# A look at geodesics through symmetries and their applications

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# Outline


1. Preliminaries
2. Surfaces and Curves
3. Geodesics
4. A Look at Surfaces of Revolution
5. Symmetries & Killing Fields
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7. Black Hole Geodesics

# Preliminaries

- **Differential geometry:** Study of smooth topological spaces called *manifolds* using tools from calculus and linear algebra.

# Surfaces

## Definition (Surface)

A regular surface $\mathcal{S} \subset \mathbb{R}^3$ is the image of a one-to-one smooth map

$$\mathbf{r} : U \subset \mathbb{R}^2 \to \mathbb{R}^3, \quad \mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle,$$

such that $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$ for all $(u, v) \in U$.

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- A surface is a 2D manifold embedded in $\mathbb{R}^3$, locally resembling $\mathbb{R}^2$.
- We write $\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}\big|_{(u_0,v_0)}$ and $\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}\big|_{(u_0,v_0)}$ for the tangent vectors at a point $\mathbf{r}(u_0, v_0)$.

# The Metric

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A *metric* on a surface $\mathcal{S}$ is a is a smoothly varying inner product on the tangent plane at each point of the surface.

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2,$$

where $E = \mathbf{r}_u \cdot \mathbf{r}_u$, $F = \mathbf{r}_u \cdot \mathbf{r}_v$, and $G = \mathbf{r}_v \cdot \mathbf{r}_v$.

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- The metric defines lengths, angles, and areas on the surface.
- The coefficients $E$, $F$, and $G$ are called the *first fundamental form* of the surface often characterized as:

$$g_{\mu\nu} = \mathbf{r}_\mu \cdot \mathbf{r}_\nu = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$$

# Curves

**Definition (Curve)**

A *curve* $\gamma$ on a surface $\mathcal{S}$ is a smooth map $\gamma : I \to \mathcal{S}$, where $I$ is an interval in $\mathbb{R}$.

$$\gamma(t) = \mathbf{r}(u(t), v(t))$$

- $\dot{\gamma}(t) = \frac{d\gamma}{dt} = \dot{u}(t)\mathbf{r}_u + \dot{v}(t)\mathbf{r}_v$

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- **Unit Speed Condition:** Under arc length re-parametrization, $\|\dot{\gamma}(t)\| = 1$.

# Curvature

## Definition (Curvature)

The *curvature* of a curve defines how much the curve deviates from being a straight line.

$$\kappa_\gamma(t) = \frac{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|}{\|\dot{\gamma}(t)\|^3} \implies \|\ddot{\gamma}(t)\|$$

- **Normal Curvature:** $\kappa_{\mathbf{N}}(t) = \kappa_\gamma(t) \cdot \mathbf{N}(t)$, where $\mathbf{N}(t)$ is the unit normal vector to the surface at $\gamma(t)$.

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- **Geodesic Curvature:** $\kappa_g(t) = \kappa_\gamma(t) - \kappa_{\mathbf{N}}(t)$, which measures the curvature of the curve relative to the surface.

# Geodesics

## Definition (Geodesic)

A *geodesic* is a curve $\gamma$ on a surface $\mathcal{S}$ that has zero geodesic curvature.

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- Often, geodesics can be considered as the shortest paths between two points on the surface.
- Geodesics are described by the geodesic equation:

$$\frac{\mathrm{d}^2 q^\sigma}{\mathrm{d}t^2} + \sum_{\mu,\nu=1}^{n} \Gamma^\sigma_{\mu\nu} \frac{\mathrm{d}q^\mu}{\mathrm{d}t}\frac{\mathrm{d}q^\nu}{\mathrm{d}t} = 0 \qquad \Gamma^\sigma_{\mu\nu} = \frac{1}{2}\sum_{\lambda=1}^{n} g^{\sigma\lambda}\left(\frac{\partial g_{\mu\nu}}{\partial q^\lambda} + \frac{\partial g_{\nu\lambda}}{\partial q^\mu} - \frac{\partial g_{\mu\lambda}}{\partial q^\nu}\right)$$

where $\Gamma^\sigma_{\mu\nu}$ are the Christoffel symbols of the second kind, describing how the coordinate system curves.

