MM5611

ADVANCED ROBOTICS

4-DOF Robot Arm

Kinematics Analysis

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1 Introduction

This project analyzes a 4 degree-of-freedom (4-DOF) robot manipulator using the Modified Denavit-Hartenberg (MDH) convention. The aim is to derive the complete kinematic equations and understand the robot's motion capabilities.

2 Robot Description

The robot has the following joint configuration:

Joint	**Type**	**Variable**
1	Revolute	θ_1
2	Prismatic	d_2
3	Revolute	θ_3
4	Revolute	θ_4

Table 1: Joint configuration of the 4-DOF robot

The robot has the following link parameters:

- L_0 – Base height (distance from ground to joint 2)
- L_1 – Link length between joint 4 and tool frame origin
- L_2 – End-effector (tool) length

2.1 Modified DH Convention

In the Modified DH convention, the transformation from frame $\{i-1\}$ to frame $\{i\}$ is performed in the following order:

1. Rotate about x_{i-1} by α_{i-1}
2. Translate along x_{i-1} by a_{i-1}
3. Rotate about z_i by θ_i
4. Translate along z_i by d_i

This gives the transformation matrix:

$$^{i-1}T_i = R_x(\alpha_{i-1}) \cdot D_x(a_{i-1}) \cdot R_z(\theta_i) \cdot D_z(d_i) \tag{1}$$

2.2 Robot Schematic

i	a_{i-1}	α_{i-1}	d_i	θ_i	Type
1	0	0	L_0	θ_1^*	Revolute
2	0	0	d_2^*	0	Prismatic
3	0	$\pi/2$	0	θ_3^*	Revolute
4	L_1	0	0	θ_4^*	Revolute
5	L_2	0	0	0	Fixed (Tool)

Table 2: Modified DH Parameters (asterisk denotes joint variable)

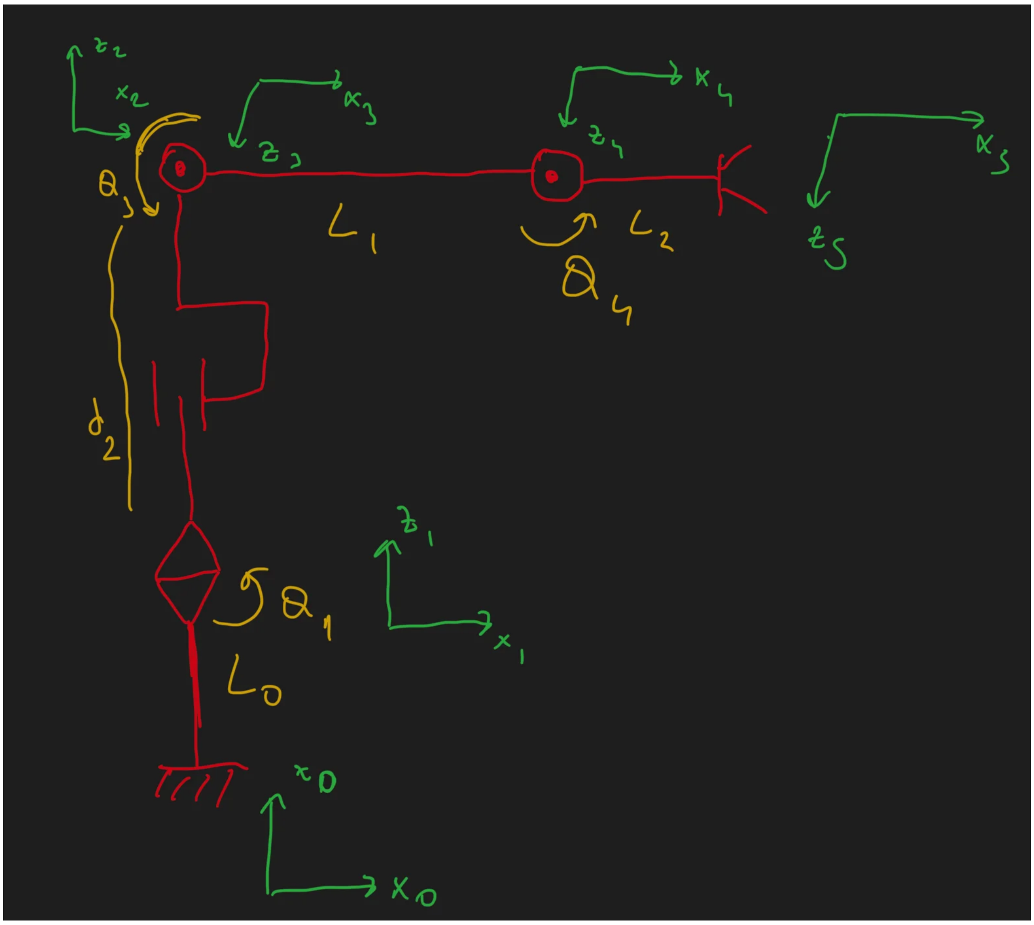

Figure 1: Schematic diagram of the 4-DOF RPRR robot with coordinate frames

3 Forward Kinematics

3.1 General MDH Transformation Matrix

For the Modified DH convention, each joint has a transformation matrix:

$$
{}^{i-1}T_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & a_{i-1} \\ \sin\theta_i\cos\alpha_{i-1} & \cos\theta_i\cos\alpha_{i-1} & -\sin\alpha_{i-1} & -\sin\alpha_{i-1}\cdot d_i \\ \sin\theta_i\sin\alpha_{i-1} & \cos\theta_i\sin\alpha_{i-1} & \cos\alpha_{i-1} & \cos\alpha_{i-1}\cdot d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}
$$

3.2 Individual Transformation Matrices

Using the DH parameters and the shorthand notation $c_i = \cos\theta_i$, $s_i = \sin\theta_i$, the individual transformation matrices are computed:

Joint 1 (Revolute): $a_0 = 0$, $\alpha_0 = 0$, $d_1 = L_0$, $\theta_1 = \theta_1^*$

$$
{}^{0}T_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & L_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}
$$

Joint 2 (Prismatic): $a_1 = 0$, $\alpha_1 = 0$, $d_2 = d_2^*$, $\theta_2 = 0$

$$^1T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

Joint 3 (Revolute): $a_2 = 0$, $\alpha_2 = \pi/2$, $d_3 = 0$, $\theta_3 = \theta_3^*$
Since $\cos(\pi/2) = 0$ and $\sin(\pi/2) = 1$:

$$^2T_3 = \begin{bmatrix} c_3 & -s_3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

Joint 4 (Revolute): $a_3 = L_1$, $\alpha_3 = 0$, $d_4 = 0$, $\theta_4 = \theta_4^*$

$$^3T_4 = \begin{bmatrix} c_4 & -s_4 & 0 & L_1 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

Tool Frame: $a_4 = L_2$, $\alpha_4 = 0$, $d_5 = 0$, $\theta_5 = 0$

$$^4T_5 = \begin{bmatrix} 1 & 0 & 0 & L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7}$$

3.3 Compound Transformation Matrices

The compound transformations are computed by sequential multiplication.

