Understanding Analysis Solutions

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Preface

Huge thanks to the math discord for answering my questions! I don't know how I'd manage without them ♡

If you don't find your exercise here check linearalgebras.com or (god forbid) chegg.

I aim to complete abbott by new years. You can see my progress here.

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Chapter 1

The Real Numbers

1.2 Some Preliminaries

Exercise 1.2.1

(a) Prove that $\sqrt{3}$ is irrational. Does a similar similar argument work to show $\sqrt{6}$ is irrational?

(b) Where does the proof break down if we try to prove $\sqrt{4}$ is irrational?

Solution

(a) Suppose for contradiction that p/q is a fraction in lowest terms, and that $(p/q)^2 = 3$. Then $p^2 = 3q^2$ implying p is a multiple of 3 since 3 is not a perfect square. Therefor we can write p as $3r$ for some r, substituting we get $(3r)^2 = 3q^2$ and $3r^2 = q^2$ implying q is also a multiple of 3 contradicting the assumption that p/q is in lowest terms.
For $\sqrt{6}$ the same argument applies, since 6 is not a perfect square.

(b) 4 is a perfect square, meaning $p^2 = 4q^2$ does not imply that p is a multiple of four as p could be 2.

Exercise 1.2.2

Show that there is no rational number satisfying $2^r = 3$

Solution

Letting $r = p/q$ we have $2^{p/q} = 3$ implying $2^p = 3^q$ which is impossible since 2 and 3 are coprime.

Exercise 1.2.3

Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all sets containing an infinite number of elements, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.

(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$.

(d) $A \cap (B \cap C) = (A \cap B) \cap C$.

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution

(a) False, consider $A_1 = \{1, 2, \dots\}, A_2 = \{2, 3, \dots\}$, ... has $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

(b) True.

(c) False, $A = \emptyset$ gives $\emptyset = C$.

(d) True, intersection is associative.

(e) True, draw a diagram.

Exercise 1.2.4

Produce an infinite collection of sets $A_1, A_2, A_3, \dots$ with the property that every A_i has an infinite number of elements, $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{\infty} A_i = \mathbf{N}$

Solution

This question is asking us to partition $\mathbf{N}$ into an infinite collection of sets. This is equivalent to asking us to unroll $\mathbf{N}$ into a square, which we can do along the diagonal

$$\begin{array}{cccccc}
1 & 3 & 6 & 10 & 15 & \cdots \\
2 & 5 & 9 & 14 & \cdots & \\
4 & 8 & 13 & \cdots & & \\
7 & 12 & \cdots & & & \\
11 & \dots & & & & \\
\vdots & & & & &
\end{array}$$

Exercise 1.2.5 (De Morgan's Laws)

Let A and B be subsets of $\mathbf{R}$.

(a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$

(b) Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$

(c) Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

Solution

(a) If $x \in (A \cap B)^c$ then $x \notin A \cap B$ so $x \notin A$ or $x \notin B$ implying $x \in A^c$ or $x \in B^c$ which is the same as $x \in A^c \cup B^c$.

(b) Let $x \in A^c \cup B^c$ implying $x \in A^c$ or $x \in B^c$ meaning $x \notin A$ or $x \notin B$ implying $x \notin A \cap B$ which is the same as $x \in (A \cap B)^c$.

(c) First let $x \in (A \cup B)^c$ implying $x \notin A \cup B$ meaning $x \notin A$ and $x \notin B$ which is the same as $x \in A^c$ and $x \in B^c$ which is just $x \in A^c \cap B^c$. Second let $x \in A^c \cap B^c$ implying $x \in A^c$ and $x \in B^c$ implying $x \notin A$ and $x \notin B$ meaning $x \notin A \cup B$ which is just $x \in (A \cup B)^c$.

Exercise 1.2.6

(a) Verify the triangle inequality in the special case where a and b have the same sign.

(b) Find an efficient proof for all the cases at once by first demonstrating $(a+b)^2 \le (|a|+|b|)^2$

(c) Prove $|a-b| \le |a-c| + |c-d| + |d-b|$ for all a, b, c, and d.

(d) Prove $||a|-|b|| \le |a-b|$. (The unremarkable identity $a = a - b + b$ may be useful.)

Solution

(a) We have equality $|a+b| = |a|+|b|$ meaning $|a+b| \le |a|+|b|$ also holds.

(b) $(a+b)^2 \le (|a|+|b|)^2$ reduces to $2ab \le 2|a||b|$ which is obviously true. and since squaring preserves inequality this implies $|a+b| \le |a|+|b|$.

(c) I would like to do this using the triangle inequality, I notice that $(a-c)+(c-d)+(d-b) = a-b$. Meaning I can use the triangle inequality for multiple terms

$$|a-b| = |(a-c)+(c-d)+(d-b)| \le |a-c| + |c-d| + |d-b|$$

The general triangle inequality is proved by repeated application of the two variable inequality

$$|(a+b)+c| \le |a+b| + |c| \le |a| + |b| + |c|$$

(d) I would like to cancel the subtraction inside $||a|-|b||$ since then the inside will be positive, and the outer absolute value will vanish. Using the suggestion let $a = (a-b)+b$

$$||a|-|b|| = ||(a-b)+b| - |b|| \stackrel{!}{\le} ||a-b| + |b| - |b|| = |a-b|$$

However this is incorrect by itself, since $|a| \le |c|$ does not imply $||a|-|b|| \le ||c|-|b||$ (draw a picture, or use the counterexample $a = 0, c = 1, b = 2$).

We can salvage this argument though, notice if $|a| \ge |b|$ then $|a| \le |c|$ does imply $||a|-|b|| \le ||c|-|b||$. And since we can swap a and b without changing anything, we can say without loss of generality assume $|a| \ge |b|$ and then apply the previous argument.

Exercise 1.2.7

Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(A) = \{f(x) : x \in A\}$.

(a) Let $f(x) = x^2$. If $A = [0,2]$ (the closed interval $\{x \in \mathbf{R} : 0 \le x \le 2\}$) and $B = [1,4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?

(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

(c) Show that, for an arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbf{R}$

(d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary function g

Solution

(a) $f(A) = [0, 4]$, $f(B) = [1, 16]$, $f(A \cap B) = [1, 4] = f(A) \cap f(B)$ and $f(A \cup B) = [0, 16] = f(A) \cup f(B)$

(b) $A = \{-1\}$, $B = \{1\}$ thus $f(A \cap B) = \emptyset$ but $f(A) \cap f(B) = \{1\}$

(c) Suppose $y \in g(A \cap B)$, then $\exists x \in A \cap B$ such that $g(x) = y$. But if $x \in A \cap B$ then $x \in A$ and $x \in B$, meaning $y \in g(A)$ and $y \in g(B)$ implying $y \in g(A) \cap g(B)$ and thus $g(A \cap B) \subseteq g(A) \cap g(B)$.

Notice why it is possible to have $x \in g(A) \cap g(B)$ but $x \notin g(A \cap B)$, this happens when something in $A \setminus B$ and something in $B \setminus A$ map to the same thing. If g is 1-1 this does not happen.

(d) I conjecture that $g(A \cup B) = g(A) \cup g(B)$. To prove this we show inclusion both ways, First suppose $y \in g(A \cup B)$. then either $y \in g(A)$ or $y \in g(B)$, implying $y \in g(A) \cup g(B)$. Now suppose $y \in g(A) \cup g(B)$ meaning either $y \in g(A)$ or $y \in g(B)$ which is the same as $y \in g(A \cup B)$ as above.

Exercise 1.2.8

Here are two important definitions related to a function $f : A \to B$. The function f is *one-to-one* $(1 - 1)$ if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. The function f is *onto* if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$ Give an example of each or state that the request is impossible:

(a) $f : \mathbf{N} \to \mathbf{N}$ that is $1 - 1$ but not onto.

(b) $f : \mathbf{N} \to \mathbf{N}$ that is onto but not $1 - 1$.

