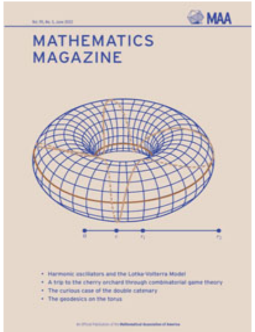


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# Why Curves Curve: The Geodesics on the Torus

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Curves on surfaces curve for two reasons:

(1) They must—the surface on which they reside is itself curved.

(2) They feel like it—they curve more than the surface forces them to.

Curves that curve as little as possible, and therefore curve only because they must, are called *geodesics*. These are the analogues of straight lines in the plane, and they minimize the distance between two sufficiently close points.

Finding the geodesics on a given surface is a central problem in the differential geometry of curves and surfaces. After the plane, the sphere provides an example where the geodesics are as simple as can be: they are all great circles. To seek a surface with more interesting and varied geodesics, it is natural to consider the torus, which indeed yields a much richer collection of geodesics. The first complete classification of the geodesics on the torus was given by G. A. Bliss in his article "The geodesics on the anchor ring" [**2**] which appeared in 1902 in the *Annals of Mathematics*. Bliss's treatment, which is based on a calculus of variations argument, is not an easy read, and it is hardly self-contained, referencing a number of nineteenth century sources that now would be difficult to find.

Over time, the simpler aspects of this classification have been distilled to the point where they appear as exercises in modern differential geometry textbooks [**3**, **8**] and are sometimes used as qualifying examination problems for graduate students in mathematics. But still this beautiful classical material, which so well illustrates the beauty, power, and perspective of differential geometry, is not easily accessible to most undergraduates. This is due, in part, to the formalities of Riemannian geometry that are normally used to explain it. Here, we arrive at a classification of the geodesics on a torus using only simple tools from standard introductory courses in single and multivariable calculus, without any specialized language or notation from differential geometry. In particular, we discern the subtle long-term behavior of a geodesic on a torus based on the angle it makes as it crosses the equator of the torus. (See Figure 1.*) Our version of this story has been used numerous times as the basis of a single, self-contained lecture on this material, successfully delivered to a multivariable calculus class. We hope that others can make similar use of these "lecture notes." More generally, we hope our

*Note that the online version of this article has color diagrams.

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study can serve as a model for clarifying and simplifying other great stories so that they also can be told effectively to students in our introductory calculus sequences.

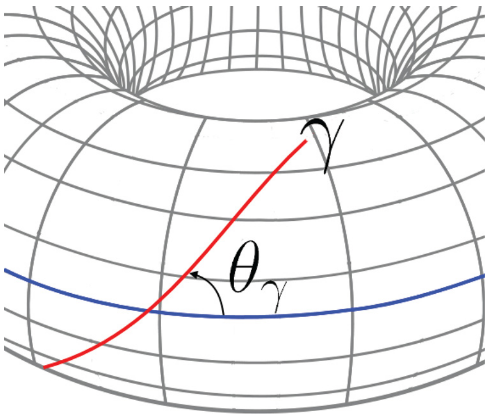



**Figure 1** A geodesic $\gamma$ (red) crossing the outer parallel (blue)

## Surfaces of revolution

A *surface of revolution* is a surface $S$ obtained by rotating a ($C^2$ smooth) curve $C$ that lies in some plane around an axis that lies in that same plane. The curve $C$ is called the *profile curve* of $S$. We take the $xz$-plane as the plane of the profile curve and the $z$-axis as the axis of rotation. (See Figure 2.)

