity of f.

(b) Define $g(a) = \lim_{x \to a} g(x)$ and $g(b) = \lim_{x \to b} g(x)$ if both limits exist, then g is continuous on $[a, b]$ meaning it is uniformly continuous on $[a, b]$ by Theorem 4.4.7, and thus is uniformly continuous the subset (a, b).

If f were uniformly continuous (a, b) Cauchy sequences are preserved meaning the sequential definition for functional limits (Theorem 4.2.3) implies the limits $\lim_{x \to a} g(x)$ and $\lim_{x \to b} g(x)$ exist.

Exercise 4.4.14

Construct an alternate proof of Theorem 4.4.7 (Continuous on K implies Uniformly Continuous on K) using the open cover characterization of compactness from the Heine-Borel Theorem (Theorem 3.3.8 (iii)).

Solution

Let f be continuous on K, and choose $\epsilon > 0$. We can create the open cover $\{V_{\delta_x/2}(x) : x \in K\}$ where δ_x is chosen so every $y \in V_\delta(x)$ has $|f(x) - f(y)| < \epsilon/2$. Now since K is compact there exists a finite subcover $O = \{V_{\delta_1/2}(x_1), \ldots, V_{\delta_n/2}(x_n)\}$ with $K \subseteq \bigcup_{k=1}^{n} V_{\delta_k/2}(x_k)$.

Let $\delta = \min\{\delta_1, \dots, \delta_n\}/2$ and choose arbitrary $x \in K$. Since O is an open cover of K, there must be some $V_{\delta_i/2}(x_i) \ni x$. Now suppose $y \in K$ so that $|x - y| < \delta$. Then

$$|x - y| < \delta \leq \delta_i/2 \text{ and } |x - x_i| < \delta_i/2 \implies |y - x_i| < \delta_i$$

Also, $|x - x_i| < \delta_i/2 < \delta_i$. Since f is continuous at x_i, this implies $|f(y) - f(x_i)| < \epsilon/2$ and $|f(x) - f(x_i)| < \epsilon/2$, so by the Triangle Inequality $|f(x) - f(y)| < \epsilon$.

4.5 The Intermediate Value Theorem

Exercise 4.5.1
Show how the Intermediate Value Theorem follows as a corollary to preservation of connected sets (Theorem 4.5.2).

Solution
Since $[a, b]$ is connected, so too must be $f([a, b])$, which must be an interval containing both $f(a)$ and $f(b)$; therefore $[f(a), f(b)]$ (or $[f(b), f(a)]$ if $f(b) < f(a)$) is a subset of $f([a, b])$, completing the proof.

Exercise 4.5.2
Provide an example of each of the following, or explain why the request is impossible

(a) A continuous function defined on an open interval with range equal to a closed interval.

(b) A continuous function defined on a closed interval with range equal to an open interval.

(c) A continuous function defined on an open interval with range equal to an unbounded closed set different from $\mathbf{R}$.

(d) A continuous function defined on all of $\mathbf{R}$ with range equal to $\mathbf{Q}$.

Solution

(a) Possible, see Exercise 4.4.8 (b)

(b) Impossible by preservation of compact sets

(c) Let $f : (0, 1) \to [2, \infty)$ be defined by

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \in (0, 1/2] \\ \frac{1}{1-x} & \text{if } x \in (1/2, 1) \end{cases}$$

This works since $[2, \infty)$ is closed, unbounded and different from $\mathbf{R}$.

(d) Impossible as this contradicts the intermediate value theorem.

Exercise 4.5.3
A function f is increasing on A if $f(x) \leq f(y)$ for all $x < y$ in A. Show that if f is increasing on $[a, b]$ and satisfies the intermediate value property (Definition 4.5.3), then f is continuous on $[a, b]$.

Solution
Let $x \in [a,b]$ and choose $\epsilon > 0$. Let $L \in (f(a), f(x)) \cap (f(x) - \epsilon, f(x))$ IVP lets us find $c \in (a,x)$ with $f(c) = L$, thus $|f(x) - f(c)| < \epsilon$. Likewise we can find $d \in (x,b)$ with $|f(d) - f(x)| < \epsilon$. Because f is increasing $f(x) - f(c) < \epsilon$ implies $f(x) - f(c') < \epsilon$ for $c' \in (c,x)$ (and likewise for d) meaning every $y \in (c,d)$ has $|f(x) - f(y)| < \epsilon$. To get a δ-neighborhood simply set $\delta = \min\{x - c, d - x\}$.

Exercise 4.5.4
Let g be continuous on an interval A and let F be the set of points where g fails to be one-to-one; that is,

$$F = \{x \in A : f(x) = f(y) \text{ for some } y \neq x \text{ and } y \in A\}.$$

Show F is either empty or uncountable.

Solution
Suppose F is nonempty, let $x, y \in A$ with $x \neq y$ and $f(x) = f(y)$. Pick $z \in (x,y)$ such that $f(z) \neq f(x)$ (if z does not exist f is constant over (x,y) and we are finished early). By the Intermediate Value Theorem every $L \in (f(x), f(z))$ has an $x' \in (x,z)$ with $f(x') = L$. And since $L \in (f(z), f(y))$ as well we can find $y' \in (z,y)$ with $f(y') = L$, thus $f(y') = f(x')$ and so f is not 1-1 at every $L \in (f(x), f(z))$ which is uncountable.

Exercise 4.5.5

(a) Finish the proof of the Intermediate Value Theorem using the Axiom of Completeness started previously.

(b) Finish the proof of the Intermediate Value Theorem using the Nested Interval Property started previously.

Solution
Let f be continuous, and let $L \in (f(a), f(b))$, we must find $c \in (a,b)$ with $f(c) = L$. (If $f(a) > f(b)$ then instead consider $f'(x) = -f(x)$ and $L' = -L$)

(a) Let $c = \sup\{x : f(x) \leq L\}$. $f(c) < L$ is not possible since we could find δ small enough that $f(c + \delta) < L$ contradicting c being an upper bound. And $f(c) > L$ is impossible since we could find δ small enough that $f(c - \delta) > L$ contradicting c being the least upper bound (since we found a smaller upper bound). Thus we must have $f(c) = L$ completing the proof.

    A detail we glossed over is $c \in (a,b)$, which can be seen since $f(b) > L$ and $f(a) < L$ has $f(a + \delta) < L$ meaning a cannot be the least upper bound.

(b) Let $I_0 = [a,b]$ and bisect into two intervals, let I_1 be the interval where L is still between f at the endpoints. Continue like this to get a sequence $I_n \subseteq I_{n-1}$, $I_n \neq \emptyset$, and let $c \in \bigcap_{n=1}^{\infty} I_n$.

    Suppose for contradiction that $f(c) < L$; then there must be some $\epsilon > 0$ so that $V_\epsilon(f(c)) < L$. Since f is continuous, there must also be some δ so that $f(V_\delta(c)) < L$ - but this contradicts our construction in that one endpoint of I_n is mapped to a number greater than L, no matter how large n gets and how small I_n is as a result. A similar argument shows $f(c)$ cannot be larger than L, and therefore $f(c) = L$.

Exercise 4.5.6
Let $f : [0, 1] \to \mathbf{R}$ be continuous with $f(0) = f(1)$

(a) Show that there must exist $x, y \in [0, 1]$ satisfying $|x - y| = 1/2$ and $f(x) = f(y)$.

(b) Show that for each $n \in \mathbf{N}$ there exist $x_n, y_n \in [0, 1]$ with $|x_n - y_n| = 1/n$ and $f(x_n) = f(y_n)$.

(c) If $h \in (0, 1/2)$ is not of the form $1/n$, there does not necessarily exist $|x - y| = h$ satisfying $f(x) = f(y)$. Provide an example that illustrates this using $h = 2/5$.

Solution

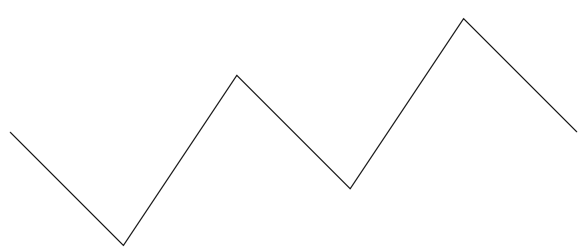
(a) Let $g(x) = f(x) - f(x + 1/2)$ and note that $g(x)$ is continuous over $[0, 1/2]$. Note also that $g(0) = f(0) - f(1/2) = -g(1/2) = f(1/2) - f(1)$, and therefore we can apply IVT to $g(x)$ over $[0, 1/2]$ to conclude that there must be a root of $g(x)$, and therefore $f(x) = f(x + 1/2)$, for some $x \in [0, 1/2]$.

(b) Let $g(x) = f(x) - f(x + 1/n)$ and note that $g(x)$ is continuous over $[0, \frac{n-1}{n}]$. Note also that $g(0) = -\sum_{i=1}^{n-1} g(i/n)$, and since $g(0)$ and $\sum_{i=1}^{n-1} g(i/n)$ have opposite sign there must be some natural number $1 \leq i \leq n - 1$ where $g(i/n)$ is opposite in sign from $g(0)$, at which point we can apply IVT in a similar fashion to part (a) and find a root of $g(x)$, completing the proof.

(c) Consider

$$f(x) = \begin{cases} -10(x - 0) + 0 & 0 < x \leq 1/5 \\ 15(x - 1/5) - 2 & 1/5 < x \leq 2/5 \\ -10(x - 2/5) + 1 & 2/5 < x \leq 3/5 \\ 15(x - 3/5) - 1 & 3/5 < x \leq 4/5 \\ -10(x - 4/5) + 2 & 4/5 < x \leq 1 \end{cases}$$

You could go through the grunt work of verifying that this meets the requirements, but it's easier to just plot the function (see graphic) - the function takes a corner every 1/5 along x.



Exercise 4.5.7
Let f be a continuous function on the closed interval $[0, 1]$ with range also contained in $[0, 1]$. Prove that f must have a fixed point; that is, show $f(x) = x$ for at least one value of $x \in [0, 1]$.

Solution
Since 0 and 1 are both in the range of f, choose a and b such that $f(a) = 0$ and $f(b) = 1$. Define $g(x) = f(x) - x$; clearly g is continuous, $g(a) = 0 - a = -a \leq 0$, and $g(b) = 1 - b \geq 0$. By IVT there must be some $c \in [a, b]$ so that $g(c) = 0$ and hence $f(c) = c$.

Exercise 4.5.8 (Inverse functions)
If a function $f : A \to \mathbf{R}$ is one-to-one, then we can define the inverse function f^{-1} on the range of f in the natural way: $f^{-1}(y) = x$ where $y = f(x)$.

Show that if f is continuous on an interval $[a, b]$ and one-to-one, then f^{-1} is also continuous.

Solution
Define a function f to be strictly increasing (decreasing) on A if $f(x) < f(y)$ ($f(x) > f(y)$) for all $x < y$ in A, and define strictly monotone to mean either strictly increasing or strictly decreasing. It's easy to show by contradiction using IVT that a one-to-one continuous function f must be strictly monotone, and similarly so is f^{-1}. Moreover, if f is strictly increasing then so is f^{-1}, and if f is strictly decreasing then so is f^{-1}.

We now show that f^{-1} satisfies the intermediate value property. Assume for now that f is strictly increasing - the proof for the case of strictly decreasing is similar. Consider $f(x_1) = y_1 < f(x_2) = y_2$ and any L between $f^{-1}(y_1)$ and $f^{-1}(y_2)$, i.e. $x_1 < L < x_2$; we need to show there is some $y_3 \in [y_1, y_2]$ such that $f^{-1}(y_3) = L$; clearly $y_3 = f(L)$ works. By Exercise 4.5.3, this shows $f^{-1}(x)$ is continuous.

4.6 Sets of Discontinuity

Exercise 4.6.1
Using modifications of Dirichlet and Thomae's functions, construct a function $f : \mathbf{R} \to \mathbf{R}$ so that

(a) $D_f = \mathbf{Z}^c$.

(b) $D_f = \{x : 0 < x \leq 1\}$.

Solution
(a) Modify the function continuous only at zero to be continuous around integers.

$$f(x) = \begin{cases} x - \text{round}(x) & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \in \mathbf{I} \end{cases}$$

where $\text{round}(x)$ rounds x to the nearest integer.

(b) Modify the function continuous only at zero to be continuous below 0 and above 1. Let

$$d(x) = \begin{cases} x & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \in \mathbf{I} \end{cases}$$

then

$$g(x) = \begin{cases} 0 & \text{if } x \le 0 \\ d(x) & \text{if } 0 < x \le 1 \\ 2 & \text{if } 1 < x \end{cases}$$

Exercise 4.6.2
Given a countable set $A = \{a_1, a_2, a_3, \ldots\}$, define $f(a_n) = 1/n$ and $f(x) = 0$ for all $x \notin A$. Find D_f.

Solution
To find D_f consider $x \in D_f$ for the two cases $x \in A$ and $x \notin A$. If $x \in A$ then f is not continuous, since $f(x) > 0$ but for any $\delta > 0$ we can find $y \in V_\delta(x)$ with $y \notin A$ (because A is countable, A^c must be dense) hence there is an unavoidable error of $|f(x) - f(y)| = f(x) > 0$.

Now consider $x \notin A$, using the sequential criterion for continuity notice every sequence $(x_n) \to x$ has $f(x_n) \to 0$ (since if $x_n \in A$ converge to 0 as $n \to \infty$, and $x_n \notin A$ are always 0), now since $f(x) = 0$ this shows f is continuous at $x \notin A$.

Together we've shown $D_f = A$. Setting $A = \mathbf{Q}$ and using a particular ordering recovers Thomae's function. Hence we can view this as a generalization of Thomae's function for arbitrary countable sets.

Exercise 4.6.3
State a similar definition for the left-hand limit

$$\lim_{x \to c^-} f(x) = L$$

Solution
For all $\epsilon > 0$ there exists a $\delta > 0$ such that $0 < c - x < \delta$ implies $|f(x) - L| < \epsilon$.

Exercise 4.6.4
Given $f : A \to \mathbf{R}$ and a limit point c of $A, \lim_{x \to c} f(x) = L$ if and only if

$$\lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

Supply a proof for this proposition.

Solution
Let $\epsilon > 0$, pick δ_1 so $0 < x - c < \delta_1$ implies $|f(x) - L| < \epsilon$, pick δ_2 so $0 < c - x < \delta_2$ implies $|f(x) - L| < \epsilon$. Finally, set $\delta = \min\{\delta_1, \delta_2\}$ to get $|f(x) - L| < \epsilon$ when $0 < |x - c| < \delta$, as desired.

Exercise 4.6.5
Prove that the only type of discontinuity a monotone function can have is a jump discontinuity.

Solution
Let f be monotone and assume f is increasing. For some c we want to show $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist.

Let $\epsilon > 0$ and set $L = \sup\{f(x) : x < c\}$, by the definition of sup, $L - \epsilon$ is not an upper bound for $\{f(x) : x < c\}$, hence there exists a $\delta_1 > 0$ with $f(c - \delta_1) > L - \epsilon$, thus $0 < c-x < \delta_1$ implies $|f(x)-L| < \epsilon$ (this is where we use the fact that f is increasing!), hence the lower limit exists. Likewise for $M = \inf\{f(x) : x > c\}$ we get $\delta_2 > 0$ with $0 < x - c < \delta_2$ implying $|f(x) - M| < \epsilon$, hence the upper limit exists.

Putting these together, we see that f is continuous at c if and only if $L = M$. In other words, the only possible discontinuity is a jump discontinuity $L \neq M$.

Exercise 4.6.6
Construct a bijection between the set of jump discontinuities of a monotone function f and a subset of $\mathbf{Q}$. Conclude that D_f for a monotone function f must either be finite or countable, but not uncountable.

Solution
In 4.6.5 we showed every $c \in D_f$ is a jump discontinuity, i.e. both sided limits L and M exist and $L \neq M$. Pick some $r \in (L, M) \cap \mathbf{Q}$ and assign $f(c) = r$. Continue like this to define a bijection $f : D_f \to Q$ where $Q \subseteq \mathbf{Q}$. Thus D_f must be finite or countable.

Exercise 4.6.7
In Section 4.1 we constructed functions where the set of discontinuity was $\mathbf{R}$ (Dirichlet's function), $\mathbf{R}\backslash\{0\}$ (modified Dirichlet function), and $\mathbf{Q}$ (Thomae's function).

(a) Show that in each of the above cases we get an F_σ set as the set where the function is discontinuous.

(b) Show that the two sets of discontinuity in Exercise 4.6.1 are F_σ sets.

Solution

(a) $\mathbf{R}$ is closed, so it is in F_σ, $\mathbf{R}\backslash\{0\} = \bigcup_{n=1}^\infty \mathbf{R}\backslash(-1/n, 1/n)$ is in F_σ since $\mathbf{R}\backslash(-1/n, 1/n)$ is closed, and finally $\mathbf{Q}$ is in F_σ since $\mathbf{Q} = \bigcup_{n=1}^\infty \{r_n\}$ (where r_n enumerate $\mathbf{Q}$, all countable sets are F_σ sets.)

(b) Recall countable unions of F_σ sets are F_σ (see 3.5.2) and that open intervals are F_σ sets, meaning $\mathbf{Z}^c = \bigcup_{z\in\mathbf{Z}}(z, z+1)$ is an F_σ set. As for $\{x : 0 < x \leq 1\} = (0, 1]$ I refer you to 3.5.3 (b).

Exercise 4.6.8
Prove that, for a fixed $\alpha > 0$, the set D_f^α is closed.

Solution
We do this by showing the complement is open. Let $x \in \left(D_f^\alpha\right)^c$, by the definition of α-continuity there exists a $\delta > 0$ such that $y, z \in V_\delta(x)$ have $|f(y) - f(z)| < \alpha$. To see openness notice $V_{\delta/2}(x) \subseteq \left(D_f^\alpha\right)^c$ since any $x' \in V_{\delta/2}(x)$ is α-continuous with $\delta' = \delta/2$.

Exercise 4.6.9
If $\alpha < \alpha'$, show that $D_f^{\alpha'} \subseteq D_f^\alpha$.

Solution
This is obvious. If $|f(y)-f(z)| < \alpha$ and $\alpha < \alpha'$ clearly $|f(y)-f(z)| < \alpha'$ as well. Increasing α only makes the condition less strict.

Exercise 4.6.10
Let $\alpha > 0$ be given. Show that if f is continuous at x, then it is α-continuous at x as well. Explain how it follows that $D_f^\alpha \subseteq D_f$.

Solution
Let $\epsilon = \alpha/2$ and use continuity to get $\delta > 0$ with $0 < |x-y| < \delta$ implying $|f(x)-f(y)| < \alpha/2$, which shows every $y, z \in V_\delta(x)$ satisfies $|f(y) - f(z)| < \alpha$ by the triangle inequality. Thus f is α-continuous at x.

The negation of "continuous at x implies α-continuous at x" is "not α-continuous at x implies not continuous at x", hence $D_f^\alpha \subseteq D_f$.

Exercise 4.6.11
Show that if f is not continuous at x, then f is not α-continuous for some $\alpha > 0$. Now explain why this guarantees that

$$D_f = \bigcup_{n=1}^{\infty} D_f^{\alpha_n}$$

where $\alpha_n = 1/n$.

Solution
Negating the definition of f being continuous at x, we see f is *not* continuous at x iff there exists an $\epsilon_0 > 0$ such that no $\delta > 0$ satisfies $|f(x) - f(y)| < \epsilon$ for all $0 < |x - y| < \delta$. Once $\alpha_n < \epsilon_0$ (i.e. $n > 1/\epsilon_0$) we will have $x \in D_f^{\alpha_n}$.

(This completes the proof that D_f is an F_σ set!)

Chapter 5

The Derivative

5.2 Derivatives and the Intermediate Value Property

Exercise 5.2.1
Supply proofs for parts (i) and (ii) of Theorem 5.2.4. (addition and scalar multiplication preserve differentiability)

Solution
Let f and g be differentiable at c. $f+g$ is differentiable at c by using the algebraic limit theorem

$$(f+g)' = \lim_{x\to c} \frac{(f+g)(x)-(f+g)(c)}{x-c} = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \lim_{x\to c} \frac{g(x)-g(c)}{x-c} = f'(c)+g'(c)$$

Likewise for $k \in \mathbf{R}$ we can apply ALT

$$(kf)' = \lim_{x\to c} \frac{kf(x)-kf(c)}{x-c} = k \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = kf'(c)$$

Exercise 5.2.2
Exactly one of the following requests is impossible. Decide which it is, and provide examples for the other three. In each case, let's assume the functions are defined on all of $\mathbf{R}$.

(a) Functions f and g not differentiable at zero but where fg is differentiable at zero.

(b) A function f not differentiable at zero and a function g differentiable at zero where fg is differentiable at zero.

(c) A function f not differentiable at zero and a function g differentiable at zero where $f+g$ is differentiable at zero.

(d) A function f differentiable at zero but not differentiable at any other point.