(c) $f : \mathbf{N} \to \mathbf{Z}$ that is $1 - 1$ and onto.

Solution

(a) Let $f(n) = n + 1$ does not have a solution to $f(a) = 1$

(b) Let $f(1) = 1$ and $f(n) = n - 1$ for $n > 1$

(c) Let $f(n) = n/2$ for even n, and $f(n) = -(n + 1)/2$ for odd n.

Exercise 1.2.9

Given a function $f : D \to \mathbf{R}$ and a subset $B \subseteq \mathbf{R}$, let $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B; that is, $f^{-1}(B) = \{x \in D : f(x) \in B\}$. This set is called the *preimage* of B.

(a) Let $f(x) = x^2$. If A is the closed interval $[0, 4]$ and B is the closed interval $[-1, 1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case? Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?

(b) The good behavior of preimages demonstrated in (a) is completely general. Show that for an arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ for all sets $A, B \subseteq \mathbf{R}$

Solution

(a) $f^{-1}(A) = [-2, 2]$, $f^{-1}(B) = [-1, 1]$, $f^{-1}(A \cap B) = [-1, 1] = f^{-1}(A) \cap f^{-1}(B)$, $f^{-1}(A \cup B) = [-2, 2] = f^{-1}(A) \cup f^{-1}(B)$

(b) First let $x \in g^{-1}(A \cap B)$ meaning $g(x) \in A \cap B$ implying $g(x) \in A$ and $g(x) \in B$ which is the same as $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$ meaning $x \in g^{-1}(A) \cap g^{-1}(B)$.

Second let $x \in g^{-1}(A) \cap g^{-1}(B)$, this is the same as $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$ which is the same as $g(x) \in A$ and $g(x) \in B$ implying $g(x) \in A \cap B$ and thus $x \in g^{-1}(A \cap B)$.

Exercise 1.2.10

Decide which of the following are true statements. Provide a short justification for those that are valid and a counterexample for those that are not:

(a) Two real numbers satisfy $a < b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

(b) Two real numbers satisfy $a < b$ if $a < b + \epsilon$ for every $\epsilon > 0$.

(c) Two real numbers satisfy $a \leq b$ if and only if $a < b + \epsilon$ for every $\epsilon > 0$.

Solution

(a) False, if $a = b$ then $a < b + \epsilon$ forall $\epsilon > 0$ but $a \not< b$

(b) False, see above

(c) True, suppose $a < b + \epsilon$ forall $\epsilon > 0$, We want to show this implies $a \leq b$. We either have $a \leq b$ or $a > b$, but $a > b$ is impossible since the gap implies there exists an ϵ small enough such that $a > b + \epsilon$. Now suppose $a \leq b$, obviously $a < b + \epsilon$ forall $\epsilon > 0$.

Exercise 1.2.11

Form the logical negation of each claim. One trivial way to do this is to simply add "It is not the case that..." in front of each assertion. To make this interesting, fashion the negation into a positive statement that avoids using the word "not" altogether. In each case, make an intuitive guess as to whether the claim or its negation is the true statement.

(a) For all real numbers satisfying $a < b$, there exists an $n \in \mathbf{N}$ such that $a + 1/n < b$

(b) There exists a real number $x > 0$ such that $x < 1/n$ for all $n \in \mathbf{N}$.

(c) Between every two distinct real numbers there is a rational number.

Solution

(a) There exist real numbers satisfying $a < b$ where $a + 1/n \geq b$ forall $n \in \mathbf{N}$ (false).

(b) For every real number $x > 0$ there exists an $n \in \mathbf{N}$ such that $x < 1/n$ (true).

(c) There exist two real numbers $a < b$ such that if $r < b$ then $r < a$ forall $r \in \mathbf{Q}$ (false).

Exercise 1.2.12
Let $y_1 = 6$, and for each $n \in \mathbf{N}$ define $y_{n+1} = (2y_n - 6)/3$

(a) Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in \mathbf{N}$.

(b) Use another induction argument to show the sequence $(y_1, y_2, y_3, \ldots)$ is decreasing.

Solution

(a) Suppose $y_n > -6$, then $y_{n+1} = (2y_n - 6)/3$ implying $y_n = (3y_{n+1} + 6)/2 > -6$ implying $y_{n+1} > -6$ by basic algebra.

(b) Suppose $y_{n+1} < y_n$ this implies $2y_{n+1} < 2y_n$ implying $2y_{n+1} - 6 < 2y_n - 6$ and finally $(2y_{n+1} - 6)/3 < (2y_n - 6)/3$ which shows $y_{n+2} < y_{n+1}$.

Exercise 1.2.13
For this exercise, assume Exercise 1.2.5 has been successfully completed.

(a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

for any finite $n \in \mathbf{N}$

(b) It is tempting to appeal to induction to conclude

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of $n \in \mathbf{N}$, but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets $B_1, B_2, B_3, \ldots$ where $\bigcap_{i=1}^{n} B_i \neq \emptyset$ is true for every $n \in \mathbf{N}$, but $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$ fails.

(c) Nevertheless, the infinite version of De Morgan's Law stated in (b) is a valid statement. Provide a proof that does not use induction.

Solution

(a) 1.2.5 Is our base case, Assume $(A_1 \cup \cdots \cup A_n)^c = A_1^c \cap \cdots \cap A_n^c$. We want to show the $n+1$ case. Using associativity we have

$$\begin{aligned}
((A_1 \cup \cdots \cup A_n) \cup A_{n+1})^c &= (A_1 \cup \cdots \cup A_n)^c \cap A_{n+1}^c \\
&= (A_1^c \cap \cdots \cap A_n^c) \cap A_{n+1}^c \\
&= A_1^c \cap \cdots \cap A_n^c \cap A_{n+1}^c
\end{aligned}$$

(b) $B_1 = \{1, 2, \ldots\}, B_2 = \{2, 3, \ldots\}, \ldots$

(c) First suppose $x \in (\bigcap_{i=1}^{\infty} A_i)^c$, then $x \notin \bigcap_{i=1}^{\infty} A_i$ meaning $x \notin A_i$ for some i, which is the same as $x \in A_i^c$ for some i, meaning $x \in \bigcup_{i=1}^{\infty} A_i^c$. This shows

$$\left(\bigcap_{i=1}^{\infty} A_i\right) \subseteq \bigcup_{i=1}^{\infty} A_i^c$$

Now suppose $x \in \bigcup_{i=1}^{\infty} A_i^c$ meaning $x \notin A_i$ for some i, which is the same as $x \notin \bigcap_{i=1}^{\infty} A_i$ implying $x \notin (\bigcap_{i=1}^{\infty} A_i)^c$. This shows inclusion the other way and completes the proof.

1.3 The Axiom of Completeness

Exercise 1.3.1

(a) Write a formal definition in the style of Definition 1.3.2 for the *infimum* or *greatest lower bound* of a set.

(b) Now, state and prove a version of Lemma 1.3.8 for greatest lower bounds.

Solution

(a) We have $i = \inf A$ if and only if

  (i) Lower bound, $a \geq i$ forall $a \in A$

  (ii) Greatest lower bound, If b is a lower bound on A then $b \leq i$

(b) Theorem: Suppose i is a lower bound for A, it is the greatest lower bound if and only if forall $\epsilon > 0$, there exists an $a \in A$ such that $i + \epsilon < a$.

Proof:
($\implies$) Suppose $i = \inf A$, then any $i + \epsilon$ cannot be a lower bound since i is defined as the greatest lower bound, and $i + \epsilon > i$.
($\impliedby$) Suppose $j > i$ is a lower bound on A, then set $\epsilon = j - i$ and we have that $i + \epsilon = j$ is not a lower bound since there exists $a \in A$ such that $j > a$. Therefor i is the greatest lower bound.

Exercise 1.3.2

Give an example of each of the following, or state that the request is impossible.

(a) A set B with $\inf B \geq \sup B$.

(b) A finite set that contains its infimum but not its supremum.

