

## Supplemental Lecture 4

### Surfaces of Zero, Positive and Negative Gaussian Curvature.

### Euclidean, Spherical and Hyperbolic Geometry.

#### Abstract

This lecture considers two dimensional surfaces embedded in three dimensional Euclidean space. The lecture begins by introducing familiar surfaces of revolution which are generated by rotating a profile curve around the  $z$  axis. Their various parametrizations, metrics and intrinsic properties such as their Gaussian curvatures are studied. Euclidean, spherical and hyperbolic geometries are introduced, contrasted and analyzed. In both spherical and hyperbolic geometries the “Parallel Axiom” of two dimensional Euclidean space is untrue. In both spherical and hyperbolic geometries their non-zero intrinsic curvatures  $K$  sets a fundamental length scale. For distances small compared to  $1/\sqrt{K}$  the curved spaces are well approximated by a flat Euclidean metric but this is untrue for distances large compared to  $1/\sqrt{K}$ . In addition, unlike Euclidean space, geodesic triangles in spherical and hyperbolic geometries which are similar are also congruent. Both the Poincare Disc and Upper-Half-Plane models of spaces of constant negative geometry are presented and studied. Their isometries, geodesics and hyperbolic triangles are introduced and analyzed.

This lecture supplements material in the textbook: Special Relativity, Electrodynamics and General Relativity: From Newton to Einstein (ISBN: 978-0-12-813720-8). The term “textbook” in these Supplemental Lectures will refer to that work.

Keywords: Non-Euclidean geometry, Poincare disc, H model, surfaces of revolution, metric, Gaussian curvature, projective geometry, geodesic triangles, isometries, Möbius transformations

## Introduction

In this lecture we look in more detail at a few topics touched upon in the textbook in its chapter on differential geometry. Although the book's discussion of general relativity uses Riemannian geometry almost exclusively, we use classical differential geometry to expose the ideas underlying more modern methods. Gaussian approaches which are visual and are based on multivariable calculus are a good match to the intended audience for the textbook, mathematics and physics undergraduates.

We begin by introducing surfaces of revolution and studying those of vanishing Gaussian curvature  $K$ , those of positive  $K$  and those of negative  $K$ . We are particularly interested in spaces of constant  $K$  where we can speak about the global geometry. The “straight” lines in these spaces (geodesics) are obtained and triangles are constructed and their properties are compared. The emphasis is on the intrinsic properties of each space so these ideas have their analogues in Riemannian geometry.

The lecture emphasizes the differences between Euclidean, spherical and hyperbolic geometries. In both spherical and hyperbolic geometries the “Parallel Axiom” of two dimensional Euclidean space is untrue. In both spherical and hyperbolic geometries their non-zero intrinsic curvatures set a fundamental length scale which is absent in Euclidean space. The Gauss-Bonnet Theorem shows that the area of a geodesic triangle in both spherical and hyperbolic geometries is determined by the deviation of the sum of their internal angles from  $\pi$ . Triangles which are similar in these spaces are also congruent. This is not true in Euclidean space where the vanishing of its intrinsic curvature leads to the fact that the sum of the internal angles in a triangle is always  $\pi$  and triangles of the same internal angles come in all sizes. In curved spaces the curvature sets a scale of length: for distances small compared to  $1/\sqrt{K}$  the space “looks” Euclidean. For larger distances compared to  $1/\sqrt{K}$  this is no longer true. The Gauss-Bonnet Theorem illustrates this point.

The hyperbolic geometries of the Poincare disc “D” model and the upper-half-plane “H” models are studied. These two spaces are isometric, and can be analyzed using the methods of complex analysis.

## Surfaces of Revolution

A surface of revolution is constructed by rotating a curve around a fixed axis. Let the curve lie in the  $x$ - $z$  plane and let the rotation axis be the  $z$ -axis, as shown in Fig. 1.

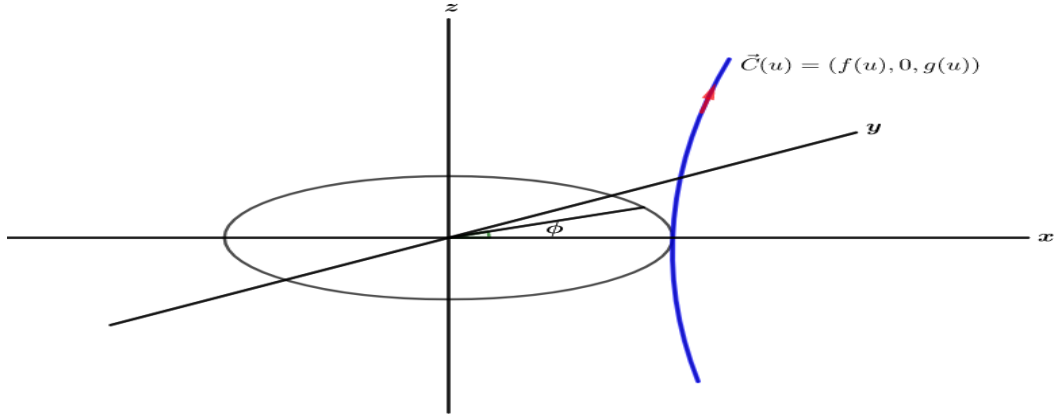


Fig. 1 Rotating a profile curve around the  $z$  axis

The curve  $\vec{C}(u)$  lies in the  $x$ - $z$  plane and is parametrized by the variable  $u$ . In terms of components  $\vec{C}(u) = (f(u), 0, g(u))$  where the functions  $f$  and  $g$  will be chosen for each application. In order to rotate the curve around the  $z$  axis, we apply a rotation matrix of angle  $\varphi$  as shown in Fig. 1.

The rotation matrix reads,

$$\begin{pmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

To check that this operator implements a rotation about the  $z$  axis, apply it to a unit vector in the  $x$  direction,

$$\begin{pmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos \varphi \\ \sin \varphi \\ 0 \end{pmatrix}$$

And similarly apply it to a unit vector in the  $y$  direction,

$$\begin{pmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\sin \varphi \\ \cos \varphi \\ 0 \end{pmatrix}$$

This checks: the vectors are rotated appropriately.