Base to Frame 2:

$$^0T_2 = {}^0T_1 \cdot {}^1T_2 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & L_0 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}$$

Base to Frame 3:

$$^0T_3 = {}^0T_2 \cdot {}^2T_3 = \begin{bmatrix} c_1c_3 & -c_1s_3 & s_1 & 0 \\ s_1c_3 & -s_1s_3 & -c_1 & 0 \\ s_3 & c_3 & 0 & L_0 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{9}$$

Base to Frame 4:
Using the shorthand $c_{34} = \cos(\theta_3 + \theta_4)$ and $s_{34} = \sin(\theta_3 + \theta_4)$:

$$^0T_4 = {}^0T_3 \cdot {}^3T_4 = \begin{bmatrix} c_1c_{34} & -c_1s_{34} & s_1 & L_1c_1c_3 \\ s_1c_{34} & -s_1s_{34} & -c_1 & L_1s_1c_3 \\ s_{34} & c_{34} & 0 & L_0 + d_2 + L_1s_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{10}$$

3.4 Complete Transformation Matrix

The complete transformation from base to end-effector (tool tip) is:

$$^0T_5 = {}^0T_4 \cdot {}^4T_5 \tag{11}$$

Since $^4T_5$ is a pure translation along x_4, the rotation part remains unchanged. The position is updated by adding L_2 times the first column of the rotation matrix (the x_4 direction in base frame).

Result

$$^0T_5 = \begin{bmatrix} c_1c_{34} & -c_1s_{34} & s_1 & c_1(L_1c_3 + L_2c_{34}) \\ s_1c_{34} & -s_1s_{34} & -c_1 & s_1(L_1c_3 + L_2c_{34}) \\ s_{34} & c_{34} & 0 & L_0 + d_2 + L_1s_3 + L_2s_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{12}$$

where $c_{34} = \cos(\theta_3 + \theta_4)$ and $s_{34} = \sin(\theta_3 + \theta_4)$

3.5 End-Effector Position Equations

From the fourth column of the transformation matrix $^0T_5$, the end-effector position equations are:

Result

$$x = \cos\theta_1 \cdot [L_1 \cos\theta_3 + L_2 \cos(\theta_3 + \theta_4)] \tag{13}$$

$$y = \sin\theta_1 \cdot [L_1 \cos\theta_3 + L_2 \cos(\theta_3 + \theta_4)] \tag{14}$$

$$z = L_0 + d_2 + L_1 \sin\theta_3 + L_2 \sin(\theta_3 + \theta_4) \tag{15}$$

Note

Important observations:

- θ_1 controls **only** the azimuthal angle in the XY plane – it appears only as $\cos\theta_1$ and $\sin\theta_1$ as multipliers.
- θ_3 and θ_4 are coupled, appearing both separately and as the sum $(\theta_3+\theta_4)$. Together they form a 2-DOF planar arm in the vertical plane.
- d_2 provides pure vertical translation and appears only in the z equation.
- The horizontal reach is $r = L_1 \cos\theta_3 + L_2 \cos(\theta_3 + \theta_4)$
- The robot has cylindrical symmetry about the z-axis due to θ_1.

3.6 End-Effector Orientation

The rotation matrix (upper-left 3×3 block of $^0T_5$) describes the end-effector orientation:

$$^0R_5 = \begin{bmatrix} c_1c_{34} & -c_1s_{34} & s_1 \\ s_1c_{34} & -s_1s_{34} & -c_1 \\ s_{34} & c_{34} & 0 \end{bmatrix} \tag{16}$$

The columns represent the end-effector frame axes expressed in the base frame:

- $\hat{x}_5 = [c_1 c_{34},\ s_1 c_{34},\ s_{34}]^T$ – pointing direction of the tool
- $\hat{y}_5 = [-c_1 s_{34},\ -s_1 s_{34},\ c_{34}]^T$
- $\hat{z}_5 = [s_1,\ -c_1,\ 0]^T$ – perpendicular to vertical plane containing the arm

4 Inverse Kinematics

4.1 Problem Statement

Given a target position (x, y, z), find the joint variables $(\theta_1, d_2, \theta_3, \theta_4)$.

From the forward kinematics equations:

$$x = \cos\theta_1 \cdot (L_1 \cos\theta_3 + L_2 \cos(\theta_3 + \theta_4)) \tag{17}$$
$$y = \sin\theta_1 \cdot (L_1 \cos\theta_3 + L_2 \cos(\theta_3 + \theta_4)) \tag{18}$$
$$z = L_0 + d_2 + L_1 \sin\theta_3 + L_2 \sin(\theta_3 + \theta_4) \tag{19}$$

Note

There are 4 unknown joint variables but only 3 position equations. This results in a **redundant system** with infinitely many solutions parameterized by d_2.

4.2 Derivation

4.2.1 Step 1: Solve for θ_1

From the x and y equations:

$$x = \cos\theta_1 \cdot r \tag{20}$$
$$y = \sin\theta_1 \cdot r \tag{21}$$

where $r = L_1 \cos\theta_3 + L_2 \cos(\theta_3 + \theta_4)$ is the horizontal reach.

Dividing these equations:

$$\frac{y}{x} = \tan\theta_1 \tag{22}$$

Result

$$\theta_1 = \operatorname{atan2}(y, x) \tag{23}$$

Note: If $x = y = 0$, the target is on the z-axis and θ_1 is arbitrary (singularity).

4.2.2 Step 2: Compute Horizontal Reach

The horizontal distance from the z-axis to the end-effector:

$$r = \sqrt{x^2 + y^2} \tag{24}$$

4.2.3 Step 3: 2R Planar Arm Subproblem

After θ_1 rotates the arm into the vertical plane containing the target, the problem reduces to a 2-link planar arm with links L_1 and L_2 reaching point (r, h) where:

- r = horizontal reach (computed above)
- $h = z - L_0 - d_2$ = vertical height above joint 3

The 2R arm equations become:

$$r = L_1 \cos\theta_3 + L_2 \cos(\theta_3 + \theta_4) \tag{25}$$
$$h = L_1 \sin\theta_3 + L_2 \sin(\theta_3 + \theta_4) \tag{26}$$

The distance from joint 3 to the target:

$$\rho = \sqrt{r^2 + h^2} \tag{27}$$

Reachability condition:

$$|L_1 - L_2| \leq \rho \leq L_1 + L_2 \tag{28}$$

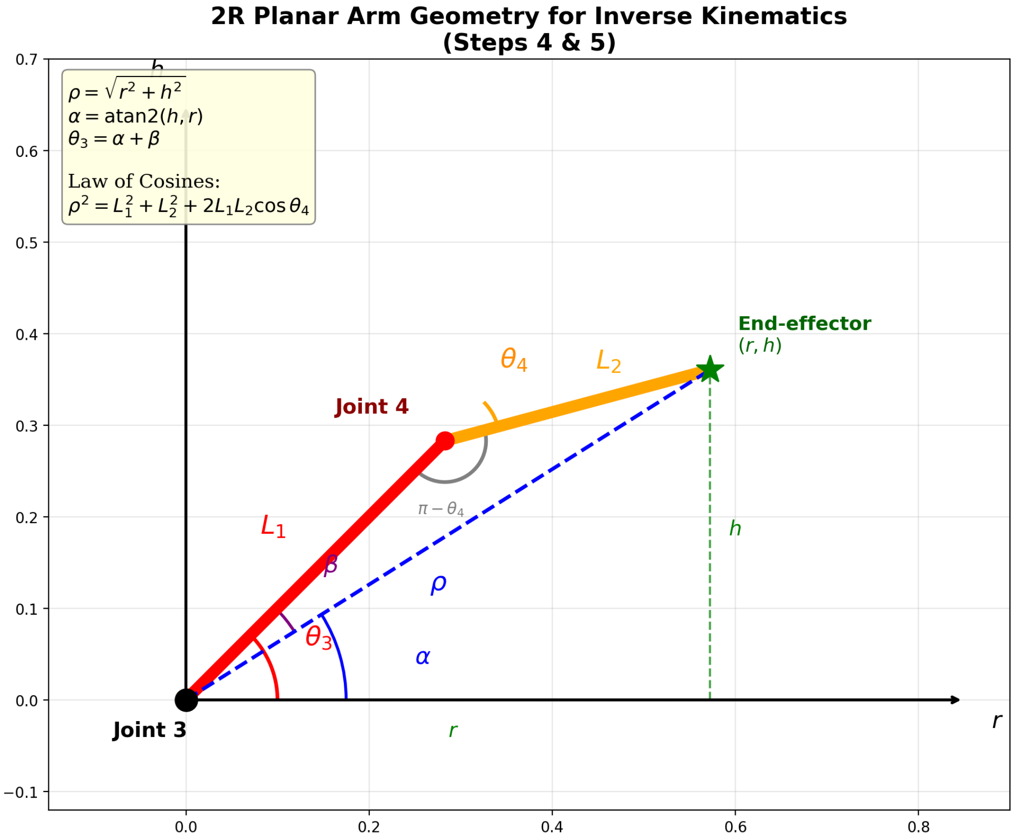

Figure 2: Geometry of the 2R planar arm for inverse kinematics. The triangle formed by Joint 3, Joint 4, and the End-effector is used to solve for θ_3 and θ_4 using the law of cosines. The angles α and β are used to compute $\theta_3 = \alpha + \beta$.