(c) A bounded subset of $\mathbf{Q}$ that contains its supremum but not its infimum.

Solution

(a) Let $B = \{0\}$ we have $\inf B = 0$ and $\sup B = 0$ thus $\inf B \leq \sup B$.

(b) Impossible, finite sets must contain their infimum and supremum.

(c) Let $B = \{r \in \mathbf{Q} \mid 1 < r \leq 2\}$ we have $\inf B = 1 \notin B$ and $\sup B = 2 \in B$.

Exercise 1.3.3

(a) Let A be nonempty and bounded below, and define $B = \{b \in \mathbf{R} : b$ is a lower bound for $A\}$. Show that $\sup B = \inf A$.

(b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

Solution

(a) By definition $\sup B$ is the greatest lower bound for A, meaning it equals $\inf A$.

(b) (a) Proves the greatest lower bound exists using the least upper bound.

Exercise 1.3.4
Let $A_1, A_2, A_3, \ldots$ be a collection of nonempty sets, each of which is bounded above.

(a) Find a formula for $\sup\left(A_1 \cup A_2\right)$. Extend this to $\sup\left(\bigcup_{k=1}^{n} A_k\right)$.

(b) Consider $\sup\left(\bigcup_{k=1}^{\infty} A_k\right)$. Does the formula in (a) extend to the infinite case?

Solution

(a) $\sup\left(\bigcup_{k=1}^{n} A_k\right) = \sup\{\sup A_k \mid k = 1, \ldots, n\}$

(b) Yes. Let $S = \{\sup A_k \mid k = 1, \ldots, \}$ and $s = \sup S$. s is obviously an upper bound for $\bigcup_{k=1}^{\infty} A_k$. to see it is the least upper bound suppose $s' < s$, then by definition there exists a k such that $\sup A_k > s'$ implying s' is not an upper bound for A_k. Therefor s is the least upper bound.

Exercise 1.3.5
As in Example 1.3.7, let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $c \in \mathbf{R}$. This time define the set $cA = \{ca : a \in A\}$.

(a) If $c \geq 0$, show that $\sup(cA) = c \sup A$.

(b) Postulate a similar type of statement for $\sup(cA)$ for the case $c < 0$.

Solution

(a) Let $s = c \sup A$. Suppose $ca > s$, then $a > \sup A$ which is impossible, meaning s is an upper bound on cA. Now suppose s' is an upper bound on cA and $s' < s$. Then $s'/c < s/c$ and $s'/c < \sup A$ meaning s'/c cannot bound A, so there exists $a \in A$ such that $s'/c > a$ meaning $s' > ca$ thus s' cannot be an upper bound on cA, and so $s = c \sup A$ is the least upper bound.

(b) $\sup(cA) = c \inf(A)$ for $c < 0$

Exercise 1.3.6
Given sets A and B, define $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Follow these steps to prove that if A and B are nonempty and bounded above then $\sup(A + B) = \sup A + \sup B$

(a) Let $s = \sup A$ and $t = \sup B$. Show $s + t$ is an upper bound for $A + B$.

(b) Now let u be an arbitrary upper bound for $A + B$, and temporarily fix $a \in A$. Show $t \leq u - a$.

(c) Finally, show $\sup(A + B) = s + t$.

(d) Construct another proof of this same fact using Lemma 1.3.8.

Solution

(a) We have $a \leq s$ and $b \leq t$, adding the equations gives $a + b \leq s + t$.

(b) $t \le u - a$ should be true since $u - a$ is an upper bound on b, meaning it is greater then or equal to the least upper bonud t. Formally $a + b \le u$ implies $b \le u - a$ and since t is the least upper bound on b we have $t \le u - a$.

(c) From (a) we know $s + t$ is an upper bound, so we must only show it is the least upper bound.

Let $u = \sup(A + B)$, from (a) we have $t \le u - a$ and $s \le u - b$ adding and rearranging gives $a + b \le 2u - s - t$. since $2u - s - t$ is an upper bound on $A + B$ it is less then the least upper bound, so $u \le 2u - s - t$ implying $s + t \le u$. and since u is the least upper bound $s + t$ must equal u.

Stepping back, the key to this proof is that $a + b \le s, \forall a, b$ implying $\sup(A + B) \le s$ can be used to transition from all $a + b$ to a single value $\sup(A + B)$, avoiding the ϵ-hackery I would otherwise use.

(d) Showing $s + t - \epsilon$ is not an upper bound for any $\epsilon > 0$ proves it is the least upper bound by Lemma 1.3.8. Rearranging gives $(s - \epsilon/2) + (t - \epsilon/2)$ we know there exists $a > (s - \epsilon/2)$ and $b > (t - \epsilon/2)$ therefor $a + b > s + t - \epsilon$ meaning $s + t$ cannot be made smaller, and thus is the least upper bound.

Exercise 1.3.7
Prove that if a is an upper bound for A, and if a is also an element of A, then it must be that $a = \sup A$.

Solution
a is the least upper bound since any smaller bound $a' < a$ would not bound a.

Exercise 1.3.8
Compute, without proofs, the suprema and infima (if they exist) of the following sets:

(a) $\{m/n : m, n \in \mathbf{N} \text{ with } m < n\}$.

(b) $\{(-1)^m/n : m, n \in \mathbf{N}\}$.

(c) $\{n/(3n + 1) : n \in \mathbf{N}\}$

(d) $\{m/(m + n) : m, n \in \mathbf{N}\}$

Solution
(a) sup $= 1$, inf $= 0$

(b) sup $= 1$, inf $= -1$

(c) sup $= 1/3$, inf $= 1/4$

(d) sup $= 1$, inf $= 0$

Exercise 1.3.9
(a) If $\sup A < \sup B$, show that there exists an element $b \in B$ that is an upper bound for A.

(b) Give an example to show that this is not always the case if we only assume $\sup A \leq \sup B$

Solution

(a) By Lemma 1.3.8 we know there exists a b such that $(\sup B) - \epsilon < b$ for any $\epsilon > 0$, We set ϵ to be small enough that $\sup A < (\sup B) - \epsilon$ meaning $\sup A < b$ for some b, and thus b is an upper bound on A.

(b) $A = \{x \mid x \leq 1\}$, $B = \{x \mid x < 1\}$ no $b \in B$ is an upper bound since $1 \in A$ and $1 > b$.

Exercise 1.3.10 (Cut Property)

The Cut Property of the real numbers is the following:

If A and B are nonempty, disjoint sets with $A \cup B = \mathbf{R}$ and $a < b$ for all $a \in A$ and $b \in B$, then there exists $c \in \mathbf{R}$ such that $x \leq c$ whenever $x \in A$ and $x \geq c$ whenever $x \in B$.

(a) Use the Axiom of Completeness to prove the Cut Property.

(b) Show that the implication goes the other way; that is, assume $\mathbf{R}$ possesses the Cut Property and let E be a nonempty set that is bounded above. Prove $\sup E$ exists.

(c) The punchline of parts (a) and (b) is that the Cut Property could be used in place of the Axiom of Completeness as the fundamental axiom that distinguishes the real numbers from the rational numbers. To drive this point home, give a concrete example showing that the Cut Property is not a valid statement when $\mathbf{R}$ is replaced by $\mathbf{Q}$.

Solution

(a) If $c = \sup A = \inf B$ then $a \leq c \leq b$ is obvious. So we must only prove $\sup A = \inf B$. If $\sup A < \inf B$ then we can find c between A and B implying $A \cup B \neq \mathbf{R}$. If $\sup A > \inf B$ then we can find a such that $a > b$ by subtracting $\epsilon > 0$ and using the least upper/lower bound facts, similarly to Lemma 1.3.8. Thus $\sup A$ must equal $\inf B$ since we have shown both alternatives are impossible.