Solution
(a) Let

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

And $g(x) = -f(x)$. Both f and g are not differentiable at 0, but $fg = 1$ (constant) is.

(b) If fg and g are differentiable at zero, then $(fg)/g = f$ is differentiable at zero provided the quotient is well defined. *However* if we let $g(x) = 0$ then $fg = 0$ is differentiable at zero regardless of f. (Note we must have $g(0) = 0$ otherwise $f = (fg)/g$ would be differentiable at zero)

(c) Impossible, since $f = (f + g) - g$ would be differentiable at zero by the differentiable limit theorem

(d) Thomae's function is a starting point

$$t(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \text{ in lowest terms} \\ x & \text{if } x \in \mathbf{I} \end{cases}$$

We have

$$t'(0) = \lim_{x\to 0} t(x)/x$$

This limit doesn't exist, but if we define $f(x) = xt(x)$ then the inside is thomae's function and so

$$f'(0) = \lim_{x\to 0} f(x)/x = \lim_{x\to 0} t(x) = 0$$

is the only place the derivative exists.

Exercise 5.2.3

(a) Use Definition 5.2.1 to produce the proper formula for the derivative of $h(x) = 1/x$.

(b) Combine the result in part (a) with the Chain Rule (Theorem 5.2.5) to supply a proof for part (iv) of Theorem 5.2.4.

(c) Supply a direct proof of Theorem 5.2.4 (iv) by algebraically manipulating the difference quotient for (f/g) in a style similar to the proof of Theorem 5.2.4 (iii).

Solution

(a)

$$h'(x) = \lim_{y\to x} \frac{1/y - 1/x}{y - x} = \lim_{y\to x} \frac{x - y}{xy(y - x)} = \lim_{y\to x} -\frac{1}{xy} = -\frac{1}{x^2}$$

(b) By chain rule $(1/g)'(x) = -g'(x)/g(x)^2$ combined with product rule gives

$$\begin{aligned}(f \cdot 1/g)'(x) = f'(x) \cdot \frac{1}{g(x)} + f(x)(1/g)'(x) &= \frac{f'(x)}{g(x)} - f(x)\frac{g'(x)}{g(x)^2} \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}\end{aligned}$$

(c)

$$(f/g)'(x) = \lim_{y\to x} \frac{(f/g)(y) - (f/g)(x)}{y - x} = \lim_{y\to x} \frac{f(y)g(x) - f(x)g(y)}{g(x)g(y)(y - x)}$$

Now the $g(x)g(y)$ goes to $g(x)^2$, we just need to evaluate the derivatives in the numerator. We do this by adding and subtracting $f(x)g(x)$ similar to the proof of the product rule, then use the functional limit theorem to finish it off

$$\lim_{y\to x}\frac{g(x)(f(y)-f(x))+f(x)(g(x)-g(y))}{g(x)g(y)(y-x)}=\frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2}$$

Exercise 5.2.4
Follow these steps to provide a slightly modified proof of the Chain Rule.

(a) Show that a function $h : A \to \mathbf{R}$ is differentiable at $a \in A$ if and only if there exists a function $l : A \to \mathbf{R}$ which is continuous at a and satisfies

$$h(x)-h(a)=l(x)(x-a) \quad \text{for all } x\in A$$

(b) Use this criterion for differentiability (in both directions) to prove Theorem 5.2.5.

Solution
(a) First suppose h is differentiable at a, then we can define

$$l(x)=\begin{cases}\frac{h(x)-h(a)}{x-a} & x\neq a\\ h'(a) & x=a\end{cases}$$

Since $\lim_{x\to a} l(x)=l(a)$, l is continuous at a.

Now suppose $l : A\to\mathbf{R}$ exists and satisfies $h(x)-h(a)=l(x)(x-a)$, dividing by $(x-a)$ gives

$$\frac{h(x)-h(a)}{x-a}=l(x) \quad (x\neq a)$$

Taking the limit of both sides as $x\to a$ gives $h'(a)=l(a)$ (keep in mind the constraint $x\neq a$ makes no difference for the limit).

(b) Let f be differentiable at a and g be differentiable at $f(a)$. we will show $g(f(x))$ is differentiable at a with derivative $g'(f(a))f'(a)$.

Multiply the top and bottom by $f(y)-f(a)$ to get

$$(g\circ f)'(a)=\lim_{y\to a}\frac{g(f(y))-g(f(a))}{f(y)-f(a)}\cdot\frac{f(y)-f(a)}{y-a}$$

We're almost done, the right hand side is $f'(a)$ we just need to evaluate the nested limit on the left. Define $l(y)=\frac{g(y)-g(f(a))}{y-f(a)}$ and $l(a)=g'(f(a))$ then we have a product of continuous functions so we can use the algebraic limit theorem

$$\lim_{y\to a} l(f(y))\cdot\frac{f(y)-f(a)}{y-a}=g'(f(a))\cdot f'(a)$$

Exercise 5.2.5
Let

$$f_a(x) = \begin{cases} x^a & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

(a) For which values of a is f continuous at zero?

(b) For which values of a is f differentiable at zero? In this case, is the derivative function continuous?

(c) For which values of a is f twice-differentiable?

Solution

(a) All $a > 0$, if $a = 0$ we get the step function and $a < 0$ gives an asymptote

(b) At zero, if the derivative exists, then the one-sided limit from above

$$\lim_{x \to 0^+} \frac{x^a}{x} = \lim_{x \to 0^+} x^{a-1}$$

must exist and be equal to zero (to match the limit from below). Thus $a > 1$ is necessary, and in this case the derivative will be continuous.

(c) The first derivative is ax^{a-1}, and its derivative at zero must be zero, so $a > 2$.

Exercise 5.2.6
Let g be defined on an interval A, and let $c \in A$.

(a) Explain why $g'(c)$ in Definition 5.2.1 could have been given by

$$g'(c) = \lim_{h \to 0} \frac{g(c+h) - g(c)}{h}.$$

(b) Assume A is open. If g is differentiable at $c \in A$, show

$$g'(c) = \lim_{h \to 0} \frac{g(c+h) - g(c-h)}{2h}.$$

Solution

(a) This is just a change of variable from the normal definition, set $h = (x - c)$ to get

$$\lim_{x \to c} \frac{g(x) - g(c)}{x - c} = \lim_{h \to 0} \frac{g(c+h) - g(c)}{h}$$

(b) Some basic algebra

$$\lim_{h \to 0} \frac{g(c+h) - g(c) + g(c) - g(c-h)}{2h} = \frac{1}{2}\left(\lim_{h \to 0} \frac{g(c+h) - g(c)}{h} + \lim_{h \to 0} \frac{g(c) - g(c-h)}{h}\right)$$

The first term is clearly $g'(c)$ and the second is $g'(c)$ with the substitution $h = c - x$ since

$$g'(c) = \lim_{x \to c} \frac{g(c) - g(x)}{c - x}$$

Thus the whole expression is $g'(c)$.

Exercise 5.2.7
Let

$$g_a(x) = \begin{cases} x^a \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Find a particular (potentially noninteger) value for a so that

(a) g_a is differentiable on $\mathbf{R}$ but such that g_a' is unbounded on $[0, 1]$.

(b) g_a is differentiable on $\mathbf{R}$ with g_a' continuous but not differentiable at zero.

(c) g_a is differentiable on $\mathbf{R}$ and g_a' is differentiable on $\mathbf{R}$, but such that g_a'' is not continuous at zero.

Solution
We need $a > 0$ to make g_a continuous at zero, for differentiation notice

$$g_a'(0) = \lim_{x\to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x\to 0} \frac{x^a \sin(1/x)}{x} = \lim_{x\to 0} x^{a-1} \sin(1/x)$$

Thus for differentiation we need $x^{a-1} \sin(1/x)$ to have a limit at zero. Keep this in mind for the problems. Also note

$$\begin{aligned} g_a'(x) &= ax^{a-1} \sin(1/x) + x^a \cos(1/x)(-1/x^2) \\ &= ax^{a-1} \sin(1/x) - x^{a-2} \cos(1/x) \end{aligned}$$

(a) To get unboundedness we need the $x^{a-2} \cos(1/x)$ term to become unbounded, so pick $a = 1.5$ to satisfy $a > 1$ (differentiable) and $a < 2$ (unbounded derivative)

(b) Pick $a = 2.5$, It's clear that $g_a'(x) = 2.5x^{1.5} \sin(1/x) - x^{0.5} \cos(1/x)$ is continuous at zero, but not differentiable since $x^{0.5} \cos(1/x)$ isn't differentiable at zero and $2.5x^{1.5} \sin(1/x)$ is (if both terms were not differentiable they could cancel eachother out, constant vigilance!).

(c) Pick $a = 3.5$, for all intents and purposes g_a' behaves like g_{a-2} (because the $x^{a-2} \cos(1/x)$ term acts as a bottleneck) meaning g_a'' exists, but isn't continuous since we get another -2 to a leading to a $x^{-0.5}$ term in g_a''.

Exercise 5.2.8
Review the definition of uniform continuity (Definition 4.4.4). Given a differentiable function $f : A \to \mathbf{R}$, let's say that f is *uniformly differentiable* on A if, given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| < \epsilon \quad \text{whenever } 0 < |x - y| < \delta.$$

(a) Is $f(x) = x^2$ uniformly differentiable on $\mathbf{R}$? How about $g(x) = x^3$?

(b) Show that if a function is uniformly differentiable on an interval A, then the derivative must be continuous on A.

(c) Is there a theorem analogous to Theorem 4.4.7 for differentiation? Are functions that are differentiable on a closed interval $[a, b]$ necessarily uniformly differentiable?

Solution

(a) We have

$$\left|\frac{x^2 - y^2}{x - y} - 2y\right| = |(x + y) - 2y| = |x - y| < \delta$$

Thus $\delta = \epsilon$ suffices to show x^2 is uniformly differentiable. Now for x^3

$$\left|\frac{x^3 - y^3}{x - y} - 3y^2\right| = \left|\frac{(x - y)(x^2 + xy + y^2)}{x - y} - 3y^2\right| = |x^2 + xy - 2y^2|$$

Let $y = (x + h)$ to get

$$|x^2 + x(x + h) - 2(x + h)^2| = |x^2 + x^2xh - 2x^2 - 4xh - 2h^2| = |3xh + 2h^2|$$

Fix $0 < h < \delta$, since x can be as big as we want we can make $|3xh + 2h^2| > \epsilon$. As no fixed δ works x^3 is not uniformly differentiable.

(b) We need to show that $\lim_{x \to y} f'(x) = f'(y)$. Now, if we choose δ small enough such that when $|x - y| < \delta$, both

$$\left|f'(x) - \frac{f(x) - f(y)}{x - y}\right| < \frac{\epsilon}{2}$$

and

$$\left|\frac{f(x) - f(y)}{x - y} - f'(y)\right| < \frac{\epsilon}{2}$$

(this is possible because f is uniformly differentiable), then we have

$$\begin{aligned} f'(x) - f'(y) &= f'(x) - \frac{f(x) - f(y)}{x - y} + \frac{f(x) - f(y)}{x - y} - f'(y) \\ &\leq \left|f'(x) - \frac{f(x) - f(y)}{x - y}\right| + \left|\frac{f(x) - f(y)}{x - y} - f'(y)\right| < \epsilon \end{aligned}$$

as desired.

(c) Consider the counterexample $f(x) = x^2 \sin(1/x)$ over $[0, 1]$ (where $f(0) = 0$). f is differentiable over $[0, 1]$ but not uniformly differentiable.

Intuitively this is because I can find x_n, y_n such that the slope between them becomes unbounded, but the derivative f' must stay bounded. To be exact set

$$x_n = \frac{1}{2\pi n + \pi/2}, \quad y_n = \frac{1}{2\pi n}$$

then

$$\begin{aligned}
\left|\frac{f(x_n)-f(y_n)}{x_n-y_n}-f'(x_n)\right| &= \left|\frac{1}{x_n-y_n}-f'(x_n)\right| \\
&= \left|\frac{(2\pi n)(2\pi n+\pi/2)}{\pi/2}-f'(x_n)\right| \\
&= |4n(2\pi n+\pi/2)-f'(x_n)|
\end{aligned}$$

Now since $f'(x_n) = 2x\sin(1/x) - \cos(1/x)$ is bounded I can defeat any δ by picking n large enough so that

$$|x_n - y_n| < \delta \quad \text{and} \quad \left|\frac{f(x_n)-f(y_n)}{x_n-y_n}-f'(x_n)\right| \geq \epsilon$$

Thus f is not uniformly differentiable.

If you try to apply the same proof as for uniform continuity you get stuck at the triangle inequality.

Exercise 5.2.9
Decide whether each conjecture is true or false. Provide an argument for those that are true and a counterexample for each one that is false.

(a) If f' exists on an interval and is not constant, then f' must take on some irrational values.

(b) If f' exists on an open interval and there is some point c where $f'(c) > 0$, then there exists a δ-neighborhood $V_\delta(c)$ around c in which $f'(x) > 0$ for all $x \in V_\delta(c)$.

(c) If f is differentiable on an interval containing zero and if $\lim_{x\to 0} f'(x) = L$, then it must be that $L = f'(0)$.

Solution

(a) If f' is not constant there exist x, y with $f'(x) < f'(y)$, since derivatives obey the intermediate value proper (Theorem 5.2.7) f' takes on the value of every irrational number in $(f'(x), f'(y))$.

(b) True if f' is continuous, False in general. Let $f(x) = x^2\sin(1/x) + x/2$ so that

$$f'(x) = 2x\sin(1/x) - \cos(1/x) + 1/2$$

Notice how f' alternates between positive and negative for small x. We have

$$\lim_{x\to 0}\frac{x^2\sin(1/x)-0}{x-0} = \lim_{x\to 0} x\sin(1/x) = 0$$

Thus $f'(0) = 1/2$, pick any δ and I can find $x \in V_\delta(0)$ with $f'(x) \leq 0$. To be explicit define $x_n = 1/(2\pi n)$ so that $f'(x_n) = -1/2$ then pick n large enough so $x_n \in V_\delta(0)$.

(c) A direct proof using L'Hospital's rule: continuity of f forces the limit of the quotient to be $0/0$.

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} f'(x) = L$$

In general a corollary of L'Hospital's rule is that derivatives can only have *essential* discontinuities, discontinuities where $\lim_{x\to c} f'(x)$ doesn't exist.

Alternatively you can see this by contradiction. Derivatives obeying the IVP rules out jump discontinuities, and removable discontinuities may be ruled out by ϵ-δing.

Exercise 5.2.10
Recall that a function $f : (a, b) \to \mathbf{R}$ is increasing on (a, b) if $f(x) \le f(y)$ whenever $x < y$ in (a, b). A familiar mantra from calculus is that a differentiable function is increasing if its derivative is positive, but this statement requires some sharpening in order to be completely accurate. Show that the function

$$g(x) = \begin{cases} x/2 + x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable on $\mathbf{R}$ and satisfies $g'(0) > 0$. Now, prove that g is not increasing over any open interval containing 0 .

In the next section we will see that f is indeed increasing on (a, b) if and only if $f'(x) \ge 0$ for all $x \in (a, b)$.

Solution
Already did this in 5.2.9 (b) as I came up with the same counterexample as abbott!

Exercise 5.2.11
Assume that g is differentiable on $[a, b]$ and satisfies $g'(a) < 0 < g'(b)$.

(a) Show that there exists a point $x \in (a, b)$ where $g(a) > g(x)$, and a point $y \in (a, b)$ where $g(y) < g(b)$.

(b) Now complete the proof of Darboux's Theorem started earlier.

Solution
(a) Since $g'(a) = \lim_{x\to a} \frac{g(x)-g(a)}{x-a} < 0$ we know

$$g'(a) - \epsilon < \frac{g(x) - g(a)}{x - a} < g'(a) + \epsilon \quad \text{when} \quad |x - a| < \delta$$

Therefor

$$g(x) < g(a) + (g'(a) + \epsilon)(x - a)$$

Pick ϵ small enough that $g'(a) + \epsilon < 0$, and since $(x - a) > 0$ we have $g(x) < g(a)$ as desired. A similar argument works for $g(y) < g(b)$.

(b) By EVT g obtains a minimum value over $[a, b]$. We just showed a and b are not minima, therefore the minimum point c must be in the interior (a, b) which has $g'(c) = 0$ by Theorem 5.2.6.

Exercise 5.2.12 (Inverse functions)
If $f : [a, b] \to \mathbf{R}$ is one-to-one, then there exists an inverse function f^{-1} defined on the range of f given by $f^{-1}(y) = x$ where $y = f(x)$. In Exercise 4.5.8 we saw that if f is continuous on $[a, b]$, then f^{-1} is continuous on its domain. Let's add the assumption that f is differentiable on $[a, b]$ with $f'(x) \neq 0$ for all $x \in [a, b]$. Show f^{-1} is differentiable with

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)} \quad \text{where } y = f(x).$$

Solution
Typically I'd use the chain rule, but to be rigorous we must prove f^{-1} is differentiable. Consider the limit

$$\left(f^{-1}\right)'(d) = \lim_{y \to d} \frac{f^{-1}(y) - f^{-1}(d)}{y - d}$$

Since both y and d are in the range of f we can substitute $y = f(x)$ and $d = f(c)$

$$\lim_{f(x) \to f(c)} \frac{x - c}{f(x) - f(c)} = \lim_{x \to c} \frac{x - c}{f(x) - f(c)} = \frac{1}{f'(c)}$$

Changing $f(x) \to f(a)$ into $x \to a$ is possible because f is continuous, meaning that x being close to a causes $f(x)$ to be close to $f(a)$. (This can be made more rigorous, but I'm lazy)

5.3 The Mean Value Theorems

Exercise 5.3.1
Recall from Exercise 4.4.9 that a function $f : A \to \mathbf{R}$ is Lipschitz on A if there exists an $M > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M$$

for all $x \neq y$ in A

(a) Show that if f is differentiable on a closed interval $[a, b]$ and if f' is continuous on $[a, b]$, then f is Lipschitz on $[a, b]$.

(b) Review the definition of a contractive function in Exercise 4.3.11. If we add the assumption that $|f'(x)| < 1$ on $[a, b]$, does it follow that f is contractive on this set?

Solution

(a) Since f' is continuous on the compact set $[a, b]$ we can set M such that $|f'(x)| \leq M$ over $[a, b]$, then pick $x, y \in [a, b]$ with $x < y$. Apply MVT on $[x, y]$ to get a $c \in (x, y)$ with

$$\frac{f(x) - f(y)}{x - y} = f'(c)$$

Which implies

$$\left| \frac{f(x) - f(y)}{x - y} \right| = |f'(c)| \leq M$$

Since x, y were arbitrary this shows f is Lipschitz.

(b) For f to be contractive we need some $c \in (0,1)$ with $|f(x) - f(y)| \leq c|x - y|$. Let x and y be arbitrary, and consider

$$c = \left|\frac{f(x) - f(y)}{x - y}\right|$$

By the mean value theorem, there must be some $d \in (0,1)$ where $|f'(d)| = c$. But since x and y are arbitrary and $|f'(d)| < 1$, we must have $c < 1$ always, and therefore f is contractive.

Exercise 5.3.2
Let f be differentiable on an interval A. If $f'(x) \neq 0$ on A, show that f is one-to-one on A. Provide an example to show that the converse statement need not be true.

Solution
Let x, y be in A with $x < y$, to show $f(x) \neq f(y)$ apply the Mean Value Theorem on $[x, y]$ to get $c \in (x, y)$ with

$$f'(c) = \frac{f(x) - f(y)}{x - y}$$

Now since $f'(c) \neq 0$ we must have $f(x) - f(y) \neq 0$, and thus $f(x) \neq f(y)$.

To see the converse is false consider how $f(x) = x^3$ is 1-1 but has $f'(0) = 0$.

Exercise 5.3.3
Let h be a differentiable function defined on the interval $[0,3]$, and assume that $h(0) = 1, h(1) = 2$, and $h(3) = 2$.

(a) Argue that there exists a point $d \in [0,3]$ where $h(d) = d$.

(b) Argue that at some point c we have $h'(c) = 1/3$.

(c) Argue that $h'(x) = 1/4$ at some point in the domain.

Solution

(a) Consider $g(x) = h(x) - x$ which is continuous and has $g(0) = 1$ and $g(3) = -1$, then apply the IVT to find $d \in (0,3)$ with $g(d) = 0$ which implies $h(d) = d$.

(b) Apply MVT on $[0,3]$ to get $c \in (0,3)$ with

$$h'(c) = \frac{h(0) - h(3)}{0 - 3} = 1/3$$

(c) We can find $c \in (0,3)$ with $h'(c) = 1/3$ and a $d \in (1,3)$ with $h'(d) = 0$. So by Darboux's theorem there exists a point $x \in (c, d)$ with $h'(x) = 1/4$.

Exercise 5.3.4
Let f be differentiable on an interval A containing zero, and assume (x_n) is a sequence in A with $(x_n) \to 0$ and $x_n \neq 0$.

(a) If $f(x_n) = 0$ for all $n \in \mathbf{N}$, show $f(0) = 0$ and $f'(0) = 0$.