Key geometric relationships:

- $\rho = \sqrt{r^2 + h^2}$ is the distance from Joint 3 to the end-effector
- $\alpha = \mathrm{atan2}(h, r)$ is the angle from horizontal to the line ρ
- β is the angle between ρ and L_1 inside the triangle
- $\theta_3 = \alpha + \beta$ (the shoulder angle)
- The interior angle at Joint 4 is $(\pi - \theta_4)$, where θ_4 is the elbow angle

4.2.4 Step 4: Solve for θ_4 (Law of Cosines)

Using the law of cosines on the triangle formed by joint 3, joint 4, and the end-effector:

$$\rho^2 = L_1^2 + L_2^2 - 2L_1L_2\cos(\pi - \theta_4) = L_1^2 + L_2^2 + 2L_1L_2\cos\theta_4 \tag{29}$$

Solving for θ_4:

Result

$$\cos\theta_4 = \frac{\rho^2 - L_1^2 - L_2^2}{2L_1L_2} \tag{30}$$

$$\theta_4 = \pm\arccos\left(\frac{\rho^2 - L_1^2 - L_2^2}{2L_1L_2}\right) \tag{31}$$

The $\pm$ sign gives **two solutions**: "elbow up" $(+)$ and "elbow down" $(-)$.

4.2.5 Step 5: Solve for θ_3

From Figure 2, we can solve for θ_3 geometrically using two angles:

- α = angle from horizontal (r-axis) to the line ρ
- β = angle between ρ and L_1 (inside the triangle at Joint 3)

Step 1: Find α directly from target coordinates:

$$\alpha = \text{atan2}(h, r) \tag{32}$$

Step 2: Find β using the Law of Cosines on the triangle (sides L_1, L_2, ρ):

$$L_2^2 = L_1^2 + \rho^2 - 2L_1\rho\cos\beta \tag{33}$$

Solving for β:

$$\cos\beta = \frac{L_1^2 + \rho^2 - L_2^2}{2L_1\rho} \tag{34}$$

$$\beta = \arccos\left(\frac{L_1^2 + \rho^2 - L_2^2}{2L_1\rho}\right) \tag{35}$$

Step 3: Combine to get θ_3:
For **elbow down** ($\theta_4 < 0$): the arm bends below the line to target

$$\theta_3 = \alpha + \beta \tag{36}$$

For **elbow up** ($\theta_4 > 0$): the arm bends above the line to target

$$\theta_3 = \alpha - \beta \tag{37}$$

Result

Simple Geometric Formula:

$$\alpha = \text{atan2}(h, r), \quad \beta = \arccos\left(\frac{L_1^2 + \rho^2 - L_2^2}{2L_1\rho}\right) \tag{38}$$

$$\theta_3 = \alpha \pm \beta \quad \text{(sign depends on elbow configuration)} \tag{39}$$

4.2.6 Step 6: Solve for d_2

From the definition $h = z - L_0 - d_2$:

Result

$$d_2 = z - L_0 - h = z - L_0 - L_1 \sin\theta_3 - L_2 \sin(\theta_3 + \theta_4) \tag{40}$$

4.3 Redundancy Resolution

Since there are infinitely many solutions parameterized by d_2, a **redundancy resolution strategy** is needed. Common approaches include:

1. **Fix** d_2: Set d_2 to a constant value (e.g., $d_2 = 0$ or mid-range), then solve for θ_3 and θ_4.
2. **Minimize** d_2: Choose the smallest valid d_2 to keep the arm low.
3. **Optimize a criterion**: Minimize joint velocities, maximize manipulability, or avoid obstacles.

For a given d_2, the valid range is determined by the reachability constraint. The point (r, h) must satisfy:

$$|L_1 - L_2| \leq \sqrt{r^2 + (z - L_0 - d_2)^2} \leq L_1 + L_2 \tag{41}$$

4.4 Summary of IK Equations

Result

Input: Target position (x, y, z)
Algorithm:

1. $\theta_1 = \text{atan2}(y, x)$
2. $r = \sqrt{x^2 + y^2}$
3. Choose d_2 from valid range (redundancy resolution)
4. $h = z - L_0 - d_2, \quad \rho = \sqrt{r^2 + h^2}$
5. $\cos\theta_4 = \dfrac{\rho^2 - L_1^2 - L_2^2}{2L_1L_2}, \quad \theta_4 = \pm\arccos(\cos\theta_4)$
6. $\alpha = \text{atan2}(h, r), \quad \beta = \arccos\left(\frac{L_1^2+\rho^2-L_2^2}{2L_1\rho}\right), \quad \theta_3 = \alpha \pm \beta$

Output: Joint values $(\theta_1, d_2, \theta_3, \theta_4)$

4.5 Numerical Example

Given:

- Target position: $(x, y, z) = (0.4, 0.2, 0.85)$ m
- Robot parameters: $L_0 = 0.5$ m, $L_1 = 0.4$ m, $L_2 = 0.3$ m

Step 1: Solve for θ_1

$$\theta_1 = \text{atan2}(0.2, 0.4) = \text{atan2}(1, 2) = 26.57 \tag{42}$$

Step 2: Compute horizontal reach

$$r = \sqrt{x^2 + y^2} = \sqrt{0.4^2 + 0.2^2} = \sqrt{0.20} = 0.447 \text{ m} \tag{43}$$

Step 3: Choose d_2 (redundancy resolution)
Select $d_2 = 0.15$ m (mid-range value).
Step 4: Compute h and ρ

$$h = z - L_0 - d_2 = 0.85 - 0.5 - 0.15 = 0.20 \text{ m} \tag{44}$$

$$\rho = \sqrt{r^2 + h^2} = \sqrt{0.447^2 + 0.20^2} = \sqrt{0.240} = 0.490 \text{ m} \tag{45}$$

Step 5: Check reachability

$$|L_1 - L_2| \leq \rho \leq L_1 + L_2 \implies |0.4 - 0.3| \leq 0.490 \leq 0.4 + 0.3 \implies 0.1 \leq 0.490 \leq 0.7 \quad \checkmark \tag{46}$$

Step 6: Solve for θ_4

$$\cos\theta_4 = \frac{\rho^2 - L_1^2 - L_2^2}{2L_1L_2} = \frac{0.490^2 - 0.4^2 - 0.3^2}{2(0.4)(0.3)} = \frac{0.240 - 0.160 - 0.090}{0.240} = \frac{-0.010}{0.240} = -0.042 \tag{47}$$

$$\theta_4 = \arccos(-0.042) = \pm 92.4 \tag{48}$$

Choosing elbow down configuration: $\theta_4 = -92.4$
Step 7: Solve for θ_3
Using the geometric method:

$$\alpha = \text{atan2}(h, r) = \text{atan2}(0.20, 0.447) = 24.1 \tag{49}$$

$$\beta = \arccos\left(\frac{L_1^2 + \rho^2 - L_2^2}{2L_1\rho}\right) = \arccos\left(\frac{0.4^2 + 0.490^2 - 0.3^2}{2(0.4)(0.490)}\right) \tag{50}$$

$$= \arccos\left(\frac{0.160 + 0.240 - 0.090}{0.392}\right) = \arccos(0.791) = 37.7 \tag{51}$$