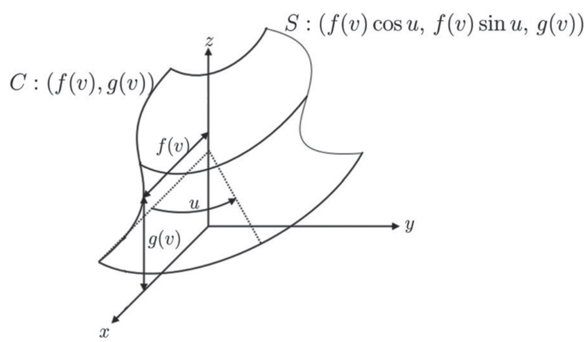



**Figure 2** A surface of revolution

We parametrize the profile curve $C$ by

$$(x, z) = (f(v), g(v))$$

for a real variable $v$. To ease calculations, we assume that the parameterization is unit speed. The resulting surface of revolution $S$ is parameterized by

$$\mathbf{X}(u, v) = (f(v)\cos u,\ f(v)\sin u,\ g(v)), \quad 0 \le u < 2\pi, \tag{1}$$

where $\|\mathbf{X}_v\| = (f')^2 + (g')^2 = 1$. The unit speed condition also implies that $\mathbf{X}_{vv} \cdot \mathbf{X}_v = 0$, as the reader can easily verify. Also note that for a surface of revolution parameterized as above, $\mathbf{X}_u \cdot \mathbf{X}_v = 0$.

The horizontal circles described by equations of the form $v \equiv$ constant are the *parallels* of $S$. The rotated copies of the profile curve described by equations of the form $u \equiv$ constant are the *meridians* of $S$.

**Example 2.1.** A sphere of radius $r$, $S^2(r)$, is a surface of revolution whose profile curve $C$ is a semicircle of radius $r$ in the $xz$-plane centered at $(x, z) = (0, 0)$. (See Figure 3a.) We parameterize $C$ by

$$(x, z) = (f(v), g(v)) = (r\cos v, r\sin v), \qquad -\frac{\pi}{2} \le v \le \frac{\pi}{2},$$

resulting in the parameterization of $S^2(r)$:

$$\mathbf{X}(u, v) = (r\cos v\cos u,\ r\sin v\sin u,\ r\sin v)\,, \qquad 0 \le u < 2\pi.$$

**Example 2.2.** A torus $T^2(r_1, r_2)$, with *inner* and *outer radii* $r_1$ and $r_2$, respectively, is a surface of revolution whose profile curve $C$ is a circle of radius $\frac{r_2 - r_1}{2}$ in the $xz$-plane centered at $(x, z) = (\frac{r_1+r_2}{2}, 0)$. (See Figure 3b.) We parametrize $C$ by

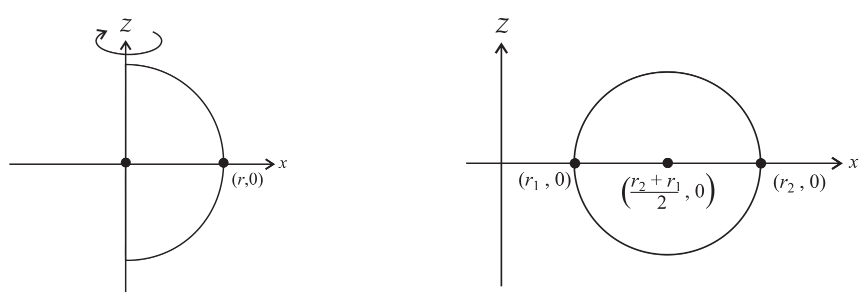
$$\begin{aligned}(x, z) &= (f(v), g(v)) \\ &= \left(\frac{r_1 + r_2}{2} + \frac{r_2 - r_1}{2}\cos v, \frac{r_2 - r_1}{2}\sin v\right), \qquad 0 \le v < 2\pi,\end{aligned}$$

and for $u \in [0, 2\pi)$, we have the resulting parametrization of $T^2(r_1, r_2)$:

$$\mathbf{X}(u, v) = (x, y, z),$$

where

$$\begin{aligned} x &= \left(\frac{r_1 + r_2}{2} + \frac{r_2 - r_1}{2}\cos v\right)\cos u, \\ y &= \left(\frac{r_1 + r_2}{2} + \frac{r_2 - r_1}{2}\cos v\right)\sin u \\ z &= \frac{r_2 - r_1}{2}\sin v. \end{aligned}$$




