Statistical Inference Assignment 7

Junhao Yuan (20307130129)

November 8, 2022

PROBLEM 1.

Suppose $X_1, \cdots, X_n \overset{iid}{\sim} \Gamma(\alpha, \lambda)$, where α is known and $\lambda > 0$. Prove that $\bar{X}/\alpha$ is the best unbiased estimator of $1/\lambda$.

SOLUTION.

The joint pdf of the sample is

$$\begin{aligned} f(X|\alpha,\lambda) &= \prod_{i=1}^{n} \frac{\lambda^\alpha x_i^{\alpha-1} e^{-\lambda x_i}}{\Gamma(\alpha)} \\ &= \frac{\lambda^{n\alpha}}{\Gamma(\alpha)^n \prod_{i=1}^n x_i} e^{-\lambda(\sum x_i)+\alpha\, log(\prod_{i=1}^n x_i)}, \end{aligned}$$

which means the joint pdf belongs to the exponential family, and has an open set in the natural parameter space. Hence, $T = (\sum x_i, \sum log(x_i))$ is complete sufficient statistics. Since $\bar{X}/\alpha = g(T)$ and $\mathbb{E}(g(T)) = 1/\lambda$, by the Lehmann-Scheff Theorem, we can conclude that $\bar{X}/\alpha$ is the best unbiased estimator of $1/\lambda$.

PROBLEM 2.

Suppose $X_1, \cdots, X_n \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$.

(a) Find the UMVUE of σ.

(b) Find the UMVUE of σ^4.

SOLUTION.

We have already known that $T = \sum(X_i - \bar{X})^2$ is the complete sufficient statistics of σ^2 and $T/\sigma^2 \sim \chi^2_{n-1}$. Therefore, we can come up with the UMVUE of σ and σ^4 based on T.

(a) Let $T_1 = T/\sigma^2$, then we have

$$\begin{aligned} \mathbb{E}(\sqrt{T_1}) &= \int_0^\infty \sqrt{t_1} \frac{t_1^{\frac{n-1}{2}-1} e^{-t_1/2}}{\Gamma\left(\frac{n-1}{2}\right) 2^{\frac{n-1}{2}}} dt_1 \\ &= \frac{2\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}. \end{aligned}$$

Hence,

$$\mathbb{E}\left(\frac{\Gamma\left(\frac{n-1}{2}\right)\sqrt{T}}{2\Gamma\left(\frac{n}{2}\right)}\right) = \mathbb{E}(g_1(T)) = \sigma.$$

Since $g_1(T)$ is an estimator based only on T, and is an unbiased estimator of σ, we know that $g_1(T)$ is the UMVUE of σ.

(b) Since

$$\mathbb{E}(T_1^2) = \int_0^\infty t_1^2 \frac{t_1^{\frac{n-1}{2}-1} e^{-t_1/2}}{\Gamma\left(\frac{n-1}{2}\right) 2^{\frac{n-1}{2}}} dt_1$$
$$= \frac{4\Gamma\left(\frac{n-1}{2}+2\right)}{\Gamma\left(\frac{n-1}{2}\right)} = (n+1)(n-1),$$

we have

$$\mathbb{E}\left(\frac{T^2}{(n+1)(n-1)}\right) = \mathbb{E}(g_2(T)) = \sigma^4.$$

Again, $g_2(T)$ is an estimator only based on T, and is an unbiased estimator of σ^4, thus, is the UMVUE of σ^4.

PROBLEM 3.

Suppose $X_1, \cdots, X_n \overset{iid}{\sim} Geom(\theta)$, where the pmf of $Geom(\theta)$ is

$$P(X = i) = \theta(1-\theta)^{i-1}, \qquad i = 1, 2, \ldots, \qquad 0 < \theta < 1. \tag{1}$$

Geometric distribution can be interpreted as number of Bernoulli trials ($B(\theta)$) needed to get one success.

(a) Show that $T = \sum_{i=1}^n X_i$ is a sufficient statistics for θ, and has pmf

$$P_\theta(T = t) = \binom{t-1}{n-1}\theta^n(1-\theta)^{t-n}, t = n, n+1, n+2, \ldots. \tag{2}$$

(b) Compute $\mathbb{E}_\theta(T)$ and use it to find the UMVUE of θ^{-1}. (Hint: $\sum_{i=0}^\infty z^i = \frac{1}{1-z}$ for $|z| < 1$.)

(c) Show that $\psi(X_1) = I(X_1 = 1)$ is an unbaised estimator of θ, use this fact to find the UMVUE of θ.

SOLUTION.

