MANA130353.01: Statistic Inference **1st Semester, 2022-2023**

Homework 5

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Note: This note is a reference answer for the homework.

Disclaimer: *This note is only used as a reference solution for the homework, and the solution to each question is not unique. If you have any questions, you can add my WeChat ID statchaij, or you can find my lecture video here. You can also come to discuss with me an hour before class every week.*

Problem 1 (5′)**.** *Let W be a statistic, show that* $\mathbb{E}_\theta(W-\theta)^2 = \operatorname{Var}(W) + (\mathbb{E}_\theta(W)-\theta)^2$.

Pf:

$$\begin{aligned}
\mathbb{E}_\theta(W-\theta)^2 &= \mathbb{E}_\theta\left(W-\mathbb{E}_\theta(W)+\mathbb{E}_\theta(W)-\theta\right)^2 \\
&= \operatorname{Var}_\theta(W) + 2\mathbb{E}_\theta\left(W-\mathbb{E}_\theta(W)\right)\left(\mathbb{E}_\theta(W)-\theta\right) + \left(\mathbb{E}_\theta(W)-\theta\right)^2 \\
&= \operatorname{Var}_\theta(W) + \left(\mathbb{E}_\theta(W)-\theta\right)^2
\end{aligned}$$

□

Problem 2 (15′)**.** $X_1,\ldots,X_n \overset{iid}{\sim} f(x \mid \mu)$ *where*

$$f(x\mid\mu) = e^{-(x-\mu)}\cdot\mathbb{I}(x>\mu), \quad \mu\in(-\infty,\infty).$$

(a). Find $\hat{\mu}_{mle}$.

(b). Use method of moments to find an unbiased estimator for μ.

(c). Compare the estimators from (a) and (b), which one has a smaller MSE?

Pf: (a). The likelihood function is

$$\mathcal{L}(\mu) = \prod_{i=1}^n e^{-(x_i-\mu)}\cdot\mathbb{I}(x_i\geq\mu) = e^{-n(\bar{x}-\mu)}\cdot\mathbb{I}\left(x_{(1)}\geq\mu\right)$$

Since $e^{-(\bar{x}-\mu)}$ is increasing of μ, and $\mu\leq x_{(1)}$, then $\mathcal{L}(\mu)$ attains the maxima at $\mu = x_{(1)}$, that is, $\hat{\mu}_{mle} = x_{(1)}$.

(b). We can calculate $\mathbb{E}(X) = \mu+1$ so that a method of moments for μ is $\hat{\mu}_{mm} = \bar{X}-1$ which is unbiased.

(c). The pdf of $X_{(1)}$ is $f_{x_{(1)}}(x) = ne^{-n(x-\mu)}\mathbb{I}\{x > \mu\}$. We can find $X_{(1)} - \mu \sim \text{Exp}(n)$ so that

$$\text{Bias}(X_{(1)}) = \mathbb{E}\left(X_{(1)} - \mu\right) = \frac{1}{n}, \text{Var}\left(X_{(1)}\right) = \text{Var}(X_{(1)} - \mu) = \frac{1}{n^2}.$$

Thus $\text{MSE}(\hat{\mu}_{mle}) = \frac{2}{n^2}$. Analogously, we can calculate

$$\text{Var}(\bar{X} - 1) = \frac{1}{n}\text{Var}(X) = \frac{1}{n},$$

then $\text{MSE}(\hat{\mu}_{mm}) = \frac{1}{n}$. Thus when $n = 1$ the mse of $\hat{\mu}_{mm}$ is smaller; when $n = 2$ the mse of $\hat{\mu}_{mm}$ and $\hat{\mu}_{mle}$ are the same; when $n > 2$ the mse of $\hat{\mu}_{mle}$ is smaller.

$\square$

Problem 3 $(5')$**.** *Let $F(x)$ and $f(x)$ be the distritbution and density functions for iid random variables $X_1, \ldots, X_n$. Show that*

$$\int \cdots \int_{a<x_1<\ldots<x_n<b} f(x_1) \cdots f(x_n)\, dx_1 \cdots dx_n = \frac{1}{n!}[F(b) - F(a)]^n.$$

