Statistical Inference Assignment 7

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Problem 1.

Suppose $X_1, \cdots, X_n \overset{iid}{\sim} \Gamma(\alpha, \lambda)$, where α is known and $\lambda > 0$. Prove that $\bar{X}/\alpha$ is the best unbiased estimator of $1/\lambda$.

Solution.

The joint pdf of the sample is

$$\begin{aligned} f(X|\alpha,\lambda) &= \prod_{i=1}^{n} \frac{\lambda^{\alpha} x_i^{\alpha-1} e^{-\lambda x_i}}{\Gamma(\alpha)} \\ &= \frac{\lambda^{n\alpha}}{\Gamma(\alpha)^n} \left(\prod_{i=1}^{n} x_i \right)^{\alpha-1} e^{-\lambda(\sum x_i)}, \end{aligned}$$

which means the joint pdf belongs to the exponential family, and has an open set in the natural parameter space. Hence, $T = \sum x_i$ is complete sufficient statistics. Since $\bar{X}/\alpha = g(T)$ and $\mathbb{E}(g(T)) = 1/\lambda$, by the Lehmann-Scheff Theorem, we can conclude that $\bar{X}/\alpha$ is the best unbiased estimator of $1/\lambda$.

Problem 2.

Suppose $X_1, \cdots, X_n \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$.

(a) Find the UMVUE of σ.

(b) Find the UMVUE of σ^4.

Solution.

The joint pdf of the sample is

$$\begin{aligned} f(X|\sigma^2) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x_i^2/(2\sigma^2)} \\ &= (2\pi\sigma^2)^{-\frac{n}{2}} e^{-(\sum x_i^2)/(2\sigma^2)} \\ &= (-\mu\pi)^{-\frac{n}{2}} e^{\mu t}, \end{aligned}$$

where $t = \sum x_i^2$ and $\mu = -1/(2\sigma^2) \in (-\infty, 0)$. Since the sample distribution belongs to the exponential family and there exists an open set in the natural parameter space, $T = \sum X_i^2$ is the complete sufficient statistics, and $T/\sigma^2 \sim \chi^2_n$. Therefore, we can come up with the UMVUE of σ and σ^4 based on T.

(a) Let $T_1 = T/\sigma^2$, then we have

$$\begin{aligned}
\mathbb{E}(\sqrt{T_1}) &= \int_0^\infty \sqrt{t_1} \frac{t_1^{\frac{n}{2}-1} e^{-t_1/2}}{\Gamma\left(\frac{n}{2}\right) 2^{\frac{n}{2}}} d\,t_1 \\
&= \frac{\sqrt{2}\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}.
\end{aligned}$$

Hence,

$$\mathbb{E}\left(\frac{\Gamma\left(\frac{n}{2}\right)\sqrt{T}}{\sqrt{2}\Gamma\left(\frac{n+1}{2}\right)}\right) = \mathbb{E}(g_1(T)) = \sigma.$$

Since $g_1(T)$ is an estimator based only on T, and is an unbiased estimator of σ, we know that $g_1(T)$ is the UMVUE of σ.

(b) Since

$$\begin{aligned}
\mathbb{E}(T_1^2) &= \int_0^\infty t_1^2 \frac{t_1^{\frac{n}{2}-1} e^{-t_1/2}}{\Gamma\left(\frac{n}{2}\right) 2^{\frac{n}{2}}} d\,t_1 \\
&= \frac{4\Gamma\left(\frac{n}{2}+2\right)}{\Gamma\left(\frac{n}{2}\right)} = n(n+2),
\end{aligned}$$

we have

$$\mathbb{E}\left(\frac{T^2}{n(n+2)}\right) = \mathbb{E}(g_2(T)) = \sigma^4.$$

Again, $g_2(T)$ is an estimator only based on T, and is an unbiased estimator of σ^4, thus, is the UMVUE of σ^4.

Problem 3.

Suppose $X_1, \cdots, X_n \overset{iid}{\sim} Geom(\theta)$, where the pmf of $Geom(\theta)$ is

$$P(X = i) = \theta(1-\theta)^{i-1}, \qquad i = 1, 2, \ldots, \qquad 0 < \theta < 1. \tag{1}$$

Geometric distribution can be interpreted as number of Bernoulli trials ($B(\theta)$) needed to get one success.

(a) Show that $T = \sum_{i=1}^n X_i$ is a sufficient statistics for θ, and has pmf

$$P_\theta(T = t) = \binom{t-1}{n-1}\theta^n(1-\theta)^{t-n}, t = n, n+1, n+2, \ldots. \tag{2}$$

(b) Compute $\mathbb{E}_\theta(T)$ and use it to find the UMVUE of θ^{-1}. (Hint: $\sum_{i=0}^\infty z^i = \frac{1}{1-z}$ for $|z| < 1$.)

(c) Show that $\psi(X_1) = I(X_1 = 1)$ is an unbaised estimator of θ, use this fact to find the UMVUE of θ.