Now apply the rotation to  $\vec{C}(u)$ ,

$$\begin{pmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f(u) \\ 0 \\ g(u) \end{pmatrix} = \begin{pmatrix} f(u)\cos \varphi \\ f(u)\sin \varphi \\ g(u) \end{pmatrix} \quad (1)$$

So, the surface of revolution is given by,

$$\vec{r}(u, \varphi) = (f(u)\cos \varphi, f(u)\sin \varphi, g(u)) \quad (2)$$

In many applications we take  $f(u) > 0$  and we take the profile curve  $\vec{C}(u) = (f(u), 0, g(u))$  to have unit speed, in the language of Chapter 12 of the textbook,

$$\dot{f}^2 + \dot{g}^2 = 1 \quad (3)$$

where the dot indicates  $d/du$ . The parameter  $u$  might be the arc length of the curve, for example.

Now we can calculate the tangent vectors to the surface at any point  $(u, \varphi)$ , visualize the tangent plane there, and calculate the metric and the intrinsic curvature. Following the notation used in the textbook where a subscript  $u$  means  $d/du$  and a subscript  $\varphi$  means  $d/d\varphi$ , we find from Eq. 2,

$$\vec{r}_u = (\dot{f} \cos \varphi, \dot{f} \sin \varphi, \dot{g}), \quad \vec{r}_\varphi = (-f \sin \varphi, f \cos \varphi, 0) \quad (4)$$

Now we can calculate the coefficients of the metric,

$$E = \vec{r}_u \cdot \vec{r}_u = \dot{f}^2 + \dot{g}^2 = 1, \quad F = \vec{r}_u \cdot \vec{r}_\varphi = 0, \quad G = \vec{r}_\varphi \cdot \vec{r}_\varphi = f^2 \quad (5)$$

So the metric reads,

$$ds^2 = du^2 + f^2(u)d\varphi^2 \quad (6)$$

We can also calculate the Gaussian curvature from the metric. Recall the more general formula for any metric with non-vanishing  $E$  and  $G$ ,

$$K = -\frac{1}{\sqrt{EG}} \left[ \frac{\partial}{\partial u} \left( \frac{1}{\sqrt{E}} \frac{\partial \sqrt{G}}{\partial u} \right) + \frac{\partial}{\partial v} \left( \frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial v} \right) \right] \quad (7)$$

where  $\vec{r}(u, v)$  sweeps out the surface in the parametrization  $(u, v)$ . Recall that Eq. 7 is more general than the present discussion of surfaces of revolution, but it applies here taking  $(u, v) \rightarrow (u, \varphi)$ . Substituting from Eq. 5,  $E = \vec{r}_u \cdot \vec{r}_u = \dot{f}^2 + \dot{g}^2$  and  $G = \vec{r}_v \cdot \vec{r}_v = f^2$ , some algebra gives,

$$K = \frac{(\dot{f}\ddot{g} - \ddot{f}\dot{g})\dot{g}}{f(\dot{f}^2 + \dot{g}^2)^2} \quad (8)$$

This result simplifies if the profile curve  $\vec{C}(u) = (f(u), 0, g(u))$  has unit speed,  $\dot{f}^2 + \dot{g}^2 = 1$ .

Differentiating Eq. 3 with respect to  $u$ , we learn that  $\dot{f}\ddot{f} + \dot{g}\ddot{g} = 0$ , which implies

$(\dot{f}\ddot{g} - \ddot{f}\dot{g})\dot{g} = -\ddot{f}(\dot{f}^2 + \dot{g}^2)$ . Substituting Eq. 3 and this result into Eq.8, we find the handy result,

$$K = -\ddot{f}/f \quad (9)$$

Now let's apply these results to surfaces which are flat, those with positive curvature and, finally, those with negative curvature. Examples of such surfaces embedded in three dimensional Euclidean space are shown in Fig. 2.

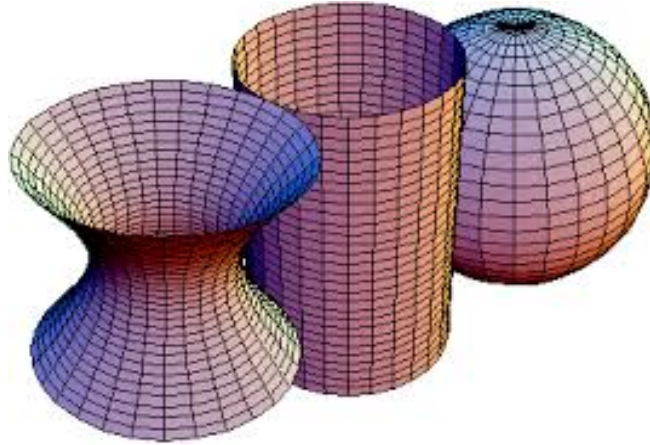


Fig. 2 A hyperboloid of rotation ( $K < 0$ ), cylinder ( $K = 0$ ), and sphere ( $K > 0$ ).

A cylinder of fixed radius  $R$  is described by the surface  $\vec{r}(z, \varphi) = (R \cos \varphi, R \sin \varphi, z)$ .

From Eq. 6 we learn that its metric is  $ds^2 = R^2 d\varphi^2 + dz^2$  which is the same as a flat two dimensional Euclidean plane, so its curvature vanishes,  $K = 0$ , as Eq. 9 predicts.

Similarly a cone with vertex at the origin is described by  $\vec{r}(z, \varphi) = (z \cos \varphi, z \sin \varphi, z)$  for  $z > 0$ . It has a metric  $ds^2 = z^2 d\varphi^2 + dz^2$  and we find  $K = 0$  again. In this case we chose the profile curve to be  $\vec{C}(u) = (z, 0, z)$ , a 45 degree line in the  $x$ - $z$  plane.

Recall from our discussions in Chapter 12 of the textbook that  $K = 0$  is expected in the cases of cylinders and cones because both surfaces can be unrolled and laid flat on a two dimensional plane: they are isometric to two dimensional Euclidean space.

Next consider a sphere of unit radius. We will rotate a semicircle around the  $z$  axis,  $\vec{C}(\theta) = (\sin \theta, 0, \cos \theta)$  for  $0 \leq \theta < \pi$ . Rotating the curve around the  $z$  axis produces the surface of the sphere,

$$\vec{r}(\theta, \varphi) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \quad (10a)$$

This produces a standard parametrization of the surface in polar coordinates shown in Fig. 3.