(b) Add the assumption that f is twice-differentiable at zero and show that $f''(0) = 0$ as well.

Solution

(a) Since $f'(0)$ exists and $f(x_n) = 0$ for all n we have

$$f'(0) = \lim \frac{f(x_n)}{x_n} = 0$$

(b) By the mean value theorem over $[0, x_n]$ there exists a $c_n \in (0, x_n)$ such that

$$f'(c_n) = \frac{f(x_n)}{x_n}$$

Then like in (a)

$$f''(0) = \lim \frac{f'(c_n)}{c_n} = 0$$

Exercise 5.3.5

(a) Supply the details for the proof of Cauchy's Generalized Mean Value Theorem (Theorem 5.3.5).

(b) Give a graphical interpretation of the Generalized Mean Value Theorem analogous to the one given for the Mean Value Theorem at the beginning of Section 5.3. (Consider f and g as parametric equations for a curve.)

Solution

(a) Let $h(x) = [f(b)-f(a)]g(x)-[g(b)-g(a)]f(x)$ and apply the MVT to h to get $c \in (a, b)$ with

$$h'(c) = \frac{h(b) - h(a)}{b - a} = \frac{[f(b) - f(a)][g(b) - g(a)] - [g(b) - g(a)][f(b) - f(a)]}{b - a} = 0$$

Thus we have

$$h'(c) = [f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c) = 0$$

Completing the proof.

(b) Rename $x = f$, $g = y$, $t = a$, then the theorem states

$$\frac{x'(t)}{y'(t)} = \frac{dx}{dy} = \frac{x(b) - x(a)}{y(b) - y(a)}$$

In other words, it's the mean value theorem for parametric curves.

Exercise 5.3.6

(a) Let $g : [0, a] \to \mathbf{R}$ be differentiable, $g(0) = 0$, and $|g'(x)| \leq M$ for all $x \in [0, a]$. Show $|g(x)| \leq Mx$ for all $x \in [0, a]$.

(b) Let $h : [0, a] \to \mathbf{R}$ be twice differentiable, $h'(0) = h(0) = 0$ and $|h''(x)| \leq M$ for all $x \in [0, a]$. Show $|h(x)| \leq Mx^2/2$ for all $x \in [0, a]$.

(c) Conjecture and prove an analogous result for a function that is differentiable three times on $[0, a]$.

Solution

(a) For $x \in [0, a]$, apply MVT to find a $c \in [0, x]$ with

$$g'(c) = \frac{g(x)}{x} \implies g(x) = g'(c)x \implies |g(x)| \leq Mx$$

(b) This is a special case of the theorem that if $f(0) = g(0) = 0$ and $f'(x) \leq g'(x)$ for all $x \in [0, a]$ then $f(x) \leq g(x)$. To prove this note how letting $h(x) = g(x) - f(x)$ changes the statement into $h'(x) \geq 0$ implying $h(x) \geq 0$. Which is true since MVT to get $c \in [0, x]$ implies $h'(c) = h(x)/x \geq 0$ thus $h(x) \geq 0$.

Now returning to $|h''(x)| \leq Mx^2/2$ apply the above result to both cases in the inequality

$$\begin{aligned} -Mx^2/2 \leq h''(x) \leq Mx^2/2 &\implies -Mx \leq h'(x) \leq Mx \\ &\implies -M \leq h(x) \leq M \\ &\implies |h(x)| \leq M \end{aligned}$$

Which proves $|h(x)| \leq M$.

(c) I conjecture $|f(x)| \leq x^3/6$ when $f(0) = f'(0) = f''(0) = 0$. The proof is the same as (b), except we differentiate one more time.

Exercise 5.3.7

A fixed point of a function f is a value x where $f(x) = x$. Show that if f is differentiable on an interval with $f'(x) \neq 1$, then f can have at most one fixed point.

Solution

Suppose for contradiction that x, y are fixed points of f with $x < y$, then apply MVT on $[x, y]$ to get

$$f'(c) = \frac{f(x) - f(y)}{x - y} = \frac{x - y}{x - y} = 1$$

But we know $f'(c) \neq 1$, therefore finding more than one fixed point is impossible.

Exercise 5.3.8

Assume f is continuous on an interval containing zero and differentiable for all $x \neq 0$. If $\lim_{x\to 0} f'(x) = L$, show $f'(0)$ exists and equals L.

Solution

Using L'Hospital's rule: Let $g(x) = f(x) - f(0)$ (and note that they have the same derivatives and are both continuous), then

$$f'(0) = g'(0) = \lim_{x\to 0} \frac{g(x)}{x} = \lim_{x\to 0} \frac{g'(x)}{1} = L$$

(A modified function is necessary to ensure $\lim_{x\to 0} g(x) = 0$.)

Exercise 5.3.9
Assume f and g are as described in Theorem 5.3.6, but now add the assumption that f and g are differentiable at a, and f' and g' are continuous at a with $g'(a) \neq 0$. Find a short proof for the 0/0 case of L'Hopital's Rule under this stronger hypothesis.

Solution
Let (x_n) be a sequence approaching a and apply MVT on $[x_n, a]$ to find $c_n, d_n \in (x_n, a)$ with

$$f'(c_n) = \frac{f(x_n)}{x_n - a} \quad \text{and} \quad g'(d_n) = \frac{g(x_n)}{x_n - a}$$

Meaning

$$\lim \frac{f(x_n)}{g(x_n)} = \lim \frac{f'(c_n)/(x_n - a)}{g'(d_n)/(x_n - a)} = \lim \frac{f'(c_n)}{g'(d_n)}$$

The continuity of f' and g' combined with $g'(a) \neq 0$ implies the limit exists

$$\lim \frac{f'(c_n)}{g'(d_n)} = \frac{f'(a)}{g'(a)} = L$$

Since we showed $\lim f(x_n)/g(x_n) = L$ *for all* sequences (x_n) the Sequential Criterion for Functional Limits (Theorem 4.2.3) implies

$$\lim_{x \to a} \frac{f(x)}{g(x)} = L$$

Which completes the proof.

Exercise 5.3.10
Let $f(x) = x \sin(1/x^4)\, e^{-1/x^2}$ and $g(x) = e^{-1/x^2}$. Using the familiar properties of these functions, compute the limit as x approaches zero of $f(x), g(x), f(x)/g(x)$, and $f'(x)/g'(x)$. Explain why the results are surprising but not in conflict with the content of Theorem 5.3.6.
1

Solution

$$\lim_{x \to 0} f(x) = 0, \quad \lim_{x \to 0} g(x) = 0, \quad \lim_{x \to 0} \frac{f(x)}{g(x)} = 0,$$

To compute the last limit we need to find f' and g' using derivative rules. Let $h(x) = x\sin(1/x^4)$ for bookkeeping.

$$\begin{aligned}
g'(x) &= \frac{d}{dx} e^{-x^{-2}} = e^{-x^{-2}}(2x^{-3}) = g(x)(2x^{-3}) \\
h'(x) &= \sin(x^{-4}) + x\cos(x^{-4})(-4x^{-5}) = \sin(x^{-4}) - 4x^{-4}\cos(x^{-4}) \\
f'(x) &= \frac{d}{dx} h(x)g(x) \\
&= h'(x)g(x) + g(x)(2x^{-3})h(x) \\
&= g(x)(h'(x) + 2x^{-3}h(x))
\end{aligned}$$

We can simplify by dividing out $g(x)$

$$\frac{f'(x)}{g'(x)} = \frac{g(x)(h'(x) + 2x^{-3}h(x))}{g(x)(2x^{-3})} = \frac{h'(x) + 2x^{-3}h(x)}{2x^{-3}} = \frac{1}{2}x^3h'(x) + h(x)$$

Now we can compute the limit

$$\begin{aligned}
\lim_{x\to 0}\left[\frac{1}{2}x^3h'(x) + h(x)\right] &= \lim_{x\to 0}\frac{1}{2}x^3h'(x) && \text{since } \lim_{x\to 0} h(x) = 0 \\
&= \frac{1}{2}\lim_{x\to 0}\left[x^3\sin(x^{-4}) - 4x^{-1}\cos(x^{-4})\right] && \text{substitute } h' \\
&= -2\lim_{x\to 0} x^{-1}\cos(x^{-4}) && \text{since } \lim_{x\to 0} x^3\sin(x^{-4}) = 0 \\
&\to \text{does not exist, just like } \lim_{x\to 0} 1/x
\end{aligned}$$

L'Hopital's rule for 0/0 would apply if f'/g' existed. But when it doesn't exist f/g may still exist. Put another way, the converse of L'Hopital's rule does not hold.

Exercise 5.3.11

(a) Use the Generalized Mean Value Theorem to furnish a proof of the 0/0 case of L'Hopital's Rule (Theorem 5.3.6).

(b) If we keep the first part of the hypothesis of Theorem 5.3.6 the same, but we assume that

$$\lim_{x\to a}\frac{f'(x)}{g'(x)} = \infty$$

does it necessarily follow that

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \infty?$$

Solution

(a) Let f, g be continuous functions with $f(a) = g(a) = 0$ and $f'(x) \neq 0$, $g'(x) \neq 0$ around a, suppose

$$\lim_{x\to a}\frac{f'(x)}{g'(x)} = L$$

We would like to show $\lim_{x\to a}\frac{f(x)}{g(x)} = L$. Choose $\epsilon > 0$ then let $\delta > 0$ be such that

$$\left|\frac{f'(x)}{g'(x)} - L\right| < \epsilon$$

Let $x \in (a, a+\delta)$ and apply the generalized mean value theorem on (a, x) to get a $c \in (a, x)$ with

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(c)}{g'(c)}$$

Subtract L from both sides and take absolute values, (and use $f(a) = g(a) = 0$) to get

$$\left|\frac{f(x)}{g(x)} - L\right| < \epsilon$$

We could do the same process starting from $x \in (a-\delta, a)$ as well, thus, for all $0 < |x| < \delta$ we have

$$\left|\frac{f(x)}{g(x)} - L\right| < \epsilon$$

Implying $\lim_{x\to a} \frac{f(x)}{g(x)} = L$ as desired.

An interesting thing to note is that *the same* δ works for both $f'(x)/g'(x)$ and for $f(x)/g(x)$. In other words, $f(x)/g(x)$ converges to L at least as fast as $f'(x)/g'(x)$ does.

(b) Choose $M > 0$ and let $\delta > 0$ be such that $0 < |x-a| < \delta$ implies $f'(x)/g'(x) > M$. Let $x \in (a, a+\delta)$ be arbitrary, then apply MVT on (a, x) to get $c \in (a, x)$ with

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$$

Since $0 < |c-a| < \delta$ we have $f'(c)/g'(c) > M$ and thus

$$\frac{f(x)}{g(x)} > M$$

for all $x \in (a, a+\delta)$, but again, we could just as easily apply this reasoning to $(a-\delta, a)$. So in general, all x with $0 < |x-a| < \delta$ satisfy

$$\frac{f(x)}{g(x)} > M$$

Which is clearly the same as saying $\lim_{x\to a} f(x)/g(x) = \infty$.

Exercise 5.3.12
If f is twice differentiable on an open interval containing a and f'' is continuous at a, show

$$\lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = f''(a).$$

(Compare this to Exercise 5.2.6(b).)

Solution
Let $\epsilon > 0$ and choose $\delta_1 > 0$ so every $|h| < \delta_1$ has

$$\left|\frac{f(a+h) - f(a)}{h} - f'(a)\right| < \epsilon$$

Choose $\delta_2 > 0$ so every $0 < |x-a| < \delta_2$ has $|f''(x) - f''(a)| < \epsilon$ (this is where we use the continuity of f'' at a.) and set $\delta = \min\{\delta_1, \delta_2\}$.

Without loss of generality assume $h > 0$, apply MVT on $(a, a+h)$ to get $c \in (a, a+h)$ with

$$\frac{f(a+h) - f(a)}{h} = f'(c)$$

Likewise MVT on $(a-h,a)$ gives $d \in (a-h,a)$ with

$$\frac{f(a)-f(a-h)}{h} = f'(d)$$

Meaning for this specific h our estimate for $f''(a)$ is

$$\frac{f(a+h)-2f(a)+f(a-h)}{h^2} = \frac{f'(c)-f'(d)}{h}$$

Note that $d < a < c$ and $|c-d| < h$, the right-hand side is essentially a central difference estimate for $f''(a)$. We can prove this using the mean value theorem on (d,c) to get a $c' \in (d,c)$ with

$$\frac{f'(c)-f'(d)}{h} = f''(c')$$

Recall our choice of δ ensures that $|x-c| < \delta$ implies $|f''(x)-f''(a)| < \epsilon$, setting $x = c'$ (note $|c'-a| < \delta$) gives (putting everything together)

$$\begin{aligned}\left|\frac{f(a+h)-2f(a)+f(a-h)}{h^2} - f''(a)\right| &= \left|\frac{f'(c)-f'(d)}{h} - f''(a)\right| \\ &= |f''(c')-f''(a)| \\ &< \epsilon.\end{aligned}$$

5.4 A Continuous Nowhere-Differentiable Function

Exercise 5.4.1
Sketch a graph of $(1/2)h(2x)$ on $[-2,3]$. Give a qualitative description of the functions

$$h_n(x) = \frac{1}{2^n}h(2^n x)$$

as n gets larger.

Solution
$(1/2)h(2x)$ looks like $h(x)$ shrunk down by a factor of 2 - both the amplitude and the period of the sawtooth are halved. Similarly, $h_n(x)$ looks like $h(x)$ shrunk down by a factor of 2^n.

Exercise 5.4.2
Fix $x \in \mathbf{R}$. Argue that the series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} h\left(2^n x\right)$$

converges and thus $g(x)$ is properly defined.

Solution
$h_n(x) \leq 1/2^n$ and we know $\sum_{n=0}^{\infty} 1/2^n$ converges, so by the comparison test (Theorem 2.7.4) $g(x)$ is properly defined.

Exercise 5.4.3
Taking the continuity of $h(x)$ as given, reference the proper theorems from Chapter 4 that imply that the finite sum

$$g_m(x) = \sum_{n=0}^{m} \frac{1}{2^n} h\left(2^n x\right)$$

is continuous on $\mathbf{R}$.

Solution
The Composition of Continuous Functions (Theorem 4.3.9) ensure $h(2^n x)$ is continuous, and the various parts of the Algebraic Continuity Theorem (Theorem 4.3.2) ensure their combinations to form $g_m(x)$ is continuous.

Exercise 5.4.4
As the graph in Figure 5.7 suggests, the structure of $g(x)$ is quite intricate. Answer the following questions, assuming that $g(x)$ is indeed continuous.

(a) How do we know g attains a maximum value M on $[0, 2]$? What is this value?

(b) Let D be the set of points in $[0, 2]$ where g attains its maximum. That is $D = \{x \in [0, 2] : g(x) = M\}$. Find one point in D.

(c) Is D finite, countable, or uncountable?

Solution
(a) $[0, 2]$ is a compact set, so by the Extreme Value Theorem g must have a maximum. Since the infinite series defining g converges (and converges absolutely, for that matter), we are free to use associativity to analyze it. Group the terms in pairs, so that $f_0(x) = h_0(x) + h_1(x)$ and in general $f_n(x) = h_{2n}(x) + h_{2n+1}(x)$. Note that over $[0, 2]$,

$$f_0(x) = \begin{cases} 2x & x \leq 1/2 \\ 1 & 1/2 \leq x \leq 3/2 \\ -2x + 4 & 3/2 \leq x \end{cases}$$

and in particular, $f_0(x)$ reaches a maximum of 1 over $[1/2, 3/2]$, an interval of length 1. Now, $f_1(x)$ looks like a repeated $f_0(x)$ scaled down a factor of 4, and therefore has a period of $2/4 = 1/2$ and a maximum value of $1/4$ over an interval of length $1/4$. Since the period of $f_1(x)$ is less than half the length of the interval $[1/2, 3/2]$, there must be one cycle of $f_1(x)$ fully within $[1/2, 3/2]$, and therefore the maximum value of $f_0(x)+f_1(x)$ is $1+1/4$. (One cycle is when the function starts at 0, goes to a maximum and plateaus, then comes back down to 0.)

A scaling argument between f_n and f_{n+1} can then be used to show that $\max g(x) > \sum_{k=0}^{n} \frac{1}{4^k}$ for all $n \in \mathbf{N}$. However,

$$g(x) = \sum_{k=0}^{\infty} f_k(x) \leq \sum_{k=0}^{\infty} \frac{1}{4^k} = 4/3$$

and therefore $\max g(x) = 4/3$.

(b) For this we'll need to track the intervals where $f_n(x)$ reaches its maximum more carefully. Note that the endpoints of each cycle in $f_n(x)$ are also endpoints of cycles in $f_{n+1}(x)$.

Some computation shows that if one of the cycles of $f_n(x)$ reaches its maximum over the interval $[a_n, a_n+b_n]$, then there will be two cycles in $f_{n+1}(x)$ which cover $[a_n, a_n+b_n/2]$ and $[a_n+b_n/2, a_n+b_n]$, leading to $f_{n+1}(x)$ having maximum intervals in $[a_n+b_n/8, a_n+3b_n/8]$ and $[a_n+5b_n/8, a_n+7b_n/8]$.

To find a point in D we can repeatedly only consider the lower maximum interval at each iteration of $f_n(x)$. Defining $a_0 = 1/2,\ b_0 = 1$, and $a_{n+1} = a_n + b_n/8,\ b_{n+1} = b_n/4$, clearly $b_n = 1/4^n$. a_n is (nearly) a geometric series, with

$$a_n = \frac{1}{2} + \frac{1}{6}\frac{4^n-1}{4^n},\ a_n + b_n = \frac{1}{2} + \frac{4^n+5}{6}$$

Both a_n and $a_n + b_n$ approach $1/2 + 1/6 = 2/3$, so we might conjecture $2/3 \in D$. Indeed, $2/3$ is in every $[a_n, a_n + b_n]$, and therefore $2/3$ is a point in D.

(c) D is uncountable; we will use an argument similar to showing that the Cantor set is uncountable - by mapping all sequences x_n of infinite 0s and 1s to a unique point in D. Construct a sequence of intervals I_n as such: If $x_n = 0$ then take the first cycle of $f_n(x)$ which is in I_{n-1} and define I_n to be the maximum interval of $f_n(x)$ in that cycle; if $x_n = 1$ then take the second cycle. (For completeness, define $I_0 = [1/2, 3/2]$.) By the Nested Interval Property the infinite intersection of these intervals yields a point in D, while clearly each unique sequence x_n will map to a unique point in D. Since the set of all x_n is uncountable, so too is D.

Exercise 5.4.5
Show that

$$\frac{g(x_m) - g(0)}{x_m - 0} = m + 1$$

and use this to prove that $g'(0)$ does not exist.

Solution
Note that $h_n(x_m) = x_m$ for $0 \le n \le m$ and $h_n(x_m) = 0$ for $n > m$, therefore $g(x_m) = (m+1)x_m$. Clearly $g(0) = 0$, so

$$\frac{g(x_m) - g(0)}{x_m - 0} = m + 1$$

If $g'(0)$ existed, then

$$\lim_{x\to 0} \frac{g(x) - g(0)}{x - 0}$$

would be well defined. But we've just identified a sequence x_m approaching 0 for which this expression grows without bound, and hence this limit cannot exist, and therefore $g'(0)$ does not exist.

Exercise 5.4.6

(a) Modify the previous argument to show that $g'(1)$ does not exist. Show that $g'(1/2)$ does not exist.

(b) Show that $g'(x)$ does not exist for any rational number of the form $x = p/2^k$ where $p \in \mathbf{Z}$ and $k \in \mathbf{N} \cup \{0\}$.

Solution

(a) Let $x_m = 1 + 1/2^m$ with $m \geq 0$. Then $h_0(x_m) = 1 - 1/2^m$, $h_n(x_m) = 1/2^m$ for $1 \leq n \leq m$, and $h_n(x_m) = 0$ for $n > m$. $g(1) = 1$, so

$$\frac{g(x_m) - g(1)}{x_m - 1} = \frac{1 - 1/2^m + m/2^m - 1}{1/2^m} = m - 1$$

and for the same reason as in Exercise 5.4.5, $g'(1)$ does not exist.

$h_0(x)$ is differentiable at $1/2$, so we can instead consider whether $(g - h_0)(x)$ is differentiable at $1/2$. But since $h_n(x) = 2h_{n+1}(x/2)$, $g(x) = 2(g - h_0)(x/2)$. Since $g'(1)$ does not exist, both $g - h_0$ and g are not differentiable at $1/2$.

(b) Note that $h_n(x)$ is only non-differentiable at points of the form $q/2^n$ where $q \in \mathbf{Z}$. Express x in lowest form, so that p is odd, and consider

$$i(x) = g(x) - \sum_{n=0}^{k-1} h_n(x) = \sum_{n=k}^{\infty} h_k(x)$$

which is differentiable at x if and only iff g is as well. Since $h_{a+b}(x) = h_a(2^b x)/2^b$,

$$i(x) = \frac{1}{2^k} \sum_{n=0}^{\infty} h_0(2^k x) = \frac{1}{2^k} g(2^k x) = \frac{1}{2^k} g(p)$$

We've shown that g is not differentiable at 0 or 1, and since g is periodic it's easy to show it's not differentiable at any $p \in \mathbf{Z}$, completing the proof.

