Statistical Inference Assignment 5

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Problem 1.

Let W be a statistics, show that $\mathbb{E}_\theta(W-\theta)^2 = \mathbb{V}_\theta(W) + (\mathbb{E}_\theta(W)-\theta)^2$.

Solution.

$$\begin{aligned}
\mathbb{E}_\theta(W-\theta)^2 &= \mathbb{E}_\theta(W-\mathbb{E}_\theta(W)+\mathbb{E}_\theta(W)-\theta)^2 \\
&= \mathbb{V}_\theta(W) + 2\mathbb{E}_\theta(W-\mathbb{E}_\theta(W))(\mathbb{E}_\theta(W)-\theta) + (\mathbb{E}_\theta(W)-\theta)^2 \\
&= \mathbb{V}_\theta(W) + 2(\mathbb{E}_\theta(W)-\mathbb{E}_\theta(W))(\mathbb{E}_\theta(W)-\theta) + (\mathbb{E}_\theta(W)-\theta)^2 \\
&= \mathbb{V}_\theta(W) + (\mathbb{E}_\theta(W)-\theta)^2.
\end{aligned}$$

Problem 2.

$X_1,\dots,X_n \overset{iid}{\sim} f(x|\mu)$ where

$$f(x|\mu) = e^{-(x-\mu)} \cdot \mathbb{I}(x \geq \mu), \qquad \mu \in (-\infty,\infty). \tag{1}$$

(a) Find $\hat{\mu}_{mle}$.

(b) Use method of moments to find an unbiased estimator for μ.

(c) Compare the estimators from (a) and (b), which one has a smaller MSE?

Solution.

(a) The likelihood function is

$$\begin{aligned}
L(\mu) &= \prod_{i=1}^{n} e^{-(x-\mu)} \cdot \mathbb{I}(x \geq \mu) \\
&= e^{-n(\bar{x}-\mu)} \cdot \mathbb{I}(x_{(1)} \geq \mu).
\end{aligned}$$

Since $e^{-(\bar{x}-\mu)}$ is monotonically increasing of μ, and $\mu \leq x_{(1)}$, we can conclude that the MLE of μ is $x_{(1)}$.

(b) We can denote $Y = X-\mu$ so that $Y \sim Exp(1)$. Since $\mathbb{E}(Y)=1$ and $\mathbb{V}(Y)=1$, we have $\mathbb{E}(X)=\mu+1$ and $\mathbb{V}(X)=1$. Therefore, by the method of moments, we can use $\bar{X}-1$ to estimate μ.

(c) Let's compute the MSE of $\mu_{mle} = x_{(1)}$ first. The pdf of $x_{(1)}$ is

$$f_{x_{(1)}}(x) = ne^{-(x-\mu)}\left(e^{-(x-\mu)}\right)^{n-1} = ne^{-n(x-\mu)}.$$

If we denote $Y = X_{(1)} - \mu$, then $Y \sim Exp(n)$. Hence,

$$\mathbb{E}(X_{(1)}) = \mathbb{E}(Y) + \mu = \frac{1}{n} + \mu,$$
$$\mathbb{V}(X_{(1)}) = \mathbb{V}(Y) = \frac{1}{n^2}.$$

Therefore, the MSE of $x_{(1)}$ is $2/n^2$.

Since

$$\mathbb{E}(\bar{X} - 1) = \mu + 1 - 1 = \mu,$$

$\bar{X} - 1$ is an unbiased estimator of μ. And its variance is

$$\mathbb{V}(\bar{X} - 1) = \mathbb{V}(\bar{X}) = \frac{\mathbb{V}(X)}{n} = \frac{1}{n}.$$

Therefore, $x_{(1)}$ has a smaller MSE when $n > 2$.

PROBLEM 3.

Let $F(x)$ and $f(x)$ be the distribution and density functions for iid random variables $X_1, \ldots, X_n$. Show that

$$\int \cdots \int_{a<x_1<\cdots<x_n<b} f(x_1)\cdots f(x_n)\, dx_1 \cdots dx_n = \frac{1}{n!}[F(b) - F(a)]^n. \tag{2}$$

SOLUTION.

For each value of $\boldsymbol{a} = (a_1, \ldots, a_n)$, there exists $n!$ permutations of $X_1, \ldots, X_n$ such that $X_{(1)} = a_1, \ldots, X_{(n)} = a_n$. Hence, we have

$$\begin{aligned}\int \cdots \int_{a<x_1<\cdots<x_n<b} f(x_1)\cdots f(x_n) dx_1 \cdots dx_n &= \frac{1}{n!}\int \cdots \int_{a<x_{(1)}<\cdots<x_{(n)}<b} f(x_{(1)})\cdots f(x_{(n)}) dx_{(1)} \cdots dx_{(n)} \\ &= \frac{1}{n!}\int_a^b f(x_{(1)})dx_{(1)} \times \cdots \times \int_a^b f(x_{(n)})dx_{(n)} \\ &= \frac{1}{n!}[F(b) - F(a)]^n.\end{aligned}$$

PROBLEM 4.

If $f(x|\theta)$ satisfies

$$\frac{d}{d\theta}\mathbb{E}_\theta\left(\frac{\partial}{\partial\theta} logf(X|\theta)\right) = \int \frac{\partial}{\partial\theta}\left[\left(\frac{\partial}{\partial\theta} logf(x|\theta)\right) f(x|\theta)\right] dx \tag{3}$$

(true for an exponential family), show that

$$\mathbb{E}_\theta \left[\left(\frac{\partial}{\partial \theta} log f(X|\theta) \right)^2 \right] = -\mathbb{E}_\theta \left(\frac{\partial^2}{\partial \theta^2} log f(X|\theta) \right). \tag{4}$$

Solution.

We have already known that

$$\mathbb{E}_\theta \left[\frac{\partial}{\partial \theta} \ln f(\mathbf{X}|\theta) \right] = 0. \tag{5}$$

Differentiate both sides of (5):

$$\begin{aligned} 0 &= \int_{\mathcal{X}} \left[\frac{\partial^2}{\partial \theta^2} \ln f(x|\theta) \right] f(x|\theta) + \left[\frac{\partial}{\partial \theta} \ln f(x|\theta) \right] \frac{\partial}{\partial \theta} f(x|\theta) \, dx \\ &= \int_{\mathcal{X}} \left[\frac{\partial^2}{\partial \theta^2} \ln f(x|\theta) \right] f(x|\theta) + \left[\frac{\partial}{\partial \theta} \ln f(x|\theta) \right]^2 f(x|\theta) \, dx \\ &= \mathbb{E}_\theta \left[\left(\frac{\partial}{\partial \theta} log f(X|\theta) \right)^2 \right] + \mathbb{E}_\theta \left(\frac{\partial^2}{\partial \theta^2} log f(X|\theta) \right), \end{aligned}$$

which is just (4).