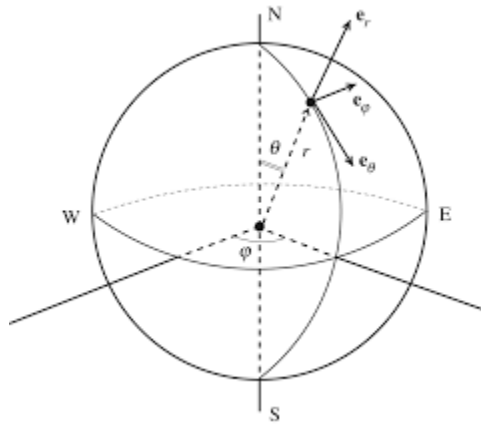


Fig. 3 A sphere with polar coordinates and unit vectors.

The sphere's metric is

$$ds^2 = d\theta^2 + \sin^2 \theta d\varphi^2 \quad (10b)$$

Since the profile curve has unit speed, the Gaussian curvature of the sphere is given by Eq. 9, which produces the expected result,  $K = 1$ , for a sphere of radius one.

Next consider a torus. The profile curve is a circle of radius one in the  $x$ - $z$  plane centered at  $x = 2$  and  $z = 0$ ,  $\vec{C}(\theta) = (2 + \cos \theta, 0, \sin \theta)$  as shown in Fig. 4.

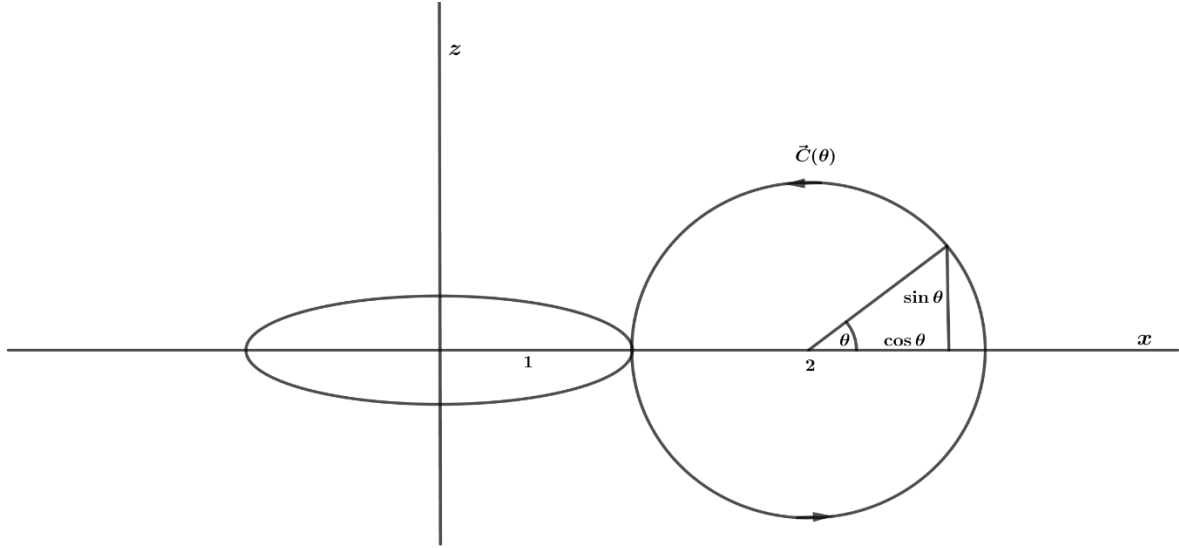


Fig. 4 The profile curve for a torus.

Then the surface of the torus becomes,

$$\vec{r}(\theta, \varphi) = ((2 + \cos \theta) \cos \varphi, (2 + \cos \theta) \sin \varphi, \sin \theta) \quad (11)$$

From this expression we can calculate the metric and Gaussian curvature. They are non-trivial functions of the angular coordinates. It is interesting to note, however, that the Gaussian curvature of the torus is positive on its outer surface and negative on its inner surface as shown in Fig. 5.

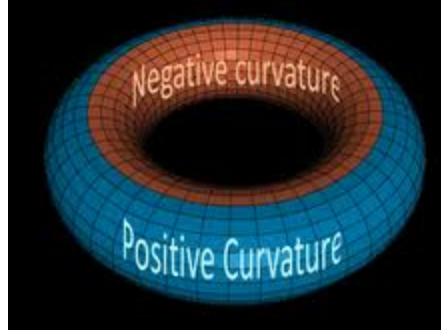


Fig. 5 A torus showing regions of positive and negative curvature.

These signs are easy to understand. Recall from Chapter 12 of the textbook that the Gaussian curvature at a point on a surface is the product of the principal curvatures at that point. For a point on the outer surface both principal curvatures clearly have the same sign while on the inner surface they have opposite signs. Using this observation it is easy to calculate the Gaussian at certain inner and outer points without even using the formulas above.

Next let's produce a surface of revolution with a constant negative Gaussian curvature  $K = -1$ . We expect the surface to resemble a saddle, where one principal curvature is positive and the perpendicular one is negative. Substituting into Eq. 9, we must solve the differential equation,

$$\frac{d^2 f}{du^2} - f = 0 \quad (12)$$

This is a familiar differential equation which is solved with exponentials,  $f(u) = ae^u + be^{-u}$ .

Let's pursue the choice  $a = 1, b = 0$ , so  $f(u) = e^u$ .

Next we can determine  $g(u)$  from the unit speed condition,  $\dot{f}^2 + \dot{g}^2 = 1$ , Eq. 3. Using integral tables to do the indefinite integral,

$$g(u) = \int \sqrt{1 - e^{2u}} \, du = \sqrt{1 - e^{2u}} - \ln(e^{-u} + \sqrt{e^{-2u} - 1}) \quad (13)$$

where the range of  $u$  must be restricted  $u \leq 0$  in order to achieve a meaningful result. This curve



is called the “tractrix”, a famous curve in the history of mathematics and differential geometry.

Finally, let’s switch to “standard” notation,  $x = f(u)$ ,  $z = g(u)$  and recall the hyperbolic identity,  $\cosh^{-1} v = \ln(v + \sqrt{v^2 - 1})$ , so that Eq. 13 becomes,

$$z = \sqrt{1 - x^2} - \cosh^{-1}(1/x) \quad (14)$$

Rotating the profile curve around the z-axis produces the “pseudo-sphere” shown in Fig. 6. Note that since  $x = e^u$  and  $u \leq 0$ , the range of  $x$  is  $0 < x \leq 1$ . Our resulting surface of constant negative curvature has a sharp edge at  $x = 1$ , as shown in Fig. 6.