Exercise 5.4.7

(a) First prove the following general lemma: Let f be defined on an open interval J and assume f is differentiable at $a \in J$. If (a_n) and (b_n) are sequences satisfying $a_n < a < b_n$ and $\lim a_n = \lim b_n = a$, show

$$f'(a) = \lim_{n \to \infty} \frac{f(b_n) - f(a_n)}{b_n - a_n}.$$

(b) Now use this lemma to show that $g'(x)$ does not exist.

Solution

(a) Keeping in mind the Sequential Criterion for Functional Limits (Theorem 4.2.3),

$$\begin{aligned}
\lim_{n\to\infty} \frac{f(b_n)-f(a_n)}{b_n-a_n} &= \lim_{n\to\infty} \frac{f(b_n)-f(a)}{b_n-a}\frac{b_n-a}{b_n-a_n} + \lim_{n\to\infty} \frac{f(a)-f(a_n)}{a-a_n}\frac{a-a_n}{b_n-a_n} \\
&= f'(a)\lim_{n\to\infty} \frac{b_n-a}{b_n-a_n} + f'(a)\lim_{n\to\infty} \frac{a-a_n}{b_n-a_n} \\
&= f'(a)\left(\lim_{n\to\infty} \frac{b_n-a+a-a_n}{b_n-a_n}\right) \\
&= f'(a)
\end{aligned}$$

(b) I claim that

$$\frac{g(y_{n+1})-g(x_{n+1})}{y_{n+1}-x_{n+1}} = \frac{g(y_n)-g(x_n)}{y_n-x_n} \pm 1 \quad \forall n \geq 0$$

To see this, note first that h_n is a straight line with slope -1 or 1 between $p_n/2^n$ and $(p_n+1)/2^n$, and therefore

$$\frac{h_n(b)-h_n(a)}{b-a} = \pm 1 \quad \forall a,b \in \left[\frac{p_n}{2^n}, \frac{p_n+1}{2^n}\right]$$

Note also that $h_n(k)=0$ when k is of the form $p/2^{n-1}$ with $p \in \mathbf{Z}$. This fact, combined with how we chose x_n and y_n so that $[x_{n+1}, y_{n+1}] \subset [x_n, y_n]$ means we can use the above constant for each term in g as it appears.

$$\begin{aligned}
\frac{g(y_{n+1})-g(x_{n+1})}{y_{n+1}-x_{n+1}} &= \sum_{k=0}^{n+1} \frac{h_k(y_{n+1})-h_k(x_{n+1})}{y_{n+1}-x_{n+1}} \\
&= \frac{h_{n+1}(y_{n+1})-h_{n+1}(x_{n+1})}{y_{n+1}-x_{n+1}} + \sum_{k=0}^{n} \frac{h_k(y_{n+1})-h_k(x_{n+1})}{y_{n+1}-x_{n+1}} \\
&= \pm 1 + \sum_{k=0}^{n} \frac{h_k(y_n)-h_k(x_n)}{y_n-x_n} \\
&= \frac{g(y_n)-g(x_n)}{y_n-x_n} \pm 1
\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \frac{g(y_n)-g(x_n)}{y_n-x_n}$$

does not exist, since the difference between consecutive elements does not converge to zero, and therefore by our lemma in part (a), g is not differentiable at x.

Exercise 5.4.8

Review the argument for the nondifferentiability of $g(x)$ at nondyadic points. Does the argument still work if we replace $g(x)$ with the summation $\sum_{n=0}^{\infty}(1/2^n)h_n(3^n x)$? Does the argument work for the function $\sum_{n=0}^{\infty}(1/3^n)h_n(2^n x)$?

Solution

The critical part to showing that $g(x)$ is not differentiable at nondyadic points was showing that

$$\frac{h_{n+1}(y_{n+1}) - h_{n+1}(x_{n+1})}{y_{n+1} - x_{n+1}}$$

does not converge to zero, preventing the limit defining the derivative to exist. For the case $\sum_{n=0}^{\infty}(1/2^n)h_n(3^n x)$, the above term would diverge to infinity, since $y_n - x_n$ would decrease by a factor of 3 on each iteration while $h_n(y_n) - h_n(x_n)$ would only decrease by a factor of 2. For similar reasons, in the case of $\sum_{n=0}^{\infty}(1/3^n)h_n(2^n x)$, the above term would converge to 0, and the argument is no longer valid.

Chapter 6

Power series

6.2 Uniform Convergence of a Sequence of Functions

Exercise 6.2.1
Let

$$f_n(x) = \frac{nx}{1+nx^2}.$$

(a) Find the pointwise limit of (f_n) for all $x \in (0, \infty)$.

(b) Is the convergence uniform on $(0, \infty)$?

(c) Is the convergence uniform on $(0, 1)$?

(d) Is the convergence uniform on $(1, \infty)$?

Solution

(a) $\lim f_n(x) = \lim \frac{nx}{1+nx^2} = \lim \frac{x}{1/n+x^2} = 1/x$

(b) Examine the difference $|f_n(x) - f(x)|$

$$\left|\frac{nx}{1+nx^2} - \frac{1}{x}\right| = \left|\frac{nx^2 - (1+nx^2)}{x(1+nx^2)}\right| = \frac{1}{x(1+nx^2)}$$

Consider $x_n = 1/n$, then

$$|f_n(x_n) - f(x_n)| = \frac{1}{(1/n)(1+n(1/n^2))} = \frac{1}{n/2} = \frac{n}{2}$$

Which shows that no matter how big n is, we can find $x = 1/n$ such that $|f_n(x) - f(x)| \geq 1/2$ meaning ϵ cannot be made smaller then $1/2$. So f isn't uniformly continuous.

(c) No, same logic as (b)

(d) Yes, because $x \geq 1$ implies

$$|f_n(x) - f(x)| = \frac{1}{x(1+nx^2)} \leq \frac{1}{n}$$

Meaning for all $\epsilon > 0$, setting $N > 1/\epsilon$ implies every $n \geq N$ has $|f_n(x) - f(x)| \leq 1/N < \epsilon$ for every $x \in (1, \infty)$.

Exercise 6.2.2

(a) Define a sequence of functions on $\mathbf{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

and let f be the pointwise limit of f_n. Is each f_n continuous at zero? Does $f_n \to f$ uniformly on $\mathbf{R}$? Is f continuous at zero?

(b) Repeat this exercise using the sequence of functions

$$g_n(x) = \begin{cases} x & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n} \\ 0 & \text{otherwise.} \end{cases}$$

(c) Repeat the exercise once more with the sequence

$$h_n(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \\ x & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n-1} \\ 0 & \text{otherwise.} \end{cases}$$

In each case, explain how the results are consistent with the content of the Continuous Limit Theorem (Theorem 6.2.6).

Solution

(a) Each f_n is continuous at zero, but f is not continuous at zero meaning (by Theorem 6.2.6) that f_n does not converge to f uniformly.

(b) Each g_n is continuous at zero, and the pointwise limit g is also continuous at zero. Since we aren't contradicting 6.2.6 the convergence may or may not be uniform.

The definitions show $|g(x) - g_n(x)| < 1/n$ for all x (max is at $x = 1/(n+1)$). Setting $N > 1/\epsilon$ gives (for all $n \geq N$ and for all $x \in \mathbf{R}$)

$$|g(x) - g_n(x)| < \epsilon$$

As desired, thus $(g_n) \to g$ uniformly.

(c) Each h_n is continuous at zero, and so is the pointwise limit h. 6.2.6 doesn't apply so we'll have to check if the convergence is uniform. Notice that if $x_n = 1/n$ then

$$|h(x_n) - h_n(x_n)| = 1 - 1/n$$

For all n, meaning no matter how big n is, we can't make $|h - h_n| < 1/2$ for all x implying h_n *does not* converge to h uniformly.

Exercise 6.2.3
For each $n \in \mathbf{N}$ and $x \in [0, \infty)$, let

$$g_n(x) = \frac{x}{1 + x^n} \quad \text{and} \quad h_n(x) = \begin{cases} 1 & \text{if } x \geq 1/n \\ nx & \text{if } 0 \leq x < 1/n \end{cases}$$

Answer the following questions for the sequences (g_n) and (h_n);

(a) Find the pointwise limit on $[0, \infty)$.

(b) Explain how we know that the convergence cannot be uniform on $[0, \infty)$.

(c) Choose a smaller set over which the convergence is uniform and supply an argument to show that this is indeed the case.

Solution
(a)

$$\lim g_n(x) = \begin{cases} x & \text{if } x \in [0, 1) \\ \frac{1}{2} & \text{if } x = 1 \\ 0 & \text{if } x \in (1, \infty) \end{cases} \quad \text{and} \quad \lim h_n(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(b) They can't converge uniformly since it would contradict Theorem 6.2.6 as both g_n and h_n are continuous but the limit functions are not.

(c) Over $[1, \infty)$ we have $h_n(x) = h(x) = 1$ for all n, thus $|h_n(x) - h(x)| = 0$ for all $x \in [1, \infty)$ so h_n converges uniformly.

Now for g_n. Let $t \in [0, 1)$, I claim $g_n(x) \to x$ uniformly over $[0, t)$ since

$$\left| \frac{x}{1 + x^n} - x \right| = \left| \frac{x - x(1 + x^n)}{1 + x^n} \right| = \left| \frac{x^{n+1}}{1 + x^n} \right| < \left| t^{n+1} \right| < \epsilon \quad \forall x$$

After setting $n > \log_t \epsilon$.

Exercise 6.2.4
Review Exercise 5.2.8 which includes the definition for a uniformly differentiable function. Use the results discussed in Section 2 to show that if f is uniformly differentiable, then f' is continuous.

Solution
The definition of f being uniformly differentiable tells us: for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| < \epsilon \quad \text{whenever } 0 < |x - y| < \delta$$

We can use this to show continuity of f' via a triangle inequality and exploiting the symmetry in x, y.

$$|f'(x) - f'(y)| < \left| f'(x) - \frac{f(x) - f(y)}{x - y} \right| + \left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| < \epsilon$$

After picking δ so that every $|x-y|<\delta$ has

$$\left|\frac{f(x)-f(y)}{x-y}-f'(y)\right|<\epsilon/2$$

Alternative proof: Let $y_n = x+1/n$, and consider the sequence of functions

$$f_n'(x)=\frac{f(x)-f(y_n)}{x-y_n}$$

Each $f_n'(x)$ is continuous, and uniform differentiability implies that f_n' uniformly converges to $f'(x)$; hence $f'(x)$ is continuous by Theorem 6.2.6.

Exercise 6.2.5
Using the Cauchy Criterion for convergent sequences of real numbers (Theorem 2.6.4), supply a proof for Theorem 6.2.5 (Cauchy Criterion for Uniform Convergence). (First, define a candidate for $f(x)$, and then argue that $f_n \to f$ uniformly.)

Solution
In the forward direction, suppose (f_n) converges uniformly to f and set N large enough that $n \geq N$ has $|f_n(x)-f(x)|<\epsilon/2$ for all x, then use the triangle inequality (where $m \geq N$ as well)

$$|f_n(x)-f_m(x)| \leq |f_n(x)-f(x)|+|f(x)-f_m(x)|<\epsilon/2+\epsilon/2=\epsilon.$$

In the reverse direction, suppose we can find an N so that every $n,m \geq N$ has $|f_n(x)-f_m(x)|<\epsilon/2$ for all x. Fix x and apply Theorem 2.6.4 to conclude the sequence $(f_n(x))$ converges to some limit L, and define $f(x)=L$. Doing this for all x gives us the pointwise limit f. Now we show $(f_n)\to f$ uniformly using the fact that $|f_n(x)-f_m(x)|<\epsilon/2$ *for all* x. Let $n \geq N$, notice that *for all* m

$$|f_n(x)-f(x)| \leq |f_n(x)-f_m(x)|+|f_m(x)-f(x)|$$

For all $m \geq N$ we have $|f_n(x)-f_m(x)|<\epsilon/2$ and

$$|f_n(x)-f(x)|<\epsilon/2+|f_m(x)-f(x)|$$

For any x we can choose m large enough to ensure $|f_m(x)-f(x)|<\epsilon/2$ (pointwise convergence), and since the inequality is for all m this implies $|f_n(x)-f(x)| \leq \epsilon$ for all x as desired.

Exercise 6.2.6
Assume $f_n \to f$ on a set A. Theorem 6.2.6 is an example of a typical type of question which asks whether a trait possessed by each f_n is inherited by the limit function. Provide an example to show that all of the following propositions are false if the convergence is only assumed to be pointwise on A. Then go back and decide which are true under the stronger hypothesis of uniform convergence.

(a) If each f_n is uniformly continuous, then f is uniformly continuous.

(b) If each f_n is bounded, then f is bounded.

(c) If each f_n has a finite number of discontinuities, then f has a finite number of discontinuities.

(d) If each f_n has fewer than M discontinuities (where $M \in \mathbf{N}$ is fixed), then f has fewer than M discontinuities.

(e) If each f_n has at most a countable number of discontinuities, then f has at most a countable number of discontinuities.

Solution

(a) False pointwise when

$$f_n(x) = \frac{1}{1+nx^2} \quad \text{and} \quad f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

Now suppose (f_n) converges to f uniformly. We have

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|$$

By uniform convergence to f, we can fix n large enough so that the first and last terms are both less than $\epsilon/3$. Then since f_n is uniformly continuous, we can choose δ so that $|x - y| < \delta$ implies the middle term is less than $\epsilon/3$, and that $|f(x) - f(y)| < \epsilon$, implying f is uniformly continuous.

(b) False pointwise when

$$f_n(x) = \begin{cases} x & \text{if } x < n \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f(x) = x$$

Now suppose $(f_n) \to f$ uniformly. We want to show f is bounded. Let M_n bound f_n, ie. $|f_n(x)| < M_n$ for all $x \in A$.

Set $\epsilon = 1$ and apply the Cauchy Criterion to get N so $m \geq n > N$ implies

$$|f_n(x) - f_m(x)| < 1$$

Setting $n = N$ and rearranging gives

$$|f_m(x)| < |f_N(x)| + \epsilon < M_N + \epsilon$$

implying f_m is bounded by M_N when $m \geq N$.

Now set $\tilde{N} > N$ large enough that $m \geq N$ implies $|f(x) - f_m(x)| < 1$ which, after rearranging gives

$$|f(x)| < 1 + |f_m(x)| < 1 + M_N$$

Implying f is bounded.

(c) False for pointwise and uniform convergence. Define

$$f_n(x) = \begin{cases} 1/m & \text{if } x = 1/m \text{ and } m \le n,\ m \in \mathbf{N} \\ 0 & \text{otherwise} \end{cases}$$

and

$$f(x) = \begin{cases} 1/m & \text{if } x = 1/m,\ m \in \mathbf{N} \\ 0 & \text{otherwise} \end{cases}$$

The reason $(f_n) \to f$ uniformly is

$$|f(x) - f_n(x)| = \left(\begin{cases} 1/m & \text{if } x = 1/m \text{ and } m > n,\ m \in \mathbf{N} \\ 0 & \text{otherwise} \end{cases} \right) < 1/(n+1)$$

Each f_n has exactly n point discontinuities, but f has countably many. Thus this is a counterexample.

Intuitively f_n adds on finer and finer details of f, which is why it converges uniformly. But discontinuities can be as small/detailed as we want without screwing up uniform convergence.

(d) False pointwise since we can have each (f_n) continuous (zero discontinuities) but have f not be continuous (see (a) for an example).

Now suppose $(f_n) \to f$ uniformly. Let x_0 be a discontinuity of f, meaning there exists an ϵ_0 such that $|f(x_0) - f(x)| > \epsilon_0$ no matter how small $|x - x_0|$ is. I'd like to show x_0 is a discontinuity of f_n for some n, i.e. that there exists an $\epsilon_0' > 0$ such that $|f_n(x_0) - f_n(x)| > \epsilon_0'$ no matter how small $|x - x_0|$ is.

Pick $\epsilon < \epsilon_0/2$ and set N large enough that $n \ge N$ implies $|f_n(x) - f(x)| < \epsilon$. Applying the three way triangle inequality gives

$$\begin{aligned} \epsilon_0 &< |f(x_0) - f(x)| \\ &\le |f(x_0) - f_n(x_0)| + |f_n(x_0) - f_n(x)| + |f_n(x) - f(x)| \\ &< 2\epsilon + |f_n(x_0) - f_n(x)| \end{aligned}$$

Letting $\epsilon_0' = \epsilon_0 - 2\epsilon > 0$ we see $|f_n(x_0) - f_n(x)| > \epsilon_0'$ meaning f_n is not continuous at x_0 for all $n \ge N$.

Now given discontinuities of f, applying the above process multiple times shows that *eventually* f_n will have every discontinuity of f, but f_n has at most M discontinuities, implying f has at most M discontinuities.

(e) False pointwise when (using a modified version of Thomae's function for f_n)

$$f_n(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{n}{n+q} & \text{if } x = p/q \text{ (in lowest terms)} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q} \end{cases}$$

Now suppose $(f_n) \to f$ uniformly. In (d) we showed every discontinuity of f is eventually a discontinuity of f_n; reworded this is saying (where D_f is the set of discontinuities of f)

$$D_f \subseteq \bigcup_{n=1}^{\infty} D_{f_n}$$

Since each D_{f_n} is countable, this implies D_f is countable.

Exercise 6.2.7

Let f be uniformly continuous on all of $\mathbf{R}$, and define a sequence of functions by $f_n(x) = f\left(x + \frac{1}{n}\right)$. Show that $f_n \to f$ uniformly. Give an example to show that this proposition fails if f is only assumed to be continuous and not uniformly continuous on $\mathbf{R}$.

Solution

Given $\epsilon > 0$ set $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. Then set $N > 1/\delta$ so that $n \geq N$ implies (since $1/n < \delta$)

$$|f(x) - f_n(x)| = |f(x) - f(x + 1/n)| < \epsilon$$

Which shows $(f_n) \to f$ uniformly.

To see this doesn't work if f is only continuous, consider $f(x) = x^2$. We have

$$|f(x) - f_n(x)| = \left| x^2 - \left(x + \frac{1}{n}\right)^2 \right| = \left| \frac{2x}{n} + \frac{1}{n^2} \right|$$

which given a fixed n, becomes arbitrarily big as x goes to infinity. Hence (f_n) does not converge uniformly.

Exercise 6.2.8

Let (g_n) be a sequence of continuous functions that converges uniformly to g on a compact set K. If $g(x) \neq 0$ on K, show $(1/g_n)$ converges uniformly on K to $1/g$.

Solution

Let's examine the difference

$$|1/g(x) - 1/g_n(x)| = \left| \frac{g(x) - g_n(x)}{g(x)g_n(x)} \right| = |g(x) - g_n(x)| \left| \frac{1}{g(x)g_n(x)} \right|$$

We'd like to bound the rightmost term.

Theorem 6.2.6 implies g is continuous, and Theorem 4.4.1 implies $g(K)$ is compact, hence $|g(K)|$ has a minimum, call it m. This allows us to bound $|1/g(x)| < 1/m$.

To bound $1/g_n$ set ϵ small enough that $m - \epsilon > 0$ then use uniform continuity to get N such that $n \geq N$ has

$$|g(x) - g_n(x)| < \epsilon$$

Since $g_n(x) \in (g(x) - \epsilon, g(x) + \epsilon)$ we have $|g_n(x)| > |g(x)| - \epsilon$ and finally $|g(x)| > m$ implies $|g_n(x)| > m - \epsilon$ thus $|1/g_n(x)| < 1/(m - \epsilon)$ and so

$$M = \frac{1}{m(m - \epsilon)} \implies |1/g(x) - 1/g_n(x)| < M|g(x) - g_n(x)|$$

Given an ϵ, setting N big enough to make $|g(x) - g_n(x)| < M/\epsilon$ gives the desired result.

Exercise 6.2.9
Assume (f_n) and (g_n) are uniformly convergent sequences of functions.

(a) Show that $(f_n + g_n)$ is a uniformly convergent sequence of functions.

(b) Give an example to show that the product $(f_n g_n)$ may not converge uniformly.

(c) Prove that if there exists an $M > 0$ such that $|f_n| \le M$ and $|g_n| \le M$ for all $n \in \mathbf{N}$, then $(f_n g_n)$ does converge uniformly.

Solution
(a) Obvious by the triangle inequality and Cauchy Criterion

(b) Let $f_n(x) = x = f(x)$ and $g_n(x) = x + 1/n$. Suppose $n, m \ge N$ for some N, Cauchy gives us

$$|f_n g_n - f_m g_m| = |x(1/n - 1/m)|$$

Making x large makes the error blow up regardless of how big N is, thus $f_n g_n$ does not converge uniformly.

