MANA130353.01: Statistic Inference **1st Semester, 2022-2023**

Homework 1

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Note: This note is a reference answer for the homework.

Disclaimer: *This note is only used as a reference solution for the homework, and the solution to each question is not unique. If you have any questions, you can add my WeChat ID statchaij.*

Problem 1. *Let $X_1, \ldots, X_n \overset{iid}{\sim} U(0,a)$. Find the joint pdf of R and V, where $R = X_{(n)} - X_{(1)}$ and $V = \frac{1}{2}\left(X_{(n)} + X_{(1)}\right)$.*

Sol: First we can easily obtain the jpdf of $(X_{(1)}, X_{(n)})$ is

$$f(x_1, x_n) = n(n-1)\frac{(x_n - x_1)^{n-2}}{a^n}\mathbb{I}(0 < x_1 < x_n < a).$$

Then we make change of variables $\begin{cases} R = X_{(n)} - X_{(1)} \\ V = \frac{1}{2}\left(X_{(n)} + X_{(1)}\right) \end{cases}$ and its corresponding Jacobi determinant

$$\left|\frac{\partial(x_1, x_n)}{\partial\left(R, V\right)}\right| = \begin{vmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 1 \end{vmatrix} = -1.$$

Then the jpdf of (R, V) is

$$g(r, v) = \begin{cases} n(n-1)\dfrac{r^{n-2}}{a^n}, 0 < r < a \text{ and } \frac{r}{2} < V < a - \frac{r}{2}. \\ 0, \text{ else} \end{cases}$$

□

Problem 2. *Let X and Y be iid $N(0,1)$ random variables. Define $Z = \min(X, Y)$. What distribution does Z follow?*

Sol: Since X, Y iid $N(0,1), X \perp\!\!\!\perp Y$, we have

$$F_Z(t) = P(Z \leq t) = 1 - P(Z > t) = 1 - P(X > t)P(Y > t) = 1 - (1 - \Phi(t))^2,$$
$$f_Z(t) = 2\phi(t) - 2\Phi(t)\phi(t),$$

where Φ and ϕ are the cdf and pdf of $N(0,1)$ respectively.

Also, we can find

$$\begin{aligned}
P(Z^2 \le t) &= P(Z > -\sqrt{t}) - P(Z > \sqrt{t}) \\
&= (P(X > -\sqrt{t}))^2 - (P(X > \sqrt{t}))^2 \\
&= (P(X > -\sqrt{t}) - P(X > \sqrt{t}))(P(X > \sqrt{-t}) + P(X > \sqrt{t})) \\
&= P(-\sqrt{t} < X \le \sqrt{t}) * 1 = P(X^2 \le t) = P(\chi^2(1) \le 1)
\end{aligned}$$

That is, $Z^2 \sim \chi^2(1)$. □

Problem 3. *A random right triangle can be constructed in the following manner. Let X be a random angle whose distribution is uniform on $(0, \pi/2)$. For each X, construct a triange as pictured below. Here Y = height of the random triangle. For a fixed constant d, find the distribution of Y and $E(Y)$.*

Sol: We find $Y = d \tan x$, then

$$F_Y(t) = P(d \tan x \le t) = P(x \le \arctan \frac{t}{d}) = \begin{cases} \frac{2}{\pi} \arctan \frac{t}{d} & t > 0 \\ 0 & \text{else} \end{cases}$$

Then its pdf is $f(t) = \frac{2}{\pi} \frac{d}{d^2+t^2}, t > 0$. Hence we obtain

$$EY = \int_0^\infty \frac{2}{\pi} \frac{d}{d^2 + t^2} \mathrm{d}x = \frac{d}{\pi} \ln(d^2 + y^2) \mid_0^\infty = \infty.$$

□

Problem 4. *X_1 and X_2 are independent $N\left(0, \sigma^2\right)$.*

(a) Find the joint distribution of Y_1 and Y_2, where

$$Y_1 = X_1^2 + X_2^2, \quad Y_2 = \frac{X_1}{\sqrt{Y_1}}.$$

(b) Show that Y_1 and Y_2 are independent. Interpret the result geometrically.

Sol: (a) Since $X_i \overset{iid}{\sim} N(0, \sigma^2), i = 1, 2$, the jpdf of (X_1, X_2) is

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+x_2^2}{2\sigma^2}}.$$

Divide the sample space into two disjoint parts

$$\Omega = \{(x_1, x_2) : x_2 \le 0\} + \{(x_1, x_2) : x_2 > 0\} \triangleq A_1 + A_2.$$

In A_1, make change of variable $X_1 = \sqrt{Y_1}Y_2, X_2 = -\sqrt{Y_1 - Y_1 Y_2^2}$ and $||J|| = \frac{1}{2\sqrt{1-y_2^2}}$.
In A_2, make change of variable $X_1 = \sqrt{Y_1}Y_2, X_2 = \sqrt{Y_1 - Y_1 Y_2^2}$ and $||J|| = \frac{1}{2\sqrt{1-y_2^2}}$.
Then we have

$$\left.\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) &= \frac{2}{2\sqrt{1-y_2^2}} \cdot \frac{1}{2\pi\sigma^2} \cdot \exp\left(-\frac{y_1 y_2^2 + y_1 - y_1 y_2^2}{2\sigma^2}\right) \\ &= \frac{1}{2\pi\sigma^2\sqrt{1-y_2^2}} \cdot \exp\left(-\frac{y_1}{2\sigma^2}\right) \end{aligned}\right., \qquad y_1 \in \mathbb{R}^+, y2 \in (-1, 1).$$

(b) Marginalize the jpdf of (Y_1, Y_2), we find their mpdf are

$$f_{Y_1}(y_1) = \frac{1}{2\sigma^2} e^{-y_1/(2\sigma^2)}, y_1 \in \mathbb{R}^+. \quad f_{Y_2}(y_2) = \frac{1}{\pi\sqrt{1-y_2^2}}, y_2 \in (-1, 1).$$

Therefore, we can write the joint pdf of Y_1 and Y_2 as

$$f_{Y_1,Y_2}(y_1, y_2) = f_{Y_1}(y_1) f_{Y_2}(y_2)$$

which means Y_1 and Y_2 are independent.

Geometric interpretation: Y_1 is the radius and Y_2 is a function of angle.

□

Problem 5. *Suppose* $X_1, \dots, X_n \overset{iid}{\sim} \chi^2_m$. *Find the distribution of* $\frac{1}{n}\sum_{i=1}^n X_i$. *(Hint: use the characteristic function.)*

Sol: Since $\forall i, X_i \sim \chi^2(n) = Ga(\frac{n}{2}, \frac{1}{2})$, its characteristic function is $\phi_i(t) = (1-2it)^{\frac{n}{2}}$. Let $S = \sum_{i=1}^n X_i$, we know $S \sim \chi^2(mn)$ and $\phi_S(t) = (1-2it)^{\frac{mn}{2}}$.

Then we can calculate the characteristic funtion of $\frac{1}{n}\sum_{i=1}^n X_i$:

$$\phi(t) = \phi_S(t/n) = (1 - \frac{2it}{n})^{\frac{mn}{2}}$$

that is, $\frac{1}{n}\sum_{i=1}^n X_i \sim Ga(\frac{mn}{2}, \frac{n}{2})$. □