(a) The profile curve of $S^2(r)$ (b) The profile curve of $T^2(r_1, r_2)$

**Figure 3** Profile curves

## Geodesics on a surface of revolution

Intuitively, a geodesic on a surface $S$ is a ($C^2$ smooth) curve $\gamma : (a, b) \to \mathbb{R}^3$ that curves only as much as it needs to in order to stay on $S$. If such a curve $\gamma$ lies on $S$, then for each $s \in (a, b)$, the vector $\ddot{\gamma}(s)$ breaks into components normal and tangent to $S$,

$$\ddot{\gamma}(s) = \ddot{\gamma}(s)^{\perp} + \ddot{\gamma}(s)^{\tan}.$$

The normal component is the curvature of $\gamma$ imposed on it by the shape of $S$, and $\kappa_n = \|\ddot{\gamma}(s)^{\perp}\|$ is called the *normal curvature*. The tangential component measures any "extra" curvature of $\gamma$ relative to $S$, and $\kappa_g = \|\ddot{\gamma}(s)^{\tan}\|$ is called the *geodesic curvature*. Recall that the *spacial curvature* of a unit-speed $\gamma$ is given by $\kappa(s) = \|\ddot{\gamma}(s)\|$. The three curvatures associated with $\gamma$ have a nice Pythagorean relationship given by

$$\kappa^2 = \kappa_n^2 + \kappa_g^2,$$

and $\gamma$ is a *geodesic* when $\kappa_g = 0$. For a surface of revolution parameterized as in equation (1):

$$\ddot{\gamma}(s)^{\tan} = \ddot{\gamma} \cdot \mathbf{X}_v + \ddot{\gamma} \cdot \mathbf{X}_u.$$

It follows that $\gamma$ is a geodesic precisely when

$$\ddot{\gamma} \cdot \mathbf{X}_v = 0 \qquad \text{and} \qquad \ddot{\gamma} \cdot \mathbf{X}_u = 0. \tag{2}$$

The geometric conditions in equation (2) can be can be rewritten as two second-order differential equations for the functions $u(s)$ and $v(s)$ that define $\gamma$. We assume the parametrization in equation (1) so that

$$\gamma(s) = \mathbf{X}(u(s), v(s)).$$

Careful application of the chain rule yields expressions for velocity and acceleration:

$$\dot{\gamma}(s) = \mathbf{X}_u \dot{u}(s) + \mathbf{X}_v \dot{v}(s),$$

and

$$\begin{aligned}
\ddot{\gamma} &= (\mathbf{X}_u)'\dot{u} + \mathbf{X}_u \ddot{u} + (\mathbf{X}_v)'\dot{v} + \mathbf{X}_v \ddot{v} \\
&= (\mathbf{X}_{uu}\dot{u} + \mathbf{X}_{uv}\dot{v})\dot{u} + \mathbf{X}_u \ddot{u} + (\mathbf{X}_{vu}\dot{u} + \mathbf{X}_{vv}\dot{v})\dot{v} + \mathbf{X}_v \ddot{v} \\
&= \mathbf{X}_{uu}\dot{u}^2 + 2\mathbf{X}_{uv}\dot{u}\dot{v} + \mathbf{X}_u \ddot{u} + \mathbf{X}_v \ddot{v} + \mathbf{X}_{vv}\dot{v}^2.
\end{aligned} \tag{3}$$

In these expressions, the vectors $\mathbf{X}_u$, $\mathbf{X}_v$, $\mathbf{X}_{uu}$, $\mathbf{X}_{vv}$, and $\mathbf{X}_{uv}$ describe the geometry of the surface at the point $\mathbf{X}(u, v)$, whereas the scalars $\dot{u}(s)$, $\dot{v}(s)$, $\ddot{u}(s)$, and $\ddot{v}(s)$ describe what $\gamma$ looks like as it passes through the point $\mathbf{X}(u(s), v(s))$.