(c) By the triangle inequality (same trick as for the product rule)

$$\begin{aligned} |f_n g_n - f_m g_m| &\le |f_n g_n - f_n g_m| + |f_n g_m - f_m g_m| \\ &= |f_n| \cdot |g_n - g_m| + |g_m| \cdot |f_n - f_m| \\ &< M|g_n - g_m| + M|f_n - f_m| \\ &< \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

After setting N big enough that $n, m \ge N$ implies $|f_n - f_m| < M/2\epsilon$ and $|g_n - g_m| < M/2\epsilon$.

Exercise 6.2.10
This exercise and the next explore partial converses of the Continuous Limit Theorem (Theorem 6.2.6). Assume $f_n \to f$ pointwise on $[a, b]$ and the limit function f is continuous on $[a, b]$. If each f_n is increasing (but not necessarily continuous), show $f_n \to f$ uniformly.

Solution
f is continuous and therefore will map $[a, b]$ to a closed interval $[c, d]$. By the Order Limit Theorem, we also know that f must be increasing - just compare the sequence $\lim_{n\to\infty} f_n(x_1)$ to $\lim_{n\to\infty} f_n(x_2)$ when $x_2 > x_1$.

The key implication of f_n and f being increasing is that there is a bound to how fast the error $|f - f_n|$ can grow with respect to x, which is proportional to how fast f grows. Intuitively, if we have $f > f_n$ and want the error to grow as fast as possible with respect to x, all we can do is hold f_n constant. Alternatively, if we have $f < f_n$, f_n can't go too far above f since f needs to catch up eventually.

More formally, for $\epsilon > 0$, define $y_1, y_2, y_3, \ldots, y_n$ to evenly split up $[c, d]$ into intervals of size at most $\epsilon_1 = \epsilon/5$. (In other words, $y_1 = c$, $y_n = d$, $y_{k+2} - y_{k+1} = y_{k+1} - y_k < \epsilon_1$.) Since f is increasing, we can define $x_1, \ldots, x_n$ so that $f(x_k) = y_k$.

Since $f_n \to f$, we can find M_k so that $m_k > M_k$ implies $|y_k - f_{m_k}(x_k)| < \epsilon_1$. Let $M = \max\{M_1, M_2, \ldots, M_n\}$. Now, let $m > M$ be arbitrary. Keeping in mind that f_m is increasing, we can bound $f_m(x_{i+1}) - f_m(x_i)$ by

$$\begin{aligned} f_m(x_{i+1}) - f_m(x_i) &= |f_m(x_{i+1}) - f_m(x_i)| \\ &\le |f_m(x_{i+1}) - y_{i+1}| + |y_{i+1} - y_i| + |y_i - f_m(x_i)| \\ &< \epsilon_1 + \epsilon_1 + \epsilon_1 = 3\epsilon_1 \end{aligned}$$

Now consider $|f(x) - f_m(x)| \quad x \in [x_i, x_{i+1}]$ with i arbitrary. Since f is increasing, $y_i \le f(x) \le y_{i+1}$, so $f(x) - y_i < \epsilon_1$. Similarly $f_m(x) - f_m(x_i) \le f_m(x_{i+1}) - f_m(x_i) < 3\epsilon_1$. Finally, we have

$$\begin{aligned} |f(x) - f_m(x)| &= |f(x) - y_i| + |y_i - f_m(x_i)| + |f_m(x) - f_m(x_i)| \\ &< \epsilon_1 + \epsilon_1 + 3\epsilon_1 = \epsilon \end{aligned}$$

Since m and i were arbitrary, this completes the proof.

Exercise 6.2.11 (Dini's Theorem)
Assume $f_n \to f$ pointwise on a compact set K and assume that for each $x \in K$ the sequence $f_n(x)$ is increasing. Follow these steps to show that if f_n and f are continuous on K, then the convergence is uniform.

(a) Set $g_n = f - f_n$ and translate the preceding hypothesis into statements about the sequence (g_n).

(b) Let $\epsilon > 0$ be arbitrary, and define $K_n = \{x \in K : g_n(x) \ge \epsilon\}$. Argue that $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$, and use this observation to finish the argument.

Solution

(a) We want $(g_n) \to 0$ uniformly, where g_n is continuous, $g_n(x)$ is decreasing and $g_n(x) \to 0$.

(b) $x \in K_{n+1}$ by definition has $g_{n+1}(x) \ge \epsilon$, since $(g_n(x))$ is decreasing we must also have $g_n(x) \ge \epsilon$. Thus $K_{n+1} \subseteq K_n$. Now, if each $K_n \neq \emptyset$ the nested compact set property (Theorem 3.3.5) would imply there exists an $x_0 \in \bigcap_{n=1}^{\infty} K_n$. But this is impossible because $g_n(x_0) \to 0$ implies eventually $g_n(x_0) < \epsilon$. Therefore since the compact set property doesn't apply, there must exist an N with $K_N = \emptyset$, implying every $n \ge N$ has (by subsets) $K_n = \emptyset$ and thus $|g_n| < \epsilon$.

Exercise 6.2.12 (Cantor Function)
Review the construction of the Cantor set $C \subseteq [0, 1]$ from Section 3.1. This exercise makes use of results and notation from this discussion.

(a) Define $f_0(x) = x$ for all $x \in [0, 1]$. Now, let

$$f_1(x) = \begin{cases} (3/2)x & \text{for } 0 \le x \le 1/3 \\ 1/2 & \text{for } 1/3 < x < 2/3 \\ (3/2)x - 1/2 & \text{for } 2/3 \le x \le 1 \end{cases}$$

Sketch f_0 and f_1 over $[0, 1]$ and observe that f_1 is continuous, increasing, and constant on the middle third $(1/3, 2/3) = [0, 1] \backslash C_1$.

(b) Construct f_2 by imitating this process of flattening out the middle third of each nonconstant segment of f_1. Specifically, let

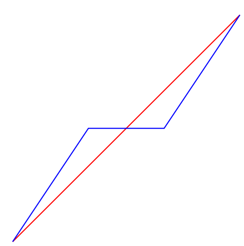
$$f_2(x) = \begin{cases} (1/2)f_1(3x) & \text{for } 0 \leq x \leq 1/3 \\ f_1(x) & \text{for } 1/3 < x < 2/3 \\ (1/2)f_1(3x-2) + 1/2 & \text{for } 2/3 \leq x \leq 1 \end{cases}$$

If we continue this process, show that the resulting sequence (f_n) converges uniformly on $[0, 1]$.

(c) Let $f = \lim f_n$. Prove that f is a continuous, increasing function on $[0, 1]$ with $f(0) = 0$ and $f(1) = 1$ that satisfies $f'(x) = 0$ for all x in the open set $[0, 1]\backslash C$. Recall that the "length" of the Cantor set C is 0 . Somehow, f manages to increase from 0 to 1 while remaining constant on a set of "length 1."

Solution

(a) f_0 in red, f_1 in blue:



(b) Define

$$f_{n+1}(x) = \begin{cases} (1/2)f_n(3x) & \text{for } 0 \leq x \leq 1/3 \\ f_n(x) & \text{for } 1/3 < x < 2/3 \\ (1/2)f_n(3x-2) + 1/2 & \text{for } 2/3 \leq x \leq 1 \end{cases}$$

We will aim to use the Cauchy Criterion to show f_n uniformly converges. I first prove through induction that $|f_n(x) - f_{n-1}(x)| < 1/2^n$. The base case of $n = 1$ is trivial by the definitions of f_1 and f_0.

Now first assume $|f_n(x) - f_{n-1}(x)| < 1/2^n$; our goal is to show $|f_{n+1}(x) - f_n(x)| < 1/2^{n+1}$. Note this is obviously true over $(1/3, 2/3)$ since $f_{n+1}(x) = f_n(x)$ in this interval. Consider $x \in [0, 1/3]$, then

$$|f_{n+1}(x) - f_n(x)| = \left| \frac{1}{2}f_n(3x) - \frac{1}{2}f_{n-1}(3x) \right| = \frac{1}{2}\left|f_n(3x) - f_{n-1}(3x)\right| < \frac{1}{2}\frac{1}{2^n} = \frac{1}{2^{n+1}}$$

The case for $[2/3, 1]$ is similar.

Given this, it should be clear that for $m \geq n$, $|f_m(x) - f_n(x)| < 1/2^n$. Since we can make $1/2^n$ arbitrarily small by increasing n, we can readily conclude that f_n satisfies Theorem 6.2.5 (Cauchy Criterion for Uniform Convergence), and hence f_n converges uniformly.

(c) $f_n(0) = 0$ and $f_n(1) = 1$ can easily be proven with induction, and imply that each f_n is continuous. Since each f_n is continuous, by the Continuous Limit Theorem f is continuous. One of the sub-results from Exercise 6.2.10 was that all f_n increasing implies f is increasing, which can be applied here to show f is increasing. Since $f_n(0)$ and $f_n(1)$ are constant with respect to n, $f(0) = 0$ and $f(1) = 1$.

Observe that for any n, $f_{n+1}(x) = f_n(x)$ for $x \in [0,1]\backslash C_n$. To prove this formally, we can use induction. By definition $f_{n+1} = f_n(x)$ over $(1/3, 2/3)$. Focusing on $[0,1/3]$, we have

$$\begin{aligned} &\frac{1}{2}f_n(3x) = f_n(x) \text{ over } [0,1/3]\backslash C_n \\ \iff &\frac{1}{2}f_n(3x) = \frac{1}{2}f_{n-1}(3x) \text{ over } [0,1/3]\backslash\left(C_{n-1}/3\right) \\ \iff &f_n(x) = f_{n-1}(x) \text{ over } [0,1]\backslash C_{n-1} \end{aligned}$$

which is the inductive hypothesis. The case over $[2/3, 1]$ is similar.

Moreover, since $C_m \subseteq C_n$ for $m \geq n$, we can make the stronger statement $f(x) = f_m(x) = f_n(x)$ for $x \in [0,1]\backslash C_n$ and $m \geq n$.

Finally, we can show by induction that $f_n(x)$ is constant over $[0,1]\backslash C_n$, implying that f is constant and $f'(x) = 0$ over $[0,1]\backslash C$.

Exercise 6.2.13

Recall that the Bolzano-Weierstrass Theorem (Theorem 2.5.5) states that every bounded sequence of real numbers has a convergent subsequence. An analogous statement for bounded sequences of functions is not true in general, but under stronger hypotheses several different conclusions are possible. One avenue is to assume the common domain for all of the functions in the sequence is countable. (Another is explored in the next two exercises.) Let $A = \{x_1, x_2, x_3, \ldots\}$ be a countable set. For each $n \in \mathbf{N}$, let f_n be defined on A and assume there exists an $M > 0$ such that $|f_n(x)| \leq M$ for all $n \in \mathbf{N}$ and $x \in A$. Follow these steps to show that there exists a subsequence of (f_n) that converges pointwise on A.

(a) Why does the sequence of real numbers $f_n(x_1)$ necessarily contain a convergent subsequence (f_{n_k}) ? To indicate that the subsequence of functions (f_{n_k}) is generated by considering the values of the functions at x_1, we will use the notation $f_{n_N} = f_{1,k}$.

(b) Now, explain why the sequence $f_{1,k}(x_2)$ contains a convergent subsequence.

(c) Carefully construct a nested family of subsequences $(f_{m,k})$, and show how this can be used to produce a single subsequence of (f_n) that converges at every point of A.

Solution

(a) Since f_n is bounded, $(f_n(x_1))$ is a bounded sequence, so by Bolzano-Weierstrauss $(f_n(x_1))$ has a convergent subsequence which uses f_{n_k}.

(b) Same reason as (a)

(c) Let $f_{2,k}$ represent the subsequence of functions generated in part (b), and keep in mind that this subsequence of functions continues to converge pointwise at x_1. We can then repeat this process, finding a subsequence of functions while considering $f_{2,k}(x_3)$, and repeat for every element in A.

If A were finite, we could simply take the last sequence and we would be done. But since A is countably infinite, there is no last element; we need to be a bit more careful. Instead of picking a particular subsequence, define a new subsequence of functions (g_n), with $g_n = f_{n,n}$ (where $f_{a,b}$ is the b'th element of the sequence of functions $f_{a,k}$). For any fixed q, the sequence $g_n(x_q)$ is guarenteed to converge, since after q elements the sequence consists solely of elements from $f_{q,k}$ which converges at x_q.

Exercise 6.2.14

A sequence of functions (f_n) defined on a set $E \subseteq \mathbf{R}$ is called equicontinuous if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f_n(x) - f_n(y)| < \epsilon$ for all $n \in \mathbf{N}$ and $|x - y| < \delta$ in E.

(a) What is the difference between saying that a sequence of functions (f_n) is equicontinuous and just asserting that each f_n in the sequence is individually uniformly continuous?

(b) Give a qualitative explanation for why the sequence $g_n(x) = x^n$ is not equicontinuous on $[0, 1]$. Is each g_n uniformly continuous on $[0, 1]$?

Solution

(a) For equicontinuous functions the same δ works for every function in the sequence, as opposed to individually being uniformly continuous where δ depends on n.

(b) Not equicontinuous since as n increases we need δ to be smaller, hence δ cannot be written independent of n. Each g_n is uniformly continuous however (since g_n is continuous on the compact set $[0, 1]$).

Exercise 6.2.15 (Arzela-Ascoli Theorem)

For each $n \in \mathbf{N}$, let f_n be a function defined on $[0, 1]$. If (f_n) is bounded on $[0, 1]$—that is, there exists an $M > 0$ such that $|f_n(x)| \leq M$ for all $n \in \mathbf{N}$ and $x \in [0, 1]$—and if the collection of functions (f_n) is equicontinuous (Exercise 6.2.14), follow these steps to show that (f_n) contains a uniformly convergent subsequence.

(a) Use Exercise 6.2.13 to produce a subsequence (f_{n_k}) that converges at every rational point in $[0, 1]$. To simplify the notation, set $g_k = f_{n_k}$. It remains to show that (g_k) converges uniformly on all of $[0, 1]$.

(b) Let $\epsilon > 0$. By equicontinuity, there exists a $\delta > 0$ such that

$$|g_k(x) - g_k(y)| < \frac{\epsilon}{3}$$

for all $|x - y| < \delta$ and $k \in \mathbf{N}$. Using this δ, let $r_1, r_2, \ldots, r_m$ be a finite collection of rational points with the property that the union of the neighborhoods $V_\delta(r_i)$ contains $[0, 1]$. Explain why there must exist an $N \in \mathbf{N}$ such that

$$|g_s(r_i) - g_t(r_i)| < \frac{\epsilon}{3}$$

for all $s, t \geq N$ and r_i in the finite subset of $[0, 1]$ just described. Why does having the set $\{r_1, r_2, \ldots, r_m\}$ be finite matter?

(c) Finish the argument by showing that, for an arbitrary $x \in [0, 1]$,

$$|g_s(x) - g_t(x)| < \epsilon$$

for all $s, t \geq N$.

Solution

(a) ...is this actually a question? The rational numbers in $[0, 1]$ are countable, so the results from Exercise 6.2.13 can be applied.

(b) r_i is a rational number, so $g_n(r_i)$ is a Cauchy sequence, and we can find N_i for each r_i where $s, t > N_i$ ensures $|g_s(r_i) - g_t(r_i)| < \epsilon/3$. Then just have $N = \max\{N_1, N_2, \ldots, N_m\}$. We need $\{r_1, r_2, \ldots, r_m\}$ to be finite so that the final operation of taking the maximum of all N_i is valid.

(c) We have

$$|g_n(x) - g_m(x)| \leq |g_n(x) - g_n(a)| + |g_n(a) - g_m(a)| + |g_m(a) - g_m(x)|$$

where a is some rational number. The first and last terms can be made less than $\epsilon/3$ by continuity of g_n and g_m and by choosing a close enough to x, while the middle term can be made less than $\epsilon/3$ from part (b).

6.3 Uniform Convergence and Differentiation

Exercise 6.3.1

Consider the sequence of functions defined by

$$g_n(x) = \frac{x^n}{n}.$$

(a) Show (g_n) converges uniformly on $[0, 1]$ and find $g = \lim g_n$. Show that g is differentiable and compute $g'(x)$ for all $x \in [0, 1]$.

(b) Now, show that (g_n') converges on $[0, 1]$. Is the convergence uniform? Set $h = \lim g_n'$ and compare h and g'. Are they the same?

Solution

(a) I claim that $(g_n) \to 0$. This can be seen by noting that for $x \in [0, 1]$, $0 \leq x^n \leq 1$ and so $0 \leq g_n(x) \leq 1/n$. Thus for any $\epsilon > 0$, any $n > N = 1/\epsilon$ will force $|g_n(x) - 0| < \epsilon$. $g(x) = 0$ is obviously differentiable, with its derivative just 0.

(b) $g'(n) = x^{n-1}$. Using a similar argument,

$$h(x) = \lim_{n\to\infty} g_n'(x) = \begin{cases} 0 & x \in [0, 1) \\ 1 & x = 1 \end{cases}$$

which is not equal to $g'(x)$ at 1. Tne convergence is not uniform. For $\epsilon = 0.5$ and any given n, choosing $1 > x > \sqrt[N]{\epsilon}$ leads to $g_n(x) > \epsilon$, preventing uniform convergence.

Exercise 6.3.2
Consider the sequence of functions

$$h_n(x) = \sqrt{x^2 + \frac{1}{n}}.$$

(a) Compute the pointwise limit of (h_n) and then prove that the convergence is uniform on $\mathbf{R}$.

(b) Note that each h_n is differentiable. Show $g(x) = \lim h_n'(x)$ exists for all x, and explain how we can be certain that the convergence is not uniform on any neighborhood of zero.

Solution

(a) As $n \to \infty$, $h_n(x) \to \sqrt{x^2} = |x|$. Now, recall that $\sqrt{a} + \sqrt{b} \geq \sqrt{a+b}$ (I think this has been proved earlier, but if not, this is easily shown by squaring both sides); alternatively $\sqrt{a+b} - \sqrt{a} \leq \sqrt{b}$, and so

$$\left| \sqrt{\left(x^2 + \frac{1}{n} \right)} - \sqrt{x^2} \right| \leq \sqrt{\frac{1}{n}} = \frac{1}{\sqrt{n}}$$

which we can clearly make less than any $\epsilon > 0$.

(b)

$$h_n'(x) = \frac{x}{\sqrt{x^2 + 1/n}}$$

which converges to

$$g(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

Define $h(x)$ to be the pointwise limit of (h_n). If the convergence to g was uniform, that would imply that $g_n = h'(x)$ (by the Differentiable Limit Theorem). But from part (a) h is not differentiable at 0, therefore the convergence to g cannot be uniform in a neighborhood around 0.

Exercise 6.3.3
Consider the sequence of functions

$$f_n(x) = \frac{x}{1 + nx^2}.$$

(a) Find the points on $\mathbf{R}$ where each $f_n(x)$ attains its maximum and minimum value. Use this to prove (f_n) converges uniformly on $\mathbf{R}$. What is the limit function?

(b) Let $f = \lim f_n$. Compute $f_n'(x)$ and find all the values of x for which $f'(x) = \lim f_n'(x)$.

Solution

(a) f_n is differentiable on $\mathbf{R}$, so by the Interior Extremum Theorem the maximum and minimum values will appear where $f_n'(x) = 0$. We have

$$f_n'(x) = \frac{1 - nx^2}{(1 + nx^2)^2}$$

which is zero at $x = \pm 1/\sqrt{n}$. Plugging these values back into $f_n(x)$ we get that $|f_n(x)| \leq \frac{1}{2\sqrt{n}}$. Clearly this forces f_n to converge uniformly to 0.

(b) $f(x) = f'(x) = 0$. We have

$$\lim_{n\to\infty} f_n'(x) = \frac{1 - nx^2}{1 + 2nx^2 + n^2x^4} = \frac{\frac{1}{n} - x^2}{\frac{1}{n} + 2x^2 + nx^4} = 0$$

and therefore $f'(x) = \lim f_n'(x)$ everywhere.

Exercise 6.3.4

Let

$$h_n(x) = \frac{\sin(nx)}{\sqrt{n}}.$$

Show that $h_n \to 0$ uniformly on $\mathbf{R}$ but that the sequence of derivatives (h_n') diverges for every $x \in \mathbf{R}$.

Solution

$|sin(x)| \leq 1$ and so $|h_n(x)| < \frac{1}{\sqrt{n}}$ which shows that $h_n \to 0$ uniformly on $\mathbf{R}$. $h_n'(x) = \frac{n\cos(nx)}{\sqrt{n}} = \sqrt{n}\cos(nx)$. Intuitively this diverges because of the unbounded $\sqrt{n}$ factor, but to prove it formally requires some thought. We want to show that for any fixed real numbers x and M we can find some n where $h_n'(x) \geq M$ (this will show that $h_n'(x)$ is unbounded and thus diverges). First let $N_1 > 4M^2$, then we just need to find some $n \geq N_1$ so that $|nx - Z\pi| \leq \pi/3$ for some integer Z; this would cause $|\cos(nx)| \geq 1/2$ and thus $h_n'(x) \geq M$.

