Statistical Inference Assignment 2

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PROBLEM 1.

Let $X_1, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$, $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$, $S_n^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2$. Suppose $X_{n+1} \sim N(\mu, \sigma^2)$ and is independent of $X_1, \ldots, X_n$, find the distribution of

$$\frac{X_{n+1} - \bar{X}}{S_n}\sqrt{\frac{n-1}{n+1}} \tag{1}$$

SOLUTION.

Since $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ and $X_{n+1} \sim N(\mu, \sigma^2)$, we have

$$\frac{X_{n+1} - \bar{X}}{\sqrt{\frac{n+1}{n}}\sigma} \sim N(0, 1),$$

and

$$\frac{nS_n^2}{\sigma^2} \sim \chi^2_{n-1}.$$

Thus, by the definition of t-distribution, we can get that

$$\frac{(X_{n+1} - \bar{X})/\sqrt{\frac{n+1}{n}\sigma^2}}{\sqrt{\frac{nS_n^2}{(n-1)\sigma^2}}} = \frac{X_{n+1} - \bar{X}}{S_n}\sqrt{\frac{n-1}{n+1}} \sim t_{n-1}.$$

PROBLEM 2.

Suppose $X_1, \ldots, X_n$ are independent and $X_i \sim N(0, \sigma^2)$ for $i = 1, \ldots, n$. Define

$$\xi = \sum_{i=1}^n \frac{(X_i - Z)^2}{\sigma_i^2} \tag{2}$$

where

$$Z = \left(\sum_{i=1}^n \frac{X_i}{\sigma_i^2}\right)\left(\sum_{i=1}^n \frac{1}{\sigma_i^2}\right)^{-1}. \tag{3}$$

Find the distribution of ξ. (Hint: Use a proper orthogonal transform.)

SOLUTION.

Rewrite ξ as

$$\xi = \sum_{i=1}^{n} \left(\frac{X_i}{\sigma_i}\right)^2 - \left(\sum_{i=1}^{n} \frac{1}{\sigma_i^2}\right) Z^2. \tag{4}$$

We can find a matrix $A \in \mathbb{R}^{n\times n}$ in the form as

$$\begin{bmatrix} \frac{1/\sigma_1}{\delta} & \frac{1/\sigma_2}{\delta} & \cdots & \frac{1/\sigma_n}{\delta} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

with rank n, where $\delta = \sqrt{\sum_{i=1}^{n} \frac{1}{\sigma_i^2}}$. Then we can use Schmidt orthogonalization to transform A into an orthogonal matrix (keep the first row of A unchanged). If we regard X_i/σ_i as a new random variable X' and define $Y := AX'$, with some simple calculation, we have $Y_1 = \delta Z$, and

$$Y_i = \sum_{j=1}^{n} a_{ij} \frac{X_j}{\sigma_j},\ (i = 2, \ldots, n).$$

It's obvious that $\mathbb{E}(Y_i) = 0$, and

$$Var(Y_i) = \sum_{j=1}^{n} a_{ij}^2 = 1,\ (i = 2. \ldots, n).$$

Thus we have $Y_i \sim N(0, 1)$ for $i = 2, \ldots, n$. Plug the results we get into (4) and we can get that

$$\begin{aligned} \xi &= \sum_{i=1}^{n} Y_i^2 - Y_1^2 \\ &= \sum_{i=2}^{n} Y_i^2, \end{aligned}$$

which means $\xi \sim \chi^2_{n-1}$.

Problem 3.

Suppose $X_1, \ldots, X_n \sim Poisson(\lambda)$. Show that

$$\frac{\bar{X} - \lambda}{\sqrt{\bar{X}/n}} \xrightarrow{D} N(0, 1) \tag{5}$$

(Hint: Use Slutsky's theorem.)

Solution.

If we define $Y = \sum_{i=1}^{n} X_i$, then by the property of Poisson distribution, we know that $Y \sim Poisson(n\lambda)$, and $\mathbb{E}(Y) = \mathbb{V}(Y) = n\lambda$. Therefore, according to the CLT, we know that

$$\frac{Y - n\lambda}{\sqrt{n\lambda}} \xrightarrow{D} N(0, 1).$$

Additionally, we have

$$Y \xrightarrow{P} n\lambda,$$

derived from LLN. And from Slutsky lemma, we know that

$$\frac{\sqrt{n\lambda}}{\sqrt{Y}} \xrightarrow{P} 1,$$

and

$$\frac{Y - n\lambda}{\sqrt{n\lambda}} \cdot \frac{\sqrt{n\lambda}}{\sqrt{Y}} \xrightarrow{D} N(0,1).$$

PROBLEM 4.