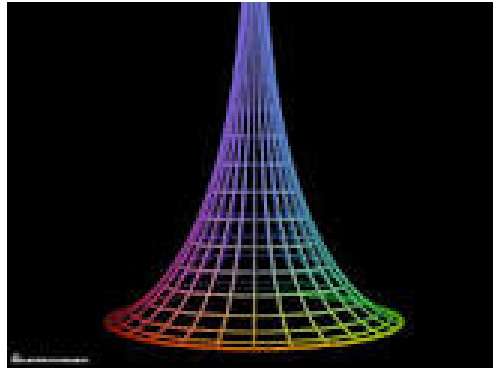


Fig. 6 The Pseudosphere

Additional properties of this surface of constant negative curvature can be found in the References [1-3]. It is curious that even though the surface has an infinite extent, the pseudosphere’s volume is finite ( $\pi/3$ ) as is its surface area ( $2\pi$ ).

When we turn to complex variables in two dimensions we will make more tractable spaces of constant negative curvature.

## Geometries: Euclidean ( $K=0$ ), Spherical ( $K=1$ ), Hyperbolic ( $K=-1$ )

### a. Euclidean Geometry

We assume that the student has a solid background in two dimensional Euclidean geometry. Our purpose here is to state a few facts that distinguish it from the other geometries, spherical and hyperbolic. A crucial issue concerns parallel lines. In 2-D Euclidean space there is one line through a given point  $P$  outside a given line  $l$  that never intersects  $l$ . In a spherical space the analogue of straight lines become great circles and any two great circles on a sphere intersect exactly twice. In hyperbolic spaces the analogues of straight lines need never intersect.

Recall other facts about Euclidean space: In Euclidean space the shortest distance between two points is a straight line, the Pythagorean Theorem holds for right triangles, and the sum of the internal angles ( $\alpha_1$ ,  $\alpha_2$  and  $\alpha_3$ ) of any triangle is  $\pi$  radians. From the wider perspective of curved spaces, this last point is a consequence of the Gauss-Bonnet Theorem,

$$\oint K dA + \int \kappa_g(s) ds = \alpha_1 + \alpha_2 + \alpha_3 - \pi \quad (15)$$

Since  $K = 0$  for Euclidean space and  $\kappa_g = 0$  for straight lines, we have the expected result: The sum of the internal angles in a Euclidean triangle is  $\pi$  radians, a results which can, of course, be obtained by much (!) more elementary methods. However, the point we want to make here is that if  $K = 0$ , *there is no intrinsic length scale in the theory*, so the area of the triangle does not appear on the left hand side of the Gauss-Bonnet Theorem, Eq. 15. Therefore, the size of the triangle *decouples* from the sum of the internal angles within it. This implies that triangles of different sizes can have the same internal angles: in Euclidean space similar triangles need not be congruent. We will see that in spherical and hyperbolic geometries the internal angles of a triangle determine the lengths of its sides so similar triangles are always congruent!

In order to decide whether two triangles are congruent, we must move them until they are on top of one another to compare the lengths of their sides. The rigid motions we employ are the global isometries of Euclidean space. These consist of translations, rotations and reflections. When we come to study spherical and hyperbolic geometries one of our earliest tasks will be the enumeration of each space's isometries. More later.

## b. Spherical Geometry

Suppose you want to make a map of the earth's northern hemisphere. One way is to project each point stereographically onto the  $u$ - $v$  plane, as shown in Fig. 7.

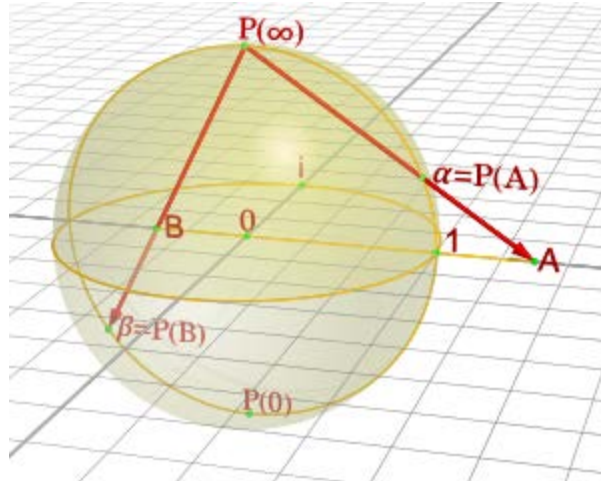


Fig. 7 The stereographic projection of the upper hemisphere onto the equatorial plane.

Here the  $u$ - $v$  plane passes through the equator and the projection of each point  $(x_1, x_2, x_3)$  produces a point  $(u, v)$  on the plane. In terms of polar angles the point  $(x_1, x_2, x_3)$  is parametrized as,

$$x_1 = \sin \theta \cos \varphi, \quad x_2 = \sin \theta \sin \varphi, \quad x_3 = \cos \theta$$

and we choose the sphere to have unit radius,  $x_1^2 + x_2^2 + x_3^2 = 1$ .

It is also convenient to use complex notation for the  $(u, v)$  plane,  $z = u + iv$ . The projection from the north pole to the  $(u, v)$  plane is shown in Fig. 8.

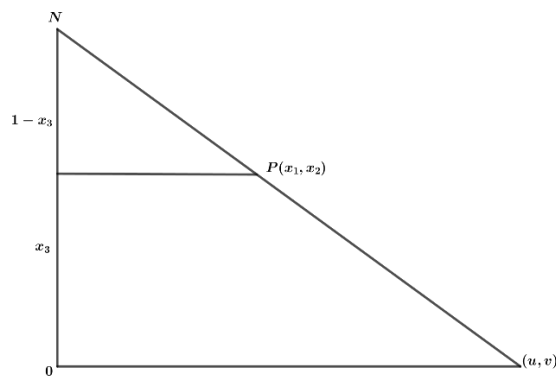


Fig. 8 The geometry of the stereographic projection

We read off Fig. 8 that,

$$z = u + iv = \frac{x_1 + ix_2}{1 - x_3} \quad (16)$$

We can solve this equation for  $(x_1, x_2, x_3)$  in terms of  $(u, v)$  subject to the constraint  $x_1^2 + x_2^2 + x_3^2 = 1$ ,

$$x_1 = \frac{2u}{1+u^2+v^2}, \quad x_2 = \frac{2v}{1+u^2+v^2}, \quad x_3 = \frac{u^2+v^2-1}{1+u^2+v^2} \quad (17)$$