To compute the inner products in equation (2) we require the following data. First, the necessary derivatives:

$$\begin{aligned}
\mathbf{X}_u &= (-f(v)\sin u,\ f(v)\cos u, 0) \\
\mathbf{X}_v &= (f'(v)\cos u,\ f'(v)\sin u,\ g'(v)) \\
\mathbf{X}_{uu} &= (-f(v)\cos u, -f(v)\sin u, 0) \\
\mathbf{X}_{vv} &= (f''(v)\cos u,\ f''(v)\sin u,\ g''(v)) \\
\mathbf{X}_{uv} = \mathbf{X}_{vu} &= (-f'(v)\sin u,\ f'(v)\cos u, 0);
\end{aligned}$$

And now the necessary inner products:

$$\begin{array}{lllllllll} \mathbf{X}_u \cdot \mathbf{X}_u & = & f^2 & \mathbf{X}_v \cdot \mathbf{X}_v & = & 1 & \mathbf{X}_u \cdot \mathbf{X}_v & = & 0 \\ \mathbf{X}_{uu} \cdot \mathbf{X}_u & = & 0 & \mathbf{X}_{vv} \cdot \mathbf{X}_u & = & 0 & \mathbf{X}_{uv} \cdot \mathbf{X}_u & = & ff' \\ \mathbf{X}_{uu} \cdot \mathbf{X}_v & = & -ff' & \mathbf{X}_{vv} \cdot \mathbf{X}_v & = & 0 & \mathbf{X}_{uv} \cdot \mathbf{X}_v & = & 0. \end{array} \tag{4}$$

Putting equations (2)–(4) together yields the *geodesic equations* for a surface of revolution:

$$2ff'\dot{u}\dot{v} + f^2\ddot{u} = 0 \tag{5}$$

and

$$\ddot{v} - ff'\dot{u}^2 = 0. \tag{6}$$

We also maintain the tacit assumption that $\gamma$ is unit speed[1], which gives

$$1 = f^2\dot{u}^2 + \dot{v}^2. \tag{7}$$

Armed with the geodesic equations (5) and (6), and the *unit speed condition* (7), we can begin our search for geodesics in earnest. The natural first question emerges: which, if any, among the meridians and parallels are geodesics? We address this question in the next two subsections.

**Meridians** Recall that on a surface of revolution parameterized as in equation (1), a meridian is a curve $\gamma(s) = \mathbf{X}(u(s), v(s))$ where $u$ is constant. But $u$ being constant implies that both $\dot{u}$ and $\ddot{u}$ are identically zero and the first geodesic equation (5) is trivially satisfied. The second geodesic equation (6) now follows from equation (7). We conclude that all meridians are geodesics.

**Parallels** On a surface of revolution, parameterized as in (1), a parallel is a curve $\gamma(s) = \mathbf{X}(u(s), v(s))$, where $v$ is constant. But if $v$ is constant, the unit speed condition (7) implies that $f\dot{u} = 1$ and so, $f \neq 0$ and $\dot{u} \neq 0$. It follows that the two geodesic equations (5) and (6) are equivalent and that $\gamma$ is a geodesic if and only if the constant $v$ is such that $f'(v) = 0$. That is, the only parallels that are geodesics are generated by critical points of the radial distance function $f$.

**Example 3.1.** On $S^2(r)$, $f(v) = r\cos v$, $-\frac{\pi}{2} < v < \frac{\pi}{2}$. Here, $f'(v) = 0$ precisely when $v = 0$. So, the only parallel that is a geodesic is the outer parallel, which is the orbit of the point $(r, 0)$.

**Example 3.2.** On $T^2(r_1, r_2)$,

$$f(v) = \frac{r_2 + r_1}{2} + \frac{r_2 - r_1}{2}\cos v \qquad (0 \leq v < 2\pi).$$

Here, $f'(v) = 0$ precisely when $v = 0$ or $v = \pi$. So, the only parallels that are geodesics are the inner and outer ones. That is, the orbits of the points $(r_1, 0)$ and $(r_2, 0)$, respectively.

[1] This can always be arranged when $\gamma$ is nonconstant. Throughout this article, "geodesic" will always mean "nonconstant geodesic."

