Geodesics on Surfaces of Revolution and Ruled Surfaces: A Study of Symmetries and Isometries

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March 6, 2025

Introduction

- **Motivation:** I like differential geometry and group theory.
- **Objective:** Study geodesics on surfaces with rotational and translational symmetries.
- **Importance:** Geodesics are fundamental to understanding curved spaces.
- **Key Question:** How do inherent symmetries simplify the geodesic equations?

Thesis Focus and Objectives

- **Derivation:** Formulate geodesic equations via variational calculus and the Euler–Lagrange framework.
- **Analysis:** Examine specific examples (sphere, cylinder) to illustrate geodesic behavior.
- **Symmetry:** Use rotational and translational symmetries to identify conserved quantities.
- **Future Work:** Extend analysis to more complex ruled surfaces and explore numerical simulations.

Differential Geometry

- **Manifolds:**
  - n-dimensional topological spaces that locally resemble Euclidean space.
  - Examples: sphere, torus, cylinder, Minkowski spacetime, Klein bottle.
- We consider surfaces (2-manifolds) embedded in $\mathbb{R}^3$.
- We define the parameterization of a surface as $\mathbf{r}(u, v)$.
- $\mathbf{r} : \mathbb{R}^2 \to \mathbb{R}^3$ is a smooth map where $u(t), v(t) \in \mathbb{R}^2$ are local coordinates.

Differential Geometry: Metric Tensor

- **Tangent Space:**

$$T_p(M) = \operatorname{span}\left\{\frac{\partial \mathbf{r}}{\partial u}, \frac{\partial \mathbf{r}}{\partial v}\right\}.$$

- **Metric Tensor:** Let $\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}$ and $\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}$.

$$g_{\mu\nu} = \mathbf{r}_\mu \cdot \mathbf{r}_\nu \quad \text{where} \quad \mu, \nu \in \{u, v\}.$$

- Let

$$E = \mathbf{r}_u \cdot \mathbf{r}_u, \quad F = \mathbf{r}_u \cdot \mathbf{r}_v, \quad G = \mathbf{r}_v \cdot \mathbf{r}_v.$$

$$E_u = \frac{\partial E}{\partial u}, \quad E_v = \frac{\partial E}{\partial v}, \quad F_u = \frac{\partial F}{\partial u}, \quad F_v = \frac{\partial F}{\partial v}, \quad G_u = \frac{\partial G}{\partial u}, \quad G_v = \frac{\partial G}{\partial v}.$$

- **Metric:**

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2.$$

Derivation of Geodesic Equations (I)

- **Action Functional:**

$$S[\gamma] = \int_a^b \mathcal{L}(t, q^\lambda, \dot{q}^\lambda)\, dt.$$

- **Euler–Lagrange Equations:** Assuming

$$\mathcal{L} = g_{\mu\nu}\, \dot{q}^\mu \dot{q}^\nu,$$

they become

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^\lambda}\right) - \frac{\partial \mathcal{L}}{\partial q^\lambda} = 0.$$

- **Geodesic Equation:** Thus,

$$\frac{\mathrm{d}^2 q^\lambda}{\mathrm{d}t^2} + \Gamma^\lambda_{\mu\nu} \frac{\mathrm{d}q^\mu}{\mathrm{d}t} \frac{\mathrm{d}q^\nu}{\mathrm{d}t} = 0.$$

Derivation of Geodesic Equations (II)

- **Christoffel Symbols:** Encode the curvature of the manifold.

$$\Gamma^u_{uu} = \frac{1}{2D}\Big[G\,E_u - F\big(2F_u - E_v\big)\Big], \quad \Gamma^u_{uv} = \Gamma^u_{vu} = \frac{1}{2D}\Big[G\,E_v - F\,G_u\Big],$$

$$\Gamma^u_{vv} = \frac{1}{2D}\Big[G\big(2F_v - G_u\big) - F\,G_v\Big], \quad \Gamma^v_{uu} = \frac{1}{2D}\Big[E\big(2F_u - E_v\big) - F\,E_u\Big],$$

$$\Gamma^v_{uv} = \Gamma^v_{vu} = \frac{1}{2D}\Big[E\,G_u - F\,E_v\Big], \quad \Gamma^v_{vv} = \frac{1}{2D}\Big[E\,G_v - F\big(2F_v - G_u\big)\Big],$$