(b) Let $B = \{x \mid e < x, \forall e \in E\}$ and let $A = B^c$. Clearly $a < b$ so the cut property applies. We have $a \leq c \leq b$ and must show the two conditions for $c = \sup E$

(i) Since $E \subseteq A$, $a \leq c$ implies $e \leq c$ thus c is an upper bound.

(ii) $c \leq b$ implies c is the smallest upper bound.

Note: Using (a) here would be wrong, it assumes the axiom of completeness so we would be making a circular argument.

(c) $A = \{r \in \mathbf{Q} \mid r^2 < 2\}$, $B = A^c$ does not satisfy the cut property in $\mathbf{Q}$ since $\sqrt{2} \notin \mathbf{Q}$

Exercise 1.3.11

Decide if the following statements about suprema and infima are true or false. Give a short proof for those that are true. For any that are false, supply an example where the claim in question does not appear to hold.

(a) If A and B are nonempty, bounded, and satisfy $A \subseteq B$, then $\sup A \leq \sup B$.

(b) If $\sup A < \inf B$ for sets A and B, then there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.

(c) If there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

Solution

(a) True. We know $a \leq \sup A$ and $a \leq \sup B$ since $A \subseteq B$. since $\sup A$ is the least upper bound on A we have $\sup A \leq \sup B$.

(b) True. Let $c = (\sup A + \inf B)/2$, $c > \sup A$ implies $a < c$ and $c < \inf B$ implies $c < b$ giving $a < c < b$ as desired.

(c) False. consider $A = \{x \mid x < 1\}$, $B = \{x \mid x > 1\}$, $a < 1 < b$ but $\sup A \not< \inf B$ since $1 \not< 1$.

1.4 Consequences of Completeness

Exercise 1.4.1
Recall that $\mathbf{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbf{Q}$, then ab and $a + b$ are elements of $\mathbf{Q}$ as well.

(b) Show that if $a \in \mathbf{Q}$ and $t \in \mathbf{I}$, then $a + t \in \mathbf{I}$ and $at \in \mathbf{I}$ as long as $a \neq 0$.

(c) Part (a) can be summarized by saying that $\mathbf{Q}$ is closed under addition and multiplication. Is $\mathbf{I}$ closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s + t$ and st?

Solution

(a) Trivial.

(b) Suppose $a + t \in \mathbf{Q}$, then by (a) $(a + t) - a = t \in \mathbf{Q}$ contradicting $t \in \mathbf{I}$.

(c) $\mathbf{I}$ is not closed under addition or multiplication. consider $(1 - \sqrt{2}) \in \mathbf{I}$ by (b), and $\sqrt{2} \in \mathbf{I}$. the sum $(1 - \sqrt{2}) + \sqrt{2} = 1 \in \mathbf{Q} \notin \mathbf{I}$. Also $\sqrt{2} \cdot \sqrt{2} = 2 \in \mathbf{Q} \notin \mathbf{I}$.

Exercise 1.4.2
Let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $s \in \mathbf{R}$ have the property that for all $n \in \mathbf{N}, s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upper bound for A. Show $s = \sup A$.

Solution
This is basically a rephrasing of Lemma 1.3.8 using the archimedean property. The most straightforward approach is to argue by contradiction:

(i) If $s < \sup A$ then there exists an n such that $s + 1/n < \sup A$ contradicting $\sup A$ being the least upper bound.

(ii) If $s > \sup A$ then there exists an n such that $s - 1/n > \sup A$ where $s - 1/n$ is not an upper bound, contradicting $\sup A$ being an upper bound.

Thus $s = \sup A$ is the only remaining possibility.

Exercise 1.4.3
Prove that $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closed for the conclusion of the theorem to hold.

Solution
Suppose $x \in \bigcap_{n=1}^{\infty}(0, 1/n)$, then we have $0 < x < 1/n$ forall n, which is impossible by the archimedean property, In other words we can always set n large enough that x lies outside the interval.

Exercise 1.4.4
Let $a < b$ be real numbers and consider the set $T = \mathbf{Q} \cap [a, b]$. Show $\sup T = b$

Solution
We must show the two conditions for a least upper bound

(i) Clearly $t \leq b$ forall $t \in T$

(ii) Let $a < b' < b$. b' Cannot be an upper bound for T since the density theorem tells us we can find $r \in \mathbf{Q} \cap [a, b]$ such that $b' < r < b$.

Exercise 1.4.5
Using Exercise 1.4.1, supply a proof for Corollary 1.4.4 by considering the real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$.

Solution
By the density theorem we first find $a < p/q < b$, then we add $(\sqrt{2})/n$ with n small enough that $a < (p/q) + (\sqrt{2})/n < b$. By (a) we know $(p/q) + (\sqrt{2})/n \in \mathbf{I}$. Thus we have found $t = (p/q) + (\sqrt{2})/n$ with $a < t < b$ and $t \in \mathbf{I}$ which proves Corollary 1.4.4.

(I'm not sure what the exercise wanted me to do with $a - \sqrt{2}$ and $b - \sqrt{2}$. I'll figure out later)
TODO Make more rigorous

Exercise 1.4.6
Recall that a set B is dense in $\mathbf{R}$ if an element of B can be found between any two real numbers $a < b$. Which of the following sets are dense in $\mathbf{R}$? Take $p \in \mathbf{Z}$ and $q \in \mathbf{N}$ in every case.

(a) The set of all rational numbers p/q with $q \leq 10$.

(b) The set of all rational numbers p/q with q a power of 2 .

(c) The set of all rational numbers p/q with $10|p| \geq q$.

Solution
(a) Dense.

(b) Dense.

(c) Not dense since we cannot make $|p|/q$ smaller then $1/10$.

Exercise 1.4.7
Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ leads to a contradiction of the fact that $\alpha = \sup T$

Solution
Recall $T = \{t \in \mathbf{R} \mid t^2 < 2\}$ and $\alpha = \sup T$. suppose $\alpha^2 > 2$, we will show there exists an $n \in \mathbf{N}$ such that $(\alpha - 1/n)^2 > 2$ contradicting the assumption that α is the least upper bound.

We expand $(\alpha - 1/n)^2$ to find n such that $(\alpha^2 - 1/n) > 2$

$$2 < (\alpha - 1/n)^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} < \alpha^2 + \frac{1 - 2\alpha}{n}$$

Then

$$2 < \alpha^2 + \frac{1-2\alpha}{n} \implies n(2-\alpha^2) < 1-2\alpha$$

Since $2-\alpha^2 < 0$ dividing reverses the inequality gives us

$$n > \frac{1-2\alpha}{2-\alpha^2}$$

This contradicts $\alpha^2 > 2$ since we have shown n can be picked such that $(\alpha^2 - 1/n) > 2$ meaning α is not the least upper bound.

Exercise 1.4.8

Give an example of each or state that the request is impossible. When a request is impossible, provide a compelling argument for why this is the case.

(a) Two sets A and B with $A \cap B = \emptyset, \sup A = \sup B, \sup A \notin A$ and $\sup B \notin B$.

(b) A sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.

(c) A sequence of nested unbounded closed intervals $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} L_n = \emptyset$. (An unbounded closed interval has the form $[a, \infty) = \{x \in R : x \geq a\}$.)

(d) A sequence of closed bounded (not necessarily nested) intervals $I_1, I_2, I_3, \ldots$ with the property that $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for all $N \in \mathbf{N}$, but $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

Solution

(a) $A = \mathbf{Q} \cap (0,1)$, $B = \mathbf{I} \cap (0,1)$. $A \cap B = \emptyset$, $\sup A = \sup B = 1$ and $1 \notin A$, $1 \notin B$.

(b) Impossible. $\bigcap_{n=1}^{\infty} J_n$ is the same as asking what happens to J_n as n goes to ∞. since every J_n is nonempty, $\bigcap_{n=1}^{\infty} = J_\infty$ must have an uncountably infinite number of elements.