For elbow down ($\theta_4 < 0$):

$$\theta_3 = \alpha + \beta = 24.1 + 37.7 = 61.8 \tag{52}$$

Result

IK Solution:

Joint	Variable	Value
1	θ_1	26.57
2	d_2	0.15 m
3	θ_3	61.8
4	θ_4	-92.4

4.5.1 Verification using Forward Kinematics

To verify, substitute the joint values back into the FK equations:

$$\theta_3 + \theta_4 = 61.8 - 92.4 = -30.6 \tag{53}$$

Check x:

$$x = \cos\theta_1 \cdot [L_1 \cos\theta_3 + L_2 \cos(\theta_3 + \theta_4)] \tag{54}$$
$$= \cos(26.57) \cdot [0.4 \times \cos(61.8) + 0.3 \times \cos(-30.6)] \tag{55}$$
$$= 0.894 \cdot [0.4 \times 0.472 + 0.3 \times 0.861] \tag{56}$$
$$= 0.894 \cdot [0.189 + 0.258] = 0.894 \times 0.447 = \mathbf{0.400}\text{ m} \quad \checkmark \tag{57}$$

Check y:

$$y = \sin\theta_1 \cdot [L_1 \cos\theta_3 + L_2 \cos(\theta_3 + \theta_4)] \tag{58}$$
$$= \sin(26.57) \times 0.447 = 0.447 \times 0.447 = \mathbf{0.200}\text{ m} \quad \checkmark \tag{59}$$

Check z:

$$z = L_0 + d_2 + L_1 \sin\theta_3 + L_2 \sin(\theta_3 + \theta_4) \tag{60}$$
$$= 0.5 + 0.15 + 0.4 \times \sin(61.8) + 0.3 \times \sin(-30.6) \tag{61}$$
$$= 0.65 + 0.4 \times 0.881 + 0.3 \times (-0.509) \tag{62}$$
$$= 0.65 + 0.352 - 0.153 = \mathbf{0.849}\text{ m} \approx 0.85\text{ m} \quad \checkmark \tag{63}$$

Note

The small error (≈ 1 mm) is due to rounding in intermediate calculations. The IK solution is verified to be correct.

4.6 Multiple Solutions

The inverse kinematics yields multiple solutions:

- **Two solutions** for each d_2 value: elbow up ($\theta_4 > 0$) and elbow down ($\theta_4 < 0$)
- **Infinite d_2 values** possible within the valid range
- **Additional solution** $\theta_1 + \pi$ if the target is on the z-axis

The two configurations (elbow up and elbow down) correspond to the $\pm$ sign in the θ_4 equation. Both configurations place the end-effector at the same target position (r, h) in the vertical plane.

4.7 Singularities

The inverse kinematics has singularities at:

- $x = y = 0$: Target is on the z-axis, θ_1 is arbitrary
- $\rho = L_1 + L_2$: Arm fully stretched, $\theta_4 = 0$ (boundary singularity)
- $\rho = |L_1 - L_2|$: Arm fully folded, $\theta_4 = \pm\pi$ (boundary singularity)

At boundary singularities, there is only one solution (elbow up and elbow down coincide).

5 Work Envelope Analysis

5.1 Cylindrical Coordinate Representation

Due to the rotational symmetry about the z-axis (from joint 1), the workspace is best analyzed in cylindrical coordinates (r, ϕ, z).

Converting the forward kinematics equations:

$$r = \sqrt{x^2 + y^2} = L_1 \cos\theta_3 + L_2 \cos(\theta_3 + \theta_4) \tag{64}$$

$$\phi = \theta_1 \quad \text{(full 360 rotation)} \tag{65}$$

$$z = L_0 + d_2 + L_1 \sin\theta_3 + L_2 \sin(\theta_3 + \theta_4) \tag{66}$$

Note

The workspace is a **solid of revolution** about the z-axis. We only need to analyze the r-z profile (2D cross-section) and revolve it around the z-axis.

5.2 2R Arm Subworkspace

For a fixed value of d_2, the end-effector traces the workspace of a 2-link planar arm with links L_1 and L_2. Defining $h = z - L_0 - d_2$:

$$r = L_1 \cos\theta_3 + L_2 \cos(\theta_3 + \theta_4) \tag{67}$$

$$h = L_1 \sin\theta_3 + L_2 \sin(\theta_3 + \theta_4) \tag{68}$$

This traces an **annulus** (ring) in the r-h plane:

- Outer radius: $L_1 + L_2$ (arm fully stretched, $\theta_4 = 0$)
- Inner radius: $|L_1 - L_2|$ (arm fully folded, $\theta_4 = \pm\pi$)

The annulus equation:

$$(L_1 - L_2)^2 \leq r^2 + h^2 \leq (L_1 + L_2)^2 \tag{69}$$

5.3 Workspace Boundary Equation

As d_2 varies from $d_{2,min}$ to $d_{2,max}$, the annulus center moves vertically. The workspace boundary in the r-z plane satisfies:

Result

$$r^2 + (z - L_0 - d_2)^2 = (L_1 + L_2)^2 \quad \text{(outer boundary)} \tag{70}$$

$$r^2 + (z - L_0 - d_2)^2 = (L_1 - L_2)^2 \quad \text{(inner boundary, if void exists)} \tag{71}$$

These describe spheres of radius $(L_1 + L_2)$ and $|L_1 - L_2|$ centered at $z = L_0 + d_2$, swept vertically as d_2 varies.

5.4 Void Condition

A **void** (hollow region) exists inside the workspace if the prismatic joint travel is insufficient to cover the inner boundary:

Result

Void Condition:

$$(d_{2,max} - d_{2,min}) < 2|L_1 - L_2| \tag{72}$$

When this condition is true, there is a toroidal void region where the robot cannot reach.

5.5 Workspace Dimensions

The workspace bounds are:

$$r_{min} = 0 \tag{73}$$
$$r_{max} = L_1 + L_2 \tag{74}$$
$$z_{min} = L_0 + d_{2,min} - (L_1 + L_2) \tag{75}$$
$$z_{max} = L_0 + d_{2,max} + (L_1 + L_2) \tag{76}$$
$$\phi \in [0, 2\pi] \tag{77}$$

The total workspace volume can be computed by revolving the r-z cross-section about the z-axis.

6 Jacobian Analysis

6.1 Definition

The Jacobian matrix relates joint velocities to end-effector velocities:

$$\begin{bmatrix} \vec{v} \\ \vec{\omega} \end{bmatrix} = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} \dot{\vec{q}} = J\dot{\vec{q}} \tag{78}$$

where $\dot{\vec{q}} = [\dot{\theta}_1, \dot{d}_2, \dot{\theta}_3, \dot{\theta}_4]^T$.

- J_v (3×4): Linear velocity Jacobian
- J_ω (3×4): Angular velocity Jacobian

6.2 Linear Velocity Jacobian J_v

The linear velocity Jacobian is obtained by differentiating the position equations with respect to each joint variable.