Express $N_1 x = 2\pi P + k$ where P is some integer and $0 \leq k < 2\pi$. (This next bit is reminiscent of arithmetic modulo 2π.) Now if $k \in [0, \pi/3]$ or $[2\pi/3, 4\pi/3]$ or $[5\pi/3, 2\pi]$ we're done. Otherwise $k \in (\pi/3, 2\pi/3)$ or $(4\pi/3, 5\pi/3)$; so consider $2N_1 x = 4\pi P + 2k$, with $2k \in (2\pi/3, 4\pi/3)$ or $(8\pi/3, 10\pi/3)$; both of these cases will have $2N_1 x$ within $\pi/3$ of a multiple of π, hence $h_n'(x)$ diverges for all x.

Exercise 6.3.5

Let

$$g_n(x) = \frac{nx + x^2}{2n}$$

and set $g(x) = \lim g_n(x)$. Show that g is differentiable in two ways:

(a) Compute $g(x)$ by algebraically taking the limit as $n \to \infty$ and then find $g'(x)$.

(b) Compute $g_n'(x)$ for each $n \in \mathbf{N}$ and show that the sequence of derivatives (g_n') converges uniformly on every interval $[-M, M]$. Use Theorem 6.3.3 to conclude $g'(x) = \lim g_n'(x)$.

(c) Repeat parts (a) and (b) for the sequence $f_n(x) = (nx^2 + 1)/(2n + x)$.

Solution

(a) By inspection $g(x) = x/2$ and $g'(x) = 1/2$.

(b) $g_n'(x) = 1/2 + \frac{x}{n}$ which approaches $\frac{1}{2}$ as $n \to \infty$. Now $\left|g_n' - \frac{1}{2}\right| = \left|\frac{x}{n}\right|$ is bounded by $\frac{M}{n}$ which goes to 0 and is not dependent on x, and therefore (g_n') converges uniformly over $[-M, M]$.

(c) $f(x) = x^2/2$, and $f'(x) = x$. We have

$$f_n'(x) = \frac{4n^2x + nx^2 - 1}{4n^2 + 4nx + x^2}$$

which approaches x as $n \to \infty$. With some algebra we have

$$|f_n'(x) - x| = \left|\frac{x^3 - 3nx^2 - 1}{4n^2 + 4nx + x^2}\right| \leq \frac{M^3 + 3nM^2 + 1}{4n^2 - 4Mn}$$

which approaches 0 as $n \to \infty$ independent of x, and therefore (f_n') converges uniformly over $[-M, M]$.

Exercise 6.3.6

Provide an example or explain why the request is impossible. Let's take the domain of the functions to be all of $\mathbf{R}$.

(a) A sequence (f_n) of nowhere differentiable functions with $f_n \to f$ uniformly and f everywhere differentiable.

(b) A sequence (f_n) of differentiable functions such that (f_n') converges uniformly but the original sequence (f_n) does not converge for any $x \in \mathbf{R}$.

(c) A sequence (f_n) of differentiable functions such that both (f_n) and (f_n') converge uniformly but $f = \lim f_n$ is not differentiable at some point.

Solution

(a) Let $g(x)$ be the continuous but nowhere-differentiable function defined in section 5.4. Then since $g(x)$ is bounded, $f_n(x) = g(x)/n$ clearly converges uniformly to 0, and is thus such a sequence.

(b) Let

$$f_n(x) = \begin{cases} 1 & \text{n is odd} \\ 0 & \text{n is even} \end{cases}$$

Clearly f_n does not converge anyewhere, but $f_n'(x)$ converges to 0.

(c) Not possible - since f_n converges uniformly, it must converge at at least one point, and since f_n' converges uniformly, we can apply Theorem 6.3.3 to show that f is differentiable everywhere.

Exercise 6.3.7
Use the Mean Value Theorem to supply a proof for Theorem 6.3.2. To get started, observe that the triangle inequality implies that, for any $x \in [a, b]$ and $m, n \in \mathbf{N}$,

$$|f_n(x) - f_m(x)| \leq |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)| .$$

Solution
Take any $\epsilon > 0$; we want to show that there is some N so that for $n, m > N$, $|f_n(x) - f_m(x)| < \epsilon$ (to use the Cauchy Criterion for Uniform Convergence).

Let $h_{n,m}(x) = f_n(x) - f_m(x)$, and note that $h'_{n,m}$ converges uniformly to 0 as n, m go to infinity (as a consequence of (f'_n) converging uniformly). More formally, for any $\epsilon_1 > 0$ we have some N_1 for which if $n, m > N_1$ then $\left|h'_{n,m}(x)\right| < \epsilon_1 \ \forall x \in [a, b]$.

By the Mean Value Theorem,

$$\frac{h_{n,m}(x) - h_{n,m}(x_0)}{x - x_0} = h'_{n,m}(x_1)$$

for some $x_1 \in [a, b]$, and therefore if $n, m > N_1$ we have

$$|(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| = |x - x_0| \left|h'_{n,m}(x_1)\right| \leq (b - a)\epsilon_1$$

This addresses the first term in the question hint.

Second, since $(f_n(x_0))$ converges as $n \to \infty$ and is thus a Cauchy sequence, we have that for any ϵ_2 there is an N_2 so that when $n, m > N_2$, $|f_n(x_0) - f_m(x_0)| < \epsilon_2$. This addresses the second term.

Setting $\epsilon_1 = \frac{\epsilon}{2(b-a)}$, $\epsilon_2 = \frac{\epsilon}{2}$, and $N = \max\{N_1, N_2\}$ completes the proof.

6.4 Series of Functions

Exercise 6.4.1
Supply the details for the proof of the Weierstrass M-Test (Corollary 6.4.5).

Solution
Let $\epsilon > 0$. Since $\sum_{n=1}^{\infty} M_n$ converges, by the Cauchy Criterion for Series there must be some N where if $n > m > N$ then $\sum_{i=m+1}^{n} M_i < \epsilon$. Then

$$\left|\sum_{i=m+1}^{n} f_i(x)\right| \leq \sum_{i=m+1}^{n} |f_i(x)| \leq \sum_{i=m+1}^{n} M_i < \epsilon$$

and applying the Cauchy Criterion, the proof is done.

Exercise 6.4.2
Decide whether each proposition is true or false, providing a short justification or counterexample as appropriate.

(a) If $\sum_{n=1}^{\infty} g_n$ converges uniformly, then (g_n) converges uniformly to zero.

(b) If $0 \leq f_n(x) \leq g_n(x)$ and $\sum_{n=1}^{\infty} g_n$ converges uniformly, then $\sum_{n=1}^{\infty} f_n$ converges uniformly.

(c) If $\sum_{n=1}^{\infty} f_n$ converges uniformly on A, then there exist constants M_n such that $|f_n(x)| \leq M_n$ for all $x \in A$ and $\sum_{n=1}^{\infty} M_n$ converges.

Solution

(a) True: applying the Cauchy Criterion with $n = m + 1$ we have that $|g_n(x)| < \epsilon$ for any $\epsilon > 0$, therefore $(g_n) \to 0$.

(b) True:

$$\left|\sum_{i=m+1}^{n} f_i(x)\right| = \sum_{i=m+1}^{n} f_i(x) \leq \sum_{i=m+1}^{n} g_i(x) = \left|\sum_{i=m+1}^{n} g_i(x)\right| < \epsilon$$

and therefore $\sum(f_i)$ converges uniformly.

(c) False: Consider the following sequence of functions, defined over $[0, 1)$:

$$g_{i,j}(x) = \begin{cases} 2^{-i} & 2^{-i}(j-1) \leq x < 2^{-i}j \\ 0 & \text{otherwise} \end{cases}$$

with $i \geq 1$ and j an integer ranging from 1 to 2^i inclusive. Each $g_{i,j}(x)$ consists of a pulse of height and width 2^{-i}, at disjoint locations for each i. Let $f_n(x)$ be obtained by iterating through each $g_{1,j}$, then through each $g_{2,j}$, then through each $g_{3,j}$, and so on.

$\sum_{n=1}^{\infty} f_n$ converges to 1 because

$$\sum_{k=1}^{2^i} g_{i,k} = 2^{-i}$$

, and uniform convergence is achieved when we include all of the $g_{i,j}$ for a given g_i. On the other hand, the upper bound (and therefore minimum value of the constant M_n) for each $g_{i,j}$ is 2^{-i}, with

$$\sum_{k=1}^{2^i} \max g_{i,k}(x) = 1$$

which implies that $\sum_{n=1}^{\infty} M_n$ will not converge.

Exercise 6.4.3

(a) Show that

$$g(x) = \sum_{n=0}^{\infty} \frac{\cos(2^n x)}{2^n}$$

is continuous on all of $\mathbf{R}$.

(b) The function g was cited in Section 5.4 as an example of a continuous nowhere differentiable function. What happens if we try to use Theorem 6.4.3 to explore whether g is differentiable?

Solution

(a) Define $g_n(x) = \frac{\cos(2^n x)}{2^n}$ and $M_n = 2^{-n} > |g_n(x)|$. By the Weierstrass M-test, $g(x)$ converges uniformly on $\mathbf{R}$. Since each $g_n(x)$ is continuous and $g(x)$ converges uniformly, $g(x)$ must also be continuous.

(b) $g_n'(x) = -\sin(2^n x)$ and thus $\sum_{n=1}^{\infty} g_n'(x)$ does not converge uniformly by Exercise 6.4.2 part a. (It might not converge pointwise either, but that seems more difficult to prove.) Therefore we cannot use Theorem 6.4.3.

Exercise 6.4.4

Define

$$g(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(1 + x^{2n})}.$$

Find the values of x where the series converges and show that we get a continuous function on this set.

Solution

Let $h_n(x) = \frac{x^{2n}}{(1+x^{2n})}$ be the terms being summed. For $|x| \geq 1$, $h_n(x)$ does not approach 0 and therefore the series does not converge. For $|x| < 1$, $|h_n(x)| \leq x^{2n}$, which forms a geometric series in x^2, which converges, so $g(x)$ converges by the Order Limit Theorem.

Note that for any $0 \leq a < 1$, $|h(x)| \leq a^{2n} = M_n$ over $[-a, a]$, and thus by the Weierstrass M-test $g(x)$ uniformly converges over $[-a, a]$ and is thus continuous over this interval. This last statement is equivalent to saying $g(x)$ is continuous over $(-1, 1)$, which is also the set where $g(x)$ is well defined.

Exercise 6.4.5

(a) Prove that

$$h(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = x + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16} + \cdots$$

is continuous on $[-1, 1]$.

(b) The series

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

converges for every x in the half-open interval $[-1, 1)$ but does not converge when $x = 1$. For a fixed $x_0 \in (-1, 1)$, explain how we can still use the Weierstrass M-Test to prove that f is continuous at x_0.

Solution

(a) For $x \in [-1, 1]$, we have

$$\left|\frac{x^n}{n^2}\right| \leq \frac{1}{n^2} = M_n$$

and since $\sum \frac{1}{n^2}$ converges (Example 2.4.4), h converges uniformly and is therefore continuous.

(b) Given a fixed x_0, we can consider the interval $(-a, a) \subset [-1, 1)$ where $-1 < -a < |x_0| < a < 1$. Then by setting $M_n = \frac{a^n}{n}$ we will have $M_n > \frac{x_0^n}{n}$ in a neighbourhood around x_0, allowing us to show via the M-Test that f is continuous at x_0.

Exercise 6.4.6
Let

$$f(x) = \frac{1}{x} - \frac{1}{x+1} + \frac{1}{x+2} - \frac{1}{x+3} + \frac{1}{x+4} - \cdots .$$

Show f is defined for all $x > 0$. Is f continuous on $(0, \infty)$? How about differentiable?

Solution
$f(x)$ converges for any $x > 0$ by the Alternating Series Test. Since f converges we are free to use associativity to group the terms in pairs, from which we get

$$\begin{aligned} f(x) &= \left(\frac{1}{x} - \frac{1}{x+1}\right) + \left(\frac{1}{x+2} - \frac{1}{x+3}\right) + \cdots \\ &= \frac{1}{x^2 + x} + \frac{1}{(x+2)^2 + (x+2)} + \cdots \\ &< \frac{1}{x^2} + \frac{1}{(x+2)^2} + \cdots \end{aligned}$$

Temporarily skipping the first term, we can use the Weierstrass M-Test with $M_n = \frac{1}{(2n)^2}$, to show that $f(x) - 1/x^2$ converges uniformly and is therefore continuous for $x > 0$. Since $1/x^2$ is also continuous for $x > 0$, f must be continuous on $(0, \infty)$.

Letting f_n' represent each term of f,

$$|f_n'(x)| = \left|\frac{(-1)^n}{(x+n-1)^2}\right| \leq \frac{1}{(n-1)^2}$$

(with the inequality only being meaningful for $n \geq 2$). Thus if we skip the first term, by the Weierstrass M-test we can be assured

$$\sum_{n=2}^{\infty} f_n'(x)$$

converges uniformly, and is therefore differentiable, and is equal to the derivative of

$$g(x) = \sum_{n=2}^{\infty} f_n(x)$$

by the Differentiable Limit Theorem. Since $f(x) = g(x) + 1/x$ and both $1/x$ and $g(x)$ are differentiable over $(0, \infty)$, we have that f is differentiable as well.

Exercise 6.4.7
Let

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^3}$$

(a) Show that $f(x)$ is differentiable and that the derivative $f'(x)$ is contimuous.

(b) Can we determine if f is twice-differentiable?

Solution

(a) Let $f_n(x) = \frac{\sin(nx)}{n^3}$. We have

$$|f_n'(x)| = \left|\frac{\cos(nx)}{n^2}\right| \leq \frac{1}{n^2}$$

and so $\sum_n^\infty f_n'(x)$ converges uniformly by the Weierstrass M-Test. We also have $f(x)$ converging at $x = 0$ (since every term is zero), so by the differentiable limit theorem we have $f(x)$ differentiable with $f'(x) = \sum_{n=1}^\infty f_n'(x)$. Since this converges uniformly, $f'(x)$ is continuous.

(b) Probably not easily - trying the same trick leaves us with trying to bound $\left|\frac{\sin(kx)}{k}\right|$ with M_n where $\sum M_n$ converges, but $M_n = 1/k$ doesn't work as $\sum_{k=1}^\infty 1/k$ diverges.

Exercise 6.4.8

Consider the function

$$f(x) = \sum_{k=1}^\infty \frac{\sin(x/k)}{k}.$$

Where is f defined? Continuous? Differentiable? Twice-differentiable?

Solution

We can use the inequality $|\sin x| \leq |x|$ to show that $f(x)$ converges uniformly by the Weierstrass M-Test over any interval $(-a, a)$, with $M_n = a/k^2$. Therefore f is defined and is continuous over all real numbers.

The derivative of each term is

$$\frac{\cos(x/k)}{k^2}$$

which can easily be shown to converge uniformly; hence by the Differentiable Limit Theorem we have that $f(x)$ is differentiable as well. A similar argument shows that f is also twice-differentiable.

Exercise 6.4.9

Let

$$h(x) = \sum_{n=1}^\infty \frac{1}{x^2 + n^2}$$

(a) Show that h is a continuous function defined on all of $\mathbf{R}$.

(b) Is h differentiable? If so, is the derivative function h' continuous?

Solution

(a) Use the M-Test with $M_n = \frac{1}{n^2}$.

(b) The termwise derivatives are

$$h_n'(x) = \frac{-2x}{x^4 + 2x^2n^2 + n^4}$$

with $|h_n'(x)| < 2x/n^4$. For any fixed $a > 0$, over the interval $(-a, a)$ we can bound $|h_n'(x)|$ with $M_n = 2a/n^4$, so by the Differentiable Limit Theorem as well as uniform convergence via the M-Test we have that h is differentiable with h' continuous.

Exercise 6.4.10
Let $\{r_1, r_2, r_3, \ldots\}$ be an enumeration of the set of rational numbers. For each $r_n \in \mathbf{Q}$, define

$$u_n(x) = \begin{cases} 1/2^n & \text{for } x > r_n \\ 0 & \text{for } x \leq r_n. \end{cases}$$

Now, let $h(x) = \sum_{n=1}^{\infty} u_n(x)$. Prove that h is a monotone function defined on all of $\mathbf{R}$ that is continuous at every irrational point.

Solution
Using $M_n = 1/2^n$, by the Weierstrass M-Test we have that h converges uniformly. Since each u_n is continuous at all irrational numbers, we have that h is continuous at all irrational numbers. Monotone-ness comes from applying the Order Limit Theorem to compare the series $\sum u_n(a)$ and $\sum u_n(b)$ for $a, b \in \mathbf{R}$.

6.5 Power Series

Exercise 6.5.1
Consider the function g defined by the power series

$$g(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots .$$

(a) Is g defined on $(-1, 1)$? Is it continuous on this set? Is g defined on $(-1, 1]$? Is it continuous on this set? What happens on $[-1, 1]$? Can the power series for $g(x)$ possibly converge for any other points $|x| > 1$? Explain.

(b) For what values of x is $g'(x)$ defined? Find a formula for g'.

Solution
(a) $g(1)$ converges by the Alternating Series Test, so the radius of convergence is at least 1, and g must be defined on at least $(-1, 1]$. Theorem 6.5.1 and Abel's Theorem together indicate that indicate that g converges absolutely on $(-1, 1]$ as well. Thus, since each term is continuous, $g(x)$ is continuous on $(-1, 1]$.

g is not defined at -1 since $g(-1)$ would otherwise be

$$\sum_{n=1}^{\infty} \frac{-1}{n}$$

which diverges.

g cannot converge at any point $|x| > 1$ because if it did, that would imply the radius of convergence is strictly larger than 1, and thus g would need to converge at -1, which it doesn't.

(b) $g'(x)$ is at least defined on $(-1, 1)$, by Theorem 6.5.7, with the derivative given by

$$g'(x) = \sum_{n=0}^{\infty}(-x)^n = \frac{1}{x+1}$$

$g'(x)$ cannot be defined at $x \leq -1$ since g isn't even defined there. To show that $g'(1)$ is defined and is also given by this formula requires a bit more care, since the infinite sum does not actually converge for 1. We return to the definition of the derivative:

$$\begin{aligned} g'(1) &= \lim_{x\to 1} \frac{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n}{1-x} = \lim_{x\to 1} \sum_{n}^{\infty} \frac{(-1)^{n+1}}{n} \frac{1-x^n}{1-x} \\ &= \lim_{x\to 1} \frac{1}{1-x} \sum_{n}^{\infty} \frac{(-1)^{n+1}}{n} (1-x^n) \end{aligned}$$

With some algebra, we can show that this converges by the alternating series test, keeping in mind that we can assume $x \in (0, 1)$. We have $\frac{1-x^n}{n} < \frac{1}{n} \to 0$, so we just need to show $\frac{1-x^n}{n} \geq \frac{1-x^{n+1}}{n+1}$:

$$\begin{aligned} \frac{1-x^n}{n} \geq \frac{1-x^{n+1}}{n+1} &\iff (1-x^n)(n+1) \geq n - nx^{n+1} \\ &\iff n - nx^n + 1 - x^n \geq n - nx^{n+1} \\ &\iff 1 - x^n \geq nx^n(1-x) \\ &\iff \frac{1-x^n}{1-x} = \sum_{i=0}^{n-1} x^i \geq \sum_{i=0}^{n-1} x^n = nx^n \end{aligned}$$

Now we know that $g'(1)$ exists. We can show that $g'(1) = \frac{1}{1+1} = 0.5$ by noting that $\frac{1}{x+1}$ is strictly decreasing on $[0, 1)$, so in order for the derivative $g'(x)$ to maintain the intermediate value property, $g'(1) = 0.5$.

Exercise 6.5.2

Find suitable coefficients (a_n) so that the resulting power series $\sum a_n x^n$ has the given properties, or explain why such a request is impossible.

(a) Converges for every value of $x \in \mathbf{R}$.

(b) Diverges for every value of $x \in \mathbf{R}$.

(c) Converges absolutely for all $x \in [-1, 1]$ and diverges off of this set.

(d) Converges conditionally at $x = -1$ and converges absolutely at $x = 1$.

(e) Converges conditionally at both $x = -1$ and $x = 1$.

Solution

(a) $a_n = 0$

(b) Impossible as $x = 0$ will always converge

(c) $a_n = \frac{1}{n^2}$. For $x = 1$ this converges, while for $x > 1$ the series diverges because

$$\frac{x^n}{n^2} < \frac{x^{2n}}{4n^2} \iff 4 < a^n$$

meaning that once $n > \log_x(4) = \ln(4)/\ln(x)$, the terms will start increasing (whereas they must approach 0 for the series to converge). A similar argument can be made for $x < -1$.

(d) Impossible because $|a_n x^n| = |a_n(-x)^n|$, and substituting $x = 1$ shows that the series at -1 is going to be the same as that at 1 considered absolutely.

