Statistical Inference

1 Probability

Theorem 1.1. *(The derivation of the pdf of order statistics in a direct way)*

$$f_{x_{(r)}}(t) = \frac{n!}{(r-1)!(n-r)!}[F_x(t)]^{r-1} f_x(t)[1-F_x(t)]^{(n-r)} \tag{1}$$

Proof. By the definition of $F_{x(r)}(t)$, we have

$$\begin{aligned} F_{x_{(r)}}(t) = \mathbb{P}(x_{(r)} \le t) &= \sum_{i=r}^{n} \mathbb{P}(\sum_{j=1}^{n} I(x_j \le t) = i) \\ &= \sum_{i=r}^{n} \binom{n}{i} [F_x(t)]^i [1-F_x(t)]^{n-i}. \end{aligned} \tag{2}$$

To get the pdf, we can take derivative of both sides of (2)

$$\begin{aligned} f_{x_{(r)}}(t) &= \sum_{i=r}^{n} \binom{n}{i} \left\{ i[F_x(t)]^{i-1} f_x(t)[1-F_x(t)]^{n-i} - (n-i)[F_x(t)]^i[1-F_x(t)]^{n-i-1} f_x(t) \right\} \\ &= \sum_{i=r}^{n} \binom{n}{i} [F_x(t)]^{i-1}[1-F_x(t)]^{n-i-1} f_x(t) \left\{ i[1-F_x(t)] - (n-i)F_x(t) \right\} \\ &= \binom{n}{r} r[F_x(t)]^{r-1}[1-F_x(t)]^{n-r} + \sum_{i=r+1}^{n} \binom{n}{i} i[F_x(t)]^{i-1}[1-F_x(t)]^{n-i} \\ &\quad - \sum_{i=r}^{n-1} \binom{n}{i} (n-i)[F_x(t)]^{i-1}[1-F_x(t)]^{n-i} \\ &= \binom{n}{r} r[F_x(t)]^{r-1}[1-F_x(t)]^{n-r} + \sum_{i=r}^{n-1} \binom{n}{i+1} (i+1)[F_x(t)]^{i}[1-F_x(t)]^{n-i-1} \\ &\quad - \sum_{i=r}^{n-1} \binom{n}{i} (n-i)[F_x(t)]^{i-1}[1-F_x(t)]^{n-i} \\ &= f_{x_{(r)}}(t) = \frac{n!}{(r-1)!(n-r)!}[F_x(t)]^{r-1} f_x(t)[1-F_x(t)]^{(n-r)} \end{aligned}$$

□

Theorem 1.2. *(the pdf of noncentral chi-squared distribution)*

$$f(x) = \begin{cases} e^{-\delta^2/2} \sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{\delta^2}{2}\right)^i \frac{x^{i+n/2-1}}{2^{i+n/2}\Gamma(n/2+i)} e^{-x/2}, & x > 0 \\ 0, & x \le 0 \end{cases} \tag{3}$$

where $\delta = \sqrt{\sum_{i=1}^{n} a_i^2}$.

Proof. To prove (3), we need to use a orthogonal transformation such that $X_1^2 + \cdots + X_n^2 = Y_1^2 + Z$, where $X_i \sim N(a_i, 1)$, $Y_1 \sim N(\delta, 1)$ and $Z \sim \chi^2_{n-1}$. The orthogonal matrix A can be

$$\begin{bmatrix} \frac{a_1}{\delta} & \frac{a_2}{\delta} & \cdots & \frac{a_n}{\delta} \\ \frac{a_2}{\sqrt{a_1^2+a_2^2}} & \frac{-a_1}{a_1^2+a_2^2} & & \\ \frac{a_2 a_3}{\sqrt{\sum_{i=1}^{3} a_i^2}} & \frac{a_1 a_3}{\sqrt{\sum_{i=1}^{3} a_i^2}} & \frac{-a_1 a_2}{\sqrt{\sum_{i=1}^{3} a_i^2}} & \\ & & & \ddots \end{bmatrix}.$$

□