To identify other geodesics on surfaces of revolution, we will tease more information out of the first geodesic equation (5). Since this equation can be written as the exact differential

$$2ff'\dot{u}\dot{v} + f^2\ddot{u} = \frac{d}{dt}(f^2\dot{u}) = 0,$$

we see that a necessary condition for a curve to be a geodesic is that its parameterizing function $u$ must satisfy $f^2\dot{u} = c$, for some constant $c$. On the other hand, let $\gamma$ be *any* unit speed curve on the surface of revolution that crosses a parallel at an angle $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. We know from vector calculus that

$$\cos\theta = \frac{|\dot{\gamma} \cdot \mathbf{X}_u|}{\|\dot{\gamma}\|\|\mathbf{X}_u\|} = \frac{|(\mathbf{X}_u\dot{u} + \mathbf{X}_v\dot{v}) \cdot \mathbf{X}_u|}{\|\mathbf{X}_u\|} = \left|\frac{f^2\dot{u}}{f}\right| = |f\dot{u}|. \tag{8}$$

Multiplying both sides of equation (8) by $f$ gives $f\cos\theta = f^2|\dot{u}|$ and so $f\cos\theta = |c|$. We have established the following theorem.

**Theorem 1** (Clairaut's relation [5][2])**.** *Let $\gamma$ be a geodesic on a surface of revolution $S$ parameterized as in (1). If $\gamma(s)$ intersects a parallel of $S$, let $\theta$ be the angle between $\gamma$ and that parallel, i.e., between $\dot{\gamma}$ and $\mathbf{X}_u$, and let $f$ be the radial distance the point of intersection is from the axis of revolution. Along $\gamma$, we then have the constant relationship*

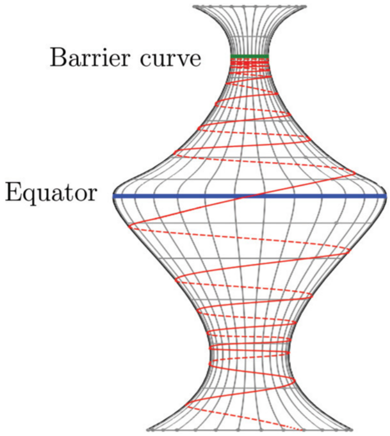
$$c = f\cos\theta. \tag{9}$$

Theorem 9 implies that for each geodesic $\gamma$ on a surface of revolution, there is a *critical distance*, $c = f\cos\theta$, which can be computed at any parallel that $\gamma$ intersects. This allows us to construct surfaces of revolution with geodesics that have prescribed behaviors. For example, suppose $\gamma$'s critical distance is $c$ and that $\gamma$ is approaching a parallel $P$ whose distance from the axis of rotation is also $c$. Can $\gamma$ intersect $P$? The answer depends on whether or not $P$ is a geodesic. More specifically, if $\gamma$ does intersect $P$, then Theorem 9 implies that it must do so tangentially ($c = c\cos\theta \implies \theta = 0$). If it happens that $P$ is a geodesic, then at the presumed point of intersection of $\gamma$ and $P$, two different geodesics would have the same tangent vector. This is not possible since a geodesic is locally uniquely determined by a point and a direction. In this situation, $\gamma$ spirals asymptotically toward $P$. For this reason, such $P$'s have elsewhere been called *barrier curves* [4, 6]. (See Figure 4(a)). If on the other hand, $P$ is not a geodesic, then $\gamma$ has reached its critical distance at this parallel and then moves in the direction of increasing radial distance from the axis of rotation. Effectively, $\gamma$ "bounces off" of $P$; see Figure 4(b), where the red geodesic is forever trapped, bouncing between the two green barrier curves. (We often call the parallel with the largest radial distance the "equator" of a surface of revolution—as noted in Figure 4.)

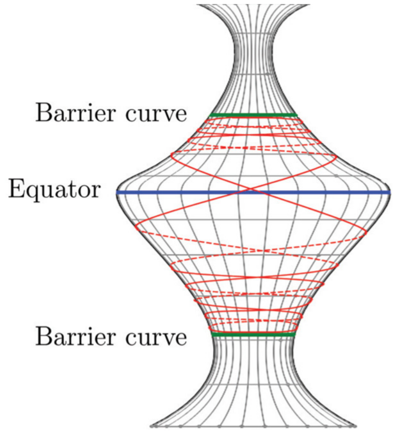
## Geodesics on the torus

The results of the previous section establish that all the meridians on a surface of revolution are geodesics. But for parallels the story is quite different. If $f$ is the function that records the radial distance from a point on the surface of revolution to the axis of revolution, then only those parallels which are orbits of local extrema of $f$ are

[2]Named after the French mathematician and geophysicist Alexis Claude Clairaut (1713–1765) who helped to establish the Newtonian claim that the earth was not a perfect sphere. Pressley [8, p. 185] explains Theorem 1 as an expression of the conservation of angular momentum about the axis of revolution when a particle slides along a geodesic under no forces other than those that keep it on the surface. See also Oprea [7, pp. 223–224].