(e) $a_n = 0$ for odd n and $a_n = (-1)^{n/2}/n$ for even n. This in effect takes only the even-powered terms of the power series, which are always positive. We then get the alternating harmonic series (scaled by 0.5) in x^2 which diverges absolutely but converges conditionally.

Exercise 6.5.3

Use the Weierstrass M-Test to prove Theorem 6.5.2.

Solution

Note that $|p| < |q|$ implies $|p^n| < |q^n|$ and so we can use the Weierstrass M-Test with $M_n = |a_n x^n|$ (which converges by the assumption of absolute convergence of $a_n x_0^n$).

Exercise 6.5.4 (Term-by-term Antidifferentiation)

Assume $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-R, R)$.

(a) Show

$$F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

is defined on $(-R, R)$ and satisfies $F'(x) = f(x)$.

(b) Antiderivatives are not unique. If g is an arbitrary function satisfying $g'(x) = f(x)$ on $(-R, R)$, find a power series representation for g.

Solution

(a) Let $N \in \mathbf{N} > R$ and split the function into

$$\begin{aligned} F(x) &= \sum_{n=0}^{N-1} \frac{a_n}{n+1} x^{n+1} + \sum_{n=N}^{N} a_n x^n \left(\frac{x}{n+1}\right) \\ &\leq \sum_{n=0}^{N-1} \frac{a_n}{n+1} x^{n+1} + \sum_{n=N}^{N} a_n x^n \left(\frac{x}{R}\right) \\ &= \sum_{n=0}^{N-1} \frac{a_n}{n+1} x^{n+1} + \left(\frac{x}{R}\right) \sum_{n=N}^{N} a_n x^n \end{aligned}$$

The first term is finite, while the second term converges by the original assumption. This shows that $F(x)$ is defined on $-R, R$, at which point we can use Theorem 6.5.7 to conclude $F'(x) = f(x)$.

(b) From Corollary 5.3.4, $g(x) = F(x) + k$ for some constant k; k gets folded into the constant term of the power series.

Exercise 6.5.5

(a) If s satisfies $0 < s < 1$, show ns^{n-1} is bounded for all $n \geq 1$.

(b) Given an arbitrary $x \in (-R, R)$, pick t to satisfy $|x| < t < R$. Use this start to construct a proof for Theorem 6.5.6.

Solution

(a) Note first that all $ns^{n-1} > 0$, and that for $n + 1 > N > \frac{1}{1-s}$ (with $N \in \mathbf{N}$, we can rearrange for s to have $\frac{n}{n+1} > s$. This implies that $ns^{n-1} > (n+1)s^n$; thus the sequence in n must be bounded by the maximum of the first N terms.

(b) Choose s satisfying $|s| = t$ and with s having the same sign as x. As a preliminary, note that $\sum_{n=0}^{\infty} a_n s^{n-1} = (1/s) \sum_{n=0}^{\infty} a_n s^n$ converges. We have

$$\sum_{n=0}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} a_n s^{n-1} n \left(\frac{x}{s}\right)^{n-1} \leq M \sum_{n=0}^{\infty} a_n s^{n-1}$$

where, denoting $p = x/s$ (with $0 < p < 1$), M is an upper bound for np^{n-1}. This completes the proof.

Exercise 6.5.6

Previous work on geometric series (Example 2.7.5) justifies the formula

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots, \qquad \text{for all } |x| < 1.$$

Use the results about power series proved in this section to find values for $\sum_{n=1}^{\infty} n/2^n$ and $\sum_{n=1}^{\infty} n^2/2^n$. The discussion in Section 6.1 may be helpful.

Solution

Let $a_n = 1$; we have

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1}{1-x}$$

with a radius of convergence of 1. By Theorem 6.5.6 we can differentiate this termwise, to get

$$\sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n = \sum_{n=0}^{\infty} x^n + \sum_{n=1}^{\infty} nx^n = \frac{1}{1-x} + \sum_{n=1}^{\infty} nx^n = \frac{1}{(1-x)^2}$$

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$$

Substituting $x = 1/2$ we have $\sum_{n=1}^{\infty} n/2^n = 2$. We can differentiate the series again to get

$$\sum_{n=1}^{\infty}(n^2+n)x^{n-1} = \sum_{n=0}^{\infty} n^2x^n + 3\sum_{n=0}^{\infty} nx^n + 2\sum_{n=0}^{\infty} x^n = \frac{2}{(1-x)^3}$$

Substituting $x = 1/2$ we have $\sum_{n=1}^{\infty} n^2/2^n = 6$.

Exercise 6.5.7
Let $\sum a_n x^n$ be a power series with $a_n \neq 0$, and assume

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

exists.

(a) Show that if $L \neq 0$, then the series converges for all x in $(-1/L, 1/L)$. (The advice in Exercise 2.7.9 may be helpful.)

(b) Show that if $L = 0$, then the series converges for all $x \in \mathbf{R}$.

(c) Show that (a) and (b) continue to hold if L is replaced by the limit.

$$L' = \lim_{n\to\infty} s_n \quad \text{where} \quad s_n = \sup\left\{\left|\frac{a_{k+1}}{a_k}\right| : k \geq n\right\}.$$

(General properties of the limit superior are discussed in Exercise 2.4.7.)

Solution
(a) Let $b_n = a_n x^n$. If $|x| < L$, we have

$$\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}x}{a_n}\right| = \lim_{n\to\infty}|Lx|$$

and thus by the ratio test, if $|Lx| < 1$ then the series $\sum_{n=1}^{\infty} a_n x^n$ converges. This implies a radius of convergence of $1/L$ if $L \neq 0$.

(b) By the same logic, if $L = 0$ then $|Lx| < 1$ regardless of x and the series converges $\forall x \in \mathbf{R}$.

(c) Since (s_n) converges to L', for any $\epsilon > 0$ we have that $|a_{k+1}/a_k| < M = L' + \epsilon$ once $k > N$ for some $N \in \mathbf{N}$. Therefore by the ratio test and similar logic to above, the radius of convergence is at least $1/M$; since ϵ is arbitrary, this is effectively a radius of convergence of $1/L$, so (a) and (b) continue to hold.

Exercise 6.5.8
(a) Show that power series representations are unique. If we have

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$

for all x in an interval $(-R, R)$, prove that $a_n = b_n$ for all $n = 0, 1, 2, \ldots$

(b) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converge on $(-R, R)$, and assume $f'(x) = f(x)$ for all $x \in (-R, R)$ and $f(0) = 1$. Deduce the values of a_n.

Solution

(a) If we substitute $x = 0$ we get that $a_0 = b_0$. If we take the termwise derivative and then substitute $x = 0$, we get that $a_1 = b_1$. We can proceed inductively by taking the termwise derivative to show that $a_n = b_n$ for all n.

(b) $f(0) = 1$ implies $a_0 = 1$. $f'(0) = f(0) = 1$ implies $na_n = 1$ for $n = 1$, or $a_1 = 1$. $f''(0) = f'(0) = 1$ implies $(2)(2-1)a_2 = (2!)a_2 = 1$. We can use induction to show in general that $a_n = 1/n!$.

Exercise 6.5.9

Review the definitions and results from Section 2.8 concerning products of series and Cauchy products in particular. At the end of Section 2.9, we mentioned the following result: If both $\sum a_n$ and $\sum b_n$ converge conditionally to A and B respectively, then it is possible for the Cauchy product,

$$\sum d_n \quad \text{where} \quad d_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$$

to diverge. However, if $\sum d_n$ does converge, then it must converge to AB. To prove this, set

$$f(x) = \sum a_n x^n, \quad g(x) = \sum b_n x^n, \quad \text{and} \quad h(x) = \sum d_n x^n.$$

Use Abel's Theorem and the result in Exercise 2.8.7 to establish this result.

Solution

By Abel's Theorem we have uniform convergence of the series defining f, g, and h over the compact set $[0, 1]$; therefore each of these functions is continuous and bounded over this set. We can thus conclude that for $x \in [0, 1]$,

$$\lim_{N\to\infty} \sum_{i=0}^{N} \sum_{j=0}^{N} \left(a_i x^i\right)\left(b_j x^j\right) = \lim_{N\to\infty} \left(\sum_{n=0}^{N} a_n x^n\right)\left(\sum_{n=0}^{N} b_n x^n\right) = f(x)g(x)$$

Since $\sum d_n$ converges, $\lim_{n\to\infty} |d_{n+1}/d_n| \le 1$ (otherwise d_n would not be bounded). But since $\sum d_n$ only converges conditionally, $\lim_{n\to\infty} |d_{n+1}/d_n| = 1$ (if it were less than 1, then we could use the Ratio Test to prove absolute convergence). We therefore have absolute convergence of the series defining $h(x)$ for $|x| < 1$ by the Ratio Test.

From the work in Section 2.8, because we have absolute convergence, informally we have a lot of leeway in how to evaluate the double summations when $|x| < 1$. In particular,

$$\lim_{N\to\infty} \sum_{i=0}^{N} \sum_{j=0}^{N} \left(a_i x^i\right)\left(b_j x^j\right) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left(a_i x^i\right)\left(b_j x^j\right) = \sum_{n=0}^{\infty} d_n x^n = h(x)$$

We now have the equality $f(x)g(x) = h(x)$, for $|x| < 1$. $h(x)$ is a power series, and Abel's Theorem implies $h(x)$ is continuous over $[0, 1]$. We also have continuity of $f(x)g(x)$ over $[0, 1]$; thus by taking limits we have that $h(1) = f(1)g(1)$, and we thus have that $AB = \sum d_n$.

Exercise 6.5.10
Let $g(x)=\sum_{n=0}^{\infty} b_n x^n$ converge on $(-R, R)$, and assume $(x_n) \rightarrow 0$ with $x_n \neq 0$. If $g\left(x_n\right)=0$ for all $n \in \mathbf{N}$, show that $g(x)$ must be identically zero on all of $(-R, R)$.

Solution
Let $f^{(n)}$ denote the n'th derivative of f (with $f^{(0)}=f$). The intermediate claims we make along the way are:

1. If a differentiable function f has a sequence $(x_n) \rightarrow 0$ satisyfing $f(x_n)=0$, then its derivative also has a sequence $(y_n) \rightarrow 0$ satisfying $f'(y_n)=0$.
2. Any function f with a bounded derivative over an interval containing 0 with some sequence $(x_n) \rightarrow 0$ satisfying $f(x_n)=0$, will also satisfy $f(0)=0$.
3. Given a power series $f(x)=\sum_{n=0}^{\infty} a_n x^n$, if $f^{(n)}(0)=0$, then $a_n=0$.

For claim 1, we apply the Mean Value Theorem to get some y_n between x_n and x_{n+1} with $f'(y_n)=0$; we thus have $|y_n| \leq \max\{x_n, x_{n+1}\}$ and therefore $(y_n) \rightarrow 0$.

For claim 2, suppose that $f(0)=\epsilon \neq 0$, and $|f'(x)|<M$. Now since $(x_n) \rightarrow 0$ we can find some x_i satisfying $|x_i|<\epsilon / M$. By the Mean Value Theorem, we then have that

$$|f'(c)|=\left|\frac{\epsilon}{\epsilon / M}\right|=M$$

for some c, violating the assumption that $f'(x)$ is bounded. Hence $f(0)=0$.

For claim 3, we differentiate termwise n times and note that all terms that still have x will evaluate to 0. We thus have

$$f^{(n)}(0)=(n!) a_n=0$$

and thus $a_n=0$.

From claim 1, we have by indution that every $g^{(i)}$ has some sequence $(x_{i, n})$ satisfying $\lim_{n \rightarrow \infty} x_{i, n} \rightarrow 0$ and $g^{(i)}(x_{i, n})=0$. Now since each of $g^{(n)}$ is bounded (by continuity over the compact set $[-R / 2, R / 2]$, for example), each of $g^{(n)}$ also has a bounded derivative, and thus we can apply claim 2 to get that $g^{(n)}(0)=0$ for all n. Finally claim 3 implies that $b_n=0$ for all n, and hence $g(x)$ must be identically 0 over $(-R, R)$.

Exercise 6.5.11
A series $\sum_{n=0}^{\infty} a_n$ is said to be Abel-summable to L if the power series

$$f(x)=\sum_{n=0}^{\infty} a_n x^n$$

converges for all $x \in[0,1)$ and $L=\lim_{x \rightarrow 1-} f(x)$.

(a) Show that any series that converges to a limit L is also Abel-summable to L.

(b) Show that $\sum_{n=0}^{\infty}(-1)^n$ is Abel-summable and find the sum.

Solution

(a) If a series $\sum_{n=0}^{\infty} a_n$ converges to L then by Abel's Theorem the power series $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly over $[0,1]$, and is therefore continuous over this interval. Hence by continuity the series is Abel-summable to L.

(b) The relevant power series here is $f(x) = \sum_{n=0}^{\infty}(-x)^n$ which has the closed-form expression $\frac{1}{1+x}$ for $|x| < 1$, and $\lim_{x\to 1^-} f(x)$ evaluates to $1/2$.

6.6 Taylor Series

Exercise 6.6.1

The derivation in Example 6.6.1 shows the Taylor series for $\arctan(x)$ is valid for all $x \in (-1,1)$. Notice, however, that the series also converges when $x = 1$. Assuming that $\arctan(x)$ is continuous, explain why the value of the series at $x = 1$ must necessarily be $\arctan(1)$. What interesting identity do we get in this case?

Solution

Abel's theorem (Theorem 6.5.4) implies the series converges uniformly on $[0,1]$. Combined with (Theorem 6.2.6) we see function the series converges to must be continuous. Taking limits shows this value must be $\arctan(1)$ giving the identity

$$\arctan(1) = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Exercise 6.6.2

Starting from one of the previously generated series in this section, use manipulations similar to those in Example 6.6.1 to find Taylor series representations for each of the following functions. For precisely what values of x is each series representation valid?

(a) $x\cos(x^2)$

(b) $x/\left(1+4x^2\right)^2$

(c) $\log(1+x^2)$

Solution

(a) We know $\cos(x) = 1 - x^2/2 + x^4/4! - \dots$ on all of $\mathbf{R}$ so

$$x\cos(x^2) = x - \frac{x^5}{2!} + \frac{x^9}{4!} - \frac{x^{13}}{6!} + \dots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+1}}{(2n)!}$$

(b) Since

$$\frac{x}{(1+4x^2)^2} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{-1/8}{1+4x^2}$$

We can use the geometric series then differentiate

$$\frac{-1/8}{1-(-4x^2)} = (-1/8)\sum_{n=0}^{\infty}(-4x^2)^n = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(4x^2)^n}{8}$$

Is valid when $|4x^2| < 1$ or $|x| < 1/2$, differentiating gives

$$\frac{x}{(1-4x^2)^2} \stackrel{?}{=} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 8x \cdot n(4x^2)^{n-1}}{8} = \sum_{n=0}^{\infty} (-1)^{n+1} xn(4x^2)^{n-1}$$

Every $|x| < 1/2$ converges by the ratio test, meaning the right-hand series converges for all $x \in (-1/2, 1/2)$. Checking endpoints $x = 1/2$ gives $\sum(-1)^{n+1} n/2$ which clearly doesn't converge, similarly $x = -1/2$ doesn't converge.

Finally, we must show our differentiated series converges to the right thing. Let $f_m = \sum_{n=0}^{m} (-1)^{n+1} xn(4^2)^{n-1}$ and $f(x) = x/(1+4x^2)$. We have $(f_m') \to f'$ uniformly (by construction via geometric series), and since (f_m) and f agree at $x = 0$ ($f_m(0) \to 0$ and $f(0) = 0$) we must have $(f_m) \to f$ uniformly.

(c) We know the Taylor series for $\log(1+x)$ is

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$$

Which converges on $(-1, 1]$. Substituting x^2 for x gives

$$\log(1+x^2) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{n}$$

Which converges on $[0, 1]$.

Exercise 6.6.3
Derive the formula for the Taylor coefficients given in Theorem 6.6.2.

Solution
We are given $f(x) = \sum_{n=0}^{\infty} a_n x^n$ defined on some nontrivial interval $(-R, R)$. $a_0 = f(0)$ by definition, now applying Theorem 6.5.7 we get

$$f'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

and $f'(0) = a_1 \cdot 1$ by definition of f', continue like this applying 6.5.7 each time to get

$$a_n = \frac{f^{(n)}(0)}{n!}$$

Exercise 6.6.4
Explain how Lagrange's Remainder Theorem can be modified to prove

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \log(2)$$

Solution

Let $E_N(x) = \log(1+x) - \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$. The proof presented of Lagrange's Remainder Theorem assumed that $x > 0$ to simplify notation, and found a value of $c \in (0, x)$, with the proof for $x < 0$ being implicit. But we can just ignore $x < 0$ to state that given $x \in (0, R)$ there exists $0 < c < x$ satisfying

$$E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1}$$

Now, $\left|f^{(N+1)}(c)\right| = N!(1+c)^{-(N+1)}$, so for $c < x = 1$ we have

$$|E_N(x)| \leq \frac{1}{N+1}$$

which converges to 0 as $N \to \infty$. Hence the Taylor series is equal to $\log(1+x)$ over at least $(0, 1)$, and we can extend this equality to $(0, 1]$ since both $\log(1+x)$ and the Taylor series are continuous at 1 (we established the latter in Exercise 6.5.1). Plugging $x = 1$ into both equations leaves us with the desired equality.

Exercise 6.6.5

(a) Generate the Taylor coefficients for the exponential function $f(x) = e^x$, and then prove that the corresponding Taylor series converges uniformly to e^x on any interval of the form $[-R, R]$.

(b) Verify the formula $f'(x) = e^x$.

(c) Use a substitution to generate the series for e^{-x}, and then informally calculate $e^x \cdot e^{-x}$ by multiplying together the two series and collecting common powers of x.

Solution

(a) $f(x) = e^x = \sum_{n=0}^{\infty} x^n/n!$ from $f^{(n)}(0) = 1$

(b) Differentiating the series is valid by Theorem 6.5.7

$$f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

(c) Let $a_n = (-1)^n/n!$, $b_n = 1/n!$, and $d_n = \sum_{k=0}^{n} a_k b_{n-k}$; the informal power series representation of $e^x \cdot e^{-x}$ becomes $\sum_{n=0}^{\infty} d_n x^n$. By plugging in 0 we have $d_0 = 0$. For $n > 0$, note that

$$(n!)d_n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}(-1)^k = \sum_{k=0}^{n} \binom{n}{k}(-1)^k = \sum_{k=0,\ k \text{ even}}^{n} \binom{n}{k} - \sum_{k=0,\ k \text{ odd}}^{n} \binom{n}{k}$$

The first term is the total number of ways to choose an even number of elements from a set of size n, while the second term is the number of ways to choose an odd number of elements. For n odd, these two terms must be equal, since for every unique subset with an even number of elements, we get a unique subset with an odd number of elements by taking the set complement. Hence for n odd we must have $d_n = 0$.

It takes a bit more work for n even. Call this set N and divide it into disjoint subsets A containing the first $n-1$ elements and B containing the remaining element. Let $\binom{P}{\text{even}}$ be the number of ways to choose an even number of elements from P and $\binom{P}{\text{odd}}$ be defined similarly. Then

$$\binom{N}{\text{odd}} = \binom{A}{\text{odd}}\binom{B}{\text{even}} + \binom{A}{\text{even}}\binom{B}{\text{odd}} = \binom{A}{\text{odd}} + \binom{A}{\text{even}}$$

while

$$\binom{N}{\text{even}} = \binom{A}{\text{even}}\binom{B}{\text{even}} + \binom{A}{\text{odd}}\binom{B}{\text{odd}} = \binom{A}{\text{even}} + \binom{A}{\text{odd}} = \binom{N}{\text{odd}}$$

and we once again have $d_n = 0$. (Incidentally, this trick does not work for $n = 0$ since it relies on being able to remove an element from N to form A.)

Putting everything together we have $e^x \cdot e^{-x} = 1$ as expected.

Exercise 6.6.6
Review the proof that $g'(0) = 0$ for the function

$$g(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

introduced at the end of this section.

(a) Compute $g'(x)$ for $x \neq 0$. Then use the definition of the derivative to find $g''(0)$.

(b) Compute $g''(x)$ and $g'''(x)$ for $x \neq 0$. Use these observations and invent whatever notation is needed to give a general description for the nth derivative $g^{(n)}(x)$ at points different from zero.

(c) Construct a general argument for why $g^{(n)}(0) = 0$ for all $n \in \mathbf{N}$.

Solution

(a) $g'(x) = 2x^{-3}e^{-x^{-2}}$. Repeatedly using L'Hospital's Rule,

$$g''(0) = \lim_{x\to 0} \frac{2x^{-3}e^{-x^{-2}}}{x} = \lim_{x\to 0} \frac{2x^{-4}}{e^{1/x^2}} = 2\lim_{x\to 0} \frac{-4x^{-5}x^3}{-2e^{1/x^2}} = 4\lim_{x\to 0} \frac{-2x^{-3}x^3}{-2e^{1/x^2}} = 0$$

(b) Explicitly computing things sounds rather tedious, so let's skip to the general form. We claim that $g^{(n)}(x)$ for $x \neq 0$ is of the form $P_n(x)e^{-1/x^2}$ where $P_n(x)$ is some polynomial in x^{-1}. We can prove this inductively, by noting $g^{(n+1)}(x) = [P_n'(x)+P_n(x)(-2x^{-3})]e^{-1/x^2}$. Differentiating a polynomial in x^{-1} with respect to x only increases the powers, and polynomials are closed under addition and multiplication, so clearly the term in square brackets continues to be a polynomial in x^{-1}.