Solution.

(a) The joint pdf of the sample is

$$f(X|\theta) = \prod_{i=1}^{n} \theta(1-\theta)^{x_i - 1} = \left(\frac{\theta}{1-\theta}\right)^n (1-\theta)^T = \left(\frac{1-e^\mu}{e^\mu}\right)^n e^{\mu T},$$

where $\mu = log(1-\theta)$ and $\mu \in (-\infty, 0)$. Hence, the sample distribution belongs to the exponential family, and T is a sufficient. Furthermore, since the natural parameter space $(-\infty, 0)$ contains an open set, T is also complete.

The probability of T equals to t can be interpreted as the number of Bernoulli trials needed to get n success, which means the t_{th} trial succeed and there had $n-1$ success in the previous $t-1$ Bernoulli trials. Therefore,

$$P_\theta(T = t) = \binom{t-1}{n-1}\theta^n(1-\theta)^{t-n}, t = n, n+1, n+2, \ldots.$$

(b)

$$\begin{aligned}
\mathbb{E}(T) &= \sum_{t=n}^{\infty} t\binom{t-1}{n-1}\theta^n(1-\theta)^{t-n} \\
&= \sum_{t=n}^{\infty}(t-n)\binom{t-1}{n-1}\theta^n(1-\theta)^{t-n} + n\sum_{t=n}^{\infty}\binom{t-1}{n-1}\theta^n(1-\theta)^{t-n} \\
&= \sum_{t=n+1}^{\infty}\binom{t-1}{n-1}\theta^n(1-\theta)^{t-n} + n \\
&= n + n\frac{1-\theta}{\theta}\sum_{t=n+1}^{\infty}\binom{t-1}{n}\theta^{n+1}(1-\theta)^{t-n-1} \\
&= \frac{n}{\theta}.
\end{aligned}$$

Let $g_1(T) = T/n$, then we have $\mathbb{E}(g_1(T)) = \theta^{-1}$. And since T is complete sufficient statistics, $g_1(T)$ is the UMVUE of θ^{-1}.

(c) Since

$$\mathbb{E}(\psi(X_1)) = P(X_1 = 1) = \theta,$$

ψ is an unbiased estimator of θ. Let $g_2(T) = \mathbb{E}(\psi(X_1)|T)$, then we have $g_2(T)$ is an unbiased estimator of θ, too. And,

$$\begin{aligned}
g_2(t) &= \mathbb{E}(\psi(X_1)|T = t) \\
&= \frac{P(x_1 = 1, \sum_{i=2}^{n} X_i = t-1)}{P(\sum_{i=1}^{n} X_i = t)} \\
&= \frac{\theta \cdot \binom{t-2}{n-2}\theta^{n-1}(1-\theta)^{t-n}}{\binom{t-1}{n-1}\theta^n(1-\theta)^{t-n}} \\
&= \frac{n-1}{t-1},
\end{aligned}$$

where $t = n, n+1, \ldots$. Since $g_2(T)$ is an unbiased estimator only based on T, a complete sufficient statistics, we can conclude that $g_2(T)$ is the UMVUE of θ.

Problem 4.

Suppose $X_1, \cdots, X_n \overset{iid}{\sim} f_a(x)$ where

$$f_a(x) = e^{-(x-a)} \mathbb{I}_{(a,\infty)}(x), \qquad -\infty < a < \infty. \tag{3}$$

Find the UMVUE of a.

Solution.

The joint pdf of the sample is

$$\begin{aligned} f(X|a) &= \prod_{i=1}^{n} f_a(x_i) \mathbb{I}_{(a,\infty)}(x) \\ &= e^{na} \mathbb{I}_{(a,\infty)}(x_{(1)}) \cdot e^{-\sum x_i}. \end{aligned}$$

By the factorization theorem, we can know that $T = X_{(1)}$ is the sufficient statistics. Then, let's prove that T is also complete.

It's easy to know that the pdf of T is

$$f(T = t) = n f_a(t)(1 - F_a(t))^{n-1} = n e^{-n(x-a)} \mathbb{I}_{(a,\infty)}(t).$$

Suppose $\phi(T)$ is any real value function satisfying $\mathbb{E}(\phi(T)) = 0$, then we have

$$\int_a^\infty \phi(t) n e^{-n(t-a)} \, dt = 0. \tag{4}$$

Take derivative with respect to a on both sides of (4), then we have

$$0 = n \int_a^\infty \phi(t) n e^{-n(t-a)} \, dt - n\phi(a) = -n\phi(a), \qquad \forall a \in \mathbb{R}.$$

which means T is also complete.

Since

$$\begin{aligned} \mathbb{E}(T) &= \int_a^\infty n t e^{-n(t-a)} dt \\ &= a + \frac{1}{n}, \end{aligned}$$

$g(T) = T - \frac{1}{n}$ is an unbiased estimator of a, and based only on T. Therefore, $X_{(1)} - \frac{1}{n}$ is the UMVUE of a.