From the forward kinematics:

$$x = \cos\theta_1 \cdot (L_1 \cos\theta_3 + L_2 \cos(\theta_3 + \theta_4)) \tag{79}$$
$$y = \sin\theta_1 \cdot (L_1 \cos\theta_3 + L_2 \cos(\theta_3 + \theta_4)) \tag{80}$$
$$z = L_0 + d_2 + L_1 \sin\theta_3 + L_2 \sin(\theta_3 + \theta_4) \tag{81}$$

Let $R = L_1 \cos\theta_3 + L_2 \cos(\theta_3 + \theta_4)$ (horizontal reach) and $H = L_1 \sin\theta_3 + L_2 \sin(\theta_3 + \theta_4)$ (vertical component).

Computing partial derivatives:

$$J_v = \begin{bmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial d_2} & \frac{\partial x}{\partial \theta_3} & \frac{\partial x}{\partial \theta_4} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial d_2} & \frac{\partial y}{\partial \theta_3} & \frac{\partial y}{\partial \theta_4} \\ \frac{\partial z}{\partial \theta_1} & \frac{\partial z}{\partial d_2} & \frac{\partial z}{\partial \theta_3} & \frac{\partial z}{\partial \theta_4} \end{bmatrix} \tag{82}$$

Result

$$J_v = \begin{bmatrix} -\sin\theta_1 \cdot R & 0 & -\cos\theta_1 \cdot H & -\cos\theta_1 \cdot L_2 \sin(\theta_3+\theta_4) \\ \cos\theta_1 \cdot R & 0 & -\sin\theta_1 \cdot H & -\sin\theta_1 \cdot L_2 \sin(\theta_3+\theta_4) \\ 0 & 1 & R & L_2 \cos(\theta_3+\theta_4) \end{bmatrix} \tag{83}$$

where $R = L_1 c_3 + L_2 c_{34}$ and $H = L_1 s_3 + L_2 s_{34}$

6.3 Angular Velocity Jacobian J_ω

The angular velocity Jacobian is constructed from the rotation axes of each joint:

- Joint 1 (revolute): rotates about $\hat{z}_0 = [0, 0, 1]^T$
- Joint 2 (prismatic): no angular contribution (column of zeros)
- Joint 3 (revolute): rotates about $\hat{z}_2 = [\sin\theta_1, -\cos\theta_1, 0]^T$
- Joint 4 (revolute): rotates about $\hat{z}_3 = [\sin\theta_1, -\cos\theta_1, 0]^T$

Note

The rotation axes $\hat{z}_2$ and $\hat{z}_3$ are parallel because both joints 3 and 4 rotate about axes perpendicular to the vertical plane containing the arm.

Result

$$J_\omega = \begin{bmatrix} 0 & 0 & \sin\theta_1 & \sin\theta_1 \\ 0 & 0 & -\cos\theta_1 & -\cos\theta_1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{84}$$

6.4 Full Jacobian Matrix

The complete 6×4 Jacobian is:

$$J = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} = \begin{bmatrix} -s_1 R & 0 & -c_1 H & -c_1 L_2 s_{34} \\ c_1 R & 0 & -s_1 H & -s_1 L_2 s_{34} \\ 0 & 1 & R & L_2 c_{34} \\ 0 & 0 & s_1 & s_1 \\ 0 & 0 & -c_1 & -c_1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{85}$$

where the shorthand notation is:

- $s_1 = \sin\theta_1, \quad c_1 = \cos\theta_1$
- $s_{34} = \sin(\theta_3+\theta_4), \quad c_{34} = \cos(\theta_3+\theta_4)$
- $R = L_1 \cos\theta_3 + L_2 \cos(\theta_3+\theta_4)$ (horizontal reach)
- $H = L_1 \sin\theta_3 + L_2 \sin(\theta_3+\theta_4)$ (vertical component)

Fully Expanded Form:

$$J = \begin{bmatrix} -\sin\theta_1(L_1c_3 + L_2c_{34}) & 0 & -\cos\theta_1(L_1s_3 + L_2s_{34}) & -\cos\theta_1 \cdot L_2\sin(\theta_3+\theta_4) \\ \cos\theta_1(L_1c_3 + L_2c_{34}) & 0 & -\sin\theta_1(L_1s_3 + L_2s_{34}) & -\sin\theta_1 \cdot L_2\sin(\theta_3+\theta_4) \\ 0 & 1 & L_1\cos\theta_3 + L_2\cos(\theta_3+\theta_4) & L_2\cos(\theta_3+\theta_4) \\ 0 & 0 & \sin\theta_1 & \sin\theta_1 \\ 0 & 0 & -\cos\theta_1 & -\cos\theta_1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{86}$$

where $s_3 = \sin\theta_3$, $c_3 = \cos\theta_3$, $s_{34} = \sin(\theta_3+\theta_4)$, $c_{34} = \cos(\theta_3+\theta_4)$.

6.5 Singularity Analysis

Singularities occur when the Jacobian J_v loses rank (rank < 3). This means the robot loses the ability to move in certain directions.

6.5.1 Case 1: $R = 0$ (Arm Vertical)

When $R = L_1\cos\theta_3 + L_2\cos(\theta_3+\theta_4) = 0$:

- The first column of J_v becomes zero
- The robot cannot move in the x-y plane by rotating θ_1
- Occurs when the arm points straight up or down

6.5.2 Case 2: $\sin(\theta_3+\theta_4) = 0$

When $\theta_3 + \theta_4 = 0$ or π:

- Columns 3 and 4 become linearly dependent
- The arm is fully stretched ($\theta_4 = 0$) or folded back ($\theta_4 = \pm\pi$)
- Lost independent control of certain motion directions

6.5.3 Case 3: Workspace Boundary

At the workspace boundary where $\rho = L_1 + L_2$ or $\rho = |L_1 - L_2|$:

- These correspond to $\theta_4 = 0$ or $\theta_4 = \pm\pi$
- The Jacobian becomes rank-deficient
- Motion is restricted to tangent of the boundary

6.6 Force/Torque Relationship

The static relationship between end-effector forces and joint torques/forces is given by:

$$\vec{\tau} = J^T \begin{bmatrix} \vec{F} \\ \vec{M} \end{bmatrix} \tag{87}$$

For pure forces using the linear velocity Jacobian:

$$\begin{bmatrix} \tau_1 \\ F_{d_2} \\ \tau_3 \\ \tau_4 \end{bmatrix} = J_v^T \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} \tag{88}$$

where τ_1, τ_3, τ_4 are joint torques (Nm) and F_{d_2} is the force on the prismatic joint (N).

6.7 Force/Torque Calculation Examples

The following examples demonstrate the calculation of joint torques from applied end-effector forces using the formula $\boldsymbol{\tau} = \mathbf{J}_v^T \mathbf{F}$.

Note

Robot Parameters:

- $L_0 = 0.5$ m (base height)
- $L_1 = 0.4$ m (link 1 length)
- $L_2 = 0.3$ m (tool/end-effector length)

6.7.1 Example 1: Gravitational Load

Given:

- Configuration: $\theta_1 = 30$, $d_2 = 0.15$ m, $\theta_3 = 45$, $\theta_4 = 30$
- Robot parameters: $L_0 = 0.5$ m, $L_1 = 0.4$ m, $L_2 = 0.3$ m
- Applied force: $\mathbf{F} = \begin{bmatrix} 0 \\ 0 \\ -10 \end{bmatrix}$ N (gravitational load)

Step 1: End-Effector Position
Using the forward kinematics equations with the given configuration:

$$\mathbf{p}_{EE} = \begin{bmatrix} 0.312 \\ 0.180 \\ 1.223 \end{bmatrix} \text{ m}$$

Step 2: Compute the Linear Velocity Jacobian
At this configuration, the linear velocity Jacobian (computed using the Robotics Toolbox) is:

$$\mathbf{J}_v = \begin{bmatrix} -0.180 & 0.000 & -0.496 & -0.251 \\ 0.312 & 0.000 & -0.286 & -0.145 \\ 0.000 & 1.000 & 0.360 & 0.078 \end{bmatrix} \tag{89}$$