If X is a random variable with pdf or pmf of the form

$$f(x|\boldsymbol{\theta}) = h(x)c(\boldsymbol{\theta})exp\left(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x)\right) \tag{6}$$

show that

(a) :

$$\mathbb{E}\left(\sum_{i=1}^{k} \frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(X)\right) = -\frac{\partial}{\partial \theta_j} log\, c(\boldsymbol{\theta}) \tag{7}$$

(b) :

$$Var\left(\sum_{i=1}^{k} \frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(X)\right) = -\frac{\partial^2}{\partial \theta_j^2} log\, c(\boldsymbol{\theta}) - \mathbb{E}\left(\sum_{i=1}^{k} \frac{\partial^2 w_i(\boldsymbol{\theta})}{\partial \theta_j^2} t_i(X)\right). \tag{8}$$

Hint:

$$\int f(x|\boldsymbol{\theta})dx = \int h(x)c(\boldsymbol{\theta})exp\left(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x)\right) = 1. \tag{9}$$

Differentiate both sides and then rearrange terms.

SOLUTION.

(a) Differentiate both sides of (9) and we can get that

$$0 = \int h(x)\left[\frac{\partial c(\boldsymbol{\theta})}{\partial \theta_j} exp\left(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x)\right) + c(\boldsymbol{\theta})exp\left(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x)\right)\sum_{i=1}^{k}\frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(x)\right] dx = 0. \tag{10}$$

Rearrange (10) and we have

$$
\begin{aligned}
0 &= \frac{1}{c(\boldsymbol{\theta})}\frac{\partial c(\boldsymbol{\theta})}{\partial \theta_j} + \mathbb{E}\left(\sum_{i=1}^{k}\frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j}t_i(X)\right) \\
&= \frac{\partial}{\partial \theta_j}\, log(c(\boldsymbol{\theta})) + \mathbb{E}\left(\sum_{i=1}^{k}\frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j}t_i(X)\right),
\end{aligned}
\tag{11}
$$

which is equation (7).

(b) Differentiate both sides of (10), and we have

$$
\begin{aligned}
0 &= \int h(x)\frac{\partial^2 c(\boldsymbol{\theta})}{\partial \theta_j^2} exp\left(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x)\right) dx \\
&\quad + 2\int h(x)\frac{\partial c(\boldsymbol{\theta})}{\partial \theta_j} exp\left(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x)\right)\sum_{i=1}^{k}\frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j}t_i(x)\, dx \\
&\quad + \int h(x)c(\boldsymbol{\theta}) exp\left(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x)\right)\left(\sum_{i=1}^{k}\frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j}t_i(x)\right)^2 dx \\
&\quad + \int h(x)c(\boldsymbol{\theta}) exp\left(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x)\right)\sum_{i=1}^{k}\frac{\partial^2 w_i(\theta)}{\partial \theta_j^2}t_i(x)\, dx \\
&= \frac{1}{c(\boldsymbol{\theta})}\frac{\partial^2 c(\boldsymbol{\theta})}{\partial \theta_j^2} + \frac{2}{c(\boldsymbol{\theta})}\frac{\partial c(\boldsymbol{\theta})}{\partial \theta_j}\mathbb{E}\left(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(X)\right) + \mathbb{E}\left(\left(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(X)\right)^2\right) \\
&\quad + \mathbb{E}\left(\sum_{i=1}^{k}\frac{\partial^2 w_i(\theta)}{\partial \theta_j^2}t_i(X)\right) \\
&= \frac{1}{c(\boldsymbol{\theta})}\frac{\partial^2 c(\boldsymbol{\theta})}{\partial \theta_j^2} - \frac{1}{c^2(\boldsymbol{\theta})}\left(\frac{\partial c(\boldsymbol{\theta})}{\partial \theta_j}\right)^2 + Var\left(\sum_{i=1}^{k}\frac{\partial^2 w_i(\theta)}{\partial \theta_j^2}t_i(X)\right) \\
&\quad + \mathbb{E}\left(\sum_{i=1}^{k}\frac{\partial^2 w_i(\theta)}{\partial \theta_j^2}t_i(X)\right) \\
&= \frac{\partial^2}{\partial \theta_j^2}\, log(c(\boldsymbol{\theta})) + Var\left(\sum_{i=1}^{k}\frac{\partial^2 w_i(\theta)}{\partial \theta_j^2}t_i(X)\right) + \mathbb{E}\left(\sum_{i=1}^{k}\frac{\partial^2 w_i(\theta)}{\partial \theta_j^2}t_i(X)\right).
\end{aligned}
$$

Thus, we can get (8).