Statistical Inference Assignment 1

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Problem 1.

Let $X_1, \ldots, X_n \overset{i.i.d.}{\sim} U(0, a)$. Find the joint pdf of R and V, where $R = X_{(n)} - X_{(1)}$ and $V = \frac{1}{2}(X_{(n)} + X_{(1)})$.

Solution.

It's easy to know that the joint pdf of $X_{(1)}$ and $X_{(n)}$ is

$$f_{x_{(1)},x_{(n)}}(x, y) = n(n-1)(y-x)^{(n-2)}.$$

Since

$$\left|\frac{\partial(R, V)}{\partial(x_{(1)}, x_{(n)})}\right| = \begin{vmatrix} -1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = 1,$$

we can derive the joint pdf of R and V from the pdf of $X_{(1)}$ and $X_{(n)}$:

$$\begin{aligned} f_{R,V}(r, v) &= f_{x_{(1)},x_{(n)}}(x(r, v), y(r, v)) \left|\frac{\partial(R, V)}{\partial(x_{(1)}, x_{(n)})}\right|^{-1} \\ &= (n-1)nr. \end{aligned}$$

Problem 2.

Let X and Y be *i.i.d.* $N(0, 1)$ random variables. Define $Z = min(X, Y)$. What distribution does Z follow?

Solution.

Since X and Y are independent and identically distributed, we have

$$\mathbb{P}(Z > z) = \mathbb{P}(X > z, Y > z) = \mathbb{P}(X > z)\mathbb{P}(Y > z) = \mathbb{P}(X > z)^2. \tag{1}$$

Hence, we can just write

$$\begin{aligned} \mathbb{P}(Z^2 > t) &= \mathbb{P}(Z > \sqrt{t}) + \mathbb{P}(Z < -\sqrt{t}) \\ &= \mathbb{P}(Z > \sqrt{t}) + 1 - \mathbb{P}(Z \geq -\sqrt{t}) \\ &= \mathbb{P}(X > \sqrt{t})^2 + 1 - \mathbb{P}(X \geq -\sqrt{t})^2 \\ &= \mathbb{P}X > \sqrt{t}^2 + 1 - (1 - \mathbb{P}(X < -\sqrt{t}))^2 \\ &= 2\mathbb{P}(X > \sqrt{t}), \end{aligned} \tag{2}$$

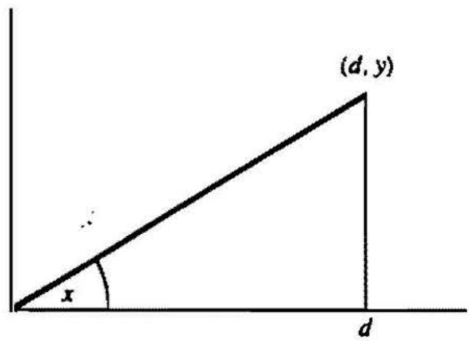
and get that

$$\mathbb{P}(Z^2 \leq t) = 1 - 2\mathbb{P}(X > \sqrt{t}) = \mathbb{P}(-\sqrt{t} < X < \sqrt{t}) = \mathbb{P}(X^2 \leq t), \tag{3}$$

which means $Z^2 \sim \chi^2(1)$.

PROBLEM 3.

A random right triangle can be constructed in the following manner. Let X be a random angle whose distribution is uniform on $(0, \frac{\pi}{2})$. For each X, construct a triangle as pictured below. Here $Y =$ *height of the random triangle*. For a fixed constant d, find the distribution of Y and $\mathbb{E}(Y)$.



SOLUTION.

Through the geometric relationship between X and Y:

$$Y = d \cdot tan(X),$$

we can get the pdf of Y:

$$\begin{aligned} f_Y(y) &= f_X(x) \left| \frac{dy}{dx} \right|^{-1} \\ &= \frac{2}{\pi} \frac{d}{d^2 + y^2}, \end{aligned}$$

which is similar to Cauchy distribution and

$$\begin{aligned} \mathbb{E}(Y) &= \int_0^\infty \frac{2}{\pi} \frac{d \cdot y}{d^2 + y^2} dy \\ &= \frac{d}{\pi} ln(d^2 + y^2)|_0^\infty = \infty. \end{aligned}$$

PROBLEM 4.

X_1 and X_2 are independent $N(0, \sigma^2)$.