(c) $L_n = [n, \infty)$ has $\bigcap_{n=1}^{\infty} L_n = \emptyset$

(d) Impossible. Let $J_n = \bigcap_{k=1}^{n} I_k$ and observe the following

  (i) Since $\bigcap_{n=1}^{N} I_n \neq \emptyset$ we have $J_n \neq \emptyset$.

  (ii) J_n being the intersection of closed intervals makes it a closed interval.

  (iii) $J_{n+1} \subseteq J_n$ since $I_{n+1} \cap J_n \subseteq J_n$

  (iv) $\bigcap_{n=1}^{\infty} J_n = \bigcap_{n=1}^{\infty} (\bigcap_{k=1}^{n} I_k) = \bigcap_{n=1}^{\infty} I_n$

By (i), (ii) and (iii) the Nested Interval Property tells us $\bigcap_{n=1}^{\infty} J_n \neq \emptyset$. Therefor by (iv) $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

1.5 Cardinality

Exercise 1.5.1
Finish the following proof for Theorem 1.5.7. Assume B is a countable set. Thus, there exists $f : \mathbf{N} \to B$, which is $1-1$ and onto. Let $A \subseteq B$ be an infinite subset of B. We must show that A is countable.

Let $n_1 = \min\{n \in \mathbf{N} : f(n) \in A\}$. As a start to a definition of $g : \mathbf{N} \to A$ set $g(1) = f(n_1)$. Show how to inductively continue this process to produce a 1-1 function g from $\mathbf{N}$ onto A.

Solution
Let $n_k = \min\{n \in \mathbf{N} \mid f(n) \in A, n \notin \{n_1, n_2, \ldots, n_{k-1}\}\}$ and $g(k) = f(n_k)$. since $g : \mathbf{N} \to A$ is 1-1 and onto, A is countable.

Exercise 1.5.2
Review the proof of Theorem 1.5.6, part (ii) showing that $\mathbf{R}$ is uncountable, and then find the flaw in the following erroneous proof that $\mathbf{Q}$ is uncountable:

Assume, for contradiction, that $\mathbf{Q}$ is countable. Thus we can write $\mathbf{Q} = \{r_1, r_2, r_3, \ldots\}$ and, as before, construct a nested sequence of closed intervals with $r_n \notin I_n$. Our construction implies $\bigcap_{n=1}^{\infty} I_n = \emptyset$ while NIP implies $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. This contradiction implies Q must therefore be uncountable.

Solution
The nested interval property is not true for $\mathbf{Q}$. Consider I_n being rational bounds for $\sqrt{2}$ with n decimal places, then $\bigcap_{n=1}^{\infty} I_n = \emptyset$ since $\sqrt{2} \notin \mathbf{Q}$.

Exercise 1.5.3

(a) Prove if $A_1, \ldots, A_m$ are countable sets then $A_1 \cup \cdots \cup A_m$ is countable.

(b) Explain why induction *cannot* be used to prove that if each A_n is countable, then $\bigcup_{n=1}^{\infty} A_n$ is countable.

(c) Show how arranging $\mathbf{N}$ into the two-dimensional array

$$\begin{array}{llllll} 1 & 3 & 6 & 10 & 15 & \cdots \\ 2 & 5 & 9 & 14 & \cdots & \\ 4 & 8 & 13 & \cdots & & \\ 7 & 12 & \cdots & & & \\ 11 & \cdots & & & & \\ \vdots & & & & & \end{array}$$

leads to a proof for the infinite case.

Solution

(a) Let B, C be disjoint countable sets. We use the same trick as with the integers and list them as

$$B \cup C = \{b_1, c_1, b_2, c_2, \ldots\}$$

Meaning $B \cup C$ is countable, and $A_1 \cup A_2$ is also countable since we can let $B = A_1$ and $C = A_2 \setminus A_1$.

Now induction: suppose $A_1 \cup \cdots \cup A_n$ is countable, $(A_1 \cup \cdots \cup A_n) \cup A_{n+1}$ is the union of two countable sets which by above is countable.

(b) Induction shows something for each $n \in \mathbf{N}$, it does not apply in the infinite case.

(c) Rearranging $\mathbf{N}$ as in (c) gives us disjoint sets C_n such that $\bigcup_{n=1}^{\infty} C_n = \mathbf{N}$. Let B_n be disjoint, constructed as $B_1 = A_1, B_2 = A_1 \setminus B_1, \ldots$ we want to do something like

$$f(\mathbf{N}) = f\left(\bigcup_{n=1}^{\infty} C_n\right) = \bigcup_{n=1}^{\infty} f_n(C_n) = \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$$

Let $f_n : C_n \to B_n$ be bijective since B_n is countable, define $f : \mathbf{N} \to \bigcup_{n=1}^{\infty} B_n$ as

$$f(n) = \begin{cases} f_1(n) & \text{if } n \in C_1 \\ f_2(n) & \text{if } n \in C_2 \\ \vdots \end{cases}$$

(i) Since each C_n is disjoint and each f_n is 1-1, $f(n_1) = f(n_2)$ implies $n_1 = n_2$ meaning f is 1-1.

(ii) Since any $b \in \bigcup_{n=1}^{\infty} B_n$ has $b \in B_n$ for some n, we know $b = f_n(x)$ has a solution since f_n is onto. Letting $x = f_n^{-1}(b)$ we have $f(x) = f_n(x) = b$ since $f_n^{-1}(b) \in C_n$ meaning f is onto.

By (i) and (ii) f is bijective and so $\bigcup_{n=1}^{\infty} B_n$ is countable. And since

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$$

We have that $\bigcup_{n=1}^{\infty} A_n$ is countable, completing the proof.

Exercise 1.5.4

(a) Show $(a, b) \sim \mathbf{R}$ for any interval (a, b).

(b) Show that an unbounded interval like $(a, \infty) = \{x : x > a\}$ has the same cardinality as $\mathbf{R}$ as well.

(c) Using open intervals makes it more convenient to produce the required 1-1, onto functions, but it is not really necessary. Show that $[0, 1) \sim (0, 1)$ by exhibiting a 1-1 onto function between the two sets.

Solution

(a) We will start by finding $f : (-1, 1) \to \mathbf{R}$ and then transform it to (a, b). Example 1.5.4 gives a suitable f

$$f(x) = \frac{x}{x^2 - 1}$$

The book says to use calculus to show f is bijective, first we will examine the derivative

$$f'(x) = \frac{x^2 - 1 - 2x^2}{(x^2-1)^2} = -\frac{x^2+1}{(x^2-1)^2}$$

The denominator and numerator are positive, so $f'(x) < 0$ forall $x \in (0,1)$. This means no two inputs will be mapped to the same output, meaning f is one to one (a rigorous proof is beyond our current ability)

To show that f is onto, we examine the limits

$$\lim_{x\to 1^-} \frac{x}{x^2-1} = -\infty$$
$$\lim_{x\to -1^+} \frac{x}{x^2-1} = +\infty$$

Then use the intermediate value theorem to conclude f is onto.

Now we shift f to the interval (a,b)

$$g(x) = f\left(\frac{2x-1}{b-a} - a\right)$$

Proving $g(x)$ is also bijective is a straightforward computation.

(b) We want a bijective $h(x)$ such that $h(x) : (a,\infty) \to (-1,1)$ because then we could compose them to get a new bijective function $f(h(x)) : (a,\infty) \to \mathbf{R}$.

Let
$$h(x) = \frac{2}{x-a+1} - 1$$

We have $h : (a,\infty) \to (1,-1)$ since $h(a) = 1$ and $\lim_{x\to\infty} h(x) = 1$.

Meaning that $f(h(x)) : (a,\infty) \to \mathbf{R}$ is our bijective map.

(c) **TODO**

Exercise 1.5.5

(a) Why is $A \sim A$ for every set A?

(b) Given sets A and B, explain why $A \sim B$ is equivalent to asserting $B \sim A$.

(c) For three sets A, B, and C, show that $A \sim B$ and $B \sim C$ implies $A \sim C$. These three properties are what is meant by saying that $\sim$ is an *equivalence relation.*

Solution

(a) The identity function $f(x) = x$ is a bijection

(b) If $f : A \to B$ is bijective then $f^{-1} : B \to A$ is bijective.