Step 3: Compute the Jacobian Transpose

$$\mathbf{J}_v^T = \begin{bmatrix} -0.180 & 0.312 & 0.000 \\ 0.000 & 0.000 & 1.000 \\ -0.496 & -0.286 & 0.360 \\ -0.251 & -0.145 & 0.078 \end{bmatrix} \tag{90}$$

Step 4: Calculate Joint Torques

$$\boldsymbol{\tau} = \mathbf{J}_v^T\mathbf{F} = \begin{bmatrix} -0.180 & 0.312 & 0.000 \\ 0.000 & 0.000 & 1.000 \\ -0.496 & -0.286 & 0.360 \\ -0.251 & -0.145 & 0.078 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -10 \end{bmatrix} \tag{91}$$

Computing each element:

$$\tau_1 = (-0.180)(0) + (0.312)(0) + (0.000)(-10) = 0.00 \text{ Nm} \tag{92}$$
$$F_2 = (0.000)(0) + (0.000)(0) + (1.000)(-10) = -10.00 \text{ N} \tag{93}$$
$$\tau_3 = (-0.496)(0) + (-0.286)(0) + (0.360)(-10) = -3.60 \text{ Nm} \tag{94}$$
$$\tau_4 = (-0.251)(0) + (-0.145)(0) + (0.078)(-10) = -0.78 \text{ Nm} \tag{95}$$

Result

Example 1 Results:

Joint	Type	Value	Interpretation
τ_1	Revolute	0.00 Nm	Force passes through rotation axis
F_2	Prismatic	−10.00 N	Supports full vertical load
τ_3	Revolute	−3.60 Nm	Shoulder resists gravity
τ_4	Revolute	−0.78 Nm	Elbow resists gravity

The prismatic joint (Joint 2) bears the entire vertical load of 10 N since its axis is aligned with the Z-axis.

6.7.2 Example 2: Horizontal Push Force

Given:

- Configuration: $\theta_1 = 0$, $d_2 = 0.15$ m, $\theta_3 = 60$, $\theta_4 = -30$
- Robot parameters: $L_0 = 0.5$ m, $L_1 = 0.4$ m, $L_2 = 0.3$ m
- Applied force: $\mathbf{F} = \begin{bmatrix} 15 \\ 0 \\ 0 \end{bmatrix}$ N (horizontal push in X-direction)

Step 1: End-Effector Position
Using the forward kinematics equations with the given configuration:

$$\mathbf{p}_{EE} = \begin{bmatrix} 0.460 \\ 0.000 \\ 1.146 \end{bmatrix} \text{ m}$$

Step 2: Compute the Linear Velocity Jacobian
At this configuration, the linear velocity Jacobian is:

$$\mathbf{J}_v = \begin{bmatrix} 0.000 & 0.000 & -0.496 & -0.150 \\ 0.460 & 0.000 & 0.000 & 0.000 \\ 0.000 & 1.000 & 0.460 & 0.260 \end{bmatrix} \tag{96}$$

Step 3: Compute the Jacobian Transpose

$$\mathbf{J}_v^T = \begin{bmatrix} 0.000 & 0.460 & 0.000 \\ 0.000 & 0.000 & 1.000 \\ -0.496 & 0.000 & 0.460 \\ -0.150 & 0.000 & 0.260 \end{bmatrix} \tag{97}$$

Step 4: Calculate Joint Torques

$$\boldsymbol{\tau} = \mathbf{J}_v^T \mathbf{F} = \begin{bmatrix} 0.000 & 0.460 & 0.000 \\ 0.000 & 0.000 & 1.000 \\ -0.496 & 0.000 & 0.460 \\ -0.150 & 0.000 & 0.260 \end{bmatrix} \begin{bmatrix} 15 \\ 0 \\ 0 \end{bmatrix} \tag{98}$$

Computing each element:

$$\tau_1 = (0.000)(15) + (0.460)(0) + (0.000)(0) = 0.00 \text{ Nm} \tag{99}$$
$$F_2 = (0.000)(15) + (0.000)(0) + (1.000)(0) = 0.00 \text{ N} \tag{100}$$
$$\tau_3 = (-0.496)(15) + (0.000)(0) + (0.460)(0) = -7.45 \text{ Nm} \tag{101}$$
$$\tau_4 = (-0.150)(15) + (0.000)(0) + (0.260)(0) = -2.25 \text{ Nm} \tag{102}$$

Result

Example 2 Results:

Joint	Type	Value	Interpretation
τ_1	Revolute	0.00 Nm	Force in radial direction
F_2	Prismatic	0.00 N	Push perpendicular to axis
τ_3	Revolute	-7.45 Nm	Shoulder bears maximum load
τ_4	Revolute	-2.25 Nm	Elbow with shorter moment arm

The shoulder joint (τ_3) experiences the highest torque due to the full moment arm created by the extended arm configuration.

7 Manipulator Dynamics

This section develops the dynamic equations of motion for the RPRR robot manipulator, relating joint torques/forces to motion (velocities and accelerations).

7.1 Link Dynamic Properties

Before deriving the dynamics equations, we define the physical properties of each link:

Link	Mass (kg)	Length (m)	COM Location	Description
0 (Base)	m_0	$L_0 = 0.5$	Fixed	Base column
1 (Prismatic)	$m_1 = 2.0$	variable d_2	Center of slider	Vertical slider
2 (Link 1)	$m_2 = 1.5$	$L_1 = 0.4$	$L_1/2$ from joint 3	Upper arm
3 (Link 2/Tool)	$m_3 = 0.8$	$L_2 = 0.3$	$L_2/2$ from joint 4	Forearm + tool

Table 3: Link mass properties for dynamic analysis

7.2 Velocity Equations

The relationship between joint velocities and end-effector velocities is established through the Jacobian matrix.

7.2.1 End-Effector Linear Velocity

The linear velocity of the end-effector is given by:

$$\dot{\mathbf{p}} = \mathbf{J}_v \dot{\mathbf{q}} \tag{103}$$

where $\dot{\mathbf{q}} = \begin{bmatrix} \dot{\theta}_1 & \dot{d}_2 & \dot{\theta}_3 & \dot{\theta}_4 \end{bmatrix}^T$ is the joint velocity vector.
Expanding using the Jacobian from Section 6:

$$\dot{x} = -R \sin\theta_1 \cdot \dot{\theta}_1 - H \cos\theta_1 \cdot \dot{\theta}_3 - L_2 \sin(\theta_3 + \theta_4) \cos\theta_1 \cdot \dot{\theta}_4 \tag{104}$$
$$\dot{y} = R \cos\theta_1 \cdot \dot{\theta}_1 - H \sin\theta_1 \cdot \dot{\theta}_3 - L_2 \sin(\theta_3 + \theta_4) \sin\theta_1 \cdot \dot{\theta}_4 \tag{105}$$
$$\dot{z} = \dot{d}_2 + R \cdot \dot{\theta}_3 + L_2 \cos(\theta_3 + \theta_4) \cdot \dot{\theta}_4 \tag{106}$$

where:

- $R = L_1 \cos\theta_3 + L_2 \cos(\theta_3 + \theta_4)$ is the horizontal reach
- $H = L_1 \sin\theta_3 + L_2 \sin(\theta_3 + \theta_4)$ is the vertical component

Note

The $\dot{d}_2$ term only appears in $\dot{z}$ because the prismatic joint moves vertically along the z-axis. The partial derivatives $\frac{\partial x}{\partial d_2} = \frac{\partial y}{\partial d_2} = 0$ since x and y do not depend on d_2.