Here  $x_3 \geq 0$  covers the hemisphere and constrains  $u^2 + v^2 \geq 1$ . The resulting parametrization of the surface of the sphere reads,

$$\vec{r}(u, v) = \frac{1}{1+u^2+v^2} (2u, 2v, u^2 + v^2 - 1) \quad (18)$$

Next we calculate the metric of the sphere in terms of these new variables. We already know the metric of the sphere when written in terms of polar angles, Eq. 10a, b. First we need the tangent plane at  $(u, v)$ ,

$$\begin{aligned} \partial \vec{r} / \partial u = \vec{r}_u &= \frac{2}{(1+u^2+v^2)^2} (-u^2 + v^2 + 1, -2uv, -2u) \\ \partial \vec{r} / \partial v = \vec{r}_v &= \frac{2}{(1+u^2+v^2)^2} (-2uv, u^2 - v^2 + 1, -2v) \end{aligned}$$

Now we can calculate the coefficients of the metric,

$$\vec{r}_u \cdot \vec{r}_u = \frac{4}{(1+u^2+v^2)^2}, \quad \vec{r}_u \cdot \vec{r}_v = 0, \quad \vec{r}_v \cdot \vec{r}_v = \frac{4}{(1+u^2+v^2)^2} \quad (19)$$

which produces the result,

$$ds^2 = \frac{4}{(1+u^2+v^2)^2} (du^2 + dv^2) \quad (20)$$

Comparing to Eq.10b, we have several different parametrizations of the metric of the sphere,

$$ds^2 = d\theta^2 + \sin^2 \theta d\varphi^2 = \frac{4}{(1+u^2+v^2)^2} (du^2 + dv^2) = \frac{4dzd\bar{z}}{(1+z\bar{z})^2}$$

We learn the surprising fact that the metric in variables  $(u, v)$  is conformal to a flat two dimensional Euclidean space. Recall from Chapter 12 of the textbook, that “conformal” means that the differentials  $du^2$  and  $dv^2$  have the same pre-factor. This implies that lines on the sphere intersect with the same angles as their image lines on the  $(u, v)$  plane. This is a useful fact in map making. However, the metric is a non-trivial function of  $(u, v)$  which means that areas on the hemisphere are distorted depending on their positions: near the equator areas are scaled by a factor of  $4/(1 + u^2 + v^2)^2$  which is near unity there, but areas near the north pole are magnified significantly. In summary, stereographic projections are angle-preserving (conformal) but not length or area preserving (not isometries).

Let’s check that our conformal metric preserves the curvature of the sphere,  $K = 1$ . We begin with Eq. 7,

$$K = -\frac{1}{\sqrt{EG}} \left[ \frac{\partial}{\partial u} \left( \frac{1}{\sqrt{E}} \frac{\partial \sqrt{G}}{\partial u} \right) + \frac{\partial}{\partial v} \left( \frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial v} \right) \right]$$

and here  $E = G = \frac{4}{(1+u^2+v^2)^2}$ . The calculation, which the student is encouraged to check, produces  $K = 1$  as it must since  $(u, v)$  provides just a different parametrization of the sphere  $(\theta, \varphi)$ .

### c. Hyperbolic Geometry

The pseudosphere turned out to have a “sharp edge” at  $z = 1$ . This limitation of the model is a manifestation of a theorem due to David Hilbert in 1901 that states that one cannot embed a surface of constant negative curvature in three dimensional Euclidean space in a complete (has no edges) and regular fashion [1-3]. Luckily, other approaches to construct negative curvature spaces exist and prove to be very instructive. They have a great deal in common with the “projective plane” approach to spherical geometry, which we reviewed above.

H. Poincare is credited with the disc model (D model) of complex variables. Begin with the conformal metric for the sphere, Eq. 17 and Eq. 20, but restrict  $u^2 + v^2 < 1$  and change a crucial sign so that the new model has constant negative Gaussian curvature,  $K = -1$ ,

$$ds^2 = \frac{4}{(1 - u^2 - v^2)^2} (du^2 + dv^2)$$

Substituting this metric into Eq. 7 for the Gaussian curvature returns the result  $K = -1$ , as the student is also encouraged to check.

If we rewrite this model in complex number notation then we can use elementary complex variable theory to learn about it with less toil, i.e. without “reinventing the wheel”. Let  $z = u + iv$  and consider the complex disc  $|z| < 1$ . Then,

$$ds^2 = \frac{4(du^2 + dv^2)}{(1 - u^2 - v^2)^2} = \frac{4dzd\bar{z}}{(1 - z\bar{z})^2} \quad (21)$$

We will see below that although the Euclidean radius of the disc is 1, its extent according to the hyperbolic metric, Eq. 21, is actually infinite.

Another handy and equivalent representation of this geometry is the upper half complex plane (H Model). Consider the complex plane labeled with variables  $H = J + iL$  and map the disc to the upper half plane with the linear fractional transformation (Möbius transformation),

$$H = J + iL = \frac{i(1+z)}{1-z} \quad (22a)$$

We can also calculate the inverse – the mapping from the H model to the D model,

$$z = \frac{H-i}{H+i} \quad (22b)$$

Here we see that the boundary of the disc D is mapped to the real axis of H. This follows from Eq. 22b because if H lies on the real axis, then it is equidistant from  $+i$  and  $-i$ , so  $|z| = 1$ . Also, as H ranges along the real axis from  $-\infty$  to  $+\infty$ , the phase of z ranges from 0 to  $2\pi$ , covering the boundary of D. Note also that  $H = +i$  maps to  $z = 0$ . This guarantees that points in the interior of H map onto interior points in D [4].