(c) Let $f : A \to B$ and $g : B \to C$, since $g \circ f : A \to C$ is bijective we have $A \sim C$.

Exercise 1.5.6

(a) Give an example of a countable collection of disjoint open intervals.

(b) Give an example of an uncountable collection of disjoint open intervals, or argue that no such collection exists.

Solution

(a) $I_1 = (0, 1)$, $I_2 = (1, 2)$ and in general $I_n = (n - 1, n)$

(b) Let A denote this set. Intuitively no such collection should exist since each I_n has nonzero length.

The key here is to try and show $A \sim \mathbf{Q}$ instead of directly showing $A \sim \mathbf{N}$.

For any nonempty interval I_n the density theorem tells us there exists an $r \in \mathbf{Q}$ such that $r \in I_n$. Assigning each $I \in A$ a rational number $r \in I$ proves $I \sim \mathbf{Q}$ and thus I is countable.

Exercise 1.5.7

Consider the open interval $(0, 1)$, and let S be the set of points in the open unit square; that is, $S = \{(x, y) : 0 < x, y < 1\}$.

(a) Find a 1-1 function that maps $(0, 1)$ into, but not necessarily onto, S. (This is easy.)

(b) Use the fact that every real number has a decimal expansion to produce a $1-1$ function that maps S into $(0, 1)$. Discuss whether the formulated function is onto. (Keep in mind that any terminating decimal expansion such as .235 represents the same real number as .234999 . . . .)

The Schröder-Bernstein Theorem discussed in Exercise 1.5.11 can now be applied to conclude that $(0, 1) \sim S$.

Solution

(a) We scale and shift up into the square. $f(x) = \frac{1}{2}x + \frac{1}{3}$

(b) Let $g : S \to (0, 1)$ be a function that interleaves decimals in the representation without trailing nines. $g(0.32, 0.45) = 0.3425$ and $g(0.1\bar{9}, 0.2) = g(0.2, 0.2) = 0.22$ etc.

Every real number can be written with two digit representations, one with trailing 9's and one without. However $g(x, y) = 0.d_1d_2 \dots \bar{9}$ is impossible since it would imply $x = 0.d_1 \dots \bar{9}$ and $y = 0.d_2 \dots \bar{9}$ but the definition of g forbids this. Therefor $g(s)$ is unique, and so g is 1-1.

Is g onto? No since $g(x, y) = 0.1$ has no solutions, since we would want $x = 0.1$ and $y = 0$ but $0 \notin (0, 1)$.

Exercise 1.5.8

Let B be a set of positive real numbers with the property that adding together any finite subset of elements from B always gives a sum of 2 or less. Show B must be finite or countable.

Solution

Notice $B \cap (a, 2)$ is finite forall $a > 0$, since if it was infinite we could make a set with sum greater then two. And since B is the countable union of finite sets $\bigcup_{n=1}^{\infty} B \cap (1/n, 2)$, B must be countable or finite.

Exercise 1.5.9

A real number $x \in \mathbf{R}$ is called algebraic if there exist integers $a_0, a_1, a_2, \ldots, a_n \in \mathbf{Z}$, not all zero, such that

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

Said another way, a real number is algebraic if it is the root of a polynomial with integer coefficients. Real numbers that are not algebraic are called *transcendental* numbers. Reread the last paragraph of Section 1.1. The final question posed here is closely related to the question of whether or not transcendental numbers exist.

(a) Show that $\sqrt{2}$, $\sqrt[3]{2}$, and $\sqrt{3} + \sqrt{2}$ are algebraic.

(b) Fix $n \in \mathbf{N}$, and let A_n be the algebraic numbers obtained as roots of polynomials with integer coefficients that have degree n. Using the fact that every polynomial has a finite number of roots, show that A_n is countable.

(c) Now, argue that the set of all algebraic numbers is countable. What may we conclude about the set of transcendental numbers?

Solution

(a) $x^2 - 2 = 0$, $x^3 - 2 = 0$ are obvious. Now consider $\sqrt{3} + \sqrt{2}$. The key is setting $x = \sqrt{3} + \sqrt{2}$ then using algebra on x to concoct an integer, and thus find the polynomial with x as a root.

We have $x^2 = 5 + 2\sqrt{6}$ meaning $x^2 - 5 = 2\sqrt{6}$ and thus $(x^2 - 5)^2 = 24$ so $(x^2 - 5)^2 - 24 = 0$ is a polynomial with $\sqrt{3} + \sqrt{2}$ as a root.

(b) Basically $A_n \sim \mathbf{Z}^n \sim \mathbf{N}^n \sim \mathbf{N}$.

(i) $A_n \sim \mathbf{Z}^n$ since integer polynomials of degree n are identical to an ordered list of n integers.

(ii) $\mathbf{Z}^n \sim \mathbf{N}^n$ since $f : \mathbf{N}^n \to \mathbf{Z}^n$ is just the piecewise application of $g : \mathbf{N} \to \mathbf{Z}$.

(iii) $\mathbf{N}^n \sim \mathbf{N}$ since it is the intersection of finite sets $\bigcup_{n=2}^{\infty} \{(a, b) : a + b = n\}$.

In general if V is countable then $V^n = (v_1, \ldots, v_n)$ is also countable.

(c) By 1.5.3 the set of all algebraic numbers $\bigcup_{n=1}^{\infty} A_n$ is countable.

Exercise 1.5.10

(a) Let $C \subseteq [0, 1]$ be uncountable. Show that there exists $a \in (0, 1)$ such that $C \cap [a, 1]$ is uncountable.

(b) Now let A be the set of all $a \in (0, 1)$ such that $C \cap [a, 1]$ is uncountable, and set $\alpha = \sup A$. Is $C \cap [\alpha, 1]$ an uncountable set?

(c) Does the statement in (a) remain true if “uncountable” is replaced by “infinite”?

Solution

(a) **TODO**

(b) **TODO**

(c) **TODO**

Exercise 1.5.11 (Schröder-Bernstein Theorem)
Assume there exists a 1-1 function $f : X \to Y$ and another 1-1 function $g : Y \to X$. Follow the steps to show that there exists a 1-1, onto function $h : X \to Y$ and hence $X \sim Y$. The strategy is to partition X and Y into components

$$X = A \cup A' \quad \text{and} \quad Y = B \cup B'$$

with $A \cap A' = \emptyset$ and $B \cap B' = \emptyset$, in such a way that f maps A onto B, and g maps B' onto A'.

(a) Explain how achieving this would lead to a proof that $X \sim Y$.

(b) Set $A_1 = X \setminus g(Y) = \{x \in X : x \notin g(Y)\}$ (what happens if $A_1 = \emptyset$?) and inductively define a sequence of sets by letting $A_{n+1} = g(f(A_n))$. Show that $\{A_n : n \in \mathbf{N}\}$ is a pairwise disjoint collection of subsets of X, while $\{f(A_n) : n \in \mathbf{N}\}$ is a similar collection in Y.

(c) Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} f(A_n)$. Show that f maps A onto B.

(d) Let $A' = X \setminus A$ and $B' = Y \setminus B$. Show g maps B' onto A'.

Solution

(a) **TODO**

(b) **TODO**

(c) **TODO**

(d) **TODO**

1.6 Cantor's theorem

Exercise 1.6.1
Show that $(0, 1)$ is uncountable if and only if $\mathbf{R}$ is uncountable.

Solution
Exercise 1.5.4 tells us $(0, 1)$ has the same cardinality as $\mathbf{R}$.
TODO Prove without using exercise 1.5.4 (probably what was intended)

Exercise 1.6.2
(a) Explain why the real number $x = .b_1b_2b_3b_4\ldots$ cannot be $f(1)$.

(b) Now, explain why $x \neq f(2)$, and in general why $x \neq f(n)$ for any $n \in \mathbf{N}$.