7.2.2 End-Effector Angular Velocity

The angular velocity of the end-effector is:

$$\boldsymbol{\omega} = \mathbf{J}_\omega \dot{\mathbf{q}} = \begin{bmatrix} (\dot{\theta}_3 + \dot{\theta}_4) \sin\theta_1 \\ -(\dot{\theta}_3 + \dot{\theta}_4) \cos\theta_1 \\ \dot{\theta}_1 \end{bmatrix} \tag{107}$$

This represents:

- $\omega_z = \dot{\theta}_1$: rotation about the base (vertical) axis
- ω_x, ω_y: combined rotation from joints 3 and 4 about their parallel axes

7.3 Acceleration Equations

The acceleration of the end-effector is obtained by differentiating the velocity equation:

$$\ddot{\mathbf{p}} = \mathbf{J}_v \ddot{\mathbf{q}} + \dot{\mathbf{J}}_v \dot{\mathbf{q}} \tag{108}$$

where $\dot{\mathbf{J}}_v$ is the time derivative of the Jacobian, computed using the chain rule:

$$\dot{\mathbf{J}}_v = \frac{\partial \mathbf{J}_v}{\partial \theta_1} \dot{\theta}_1 + \frac{\partial \mathbf{J}_v}{\partial d_2} \dot{d}_2 + \frac{\partial \mathbf{J}_v}{\partial \theta_3} \dot{\theta}_3 + \frac{\partial \mathbf{J}_v}{\partial \theta_4} \dot{\theta}_4 \tag{109}$$

Analysis of each partial derivative:

- $\dfrac{\partial \mathbf{J}_v}{\partial \theta_1} \neq \mathbf{0}$: The Jacobian contains $\sin\theta_1$ and $\cos\theta_1$ terms
- $\dfrac{\partial \mathbf{J}_v}{\partial d_2} = \mathbf{0}$: The Jacobian does not contain d_2

- $\frac{\partial \mathbf{J}_v}{\partial \theta_3} \neq \mathbf{0}$: The Jacobian contains θ_3 through R and H
- $\frac{\partial \mathbf{J}_v}{\partial \theta_4} \neq \mathbf{0}$: The Jacobian contains θ_4 through R, H, and $\sin(\theta_3 + \theta_4)$

The $\dot{\mathbf{J}}_v \dot{\mathbf{q}}$ term represents centripetal and Coriolis accelerations that arise even when joint accelerations are zero.

7.4 Determination of Inertial Forces

7.4.1 Link Inertia Tensors

For slender rod approximation, the moment of inertia about the center of mass is:

$$I = \frac{1}{12} m L^2 \tag{110}$$

Link	Mass (kg)	$I_{xx} = I_{yy}$ (kg·m²)	COM from joint (m)
1 (Slider)	2.0	≈ 0	0
2 (Upper arm)	1.5	0.020	$L_1/2 = 0.2$
3 (Forearm)	0.8	0.006	$L_2/2 = 0.15$
Total	**4.3**	-	-

7.4.2 Linear Inertial Forces

The inertial force on each link is:

$$\mathbf{F}_i = m_i \mathbf{a}_{ci} \tag{111}$$

7.4.3 Rotational Inertial Torques

The inertial torque on each link follows Euler's equation:

$$\mathbf{N}_i = \mathbf{I}_{ci} \dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times (\mathbf{I}_{ci} \boldsymbol{\omega}_i) \tag{112}$$

7.5 Numerical Examples

7.5.1 Example 1: Constant Velocity Motion

Given:

- Configuration: $\theta_1 = 30$, $d_2 = 0.15$ m, $\theta_3 = 45$, $\theta_4 = 30$
- Joint velocities: $\dot{\theta}_1 = 10/\text{s}$, $\dot{d}_2 = 0.05$ m/s, $\dot{\theta}_3 = 20/\text{s}$, $\dot{\theta}_4 = 15/\text{s}$
- Joint accelerations: $\ddot{\mathbf{q}} = \mathbf{0}$

Results:

$$\dot{\mathbf{p}} = \begin{bmatrix} -0.270 \\ -0.083 \\ 0.196 \end{bmatrix} \text{ m/s}, \quad |\dot{\mathbf{p}}| = 0.344 \text{ m/s} \tag{113}$$

$$\ddot{\mathbf{p}} = \dot{\mathbf{J}}_v \dot{\mathbf{q}} = \begin{bmatrix} -0.061 \\ -0.136 \\ -0.103 \end{bmatrix} \text{ m/s}^2, \quad |\ddot{\mathbf{p}}| = 0.181 \text{ m/s}^2 \tag{114}$$

Note

Even with zero joint accelerations, the end-effector experiences acceleration due to Coriolis and centripetal effects ($\dot{\mathbf{J}}\dot{\mathbf{q}}$ term).

7.5.2 Example 2: Accelerating from Rest

Given:

- Configuration: $\theta_1 = 0$, $d_2 = 0.10$ m, $\theta_3 = 0$, $\theta_4 = 0$
- Joint velocities: $\dot{\mathbf{q}} = \mathbf{0}$
- Joint accelerations: $\ddot{\theta}_1 = 30/\text{s}^2$, $\ddot{d}_2 = 0.1$ m/s^2, $\ddot{\theta}_3 = 45/\text{s}^2$, $\ddot{\theta}_4 = 30/\text{s}^2$

Results:

$$\ddot{\mathbf{p}} = \mathbf{J}_v\ddot{\mathbf{q}} = \begin{bmatrix} 0.000 \\ 0.367 \\ 0.807 \end{bmatrix} \text{ m/s}^2, \quad |\ddot{\mathbf{p}}| = 0.886 \text{ m/s}^2 \tag{115}$$

Inertial Forces:

$$\mathbf{F}_1 = m_1\ddot{d}_2\hat{\mathbf{z}} = 0.20 \text{ N (vertical)} \tag{116}$$

$$\mathbf{F}_2 = m_2\mathbf{a}_{c2} = \begin{bmatrix} -0.157 \\ 0 \\ 0.386 \end{bmatrix} \text{ N} \tag{117}$$

7.6 Equations of Motion

The complete dynamic model of the RPRR manipulator is expressed in the standard form:

$$\boxed{\boldsymbol{\tau} = \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q})} \tag{118}$$

where:

- $\boldsymbol{\tau} = [\tau_1, F_2, \tau_3, \tau_4]^T$ – Joint torques/forces (τ in Nm, F in N)
- $\mathbf{M}(\mathbf{q}) \in \mathbb{R}^{4\times4}$ – Mass/Inertia matrix (symmetric, positive definite)
- $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbb{R}^{4\times4}$ – Coriolis/Centrifugal matrix
- $\mathbf{g}(\mathbf{q}) \in \mathbb{R}^4$ – Gravity vector

7.6.1 Mass/Inertia Matrix $\mathbf{M}(\mathbf{q})$

At configuration $\theta_1 = 30$, $d_2 = 0.15$ m, $\theta_3 = 45$, $\theta_4 = 30$:

$$\mathbf{M}(\mathbf{q}) = \begin{bmatrix} 0.1232 & 0 & 0 & 0 \\ 0 & 4.3000 & 0.4695 & 0.0311 \\ 0 & 0.4695 & 0.3151 & 0.0656 \\ 0 & 0.0311 & 0.0656 & 0.0240 \end{bmatrix} \tag{119}$$

Properties:

- Symmetric: $\mathbf{M} = \mathbf{M}^T$
- Positive definite: $\mathbf{x}^T\mathbf{M}\mathbf{x} > 0$ for all $\mathbf{x} \neq 0$
- $M_{22} = 4.3$ kg is the total mass (prismatic joint supports all links)
- Off-diagonal terms represent coupling between joints

7.6.2 Coriolis/Centrifugal Matrix $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$

At the same configuration with $\dot{\theta}_1 = 20/\text{s}$, $\dot{d}_2 = 0.05$ m/s, $\dot{\theta}_3 = 30/\text{s}$, $\dot{\theta}_4 = 20/\text{s}$:

$$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} -0.0929 & 0 & -0.0529 & -0.0135 \\ 0 & 0 & -0.3307 & -0.1012 \\ 0.0529 & 0 & -0.0084 & -0.0209 \\ 0.0135 & 0 & 0.0126 & 0 \end{bmatrix} \tag{120}$$

The Coriolis matrix satisfies: $\dot{\mathbf{M}} - 2\mathbf{C}$ is skew-symmetric (energy conservation).