- **Geodesic Equations for u and v:**

$$\ddot{u} + \Gamma^u_{uu}\,\dot{u}^2 + 2\,\Gamma^v_{uv}\,\dot{u}\dot{v} + \Gamma^u_{vv}\,\dot{v}^2 = 0, \tag{1}$$

$$\ddot{v} + \Gamma^v_{uu}\,\dot{u}^2 + 2\,\Gamma^v_{uv}\,\dot{u}\dot{v} + \Gamma^v_{vv}\,\dot{v}^2 = 0. \tag{2}$$

Surfaces of Revolution

- **Definition:** Obtained by rotating a profile curve about an axis.
- **Parametrization:**

$$\mathbf{r}(u, v) = \begin{bmatrix} x(v)\cos(u) \\ x(v)\sin(u) \\ z(v) \end{bmatrix},$$

where: u is the angular coordinate and v parameterizes the profile curve.

- **Metric Components:**

$$E = x(v)^2, \quad F = 0, \quad G = \dot{x}(v)^2 + \dot{z}(v)^2.$$

- **Geodesic Equations:** The geodesic equations for this surface are

$$\ddot{u} + \frac{2\dot{x}}{x}\,\dot{v}\,\dot{u} = 0,$$

$$\ddot{v} - \frac{x\dot{x}}{\dot{x}^2 + \dot{z}^2}\,\dot{u}^2 + \frac{\ddot{x}\,\dot{x} + \ddot{z}\,\dot{z}}{\dot{x}^2 + \dot{z}^2}\,\dot{v}^2 = 0.$$

- **Advantage:** Rotational symmetry yields conserved quantities simplifying the equations.

Meridians and Parallels

- **Meridians:**
  - Defined by constant u ($\dot{u} = 0$); reduce to a linear equation in v.
- **Parallels:**
  - Defined by constant v; geodesic only if $x'(v_0) = 0$ (local extremum of $x(v)$).
- **Interpretation:** Illustrate how symmetry reduces the complexity of geodesic equations.

Examples: Sphere and Cylinder

Sphere:

- **Parametrization:**

$$\mathbf{r}(\theta, \phi) = \begin{bmatrix} R\cos\theta\sin\phi \\ R\sin\theta\sin\phi \\ R\cos\phi \end{bmatrix}.$$

- **Metric:**

$$ds^2 = R^2\sin^2\phi\, d\theta^2 + R^2\, d\phi^2.$$

- **Geodesics:**

$$\ddot{\theta} - 2\tan\theta\,\dot{\phi}\,\dot{\theta} = 0,$$
$$\ddot{\phi} - \cos\theta\sin\theta\,\dot{\theta}^2 = 0.$$

Great circles represent the shortest paths.

Cylinder: Cylinder:

- **Parametrization:**

$$\mathbf{r}(u, v) = \begin{bmatrix} R\cos u \\ R\sin u \\ v \end{bmatrix}.$$

- **Metric:**

$$ds^2 = R^2\,du^2 + dv^2.$$

- **Geodesics:**

$$\ddot{u} = 0, \quad \ddot{v} = 0.$$

$$u = at + b, \quad v = ct + d.$$

Helices along the cylinder's surface.

Ruled Surfaces

- **Definition:** Generated by moving a straight line (ruling) along a base curve.
- **Parametrization:**

$$\mathbf{r}(u, v) = \gamma(u) + v\,\mathbf{d}(u),$$

  where:
  - $\gamma(u)$ is the base curve.
  - $\mathbf{d}(u)$ is the direction vector.
- **Analysis:** Lacks full rotational symmetry, making the geodesic analysis more complex.
- **Future Work:** Focus on exploiting partial symmetries to simplify the analysis.

Symmetries and Isometries

- **Symmetries:**
  - Provide conserved quantities (e.g., Clairaut's relation).
  - Lower the order of geodesic differential equations.
  - Focus on Lie groups and algebras and their actions.
  - $SO(3)$ for rotational symmetry, $SE(3)$ for translational symmetry.
- **Isometries:**
  - Transformations that preserve distances.
  - Enable mapping of complex problems to simpler, equivalent ones.
- **Outcome:** Both are key to understanding and simplifying geodesic behavior.

Challenges and Open Questions

- **Analytical Complexity:** How can we further simplify the geodesic equations on less symmetric surfaces?
- **Numerical Approaches:** What are the most effective numerical methods to simulate geodesics on complex surfaces?
- **Extension to Higher Dimensions:** Can these techniques be generalized to manifolds beyond $\mathbb{R}^3$?
- **Interdisciplinary Applications:** How can the study of geodesics impact fields such as computer graphics and physical modeling?

References

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Questions and Discussion

Thank you!

Any Questions?