Statistical Inference

1 Probability

Theorem 1.1. *(The derivation of the pdf of order statistics by a direct way)*

$$f_{x_{(r)}}(t) = \frac{n!}{(r-1)!(n-r)!}[F_x(t)]^{r-1} f_x(t)[1-F_x(t)]^{(n-r)} \tag{1}$$

Proof. By the definition of $F_{x(r)}(t)$, we have

$$\begin{aligned} F_{x_{(r)}}(t) = \mathbb{P}(x_{(r)} \le t) &= \sum_{i=r}^{n} \mathbb{P}(\sum_{j=1}^{n} I(x_j \le t) = i) \\ &= \sum_{i=r}^{n} \binom{n}{i} [F_x(t)]^i [1-F_x(t)]^{n-i}. \end{aligned} \tag{2}$$

To get the pdf, we can take derivative of both sides of (2)

$$\begin{aligned} f_{x_{(r)}}(t) &= \sum_{i=r}^{n} \binom{n}{i} \left\{ i[F_x(t)]^{i-1} f_x(t)[1-F_x(t)]^{n-i} - (n-i)[F_x(t)]^i [1-F_x(t)]^{n-i-1} f_x(t) \right\} \\ &= \sum_{i=r}^{n} \binom{n}{i} [F_x(t)]^{i-1}[1-F_x(t)]^{n-i-1} f_x(t) \left\{ i[1-F_x(t)] - (n-i)F_x(t) \right\} \\ &= \binom{n}{r} r[F_x(t)]^{r-1}[1-F_x(t)]^{n-r} + \sum_{i=r+1}^{n} \binom{n}{i} i[F_x(t)]^{i-1}[1-F_x(t)]^{n-i} \\ &\quad - \sum_{i=r}^{n-1} \binom{n}{i} (n-i)[F_x(t)]^{i-1}[1-F_x(t)]^{n-i} \\ &= \binom{n}{r} r[F_x(t)]^{r-1}[1-F_x(t)]^{n-r} + \sum_{i=r}^{n-1} \binom{n}{i+1} (i+1)[F_x(t)]^{i}[1-F_x(t)]^{n-i-1} \\ &\quad - \sum_{i=r}^{n-1} \binom{n}{i} (n-i)[F_x(t)]^{i-1}[1-F_x(t)]^{n-i} \\ &= f_{x_{(r)}}(t) = \frac{n!}{(r-1)!(n-r)!}[F_x(t)]^{r-1} f_x(t)[1-F_x(t)]^{(n-r)} \end{aligned}$$

□