(a) The joint pdf of the sample is

$$f(X|\theta) = \prod_{i=1}^n \theta(1-\theta)^{x_i - 1} = \left(\frac{\theta}{1-\theta}\right)^n (1-\theta)^T = \left(\frac{1-e^\mu}{e^\mu}\right)^n e^{\mu T},$$

where $\mu = log(1-\theta)$ and $\mu \in (-\infty, 0)$. Hence, the sample distribution belongs to the exponential family, and T is a sufficient. Furthermore, since the natural parameter space $(-\infty, 0)$ contains an open set, T is also complete.

The probability of T equals to t can be interpreted as the number of Bernoulli trials needed to get n success, which means the t_{th} trial succeed and there had $n-1$ success in the previous

$t-1$ Bernoulli trials. Therefore,

$$P_\theta(T=t) = \binom{t-1}{n-1}\theta^n(1-\theta)^{t-n}, t = n, n+1, n+2, \ldots.$$

(b)

$$\begin{aligned}
\mathbb{E}(T) &= \sum_{t=n}^{\infty} t\binom{t-1}{n-1}\theta^n(1-\theta)^{t-n} \\
&= \sum_{t=n}^{\infty}(t-n)\binom{t-1}{n-1}\theta^n(1-\theta)^{t-n} + n\sum_{t=n}^{\infty}\binom{t-1}{n-1}\theta^n(1-\theta)^{t-n} \\
&= \sum_{t=n+1}^{\infty}\binom{t-1}{n-1}\theta^n(1-\theta)^{t-n} + n \\
&= n + n\frac{1-\theta}{\theta}\sum_{t=n+1}^{\infty}\binom{t-1}{n}\theta^{n+1}(1-\theta)^{t-n-1} \\
&= \frac{n}{\theta}.
\end{aligned}$$

Let $g_1(T) = T/n$, then we have $\mathbb{E}(g_1(T)) = \theta^{-1}$. And since T is complete sufficient statistics, $g_1(T)$ is the UMVUE of θ^{-1}.

(c) Since

$$\mathbb{E}(\psi(X_1)) = P(X_1 = 1) = \theta,$$

ψ is an unbiased estimator of θ. Let $g_2(T) = \mathbb{E}(\psi(X_1)|T)$, then we have $g_2(T)$ is an unbiased estimator of θ, too. And,

$$\begin{aligned}
g_2(t) &= \mathbb{E}(\psi(X_1)|T=t) \\
&= \frac{P(x_1 = 1, \sum_{i=2}^n X_i = t-1)}{P(\sum_{i=1}^n X_i = t)} \\
&= \frac{\theta \cdot \binom{t-2}{n-2}\theta^{n-1}(1-\theta)^{t-n}}{\binom{t-1}{n-1}\theta^n(1-\theta)^{t-n}} \\
&= \frac{n-1}{t-1},
\end{aligned}$$

where $t = n, n+1, \ldots$. Since $g_2(T)$ is an unbiased estimator only based on T, a complete sufficient statistics, we can conclude that $g_2(T)$ is the UMVUE of θ.

PROBLEM 4.

Suppose $X_1, \cdots, X_n \overset{iid}{\sim} f_a(x)$ where

$$f_a(x) = e^{-(x-a)}\mathbb{I}_{(a,\infty)}(x), \qquad -\infty < a < \infty. \tag{3}$$

Find the UMVUE of a.

SOLUTION.

The joint pdf of the sample is

$$
\begin{aligned}
f(X|a) &= \prod_{i=1}^{n} f_a(x_i)\mathbb{I}_{(a,\infty)}(x) \\
&= e^{na}\mathbb{I}_{(a,\infty)}(x_{(1)}) \cdot e^{-\sum x_i}.
\end{aligned}
$$

By the factorization theorem, we can know that $T = X_{(1)}$ is the sufficient statistics. Then, let's prove that T is also complete.

It's easy to know that the pdf of T is

$$
f(T = t) = nf_a(t)(1 - F_a(t))^{n-1} = ne^{-n(x-a)}\mathbb{I}_{(a,\infty)}(t).
$$

Suppose $\phi(T)$ is any real value function satisfying $\mathbb{E}(\phi(T)) = 0$, then we have

$$
\int_a^\infty \phi(T)ne^{-n(t-a)}\,d\,t = 0. \tag{4}
$$

Take derivative with respect to a on both sides of (4), then we have

$$
\phi(a) = 0,\ \forall a \in \mathbb{R},
$$

which means T is also complete.

Since

$$
\begin{aligned}
\mathbb{E}(T) &= \int_a^\infty nte^{-n(t-a)}dt \\
&= a + \frac{1}{n},
\end{aligned}
$$

$g(T) = T - \frac{1}{n}$ is an unbiased estimator of a, and based only on T. Therefore, $g(T) = X_{(1)} - \frac{1}{n}$ is the UMVUE of a.