(a) Find the joint distribution of Y_1 and Y_2, where

$$Y_1 = X_1^2 + X_2^2, \qquad Y_2 = \frac{X_1}{\sqrt{Y_1}}.$$

(b) Show that Y_1 and Y_2 are independent. Interpret the result geometrically.

SOLUTION.

(a) Since X_1 and X_2 are independent, the joint pdf of them is

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sigma^2}e^{-\frac{x_1^2+x_2^2}{2\sigma^2}}.$$

Considering that

$$\begin{aligned}\{Y_1 = y_1, Y_2 = y_2\} &= \{X_1^2 + X_2 = y_1, \frac{X_1}{\sqrt{y_1}} = y_2\} \\ &= \{X_1 = x_1, X_2 = x_2\} \cup \{X_1 = x_1, X_2 = -x_2\},\end{aligned}$$

where $x_1 = y_2\sqrt{y_1}$, $x_2 = \sqrt{y_1(1-y_2^2)}$, we have

$$\begin{aligned}f_{Y_1,Y_2}(y_1,y_2) &= [f_{X_1,X_2}(x_1,x_2) + f_{X_1,X_2}(x_1,x_2)]\left|\frac{\partial(Y_1,Y_2)}{\partial(X_1,X_2)}\right|^{-1} \\ &= \frac{1}{2\sigma^2}e^{-2\sigma^{-2}y_1} \cdot \frac{1}{\pi\sqrt{1-y_2^2}}.\end{aligned}$$

(b) If we denote $Y = Y_1/\sigma$, it's obvious that $Y \sim \chi^2(2)$. So the pdf of Y is

$$f_Y(y) = \frac{1}{2}e^{-y/2}.$$

Thus the pdf of Y_1 is

$$f_{Y_1}(y_1) = \frac{1}{2\sigma^2}e^{-2\sigma^{-2}y_1}.$$

Therefore, we can write the joint pdf of Y_1 and Y_2 as

$$f_{Y_1,Y_2}(y_1,y_2) = f_{Y_1}(y_1)f_{Y_2}(y_2),$$

which means Y_1 and Y_2 are independent. We can Interpret the result geometrically in a Cartesian coordinate system. From a geometric point of view, we can see (X_1, X_2) as the coordinates of a point on a circle, and Y_1 denotes the square of the radius, Y_2 denotes cosine of the angle between the x-axis and the line connecting this point to the origin. When one of Y_1 and Y_2 is fixed, the other can still change in its domain.

PROBLEM 5.

Suppose $X_1, \dots, X_m \overset{i.i.d.}{\sim} \chi_n^2$. Find the distribution of $\frac{1}{n}\sum_{i=1}^n X_i$. Hint: use the characteristic function.

SOLUTION.

Suppose $X \sim Ga(\alpha, \lambda)$, then the characteristic function of X is

$$\begin{aligned} M_X(t) = \mathbb{E}(e^{itx}) &= \int_0^\infty e^{itx} \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} dx \\ &= \int_0^\infty \frac{\left[\left(1 - \frac{it}{\lambda}\right)\lambda\right]^\alpha x^{\alpha-1} e^{-\left(1-\frac{it}{\lambda}\right)\lambda x}}{\left(1-\frac{it}{\lambda}\right)^\alpha \Gamma(\alpha)} \\ &= \left(1 - \frac{it}{\lambda}\right)^{-\alpha}. \end{aligned} \tag{4}$$

Since Chi-square distribution is just a special case of Gamma distribution, we can get the characteristic function of chi-square distribution simply by substituting (α, λ) with $(\frac{n}{2}, \frac{1}{2})$ in (4). Thus, the characteristic function of X_i is

$$M_i(t) = (1 - 2it)^{\frac{n}{2}}. \tag{5}$$

Denote $Y = \sum_{i=1}^n X_i$, then $Y \sim \chi^2_{mn}$, and

$$M_Y(t) = (1 - 2it)^{\frac{mn}{2}}. \tag{6}$$

And by the property of characteristic function, we can calculate that the characteristic function of $Z = \frac{Y}{n}$ is

$$M_Z(t) = M_Y(t/n) = \left(1 - \frac{it}{n/2}\right)^{-\frac{mn}{2}}, \tag{7}$$

which means $Z \sim Ga\left(\frac{mn}{2}, \frac{n}{2}\right)$.