(c) Point out the contradiction that arises from these observations and conclude that $(0, 1)$ is uncountable.

TODO Make question self contained

Solution
(a) The first digit is different

(b) The nth digit is different

(c) Therefor x is not in the list, since the nth digit would be different by definition

Exercise 1.6.3
Supply rebuttals to the following complaints about the proof of Theorem 1.6.1.

(a) Every rational number has a decimal expansion, so we could apply this same argument to show that the set of rational numbers between 0 and 1 is uncountable. However, because we know that any subset of $\mathbf{Q}$ must be countable, the proof of Theorem 1.6.1 must be flawed.

(b) Some numbers have two different decimal representations. Specifically, any decimal expansion that terminates can also be written with repeating 9's. For instance, $1/2$ can be written as .5 or as $.4999\ldots$ Doesn't this cause some problems?

Solution
(a) False, since the constructed number has an infinite number of decimals it is irrational.

(b) No, since if we have $9999\ldots$ and change the nth digit $9992999 = 9993$ is still different.

Exercise 1.6.4
Let S be the set consisting of all sequences of 0 's and 1 's. Observe that S is not a particular sequence, but rather a large set whose elements are sequences; namely,

$$S = \{(a_1, a_2, a_3, \ldots) : a_n = 0 \text{ or } 1\}$$

As an example, the sequence $(1, 0, 1, 0, 1, 0, 1, 0, \ldots)$ is an element of S, as is the sequence $(1, 1, 1, 1, 1, 1, \ldots)$. Give a rigorous argument showing that S is uncountable.

Solution
We flip every bit in the diagonal just like with $\mathbf{R}$. Another way would be to show $S \sim \mathbf{R}$ by writing real numbers in base 2.

Exercise 1.6.5

(a) Let $A = \{a, b, c\}$. List the eight elements of $P(A)$. (Do not forget that $\emptyset$ is considered to be a subset of every set.)

(b) If A is finite with n elements, show that $P(A)$ has 2^n elements.

Solution

(a) $A = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

(b) There are n elements, we can include or exclude each element so there are 2^n subsets.

Exercise 1.6.6

(a) Using the particular set $A = \{a, b, c\}$, exhibit two different $1-1$ mappings from A into $P(A)$.

(b) Letting $C = \{1, 2, 3, 4\}$, produce an example of a $1-1$ map $g : C \to P(C)$.

(c) Explain why, in parts (a) and (b), it is impossible to construct mappings that are onto.

Solution

(a) $f(x) = \{x\}$, $f(x) = \{x, b\}$ for $x \neq b$ and $f(x) = \{a, b, c\}$ for $x = b$.

(b) $f(x) = \{x\}$.

(c) We can hit at most n elements in the power set out of the 2^n total elements.

Theorem (Cantor's Theorem)
Given any set A, there does not exist a function $f : A \to P(A)$ that is onto.

Proof
Suppose $f : A \to P(A)$ is onto. We want to use the self referential nature of $P(A)$ to find a contradiction. Define

$$B = \{a : a \notin f(a)\}$$

Since f is onto we must have $f(a) = B$ for some $a \in A$. Then either

(i) $a \in B$ implies $a \in f(a)$ which by the definition of B implies $a \notin B$, so $a \in B$ is impossible.

(ii) $a \notin B$ implies $a \notin f(a)$ since $f(a) = B$. but if $a \notin f(a)$ then $a \in B$ by the definition of B, contradicting $a \notin B$.

Therefor f cannot be onto, since we have found a $B \in P(A)$ where $f(a) = B$ is impossible.

Stepping back, the pearl of the argument is that if $B = f(a)$ then $B = \{a : a \notin B\}$ is undecidable/impossible.

Exercise 1.6.7
See the proof of Cantor's theorem above (the rest is a computation)

Exercise 1.6.8
See the proof of Cantor's theorem above

Exercise 1.6.9
Using the various tools and techniques developed in the last two sections (including the exercises from Section 1.5), give a compelling argument showing that $P(\mathbf{N}) \sim \mathbf{R}$.

Solution
TODO

Exercise 1.6.10
As a final exercise, answer each of the following by establishing a $1-1$ correspondence with a set of known cardinality.

(a) Is the set of all functions from $\{0,1\}$ to $\mathbf{N}$ countable or uncountable?

(b) Is the set of all functions from $\mathbf{N}$ to $\{0,1\}$ countable or uncountable?

(c) Given a set B, a subset $\mathcal{A}$ of $P(B)$ is called an antichain if no element of $\mathcal{A}$ is a subset of any other element of $\mathcal{A}$. Does $P(\mathbf{N})$ contain an uncountable antichain?

Solution

(a) The set of functions from $\{0,1\}$ to $\mathbf{N}$ is the same as $\mathbf{N}^2$ which we found was countable in Exercise 1.5.9.

(b) This is the same as an infinite list of zeros and ones which we showed was uncountable in Exercise 1.6.4.

(c) **TODO**

Chapter 2

Sequences and Series

2.2 The Limit of a Sequence

Exercise 2.2.1
What happens if we reverse the order of the quantifiers in Definition 2.2.3?

Definition: A sequence (x_n) *verconges* to x if *there exists* an $\epsilon > 0$ such that *for all* $N \in \mathbf{N}$ it is true that $n \geq N$ implies $|x_n - x| < \epsilon$

Give an example of a vercongent sequence. Is there an example of a vercongent sequence that is divergent? Can a sequence verconge to two different values? What exactly is being described in this strange definition?

Solution
Firstly, since we have *for all* $N \in \mathbf{N}$ we can remove N entirely and just say $n \in \mathbf{N}$. Our new definition is

Definition: A sequence (x_n) *verconges* to x if *there exists* an $\epsilon > 0$ such that *for all* $n \in \mathbf{N}$ we have $|x_n - x| < \epsilon$.

In other words, a series (x_n) *verconges* to x if $|x_n - x|$ is bounded. This is a silly definition though since if $|x_n - x|$ is bounded, then $|x_n - x'|$ is bounded for all $x' \in \mathbf{R}$, meaning if a sequence is vercongent it verconges to every $x' \in \mathbf{R}$.

Put another way, a sequence is vercongent *if and only if* it is bounded.

Exercise 2.2.2
Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

(a) $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$.

(b) $\lim \frac{2n^2}{n^3+3} = 0$.

(c) $\lim \frac{\sin\left(n^2\right)}{\sqrt[3]{n}} = 0$.

Solution
(a) We have

$$\left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| = \left|\frac{5(2n+1) - 2(5n+4)}{5(5n+4)}\right| = \left|\frac{-3}{5(5n+4)}\right| = \frac{3}{5(5n+4)} < \epsilon$$

We now find n such that the distance is less then ϵ

$$\frac{3}{5(5n+4)} < \frac{1}{n} < \epsilon \implies n > \frac{1}{\epsilon}$$

You could also solve for the smallest n, which would give you

$$\frac{3}{5(5n+4)} < \epsilon \implies 5n+4 > \frac{3}{5\epsilon} \implies n > \frac{3}{25\epsilon} - \frac{4}{5}$$

I prefer the first approach, the second is better if you were doing numerical analysis and wanted a precise convergence rate.

(b) We have

$$\left|\frac{2n^2}{n^3+3} - 0\right| = \frac{2n^2}{n^3+3} < \frac{2n^2}{n^3} = \frac{2}{n} < \epsilon \implies n > \frac{2}{\epsilon}$$

(c) We have

$$\frac{\sin(n^2)}{n^{1/3}} \le \frac{1}{n^{1/3}} < \epsilon \implies n > \frac{1}{\epsilon^3}$$

Really slow convergence! if $\epsilon = 10^{-2}$ we would require $n > 10^6$

Exercise 2.2.3

Describe what we would have to demonstrate in order to disprove each of the following statements.

(a) At every college in the United States, there is a student who is at least seven feet tall.