(a) Barrier curve is a geodesic




(b) Barrier curves are not geodesics

**Figure 4** Barrier curves

geodesics. For the torus, this means that the only parallels which are geodesics are the inner and outer ones. To find other geodesics on the torus, the following theorem is fundamental.

**Theorem 2.** *Except for the inner parallel, every geodesic on the torus must intersect the outer parallel.*

*Proof.* All meridians intersect the outer parallel at right angles. We therefore assume that a geodesic $\gamma$ intersects the parallels of $T^2$ with a radial distance function $f$ and an angular function $\theta \neq \frac{\pi}{2}$. Now follow $\gamma$ in the direction of increasing $f$—which is monotonically increasing and bounded above by $r_2$. We need only show that $\gamma$ cannot asymptotically approach the outer parallel. If that were to happen, we would have $\lim_{f\to r_2} \theta = 0$. But, by equation (9), $f$ and $\theta$ increase together. So, $\gamma$ must intersect the outer parallel. ∎

Proceeding to the classification of the geodesics on $T^2(r_1, r_2)$, we first define the *critical angle* of the torus:

$$\theta_C = \cos^{-1}\left(\frac{r_1}{r_2}\right).$$

We can now classify the geodesics on the torus according to the relationship between the geodesic's crossing angle and the critical angle of the torus. We do this in the following subsections, where we also note the long-term behavior of the geodesics.

**Classification of geodesics on the Torus** Let $T^2(r_1, r_2)$ be a torus with inner and outer radii, $r_1$ and $r_2$, respectively. All meridians are geodesics and of the parallels, only the inner and outer parallel are geodesics. If $\gamma$ a geodesic which crosses the outer parallel at an angle, $\theta_\gamma \in (0, \frac{\pi}{2})$, we are reduced to the following three cases treated in the following subsections.

*Case 1.* $0 < \theta_\gamma < \theta_C$ Since cosine is a decreasing function, $\cos\theta_\gamma > \cos\theta_C = \frac{r_1}{r_2}$ and so $r_2 \cos\theta_\gamma > r_1$. But since $r_2$ is the distance where $\gamma$ crosses the outer parallel, Theorem 1 implies that $\gamma$'s critical distance $c$ is greater than $r_1$. (See Figure 5.)

*Case 2.* $\theta_\gamma = \theta_C$ This case leads to $\cos\theta_\gamma = \cos\theta_C = \frac{r_1}{r_2}$. Theorem 1 implies that

$$c = r_2 \cos\theta_\gamma = r_2 \frac{r_1}{r_2} = r_1.$$

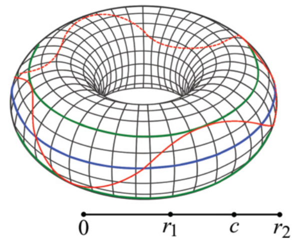



**Figure 5** Case 1 geodesic oscillating between two barrier curves

So, $c = r_1$ and since the inner parallel is a geodesic, $\gamma$ spirals asymptotically toward it. (See Figure 6.)

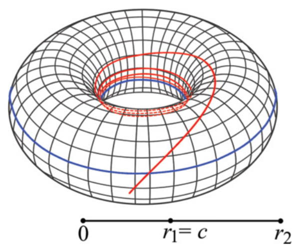



**Figure 6** Case 2 geodesic asymptotically approaching the inner parallel

*Case 3.* $\theta_C < \theta_\gamma < \frac{\pi}{2}$ This case leads to $\cos\theta_\gamma < \cos\theta_C$. Theorem 1 implies that

$$c = r_2 \cos\theta_\gamma < r_2 \cos\theta_c = r_1.$$

So, $c < r_1$, and $\gamma$ can never realize its critical distance by either intersection or asymptotic approach. (See Figure 7.)