Pf: **Method I:** Let $\pi : [n] \to [n]$ be any permutation and $\pi(\mathbf{X}) = (X_{\pi(1)}, \cdots, X_{\pi(n)})$ where $[n]$ denote the index set $\{1, 2, \cdots, n\}$. Since X_i is iid, then the jpdf of $\mathbf{X}$ is the same as $\pi(\mathbf{X})$ which implies $P(a < X_1 < \cdots < X_n < b) = P(a < X_{\pi(1)} < \cdots < X_{\pi(n)} < b)$. Note that $\{a < X_1 < \cdots < X_n < b\} \cap \{a < X_{\pi(1)} < \cdots < X_{\pi(n)} < b\} = \emptyset$ iff $\pi(\mathbf{X}) \neq \mathbf{X}$. Since there are $n!$ different permutations and $\int_a^b \cdots \int_a^b f(x_1) \cdots f(x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n = [F(b) - F(a)]^n$, we have

$$\begin{aligned}
\int_a^b \cdots \int_a^b f(x_1) \cdots f(x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n &= \sum_{\{\text{all possible permutations}\}} \int \cdots \int_{a<x_1<\ldots<x_n<b} f(x_1) \cdots f(x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n \\
&= n! \int \cdots \int_{a<x_1<\ldots<x_n<b} f(x_1) \cdots f(x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n
\end{aligned}$$

that is,

$$\int \cdots \int_{a<x_1<\ldots<x_n<b} f(x_1) \cdots f(x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n = \frac{1}{n!}[F(b) - F(a)]^n$$

Method II:

$$
\begin{aligned}
LHS &= \int_a^b \int_{x_1}^b \int_{x_2}^b \cdots \int_{x_{n-1}}^b f(x_1)\cdots f(x_n) dx_1 \cdots dx_n \\
&= \int_a^b \int_{x_1}^b \int_{x_2}^b \cdots \int_{x_{n-2}}^b f(x_1)\cdots f(x_{n-1})[F(b)-F(x_{n-1})] dx_1 \cdots dx_{n-1} \\
&= \int_a^b \int_{x_1}^b \int_{x_2}^b \cdots \int_{x_{n-3}}^b f(x_1)\cdots f(x_{n-2})\frac{1}{2}[F(b)-F(x_{n-2})]^2 dx_1 \cdots dx_{n-2} \\
&= \cdots \\
&= \frac{1}{n!}[F(b)-F(a)]^n = RHS.
\end{aligned}
$$

$\square$

Problem 4 (5′). *If $f(x \mid \theta)$ satisfies*

$$\frac{d}{d\theta}\mathbb{E}_\theta\left(\frac{\partial}{\partial\theta}\log f(X \mid \theta)\right) = \int \frac{\partial}{\partial\theta}\left[\left(\frac{\partial}{\partial\theta}\log f(x \mid \theta)\right) f(x \mid \theta)\right] dx$$

(true for an exponential family), show that

$$\mathbb{E}_\theta\left[\left(\frac{\partial}{\partial\theta}\log f(X \mid \theta)\right)^2\right] = -\mathbb{E}_\theta\left(\frac{\partial^2}{\partial\theta^2}\log f(X \mid \theta)\right).$$

Sol: Since

$$\mathbb{E}_\theta\left[\frac{\partial}{\partial\theta}\log f(x \mid \theta)\right] = \int_{\mathcal{X}} \frac{\partial}{\partial\theta}\log f(x \mid \theta) f(x \mid \theta)\mathrm{d}x = \int_{\mathcal{X}} \frac{\partial}{\partial\theta} f(x \mid \theta)\mathrm{d}x = 0,$$

then we take derivative of both sides to obtain

$$
\begin{aligned}
0 = \frac{d}{d\theta}\mathbb{E}_\theta\left(\frac{\partial}{\partial\theta}\log f(\mathbf{X} \mid \theta)\right) &= \int \frac{\partial}{\partial\theta}\left[\left(\frac{\partial}{\partial\theta}\log f(x \mid \theta)\right) f(x \mid \theta)\right] dx \\
&= \int_{\mathcal{X}} \left[\frac{\partial^2}{\partial\theta^2}\log f(x \mid \theta)\right] f(x \mid \theta) + \left[\frac{\partial}{\partial\theta}\log f(x \mid \theta)\right] \frac{\partial}{\partial\theta} f(x \mid \theta) dx \\
&= \mathbb{E}_\theta\left[\frac{\partial^2}{\partial\theta^2}\log f(x \mid \theta)\right] + \int_{\mathcal{X}} \left[\frac{\partial}{\partial\theta}\log f(x \mid \theta)\right]^2 f(x \mid \theta) dx \\
&= \mathbb{E}_\theta\left(\frac{\partial^2}{\partial\theta^2}\log f(X \mid \theta)\right) + \mathbb{E}_\theta\left[\left(\frac{\partial}{\partial\theta}\log f(X \mid \theta)\right)^2\right],
\end{aligned}
$$

that is

$$\mathbb{E}_\theta\left[\left(\frac{\partial}{\partial\theta}\log f(X \mid \theta)\right)^2\right] = -\mathbb{E}_\theta\left(\frac{\partial^2}{\partial\theta^2}\log f(X \mid \theta)\right).$$

$\square$