MANA130353.01: Statistic Inference	**1st Semester, 2022-2023**

Homework 4

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Note: This note is a reference answer for the homework.

Disclaimer: *This note is only used as a reference solution for the homework, and the solution to each question is not unique. If you have any questions, you can add my WeChat ID statchaij, or you can find my lecture video here. You can also come to discuss with me an hour before class every week.*

Problem 1 $(10')$**.** *Let* $X_1, \ldots, X_n \overset{iid}{\sim} N\left(\theta, a\theta^2\right)$*, where* a *is a known positive constant and* $\theta > 0$.

(a). Show that the parameter space does not contain a two-dimensional open set.

(b). Show that the statistic $T = \left(\bar{X}, S^2\right)$ *is a sufficient statistic for* θ*, but the family of distributions is not complete.*

Pf: (a). The parameter space is $\Theta = \{(\mu, \sigma^2) : \sigma^2 = a\mu^2\}$ which is a quadratic curve in $\mathbb{R}^2$, so it does not contain a two-dimensional open set.

(b). The jpdf of $(X_1, \cdots, X_n)$ is

$$f(x, \theta) = \left(2\pi a\theta^2\right)^{-\frac{n}{2}} \cdot \exp\left(-\sum_{i=1}^{n} \frac{(x_i - \theta)^2}{2a\theta^2}\right)$$
$$= \left(2\pi a\theta^2\right)^{-\frac{n}{2}} \cdot \exp\left(-\frac{n}{2a}\right) \cdot \exp\left(-\frac{\sum_{i=1}^{n} x_i^2}{2a\theta^2} + \frac{\sum_{i=1}^{n} x_i}{a\theta}\right).$$

By factorization criterion we have $(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i^2)$ is sufficient for θ. Since there is a bijection between $(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i^2)$ and $(\bar{X}, S^2)$, then $(\bar{X}, S^2)$ is also sufficient for θ.

However, we can show

$$E\left(\bar{X}^2\right) = \frac{n+a}{n}\theta^2, E\left(S^2\right) = a\theta^2.$$

Let $g((\bar{x}, s^2) = \frac{n}{n+a}\bar{x}^2 - \frac{1}{a}s^2$, then $\forall\theta$, $E_\theta g\left(\bar{x}, s^2\right) = 0$ but

$$P_\theta\left(\frac{n}{n+a}\bar{X}^2 - \frac{1}{a}S^2 = 0\right) \neq 1, \quad \forall\theta$$

Hence the family of distributions is not complete.

$\square$

Problem 2 $(10')$**.** *Let $X_1, \ldots, X_n$ be a random sample from the following population:*

$$f(x, \theta) = \theta x^{\theta-1}, \quad 0 < x < 1, \quad \theta > 0.$$

(a). Is $\sum_{i=1}^n X_i$ sufficient for θ ?

(b). Find a complete sufficient statistic for θ.

Pf: (a). The jpdf of $(X_!, \cdots, X_n)$ is

$$f(x, \theta) = \theta^n \prod_{i=1}^n x_i^{\theta-1} \mathbb{I} \prod_{i=1}^n (0 < x_i < 1) = \theta^n \exp\left(\theta \sum_{i=1}^n \ln x_i\right) \prod_{i=1}^n \frac{1}{x_i} (0 < x_i < 1)$$

Suppose $T(X) = \sum_{i=1}^n X_i$ is sufficient for θ, there is a function g such that

$$\theta^n \prod_{i=1}^n x_i^{\theta-1} \mathbb{I} \prod_{i=1}^n (0 < x_i < 1) = f(\sum_{i=1}^n x_i, \theta) h(x)$$

where $h(x) = \mathbb{I} \prod_{i=1}^n (0 < x_i < 1)$. We can find to sample point $X = (\frac{1}{2}, \frac{1}{2}, x_3, \cdots, x_n)$ and $X' = (\frac{1}{4}, \frac{3}{4}, x_3, \cdots, x_n)$ such that $T(X) = T(X')$ but $\frac{1}{2} \cdot \frac{1}{2} \prod_{i=3}^n x_i^{\theta-1} \neq \frac{1}{4} \cdot \frac{3}{4} \prod_{i=3}^n x_i^{\theta-1}$ for $0 < \theta < 1$, which makes a contradiction. Hence $\sum_{i=1}^n X_i$ is not sufficient for θ.

(b). By factorization criterion we have $\sum_{i=1}^n \ln X_i$ is sufficient for θ. Since the distribution family is exponential family with natural parameter θ and its corresponding parameter space $\Theta = \{\theta, \theta > 0\}$ have some open set, then $\sum_{i=1}^n \ln X_i$ is complete for θ.

$\square$

Problem 3 $(5')$**.** *Suppose $X_1, \ldots, X_n$ are independently sampled from the following pmf*

$$P(X = k) = -\frac{1}{\ln(1-p)} \frac{p^k}{k}, \quad 0 < p < 1, k = 1, 2, \ldots$$

Use the method of moment to find an estimator for p.

Pf: Since

$$E(X) = \sum_{k=1}^\infty -\frac{p^k}{\ln(1-p)} = -\frac{1}{\ln(1-p)} \cdot \frac{p}{1-p}$$

$$E\left(X^2\right) = \sum_{k=1}^\infty -\frac{k \cdot p^k}{\ln(1-p)} = -\frac{1}{\ln(1-p)} \cdot \frac{p}{(1-p)^2}.$$

Then $p = 1 - \frac{E(X)}{E(X^2)}$ so that a method of moment estimator for p is $\widehat{p} = 1 - \frac{\sum_{i=1}^n X_i}{\sum_{i=1}^1 X_i^2}$. $\square$

Problem 4 (5′). *Suppose* $X_1, \ldots, X_n \overset{iid}{\sim} N\left(\mu, \sigma^2\right)$. *Find a method of moment estimator for* $P(X > 1)$.

Sol: **Method I**: Since $E(\mathbb{I}\{X > 1\}) = P(X > 1)$, then a method of moment estimator for $P(X > 1)$ is

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}\{X_i > 1\}.$$

Method II: Since $P(X > 1) = 1 - \Phi\left(\frac{1-\mu}{\sigma}\right)$, a method of moment estimator for $P(X > 1)$ is

$$\hat{\theta} = 1 - \Phi(\frac{1 - \bar{X}}{\sqrt{S_n^2}}),$$

where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, $S_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$.

$\square$