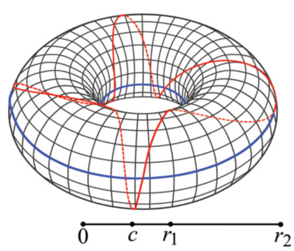



**Figure 7** Case 3 geodesic with no barrier curve

## Afterword and looking ahead

Except for the meridians and the inner and outer parallels of a torus $T^2(r_1, r_2)$, each geodesic $\gamma$ must intersect the equator of the torus at the geodesics's crossing angle $0 < \theta_\gamma < \frac{\pi}{2}$, and $\gamma$'s eventual behavior is determined by the relationship between $\theta_\gamma$ and the critical angle of the torus defined by $\theta_C = \cos^{-1}(\frac{r_1}{r_2})$. We are left with the three cases of the previous section. We will examine Case 1 and Case 3 more closely.

In Case 1 ($\theta_\gamma < \theta_C$), the geodesic does not have enough "energy" (see Jantzen [6]) to encircle the outer parallel, but oscillates back and forth across the equator and between its barrier curves as it encircles the $z$-axis. In Case 3 ($\theta_\gamma > \theta_C$), the geodesic's crossing angle is steep enough, and so its energy sufficient, that it can repeatedly encircle the outer parallel as it winds around the $z$-axis. In some instances, these geodesics may be *closed*. That is, there is an $s_0 \in \mathbb{R}$ such that $\gamma(s_0 + s) = \gamma(s)$, for all values of $s$. Here, such a closed geodesic winds $p$ times around the $z$-axis while making $q$-oscillations ($2q$ crossings) across the equatorial geodesic, before returning to its "starting point," which we take to be an arbitrary point where $\gamma$ intersects the equator. In this way, closed geodesics on the torus are indexed by pairs of coprime integers $p$ and $q$. We call the ordered pair $(p, q)$ the *index* of the closed geodesic.

In a follow-up note, and again using elementary means, we determine the permissible index of a geodesic on $T^2(r_1, r_2)$, by examining allowable quotients $p/q$ in terms of the tori's defining parameters $r_1$ and $r_2$ (see Alexander [1]). Here, another angle takes center stage—the *return angle* of a closed geodesic. More specifically, and to illustrate, let $\gamma$ be a closed Case 1 geodesic, which intersects the equator at two points $p_1$ and $p_2$. The return angle of $\gamma$ is denoted by $\gamma_r$ and is defined as the central angle subtended by the rays defined by the origin and the points $p_1$ and $p_2$; see Figure 8, where a portion of a Case 1 geodesic is depicted in red.

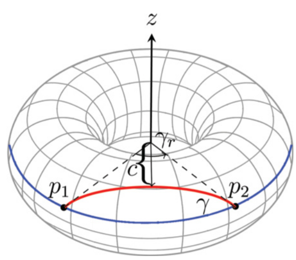



**Figure 8** The return angle $\gamma_r$

We find, in particular, that $T^2(r_1, r_2)$ has a closed Case 1 geodesic $\gamma$ of index $(p, q)$ precisely when the quotient $\frac{p}{q}$ is bounded below (and related to $\gamma$'s return angle $\gamma_r$) as follows:

$$\sqrt{\frac{r_2 - r_1}{2r_2}} < \frac{p}{q} = \frac{\gamma_r}{\pi}.$$

**Remark.** *The details of this section are beyond the scope of this paper and will appear elsewhere.*

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**Summary.** Based on an intuitively appealing way of defining what is meant by a geodesic on a two-dimensional surface in three-dimensional Euclidean space, we describe the general behavior of the geodesics on surfaces of revolution and give a simple yet precise characterization of the geodesics on a torus. We use only simple tools from the standard introductory courses in single and multivariable calculus, and no specialized language or notation from differential geometry, so our presentation is self-contained for anyone who is familiar with the contents of these courses. In particular, we are able to develop Clairaut's relation with relative ease.

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