The mapping Eq. 22a and 22b relates the metrics in the two spaces. Given Eq. 22a let's find the induced metric in H. We need the quantities  $dz$ ,  $d\bar{z}$  and  $z\bar{z}$  in terms of H. First from Eq. 22b,

$$\frac{dz}{dH} = \frac{2i}{(H+i)^2}$$

and, doing some straight-forward algebra,

$$1 - z\bar{z} = 1 - \frac{|H-i|^2}{|H+i|^2} = \frac{4\text{Im}H}{|H+i|^2}$$

Finally,

$$dzd\bar{z} = \frac{dz}{dH} \frac{d\bar{z}}{d\bar{H}} dHd\bar{H} = + \frac{4}{|H+i|^4} dHd\bar{H}$$

Collecting everything, Eq. 21 gives,

$$ds^2 = \frac{4dzd\bar{z}}{(1-z\bar{z})^2} = \frac{4 \cdot 4}{|H+i|^4} \frac{|H+i|^4}{4 \cdot 4 \text{Im}H} dHd\bar{H} = \frac{dHd\bar{H}}{(\text{Im}H)^2} \quad (23)$$

In term of  $u$  and  $v$  ( $z = u + iv$ ) and  $J$  and  $L$  ( $H = J + iL$ ),

$$\frac{4(du^2+dv^2)}{(1-u^2-v^2)^2} = \frac{dJ^2+dL^2}{L^2} \quad (24)$$

where  $u$  and  $v$  are confined to  $D$  ( $u^2 + v^2 < 1$ ) and  $J$  and  $L$  are confined to  $H$  ( $L > 0$ ).

Note that both metrics are conformal to the flat Euclidean plane. The mapping between  $D$  and  $H$  is an isometry by construction so it preserves the hyperbolic length and the angles between them. (Any map represented by an analytic function has this last property [4].)

It is instructive to check that the Gaussian curvature of the  $D$  or  $H$  model is  $-1$ . For the  $H$  model the metric coefficients are,

$$E = L^{-2}, \quad F = 0, \quad G = L^{-2}$$

So, from Eq. 7,

$$K = -\frac{1}{\sqrt{EG}} \left[ \frac{\partial}{\partial J} \left( \frac{1}{\sqrt{E}} \frac{\partial \sqrt{G}}{\partial J} \right) + \frac{\partial}{\partial L} \left( \frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial L} \right) \right] = -L^{-2} \frac{\partial}{\partial L} \left( L \frac{\partial L^{-1}}{\partial L} \right)$$

$$K = -L^{-2} \frac{\partial}{\partial L} (-L^{-1}) = -\frac{L^2}{L^2} = -1$$

as claimed.

To understand the geometry of the D or H models we need to understand how to move figures around while leaving their shapes and sizes, as measured by the hyperbolic metric Eq. 24, unchanged. In Euclidean space we can translate, rotate and reflect objects and preserve their lengths and angles. On the surface of a sphere we can rotate and reflect objects. In a rotation we are effectively sliding the object around the surface of constant curvature. What are the isometries of H? The non-trivial form of the metric Eq. 24 presents a challenge. Consider a translation of  $H = J + iL$  at constant  $ImH = L$ ,  $H \rightarrow H + a$ , where  $a$  is a real number. This is certainly an isometry since  $ImH$  is held constant. It also maps the upper half plane onto itself. Next, a dilation  $H \rightarrow aH$  is also an isometry if  $a$  is a real constant. This is so because it leaves the metric unchanged and such dilations leave the point in the upper half plane if  $a$  is real and positive. Finally, there are inversions,  $H \rightarrow -1/H$ . The minus sign is required here so that the image of H remains in the upper half plane. The less trivial part of understanding inversions is to check that they leave the metric unchanged. Suppose that  $P = -1/H$ . Then  $dP = dH/H^2$  and  $dPd\bar{P} = dHd\bar{H}/|H|^4$ . Next we need the imaginary part of P,  $ImP = Im(-1/H) = Im(-\bar{H}/|H|^2) = ImH/|H|^2$ . Collecting these results,

$$\frac{dHd\bar{H}}{(ImH)^2} = \frac{|H|^4 dPd\bar{P}}{(|H|^2 ImP)^2} = \frac{dPd\bar{P}}{(ImP)^2}$$

which proves the point.

These three symmetries can be applied sequentially to produce a linear fractional transformation (Möbius transformation) of the form,

$$H \rightarrow \frac{aH+b}{cH+d} \quad (25)$$

where the coefficients  $(a, b, c, d)$  are all real and the determinant of the Möbius transformation,  $ad - bc$  is positive (which can also be chosen to be unity without loss of generality). These mappings are composed only of isometries and so are themselves isometries. Recall another crucial property of Möbius transformations: they map lines and circles into lines and circles and they preserve the angles between intersecting curves [4]. So, if  $G$  is a mapping of the form Eq. 25, then it takes the imaginary axis  $iK$  to another line which must be perpendicular to the real axis or a circle which is also perpendicular to the real axis. Note also that  $G$  maps the real axis onto itself. These facts are visualized in Fig. 9.



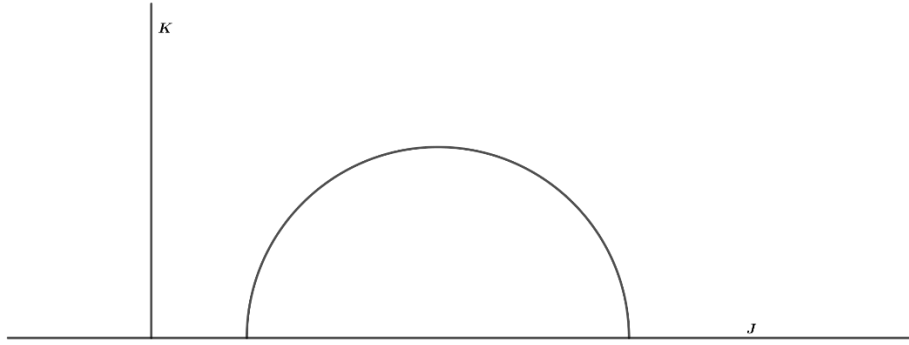


Fig. 9 Geodesics in the H model: vertical straight line, semi-circle with center on the real axis.

In the figure the curve must be a semi-circle in order that it intersect the real axis at right angles.