7.6.3 Gravity Vector $\mathbf{g}(\mathbf{q})$

$$\mathbf{g}(\mathbf{q}) = \begin{bmatrix} 0 \\ 42.183 \\ 4.605 \\ 0.305 \end{bmatrix} \quad \text{(units: Nm for } \tau\text{, N for } F\text{)} \tag{121}$$

The large value $g_2 = 42.18$ N represents the gravitational force on the prismatic joint supporting the total mass ($m_1 + m_2 + m_3 = 4.3$ kg, and $4.3 \times 9.81 = 42.18$ N).

7.6.4 Example: Full Dynamics Calculation

Given:

- $\mathbf{q} = [0, 0.10 \text{ m}, 30, 0]^T$
- $\dot{\mathbf{q}} = [10/\text{s}, 0.02 \text{ m/s}, 15/\text{s}, 10/\text{s}]^T$
- $\ddot{\mathbf{q}} = [50/\text{s}^2, 0.1 \text{ m/s}^2, 60/\text{s}^2, 40/\text{s}^2]^T$

Result:

Joint	$\mathbf{M}\ddot{\mathbf{q}}$	$\mathbf{C}\dot{\mathbf{q}}$	$\mathbf{g}$	$\boldsymbol{\tau}$
τ_1	0.215 Nm	−0.015 Nm	0 Nm	**0.200 Nm**
F_2	1.174 N	−0.033 N	42.183 N	**43.324 N**
τ_3	0.458 Nm	0.004 Nm	6.287 Nm	**6.749 Nm**
τ_4	0.103 Nm	0.001 Nm	1.020 Nm	**1.123 Nm**

Note

For this RPRR robot, gravity dominates the required torques/forces. The Coriolis terms are relatively small at typical operating speeds. The inertia terms become significant during rapid accelerations.

8 Trajectory Generation

Trajectory generation creates smooth motion profiles that move the robot from an initial configuration q_0 to a final configuration q_f over a specified time interval T.

8.1 Cubic Polynomial Trajectory

A cubic polynomial is the simplest trajectory that satisfies position constraints at both endpoints with zero velocity (smooth start and stop):

$$\boxed{q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3} \tag{122}$$

8.1.1 Boundary Conditions

The trajectory must satisfy four constraints:

$$q(0) = q_0 \quad \text{(start position)} \tag{123}$$
$$q(T) = q_f \quad \text{(end position)} \tag{124}$$
$$\dot{q}(0) = 0 \quad \text{(zero initial velocity)} \tag{125}$$
$$\dot{q}(T) = 0 \quad \text{(zero final velocity)} \tag{126}$$

8.1.2 Solving for Coefficients

From the boundary conditions:

At $t = 0$:

$$q(0) = a_0 = q_0 \tag{127}$$
$$\dot{q}(0) = a_1 = 0 \tag{128}$$

At $t = T$:

$$q(T) = a_0 + a_2T^2 + a_3T^3 = q_f \tag{129}$$
$$\dot{q}(T) = 2a_2T + 3a_3T^2 = 0 \tag{130}$$

Solving the system of equations:

$$a_2T^2 + a_3T^3 = q_f - q_0 \tag{131}$$
$$2a_2T + 3a_3T^2 = 0 \implies a_2 = -\frac{3}{2}a_3T \tag{132}$$

Substituting:

$$-\frac{3}{2}a_3T^3 + a_3T^3 = q_f - q_0 \tag{133}$$
$$-\frac{1}{2}a_3T^3 = q_f - q_0 \tag{134}$$
$$a_3 = \frac{-2(q_f - q_0)}{T^3} \tag{135}$$

And therefore:

$$a_2 = -\frac{3}{2} \cdot \frac{-2(q_f - q_0)}{T^3} \cdot T = \frac{3(q_f - q_0)}{T^2} \tag{136}$$

8.1.3 Final Coefficients

Result

Cubic Polynomial Coefficients:

$$\boxed{\begin{aligned} a_0 &= q_0 \\ a_1 &= 0 \\ a_2 &= \frac{3(q_f - q_0)}{T^2} \\ a_3 &= \frac{-2(q_f - q_0)}{T^3} \end{aligned}} \tag{137}$$

8.1.4 Velocity and Acceleration Profiles

Velocity:

$$\dot{q}(t) = 2a_2t + 3a_3t^2 = \frac{6(q_f - q_0)}{T^2}t - \frac{6(q_f - q_0)}{T^3}t^2 \tag{138}$$

Maximum velocity occurs at $t = T/2$:

$$\dot{q}_{max} = \dot{q}\left(\frac{T}{2}\right) = \frac{3(q_f - q_0)}{2T} \tag{139}$$

Acceleration:

$$\ddot{q}(t) = 2a_2 + 6a_3t = \frac{6(q_f - q_0)}{T^2} - \frac{12(q_f - q_0)}{T^3}t \tag{140}$$

8.1.5 Numerical Example

Given: Move joint from $q_0 = 0$ to $q_f = 60$ in $T = 2$ seconds.

Coefficients:

$$a_0 = 0 \tag{141}$$

$$a_1 = 0 \tag{142}$$

$$a_2 = \frac{3(60 - 0)}{2^2} = 45\ ^\circ/\mathrm{s}^2 \tag{143}$$

$$a_3 = \frac{-2(60 - 0)}{2^3} = -15\ ^\circ/\mathrm{s}^3 \tag{144}$$

Trajectory equation:

$$q(t) = 45t^2 - 15t^3 \quad \text{(degrees)} \tag{145}$$

Verification:

t (s)	$q(t)$ (°)	$\dot{q}(t)$ (°/s)	$\ddot{q}(t)$ (°/s²)
0.0	0.0	0.0	90.0
0.5	9.4	33.8	45.0
1.0	30.0	45.0	0.0
1.5	50.6	33.8	-45.0
2.0	60.0	0.0	-90.0

8.2 Cartesian Trajectory with Cubic Scaling

For Cartesian space trajectories (straight line, circular, sinusoidal), we use cubic time scaling:

$$s(t) = 3\left(\frac{t}{T}\right)^2 - 2\left(\frac{t}{T}\right)^3, \quad s \in [0, 1] \tag{146}$$

Straight Line:

$$\mathbf{p}(t) = \mathbf{p}_0 + s(t) \cdot (\mathbf{p}_f - \mathbf{p}_0) \tag{147}$$

Circular Arc (in XZ plane):

$$x(t) = x_c + R\cos(\theta_0 + s(t) \cdot \Delta\theta) \tag{148}$$

$$z(t) = z_c + R\sin(\theta_0 + s(t) \cdot \Delta\theta) \tag{149}$$

Sinusoidal Path:

$$x(t) = x_0 + s(t) \cdot (x_f - x_0) \tag{150}$$

$$z(t) = z_c + A\sin(2\pi n \cdot s(t)) \tag{151}$$

where A is amplitude and n is number of waves.

Note

For all Cartesian trajectories, inverse kinematics is used at each time step to convert the desired end-effector position $\mathbf{p}(t)$ to joint angles $\mathbf{q}(t)$.