# Surfaces of Revolution

## Definition (Surface of Revolution)

A *surface of revolution* is a surface generated by rotating a curve $\gamma$ about an axis. Under rotation about the $z$-axis, the surface can be described as:

$$\mathcal{S} = \mathbf{r}(u, v) = \langle x(u)\cos(v),\ x(u)\sin(v),\ z(u) \rangle \qquad u \in I,\ v \in [0, 2\pi]$$

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- The metric on a surface of revolution is given by:

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- By the unit speed condition, the following equation hold:

$$1 = \left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 + x(u)^2 \left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)^2$$

# Geodesics on Surfaces of Revolution

- The geodesic equation for a surface where $F = 0$ is given by:

$$\ddot{u} + \frac{E_u}{2E}\dot{u}^2 + 2\frac{E_v}{2E}\dot{u}\dot{v} - \frac{G_u}{2E}\dot{v}^2 = 0$$

$$\ddot{v} - \frac{E_v}{2G}\dot{u}^2 + 2\frac{G_u}{2G}\dot{u}\dot{v} + \frac{G_v}{2G}\dot{v}^2 = 0$$

Where $E_u = \frac{\partial E}{\partial u}$, $E_v = \frac{\partial E}{\partial v}$, $G_u = \frac{\partial G}{\partial u}$, and $G_v = \frac{\partial G}{\partial v}$.

$$\ddot{u} + \frac{x_u x_{uu} + z_u z_{uu}}{x_u^2 + z_u^2}\,\dot{u}^2 - \frac{x x_u}{x_u^2 + z_u^2}\,\dot{v}^2 = 0 \implies \ddot{u} - x x_u\,\dot{v}^2 = 0$$

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- This corresponds to the geodesic equations for a surface of revolution:

$$\ddot{u} + \frac{x_u x_{uu} + z_u z_{uu}}{x_u^2 + z_u^2}\,\dot{u}^2 - \frac{x x_u}{x_u^2 + z_u^2}\,\dot{v}^2 = 0 \implies \ddot{u} - x x_u\,\dot{v}^2 = 0$$

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## Definition (Isometry)

An *isometry* is a symmetry that preserves the metric, that is, it preserves lengths and angles:

$$g(\phi(p), \phi(q)) = g(p, q)$$

# Example: Symmetries & Isometries

## Definition (Group)

A *group* is a set $G$ with a binary operation $\circ$ that satisfies the following properties:

- **Closure:** For all $a, b \in G$, $a \circ b \in G$.
- **Associativity:** For all $a, b, c \in G$, $(a \circ b) \circ c = a \circ (b \circ c)$.
- **Identity:** There exists an element $e \in G$ such that for all $a \in G$, $e \circ a = a \circ e = a$.
- **Inverse:** For each element $a \in G$, there exists an element $a^{-1} \in G$ such that $a \circ a^{-1} = e$.

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- **Killing Vector Fields:** A group of isometries that generate symmetries of the metric.

# Killing Vector Fields

## Definition (Killing Vector Field)

A *Killing vector field* is a vector field $K^\mu$ on a manifold $\mathcal{M}$ that satisfies the Killing equation:

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0$$

where $\nabla_\mu K_\nu = \frac{\partial K_\nu}{\partial x^\mu} + \Gamma^\lambda_{\mu\nu} K_\lambda$ is the covariant derivative of the vector field.

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- Easiest way to find Killing vector fields is to look for symmetries of the metric, (i.e. the coordinates that the metric is independent of).
- The number of independent Killing vector fields is related to the number of symmetries of the manifold.

## Schwarzschild Metric

The Schwarzschild solution describes the geometry of spacetime around a non-rotating, spherically symmetric mass:

$$ds^2 = -(1 - 2GM/r)dt^2 + (1 - 2GM/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

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- Time-translation symmetry $\to$ Killing vector $\partial_t$ $\to$ conserved energy $E$
- Rotational symmetry $\to$ Killing vector $\partial_\phi$ $\to$ conserved angular momentum $L$
- These symmetries significantly simplify the geodesic equations

# Effective Potential

Using the conserved quantities, the geodesic motion can be reduced to a one-dimensional problem:

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}}(r) = \frac{1}{2}E^2,$$

$$V_{\text{eff}}(r) = (1 - 2GM/r)\left(\frac{L^2}{r^2} + \epsilon\right).$$

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    - Bound orbits: $E^2 < V_{\text{eff}}(r \to \infty)$
    - Unbound orbits: $E^2 > V_{\text{eff}}(r \to \infty)$

# Circular Orbits & Photon Sphere

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  - These special orbits are critical for understanding accretion disks and gravitational lensing around black holes