Next let's check that vertical lines, like the imaginary axis, are geodesics. To see this, consider two points  $iK_1$  and  $iK_2$  on the imaginary axis. Then the hyperbolic distance between them is, applying Eq. 24,

$$d(iK_1, iK_2) = \int_{K_1}^{K_2} \frac{dK}{K} = \ln(K_2/K_1) \quad (26)$$

If we choose another path between points  $iK_1$  and  $iK_2$ , we obtain a hyperbolic distance larger than Eq. 26 because the additional contributions to the integral are all positive. The important point is that vertical lines are geodesics and so are circles which intersect the real axis at 90 degrees since they are the images of the imaginary axis under mappings  $G$  that are isometries. If we consider the mapping  $G$  that takes the imaginary axis to a circle  $C$ , the distances between points on the circle can be computed,

$$d(iK_1, iK_2) = d(G(iK_1), G(iK_2))$$

Given two points in  $H$ ,  $H_1$  and  $H_2$ , it is easy to find the geodesic between them. Let's do this for  $H_1 = v_1 + iw_1$  and  $H_2 = v_2 + iw_1$ , two points with the same imaginary parts. The geodesic

will be the semicircle that passes through them and has its center on the real axis. To find the circle, we draw a straight line between the two points, bisect it and drop a perpendicular to the real axis. The intersection point is the center of the circle. We choose its radius to be the distance from the center of the circle to either point  $H_1$  or  $H_2$ . The construction is shown in Fig. 10 where the center of the circle is labeled  $J_0$  and its perpendicular intersections with the real axis are  $J_1$  and  $J_2$ .

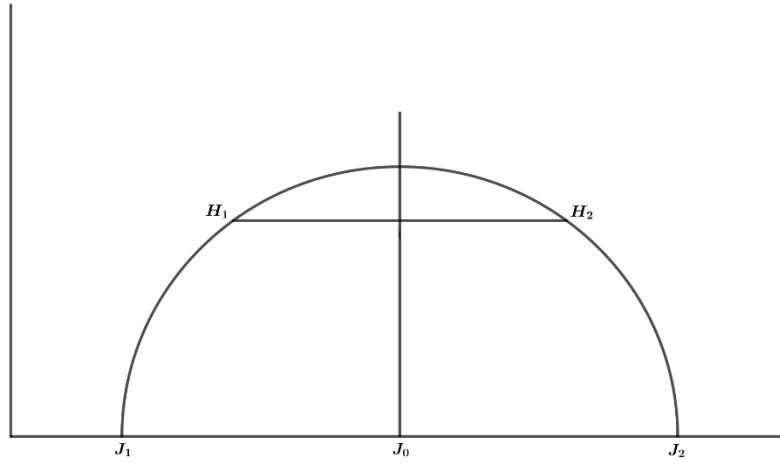


Fig. 10. Construction of the semi-circular geodesic between points  $H_1$  and  $H_2$

The mapping that takes the circle to a vertical geodesic is,

$$G(H) = \frac{H-J_2}{H-J_1} \quad (27)$$

because  $G(J_2) = 0$ , and  $G(J_1) = \infty$  implying that the image of the circle is a vertical straight line. The hyperbolic distance between  $H_1$  and  $H_2$  is then  $d(H_1, H_2) = d(G(H_1), G(H_2))$ . Here  $G(H_1)$  and  $G(H_2)$  have the same real parts and so by translation by a real number, which is an isometry, can be brought to the imaginary axis and Eq. 26 can be applied.

It might appear surprising that the geodesic between  $H_1$  and  $H_2$  is the circular arc show in the figure and not the straight Euclidean line also shown there. But recall that the metric is proportional to  $(ImH)^{-2}$  so paths that bow outward from the real axis lend less weight to the

hyperbolic distance than straight ones of constant height. Instead of using the trickery of complex variables one could calculate the shapes of the geodesics in  $H$  directly from the metric  $dHd\bar{H}/(ImH)^2$  [2].

Let's consider the properties of geodesics (hyperbolic) triangles in  $H$ . We can apply the Gauss-Bonnet Theorem here, although elementary geometry will also yield the results we want. Recall the statement of the Gauss-Bonnet Theorem for a piecewise smooth, simple curve as shown in Fig. 11.

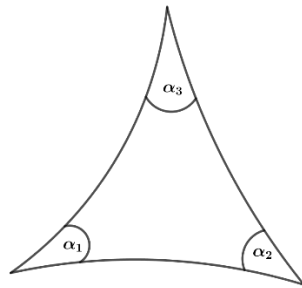


Fig. 11 A hyperbolic triangle,  $\alpha_1 + \alpha_2 + \alpha_3 < \pi$ .

From Eq. 20 of the Supplementary Lecture “The Gauss-Bonnet Theorem”,

$$\oint K dA + \oint \kappa_g(s) ds = \alpha_1 + \alpha_2 + \alpha_3 - \pi \quad (28)$$

where the double integral goes over the enclosed area and the single integral goes over the three smooth portions of its boundary. For the Geodesic (hyperbolic) triangle in  $H$ ,  $K = -1$  and  $\kappa_g = 0$ , so Eq. 28 reduces to,

$$\Delta = \pi - (\alpha_1 + \alpha_2 + \alpha_3) \quad (29)$$

where  $\Delta$  is the area of the triangle. We learn that the sum of the internal angles in the triangle is less than  $\pi$  and the “deficit” tells us the area of the triangle itself. Note that we have taken  $K = -1$  and have suppressed length dimensions here.

Let's suppose that  $\Delta$ , the area of the triangle, is small on the scale set by the curvature  $K = -1$ . If  $\Delta \ll 1$ , then the sum of the internal angles is almost  $\pi$  and the triangle is "almost" Euclidean. This example illustrates a more general feature about Riemannian geometry: for small distances the space is well approximated by a Euclidean metric. On size scales comparable and large compared to  $1/\sqrt{|K|}$ , the global curvature is numerically significant and deviations from Euclidean estimates are large. This fact is apparent in the Gauss-Bonnet Theorem, Eq. 29. As  $\Delta$  increase from unity to its maximum allowable,  $\pi$ , the sum of the internal angles falls to zero and the triangle, Fig. 11, is highly distorted from its Euclidean relatives.

Another distinction between Euclidean and the geometries with constant non-zero curvatures concerns parallelism. In spherical geometry, two spherical lines meet at two points: visualize two great circles on a globe. By contrast, on a Euclidean plane, two distinct lines meet in one point if and only if they are not parallel. Through any point  $P$  not on a line  $l$ , there is only one line parallel to  $l$ , consistent with the famous postulate of Euclid. In hyperbolic geometry, the situation is different. Let's illustrate this with figures on the disc model  $D$ . Since  $D$  and  $H$  are isometric, the same conclusions hold in  $H$ , but the figures are more manageable on  $D$ . On  $D$  the hyperbolic lines are rays through the origin and circles which intersect the boundary at 90 degrees. On the disc, two hyperbolic lines are said to be parallel if they only meet at the boundary as shown in Fig. 12. Since the hyperbolic distance from the origin  $z = 0$  to the edge of the disc at  $|z| = 1$  is infinite, the lines in Fig. 12 only "meet at infinity": hence the term parallel.