(c) Let $P_n(x) = \sum a_n x^{-b_n}$ with a_n constant and $b_n \geq 0$. We'll compute the formula for the derivative termwise, and use induction.

$$\lim_{x\to 0} \frac{a_n x^{-b_n} e^{-1/x^2}}{x} = a_n \lim_{x\to 0} \frac{x^{-(b_n+1)}}{e^{1/x^2}}$$

It's easy to show that this is equal to 0, since every time we apply L'Hospital's rule, the denominator doesn't change (and continues to go to infinity) while the exponent in the numerator increases by 2 each time, eventually becoming non-negative and preventing the numerator from also going to infinity.

Exercise 6.6.7
Find an example of each of the following or explain why no such function exists.

(a) An infinitely differentiable function $g(x)$ on all of $\mathbf{R}$ with a Taylor series that converges to $g(x)$ only for $x \in (-1, 1)$.

(b) An infinitely differentiable function $h(x)$ with the same Taylor series as $\sin(x)$ but such that $h(x) \neq \sin(x)$ for all $x \neq 0$.

(c) An infinitely differentiable function $f(x)$ on all of $\mathbf{R}$ with a Taylor series that converges to $f(x)$ if and only if $x \leq 0$.

Solution
(a) $g(x) = 1/(1+x^2)$. We already have that the Taylor series for this is $\sum_{n=0}^{\infty}(-1)^n x^2 n$, and that it converges to $g(x)$ for $|x| < 1$, but it does not converge at all at ± 1.

(b) Let $a(x)$ be the counterexample function introduced at the end of this section. Then set $h(x) = \sin(x) + a(x)$; since the Taylor series of $a(x)$ is identically 0, $h(x)$ has the same Taylor series as $\sin(x)$.

(c)
$$f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-1/x^2} & x > 0 \end{cases}$$

has Taylor series identically 0.

Exercise 6.6.8
Here is a weaker form of Lagrange's Remainder Theorem whose proof is arguably more illuminating than the one for the stronger result.

(a) First establish a lemma: If g and h are differentiable on $[0, x]$ with $g(0) = h(0)$ and $g'(t) \leq h'(t)$ for all $t \in [0, x]$, then $g(t) \leq h(t)$ for all $t \in [0, x]$.

(b) Let f, S_N, and E_N be as Theorem 6.6.3, and take $0 < x < R$. If $\left|f^{(N+1)}(t)\right| \leq M$ for all $t \in [0, x]$, show

$$|E_N(x)| \leq \frac{M x^{N+1}}{(N+1)!}$$

Solution

(a) Let $e(t) = h(t) - g(t)$. If $g(t) > h(t)$ for some $t \in [0, x]$ then we could apply the Mean Value Theorem on $e(t)$ between 0 and t to find $c \in (0, t) \subseteq [0, x]$ with $e'(c) = h'(t) - g'(t) < 0$, a contradiction since $h'(t) \geq g'(t)$.

(b) Since $S_N^{(N+1)} = 0$, $f^{(N+1)}(t) = E^{(N+1)}(t)$. We then have $\left|E_N^{(N+1)}(t)\right| \leq E_N^{(N+1)} \leq M$, and by repeated application of the lemma in part (a) we have $E_N(x) \leq \frac{Mx^{N+1}}{(N+1)!}$. A similar argument holds for $E_N(x) \geq -\frac{Mx^{N+1}}{(N+1)!}$, and we can combine these succiently as

$$|E_N(x)| \leq \frac{Mx^{N+1}}{(N+1)!}$$

Exercise 6.6.9 (Cauchy's Remainder Theorem)
Let f be differentiable $N + 1$ times on $(-R, R)$. For each $a \in (-R, R)$, let $S_N(x, a)$ be the partial sum of the Taylor series for f centered at a; in other words, define

$$S_N(x, a) = \sum_{n=0}^{N} c_n (x - a)^n \quad \text{where} \quad c_n = \frac{f^{(n)}(a)}{n!}.$$

Let $E_N(x, a) = f(x) - S_N(x, a)$. Now fix $x \neq 0$ in $(-R, R)$ and consider $E_N(x, a)$ as a function of a.

(a) Find $E_N(x, x)$.

(b) Explain why $E_N(x, a)$ is differentiable with respect to a, and show

$$E'_N(x, a) = \frac{-f^{(N+1)}(a)}{N!}(x - a)^N.$$

(c) Show

$$E_N(x) = E_N(x, 0) = \frac{f^{(N+1)}(c)}{N!}(x - c)^N x$$

for some c between 0 and x. This is Cauchy's form of the remainder for Taylor series centered at the origin.

Solution

(a)

$$E_N(x, x) = f(x) - S_N(x, x) = f(x) - c_0 = f(x) - f(x) = 0$$

(b)

$$\begin{aligned} E'_N(x, a) = -S'_N(x, a) &= \sum_{n=1}^{N} \frac{f^{(n)}(a)}{(n-1)!}(x - a)^{n-1} - \sum_{n=0}^{N} \frac{f^{(n+1)}(a)}{n!}(x - a)^n \\ &= \sum_{n=0}^{N-1} \frac{f^{(n+1)}(a)}{(n)!}(x - a)^n - \sum_{n=0}^{N} \frac{f^{(n+1)}(a)}{n!}(x - a)^n = \frac{-f^{(N+1)}(a)}{N!}(x - a)^N \end{aligned}$$

(c) By the Mean Value Theorem

$$\frac{E_N(x,x) - E_N(x,0)}{x} = E_N'(x,c)$$

for some $c \in (0,x)$. Plugging in $E_n(x,x) = 0$ and the expression for E_N' derived in part (b) leaves us with the desired result.

Exercise 6.6.10
Consider $f(x) = 1/\sqrt{1-x}$.

(a) Generate the Taylor series for f centered at zero, and use Lagrange's Remainder Theorem to show the series converges to f on $[0,1/2]$. (The case $x < 1/2$ is more straightforward while $x = 1/2$ requires some extra care.) What happens when we attempt this with $x > 1/2$?

(b) Use Canchy's Remainder Theorem proved in Exercise 6.6.9 to show the series representation for f holds on $[0,1)$.

Solution
(a) We have

$$f^{(n)}(x) = \frac{\prod_{i=1}^n (2i-1)}{2^n} \left(\frac{1}{1-x}\right)^N \sqrt{\frac{1}{1-x}}$$

The Taylor series is

$$\sum_{n=0}^N \frac{\prod_{i=1}^n (2i-1)}{2^n n!} x^n = \sum_{n=0}^N \left(\frac{x^n}{2^n} \prod_{i=1}^n \frac{2i-1}{i}\right) \leq \sum_{n=0}^N \left(\frac{x^n}{2^n} \prod_{i=1}^n 2\right) = \sum_{n=0}^N x^n$$

so we know that the Taylor series at least converges to something for $x \in [0,1)$.

Lagrange's Remainder Theorem gives us

$$E_N(x) = \frac{\left(\prod_{i=1}^{N+1} (2i-1)\right) \left(\frac{1}{1-c}\right)^{N+1} \sqrt{\frac{1}{1-c}} x^{N+1}}{2^{N+1}(N+1)!}$$

for some $c \in (0,x)$. For $x = 1/2$ and $c < x$:

$$\begin{aligned}|E_N(x)| &\leq \left(\prod_{i=1}^{N+1} \frac{2i-1}{i}\right) \left(\frac{1}{2-2c}\right)^{N+1} \left(\sqrt{\frac{1}{1-c}}\right) \frac{1}{2^{N+1}} \\ &\leq 2^{N+1} d^{N+1} \left(\sqrt{\frac{1}{1-c}}\right) \frac{1}{2^{N+1}} = d^{N+1} \sqrt{\frac{1}{1-c}}\end{aligned}$$

where $d = 1/(2-2c) < 1$; this shows E_N converges to 0 over $[0,1/2]$.

(b) Plugging in Cauchy's Remainder Theorem,

$$\begin{aligned} E_N(x) &= \left(\prod_{i=1}^{N} \frac{2i-1}{i} \right) \frac{(2N+1)(x-c)^N x}{2^N(1-c)^{N+1}} \sqrt{\frac{1}{1-c}} \\ &\leq (2N+1)\,(d)^N \, \frac{x}{1-c} \sqrt{\frac{1}{1-c}} \end{aligned}$$

where $d = \frac{x-c}{1-c} < 1$. The first term is linear in N, the second is exponentially decaying in N, and the last two terms are constant, so the behaviour is dominated by exponential decay and $E_N(x)$ converges to 0.

6.7 The Weierstrauss Approximation Theorem

Exercise 6.7.1
Assuming WAT, show that if f is continuous on $[a, b]$, then there exists a sequence (p_n) of polynomials such that $p_n \to f$ uniformly on $[a, b]$.

Solution
Repeatedly apply WAT with $\epsilon = 1/n$.

Exercise 6.7.2
Prove Theorem 6.7.3.

Solution
Recall Theorem 4.4.7, which states that a continuous functions over a compact set is uniformly continuous over that set. Given $\epsilon > 0$, apply uniform continuity on f with $\epsilon/2$ to obtain some $\delta > 0$, and partition $[a, b]$ into uniform segments, with each segment length lower than δ. Define $\phi(x)$ at the endpoints of each segment to be equal to $f(x)$, and to linearly interpolate between segment endpoints.

For any $x \in (a, b)$, let q be the largest segment endpoint less than x, and r be the following segment endpoint. (If $x = a$ or $x = b$ then these aren't necessarilly defined, but then $\phi(x) = f(x)$ so there's nothing to worry about.) Since $|x - q| < \delta$ we have that $|f(x) - \phi(q)| < \epsilon/2$. We similarly also have $|\phi(q) - \phi(r)| < \epsilon/2$. Also, note that $\phi(x)$ must lie between $\phi(q)$ and $\phi(r)$, so $|\phi(q) - \phi(x)| \leq |\phi(q) - \phi(r)| < \epsilon/2$. Applying the triangle inequality leaves us with $|f(x) - \phi(x)| < \epsilon$ as desired.

Exercise 6.7.3

(a) Find the second degree polynomial $p(x) = q_0 + q_1 x + q_2 x^2$ that interpolates the three points $(-1, 1), (0, 0)$, and $(1, 1)$ on the graph of $g(x) = |x|$. Sketch $g(x)$ and $p(x)$ over $[-1, 1]$ on the same set of axes.

(b) Find the fourth degree polynomial that interpolates $g(x) = |x|$ at the points $x = -1, -1/2, 0, 1/2$, and 1 . Add a sketch of this polynomial to the graph from (a).

Solution

(a) $p(x) = x^2$

(b) $p(x) = \frac{7}{3}x^2 - \frac{4}{3}x^4$

Exercise 6.7.4
Show that $f(x) = \sqrt{1-x}$ has Taylor series coefficients a_n where $a_0 = 1$ and

$$a_n = \frac{-1 \cdot 3 \cdot 5 \cdots (2n-3)}{2 \cdot 4 \cdot 6 \cdots 2n}$$

for $n \geq 1$.

Solution
For $n \geq 1$:

$$f^{(n)} = -\frac{\prod_{i=1}^{N-1}(2i-1)}{2^n}(1-x)^{-\frac{2n-1}{2}}$$

$$a_n = \frac{f^{(n)}(0)}{n!} = -\frac{\prod_{i=1}^{N-1}(2i-1)}{\left(\prod_{i=1}^{N} 2\right)\left(\prod_{i=1}^{N} i\right)} = \frac{-1 \cdot 3 \cdot 5 \cdots (2n-3)}{2 \cdot 4 \cdot 6 \cdots 2n}$$

Exercise 6.7.5
(a) Follow the advice in Exercise 6.6.9 to prove the Cauchy form of the remainder:

$$E_N(x) = \frac{f^{(N+1)}(c)}{N!}(x-c)^N x$$

for some c between 0 and x.

(b) Use this result to prove equation (1) is valid for all $x \in (-1, 1)$.

Solution
(a) See solution to Exercise 6.6.9

(b)

$$E_N(x) = \frac{-x(1-x)^{-\frac{1}{2}}}{2}\left(\prod_{i=1}^{N} \frac{2i-1}{2i}\right)\left(\frac{x-c}{1-c}\right)^N$$

For $0 < c < x < 1$, the first term is constant, the second term is less than 1, and the last term converges to 0.

Exercise 6.7.6
(a) Let

$$c_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}$$

for $n \geq 1$. Show $c_n < \frac{2}{\sqrt{2n+1}}$.

(b) Use (a) to show that $\sum_{n=0}^{\infty} a_n$ converges (absolutely, in fact) where a_n is the sequence of Taylor coefficients generated in Exercise 6.7.4.

(c) Carefully explain how this verifies that equation (1) holds for all $x \in [-1, 1]$

Solution

(a) We can show this by induction, if a bit inelegantly. The base case is trivial matter of computation. For the inductive case, we want to show $c_{n+1} = c_n \frac{2n+1}{2n+2} \leq \frac{2}{\sqrt{2n+1}} \sqrt{\frac{2n+1}{2n+3}}$. If we work through the algebra of the claim $\frac{2n+1}{2n+1} < \sqrt{\frac{2n+1}{2n+3}}$, we find that this is equivalent to

$$8n^3 + 20n^2 + 14n + 3 < 8n^3 + 20n^2 + 16n + 4$$

which is clearly true for $n \geq 1$, and the inductive step is done.

(b) $|a_n| = c_n/(2n-1) < \frac{2}{(2n-1)\sqrt{2n+1}} \leq 2 \cdot (2n-1)^{2/3}$ which implies absolute convergence by comparison against an appropriate geometric series.

(c) (b) implies that the Taylor series of $\sqrt{1-x}$ converges absolutely at 1. With Theorem 6.5.2, the Taylor series converges uniformly over $[-1,1]$ and is therefore continuous. We also have that the Taylor series converges to $\sqrt{1-x}$ for $x \in (-1,1)$, which is also continuous. Therefore taking limits as both functions approach ± 1 gets us that they are equal over $[-1,1]$.

Exercise 6.7.7

(a) Use the fact that $|a| = \sqrt{a^2}$ to prove that, given $\epsilon > 0$, there exists a polynomial $q(x)$ satisfying

$$||x| - q(x)| < \epsilon$$

for all $x \in [-1,1]$.

(b) Generalize this conclusion to an arbitrary interval $[a,b]$.

Solution

(a) Let the polynomial $p(x)$ be the partial sum of the Taylor series of $\sqrt{1-x}$ which satisfies $\left|p(x) - \sqrt{1-x}\right| < \epsilon$, and let $q(x) = p(1-x^2)$. We then have

$$\left|q(x) - \sqrt{1-(1-x^2)}\right| = ||x| - q(x)| < \epsilon$$

as desired.

(b) Let $c = \max\{|a|, |b|\}$, and let $p(x)$ satisfy $||x| - p(x)| < \epsilon/c$. Then

$$\left|\left|\frac{x}{c}\right| - p\left(\frac{x}{c}\right)\right| < \frac{\epsilon}{c}$$

$$\left||x| - c \cdot p\left(\frac{x}{c}\right)\right| < \epsilon$$

for $x \in [-c,c] \supseteq [a,b]$, so we can use the polynomial $c \cdot p(\frac{x}{c})$.

Exercise 6.7.8

(a) Fix $a \in [-1,1]$ and sketch

$$h_a(x) = \frac{1}{2}(|x-a| + (x-a))$$

over $[-1,1]$. Note that h_a is polygonal and satisfies $h_a(x) = 0$ for all $x \in [-1,a]$.

(b) Explain why we know $h_a(x)$ can be uniformly approximated with a polynomial on $[-1, 1]$.

(c) Let ϕ be a polygonal function that is linear on each subinterval of the partition

$$-1 = a_0 < a_1 < a_2 < \cdots < a_n = 1.$$

Show there exist constants $b_0, b_1, \ldots, b_{n-1}$ so that

$$\phi(x) = \phi(-1) + b_0 h_{a_0}(x) + b_1 h_{a_1}(x) + \cdots + b_{n-1} h_{a_{n-1}}(x)$$

for all $x \in [-1, 1]$.

(d) Complete the proof of WAT for the interval $[-1, 1]$, and then generalize to an arbitrary interval $[a, b]$.

Solution

(a) Left as an application for your favourite graphing calculator

(b) $|x - a|$ can be uniformly approximated, and multiplication by a constant and addition of polynomials preserves the ability to be uniform approximated.

(c) $b_0 = \frac{\phi(a_1)-\phi(a_0)}{a_1-a_0}$, and for $n \geq 1$, $b_n = \frac{\phi(a_{n+1})-\phi(a_n)}{a_{n+1}-a_n} - b_{n-1}$

(d) Fix $\epsilon > 0$. For a function f continuous over $[-1, 1]$, apporixmate it uniformly within $\epsilon/2$ with a polygonal function $\phi(x)$, and approximate $\phi(x)$ uniformly within $\epsilon/2$ with a polynomial. The triangle inequality ensures that this polynomial uniformly approximates f within ϵ. To generalize over $[a, b]$ the same technique in Exercise 6.6.7 of scaling x and f can be used.

Exercise 6.7.9

(a) Find a counterexample which shows that WAT is not true if we replace the closed interval $[a, b]$ with the open interval (a, b).

(b) What happens if we replace $[a, b]$ with the closed set $[a, \infty)$. Does the theorem still hold?

Solution

(a) $1/x$ over $(0, 1)$, since $1/x$ is unbounded while any approximating polynomial must be bounded over $(0, 1)$.

(b) e^x, since exponentials grow faster than polynomials, and therefore the difference between e^x and any polynomial is unbounded over $[0, \infty)$.

Exercise 6.7.10

Is there a countable subset of polynomials $\mathcal{C}$ with the property that every continuous function on $[a, b]$ can be uniformly approximated by polynomials from $\mathcal{C}$?

Solution

Yes - we will approach this by adapting the logic used to prove WAT.

We start by choosing some sequence of polynomials which converge uniformly to $|x|$. The set of all polynomials that appear in this sequence is countable; denote this set by $\mathcal{A}$. For a fixed $a \in [-1, 1]$, we can turn this into a sequence of polynomials approaching h_a as described in Exercise 6.7.8; denote this countable set of polynomials $\mathcal{A}_a$.

The rationals in $[-1, 1]$ are countable, and therefore the union of all $\mathcal{A}_{\dashv}$ for rational a is also countable; denote this set as $\mathcal{B}$. For a similar reason, the set of all polynomials of the form $a \cdot p(x/b)$ where a, b are rational numbers and $p \in \mathcal{B}$ is also countable; denote this set $\mathcal{D}$.

Because rationals are dense in $\mathbf{R}$, it's easy to adapt the proof of Theorem 6.7.3 (Exercise 6.7.2) to only work with polygonal functions whose segment endpoints are rational (i.e. both the endpoint and the value of the function at the endpoint are rational). Let $\mathcal{P}$ denote the set of polygonal functions over $[-1, 1]$ with rational segment endpoints only, and $\mathcal{P}_n$ be the elements in $\mathcal{P}$ with n segments. Any element in $\mathcal{P}_n$ can be uniformly approximated by the sum of n elements from $\mathcal{D}$ plus some rational constant c; call the set of polynomials that can be generated this way $\mathcal{D}_n$. Since n is finite, $\mathcal{D}_n$ is also countable, and so is $\bigcup_{i=1}^{\infty} \mathcal{D}_i$, meaning we have a countable set of polynomials which can uniformly approximate any function from $\mathcal{P}$.

This implies any continuous function can be uniformly apprxoimated over $[-1, 1]$ by a countable set of polynomials. The scaling trick used to extend the original proof of WAT can be used here (except limited to scaling factors of rational numbers) to complete the proof.

Exercise 6.7.11

Assume that f has a continuous derivative on $[a, b]$. Show that there exists a polynomial $p(x)$ such that

$$|f(x) - p(x)| < \epsilon \quad \text{and} \quad |f'(x) - p'(x)| < \epsilon$$

for all $x \in [a, b]$.

Solution

Let $q(x)$ be a polynomial which uniformly approximates $f'(x)$ to within $\frac{\epsilon}{(b-a)}$, and $p(x)$ to be the antiderivative of $q(x)$ which passes through $(a, f(a))$. Let $e(x) = f(x) - p(x)$ be the error between f and p, and note that $|e'(x)| < \frac{\epsilon}{b-a}$.

By the Mean Value Theorem, for any $x \in [a, b]$,

$$|e(x) - e(a)| = |(x - a)e'(c)| < (b - a)\frac{\epsilon}{(b - a)} = \epsilon$$

for some $c \in [a, b]$, completing the proof.