# Statistical Inference Assignment 8

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Problem 1.

Suppose $X_1, \ldots, X_{11}$ are iid $\mathcal{N}(\mu_1, \sigma^2)$ and $Y_1, \ldots, Y_{21}$ are iid $\mathcal{N}(\mu_2, \sigma^2)$. Suppose we compute $\bar{X} = 1.1$, $S_X^2 = 1.25$, $\bar{Y}^2 = 1.9$, $S_Y^2 = 1.21$. Construct an EXACT 95% confidence interval for $\mu_1 - \mu_2$.

Solution.

We have already known that

$$\sqrt{\frac{mn}{m+n}} \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{S_w} \sim t_{m+n-2},$$

where $m = 11$, $n = 21$ and

$$S_w^2 = \frac{(m-1)S_X^2 + (n-1)S_Y^2}{m+n-2}.$$

Therefore, the 95% confidence interval is

$$\left[(\bar{X} - \bar{Y}) - \sqrt{\frac{1}{m} + \frac{1}{n}} S_w \, t_{m+n-2}(0.025),\ (\bar{X} - \bar{Y}) - \sqrt{\frac{1}{m} + \frac{1}{n}} S_w \, t_{m+n-2}(0.025)\right].$$

Then, using the given statistics, we have the confidence interval is

$$[-1.641, 0.041].$$

Problem 2.

Among 1000 random selected voters 450 say they will vote for candidate A. Could you provide a 95% confidence interval for the true supporting rate of candidate A? Does your confidence interval cover 0.5?

Solution.

Denote $X_i$ as the event that whether the $i_{th}$ random selected voter support candidate A or not, and $X_i \overset{iid}{\sim} B(1, p)$, where $p$ is the probability that the voter support candidate A. We have already known that

$$\frac{\sqrt{n}(\bar{X} - p)}{\sqrt{p(1-p)}} \xrightarrow{L} \mathcal{N}(0, 1),$$

and

$$\frac{\bar{X}}{p} \xrightarrow{P} 1.$$

Therefore, by the Slutsky Lemma, we have

$$\frac{\sqrt{n}(\bar{X}-p)}{\sqrt{\bar{X}(1-\bar{X})}} \xrightarrow{L} \mathcal{N}(0,1),$$

and we can construct the 95% confidence interval like

$$\left[\bar{X} - \sqrt{\frac{\bar{X}(1-\bar{X})}{n}}u(0.025),\ \bar{X} + \sqrt{\frac{\bar{X}(1-\bar{X})}{n}}u(0.025)\right].$$

And since $\bar{X} = 0.45$, we have the 95% confidence interval is

$$[0.419,\ 0.481],$$

which doesn't cover 0.5.

PROBLEM 3.

$X_1, \dots, X_m \overset{iid}{\sim} \mathcal{N}(a, \sigma_1^2)$, $Y_1, \dots, Y_n \overset{iid}{\sim} \mathcal{N}(b, \sigma_2^2)$. Suppose $\sigma_2^2/\sigma_1^2 = \lambda$ and $\lambda$ is known, find a $1-\alpha$ confidence interval for $b-a$.

SOLUTION.

We have already known that

$$\bar{X} \sim \mathcal{N}\left(a,\ \frac{\sigma_1^2}{m}\right), \qquad \frac{S_X^2}{\sigma_1^2} \sim \chi_{m-1}^2,$$

and

$$\bar{Y} \sim \mathcal{N}\left(b,\ \frac{\sigma_2^2}{n}\right), \qquad \frac{S_Y^2}{\sigma_2^2} \sim \chi_{n-1}^2,$$

where

$$S_X^2 = \frac{\sum_{i=1}^m (X_i - \bar{X})^2}{m-1}, \qquad S_Y^2 = \frac{\sum_{j=1} (Y_j - \bar{Y})^2}{n-1}.$$

Therefore,

$$\frac{\left[(\bar{Y}-\bar{X})-(b-a)\right]/\sqrt{\frac{\sigma_1^2}{m}+\frac{\sigma_2^2}{n}}}{\sqrt{\left(\frac{(m-1)S_X^2}{\sigma_1^2}+\frac{(n-1)S_Y^2}{\sigma_2^2}\right)/(m+n-2)}} = \sqrt{\frac{m+n-2}{1/m+\lambda/n}}\frac{(\bar{Y}-\bar{X})-(b-a)}{\sqrt{(m-1)S_X^2+(n-1)S_Y^2/\lambda}} \sim t_{m+n-2},$$

and a $1-\alpha$ confidence interval for $b-a$ is

$$\left[(\bar{Y}-\bar{X}) \pm \sqrt{(m-1)S_X^2 + \frac{(n-1)S_Y^2}{\lambda}}\sqrt{\frac{\frac{1}{m}+\frac{\lambda}{n}}{m+n-2}}t_{m+n-2}\left(\frac{\alpha}{2}\right)\right].$$

Problem 4.

Suppose $X_1, \ldots, X_n$ are iid $\mathcal{N}(\mu, 16)$, if we want to construct a $1-\alpha$ level confidence interval for $\mu$ with length less than $L$. How large must $n$ be?

Solution.

Since $\bar{X} \sim \mathcal{N}(\mu, \frac{16}{n})$, the shortest $1-\alpha$ confidence interval for $\mu$ is

$$\left[\bar{X} - \frac{4u(\alpha/2)}{\sqrt{n}}, \qquad \bar{X} + \frac{4u(\alpha/2)}{\sqrt{n}}\right],$$

and the length is $8u(\alpha/2)/\sqrt{n}$. To make $8u(\alpha/2)/\sqrt{n}$ less than $L$, we must have

$$n > \frac{64u^2(\alpha/2)}{L^2},$$

which means

$$n \geq \left[\frac{64u^2(\alpha/2)}{L^2}\right] + 1,$$

where $[C]$ denotes the integer part of $C$.