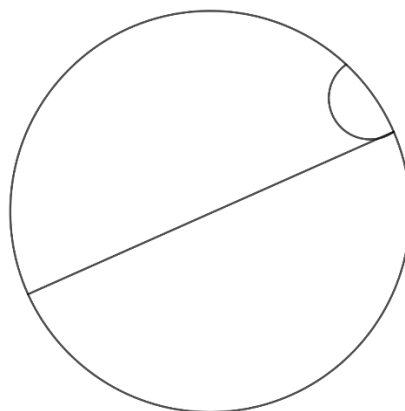


Fig. 12 Two parallel lines on the Poincare disc

But there are hyperbolic lines like those shown in Fig. 13 which never meet. They are called “ultra-parallel”.

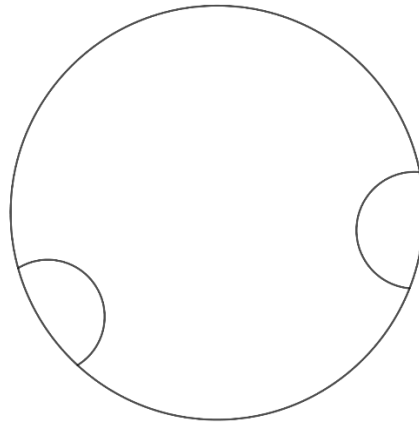


Fig. 12 Two ‘ultra parallel’ lines in the Poincare disc model of non-Euclidean geometry.

There are more properties of spaces of negative curvature which are quite surprising to denizens of Euclidean three space. In Euclidean space, triangles with the same angles are similar but may come in different sizes, i. e. they are not necessarily congruent. In models with non-vanishing curvatures, such as the H or D models, triangles which are similar are necessarily congruent. This means that one can find an isometry which maps their vertices onto one another. In fact the hyperbolic length of any side of a hyperbolic triangle is determined by only its angles. We already learned from the Gauss-Bonnet Theorem that the area of such a triangle is determined just by the sum of its internal angles. Here’s the result. Consider the hyperbolic triangle in Fig. 14 where  $(a, b, c)$  label the vertices,  $(\alpha, \beta, \gamma)$  label the angles and  $(A, B, C)$  label the hyperbolic lengths of the sides.

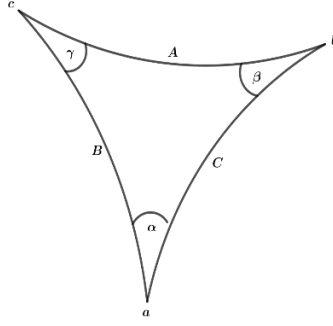


Fig. 14. A hyperbolic triangle with vertices, internal angles and lengths labeled.

Then the hyperbolic length of side C is given in terms of the angles by [2],

$$\cosh C = (\cos \gamma + \cos \alpha \cos \beta) / \sin \alpha \sin \beta \quad (30)$$

with similar equations for sides A and B.

The spherical geometry analogues for Eq. 29 and 30 are more familiar. First, the Gauss-Bonnet Theorem applied to geodesic triangles on a sphere gives the area in terms of the angular deficit,

$$\Delta = (\alpha_1 + \alpha_2 + \alpha_3) - \pi \quad (31)$$

So the sum of the internal angles in a spherical triangle must be greater than  $\pi$  and the “deficit” determines the area of the triangle without information about the length of any of its sides. We used this result in the textbook to better understand parallel transport and other topics in differential geometry. And finally, the spherical analog of Eq. 30, the length of the side of a spherical triangle in terms of its internal angles, reads,

$$\cos C = (\cos \gamma + \cos \alpha \cos \beta) / \sin \alpha \sin \beta \quad (32)$$

where the hyperbolic  $\cosh C$  has been replaced by the trigonometric  $\cos C$  when passing from hyperbolic to spherical geometry

These results and formulas are derived in the references [1-3] by elementary calculations.

Finally, let's consider distances and the metric in the D model. To begin use the metric in Eq. 21 to calculate the hyperbolic distance along a ray in the D model from the origin to  $z = r e^{i\varphi}$ . Now  $u = r \cos \varphi$  and  $v = r \sin \varphi$ , which produces the hyperbolic metric in plane polar coordinates,

$$ds^2 = \frac{4(du^2 + dv^2)}{(1 - u^2 - v^2)^2} = \frac{4(dr^2 + r^2 d\varphi^2)}{(1 - r^2)^2} \quad (33)$$

So, the radial hyperbolic distance is,

$$\rho(0, r e^{i\varphi}) = \rho(0, r) = \int_0^r \frac{2dt}{1-t^2} = 2 \tanh^{-1} r \quad (34)$$

So,

$$r = \tanh(\rho/2) \quad (35)$$

This result allows us to write the metric in terms of hyperbolic polar coordinates  $(\rho, \varphi)$  instead of the Euclidean variables  $(r, \varphi)$ , Substitute  $r = \tanh(\rho/2)$  into Eq. 33. An exercise in hyperbolic functions produces the following intermediate results,

$$\begin{aligned} dr &= \frac{d\rho}{2 \cosh^2(\rho/2)} \\ \frac{1}{(1 - r^2)^2} &= \cosh^4(\rho/2) \\ \cosh^4(\rho/2) \tanh^2(\rho/2) &= \frac{1}{2} \sinh^2 \rho \end{aligned}$$

Using these results we substitute into Eq.33 and find the metric of the D model in terms of the hyperbolic distance  $\rho$ ,

$$ds^2 = d\rho^2 + \sinh^2 \rho d\varphi^2 \quad (36)$$

This is a slick result. First, it is easy to calculate  $K = -1$  from this metric. And second, it is instructive to compare this metric with that of the spherical model in polar coordinates,

$$ds^2 = d\theta^2 + \sin^2 \theta d\varphi^2$$

So, to pass from a sphere to a pseudo-sphere one makes the translation,  $\sin \theta \rightarrow \sinh \rho$ , hence the name “hyperbolic geometry”. And finally, we note from Eq. 34 that as the Euclidean  $r \rightarrow 1$ , the hyperbolic distance  $\rho \rightarrow \infty$ , which confirms our earlier claim that the hyperbolic extent of the D model is infinite.

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