(b) For all colleges in the United States, there exists a professor who gives every student a grade of either A or B.

(c) There exists a college in the United States where every student is at least six feet tall.

Solution

(a) Find a collage in the United States with no students over seven feet tall.

(b) Find a collage in the United States with a professor who has given a grade other then an A or B.

(c) Find a collage in the united States with at least one student under six feet tall.

Exercise 2.2.4

Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

(a) A sequence with an infinite number of ones that does not converge to one.

(b) A sequence with an infinite number of ones that converges to a limit not equal to one.

(c) A divergent sequence such that for every $n \in \mathbf{N}$ it is possible to find n consecutive ones somewhere in the sequence.

Solution

(a) $a_n = (-1)^n$

(b) Impossible, if $\lim a_n = a \neq 1$ then for any $n \geq N$ we can find a n with $a_n = 1$ meaning $\epsilon < |1 - a|$ is impossible.

(c) $a_n = (1, 2, 1, 1, 3, 1, 1, 1, \dots)$

Exercise 2.2.5

Let $[[x]]$ be the greatest integer less than or equal to x. For example, $[[\pi]] = 3$ and $[[3]] = 3$. For each sequence, find $\lim a_n$ and verify it with the definition of convergence.

(a) $a_n = [[5/n]]$,

(b) $a_n = [[(12 + 4n)/3n]]$.

Reflecting on these examples, comment on the statement following Definition 2.2.3 that "the smaller the ϵ-neighborhood, the larger N may have to be."

Solution

(a) For all $n > 5$ we have $[[5/n]] = 0$ meaning $\lim a_n = 0$.

(b) The inside clearly converges to $4/3$ from above, so $\lim a_n = 1$.

Some sequences eventually reach their limit, meaning N no longer has to increase.

Exercise 2.2.6

Theorem 2.2.7 (Uniqueness of Limits). *The limit of a sequence, when it exists, must be unique.*

Prove Theorem 2.2.7. To get started, assume $(a_n) \to a$ and also that $(a_n) \to b$. Now argue $a = b$

Solution

If $a \neq b$ then we can set ϵ small enough that having both $|a_n - a| < \epsilon$ and $|a_n - b| < \epsilon$ is impossible. Therefor $a = b$.

(Making this rigorous is trivial and left as an exercise to the reader)

Exercise 2.2.7

Here are two useful definitions:

(i) A sequence (a_n) is *eventually* in a set $A \subseteq \mathbf{R}$ if there exists an $N \in \mathbf{N}$ such that $a_n \in A$ for all $n \geq N$.

(ii) A sequence (a_n) is *frequently* in a set $A \subseteq \mathbf{R}$ if, for every $N \in \mathbf{N}$, there exists an $n \geq N$ such that $a_n \in A$.

   (a) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?

   (b) Which definition is stronger? Does frequently imply eventually or does eventually imply frequently?

   (c) Give an alternate rephrasing of Definition 2.2.3B using either frequently or eventually. Which is the term we want?

(d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2 . Is (x_n) necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?

Solution

(a) Frequently, but not eventually.

(b) Eventually is stronger, it implies frequently.

(c) $(x_n) \to x$ *if and only if* x_n is eventually in any ϵ-neighborhood around x.

(d) (x_n) is frequently in $(1.9, 2.1)$ but not necessarily eventually (consider $x_n = 2(-1)^n$).

Exercise 2.2.8

For some additional practice with nested quantifiers, consider the following invented definition:

Let's call a sequence (x_n) zero-heavy if there exists $M \in \mathbf{N}$ such that for all $N \in \mathbf{N}$ there exists n satisfying $N \leq n \leq N + M$ where $x_n = 0$

(a) Is the sequence $(0, 1, 0, 1, 0, 1, \ldots)$ zero heavy?

(b) If a sequence is zero-heavy does it necessarily contain an infinite number of zeros? If not, provide a counterexample.

(c) If a sequence contains an infinite number of zeros, is it necessarily zeroheavy? If not, provide a counterexample.

(d) Form the logical negation of the above definition. That is, complete the sentence: A sequence is not zero-heavy if ....

Solution

(a) No.

(b) Yes. as any finite number of zeros K would lead to a contradiction when $M > K$.

(c) No, consider $(0, 1, 0, \ldots)$ from (a).

(d) A sequence is not zero-heavy if there exists an $M \in \mathbf{N}$ such that for all $N \in \mathbf{N}$ there exists an $n \in \mathbf{N}$ such that $N \leq n \leq N + M$ but $x_n \neq 0$.

2.3 The Algebraic and Order Limit Theorems

Exercise 2.3.1
Let $x_n \geq 0$ for all $n \in \mathbf{N}$.

(a) If $(x_n) \to 0$, show that $\left(\sqrt{x_n}\right) \to 0$.

(b) If $(x_n) \to x$, show that $\left(\sqrt{x_n}\right) \to \sqrt{x}$.

Solution

(a) Setting $x_n < \epsilon^2$ implies $\sqrt{x_n} < \epsilon$ (for all $n \geq N$ of course)

(b) We want $|\sqrt{x_n} - \sqrt{x}| < \epsilon$ multiplying by $(\sqrt{x_n} + \sqrt{x})$ gives $|x_n - x| < (\sqrt{x_n} + \sqrt{x})\epsilon$ since x_n is convergent, it is bounded $|x_n| \leq M$ implying $\sqrt{|x_n|} \leq \sqrt{M}$, multiplying gives

$$|x_n - x| < \left(\sqrt{x_n} + \sqrt{x}\right)\epsilon \leq \left(\sqrt{M} + \sqrt{x}\right)\epsilon$$

Since $|x_n - x|$ can be made arbitrarily small we can make this true for some $n \geq N$. Now dividing by $\sqrt{M} + \sqrt{x}$ gives us

$$|\sqrt{x_n} - \sqrt{x}| \leq \frac{|x_n - x|}{\sqrt{M} + \sqrt{x}} < \epsilon$$

Therefor $|\sqrt{x_n} - \sqrt{x}| < \epsilon$ completing the proof.

Exercise 2.3.2
Using only Definition 2.2.3, prove that if $(x_n) \to 2$, then

(a) $\left(\frac{2x_n - 1}{3}\right) \to 1$;

(b) $(1/x_n) \to 1/2$.

(For this exercise the Algebraic Limit Theorem is off-limits, so to speak.)

Solution

(a) We have $\left|\frac{2}{3}x_n - \frac{4}{3}\right| = \frac{2}{3}|x_n - 2| < \epsilon$ which can always be done since $|x_n - 2|$ can be made arbitrarily small.

(b) Want $|(1/x_n) - 1/2| < \epsilon$ have $|x_n - 2| < \epsilon$ **TODO**

Exercise 2.3.3 (Squeeze Theorem)
Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbf{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.

Solution
Let $y = \lim y_n$. By the order limit theorem we have $l \leq y \leq l$ implying $y = l$.

Exercise 2.3.4
Let $(a_n) \to 0$, and use the Algebraic Limit Theorem to compute each of the following limits (assuming the fractions are always defined):

(a) $\lim \left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right)$

(b) $\lim \left(\frac{(a_n+2)^2-4}{a_n}\right)$

(c) $\lim \left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right)$.

Solution

(a) I'm not sure how much work I have to show, many of these steps are obvious

$$\begin{aligned}\lim \left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right) &= \lim \left(\frac{1}{1+3a_n-4a_n^2}\right) + 2\lim \left(\frac{a_n}{1+3a_n-4a_n^2}\right) \\ &= 2\lim \left(\frac{1}{1/a_n+3-4a_n}\right) \\ &= 0\end{aligned}$$

TODO Show this more rigorously

(b)

$$\lim \left(\frac{(a_n+2)^2-4}{a_n}\right) = \lim \left(\frac{a_n^2+2a_n}{a_n}\right) = \lim (a_n+2) = \infty$$

(c) This one is a straightforward application of the algebraic limit theorem

$$\lim \left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right) = 3/5$$