Homework 2

Due Sept 28th, 2022

Problem 1

Let $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$, $S_n^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2$. Suppose $X_{n+1} \sim N(\mu, \sigma^2)$ and is independent of $X_1, \dots, X_n$, find the distribution of

$$\frac{X_{n+1} - \bar{X}}{S_n}\sqrt{\frac{n-1}{n+1}}.$$

Problem 2

Suppose $X_1, \dots, X_n$ are independent and $X_i \sim N(0, \sigma_i^2)$ for $i = 1, \dots, n$. Define

$$\xi = \sum_{i=1}^n \frac{(X_i - Z)^2}{\sigma_i^2}$$

where

$$Z = \left(\sum_{i=1}^n \frac{X_i}{\sigma_i^2}\right)\left(\sum_{i=1}^n \frac{1}{\sigma_i^2}\right)^{-1}.$$

Find the distribution of ξ. (Hint: Use a proper orthogonal transform.)

Problem 3

Suppose $X_1, \dots, X_n \sim Poissson(\lambda)$. Show that

$$\frac{\bar{X} - \lambda}{\sqrt{\bar{X}/n}} \xrightarrow{D} N(0, 1).$$

(Hint: Use Slutsky's theorem.)

Problem 4

If X is a random variable with pdf or pmf of the form

$$f(x|\boldsymbol{\theta}) = h(x)c(\boldsymbol{\theta})\exp\left(\sum_{i=1}^k w_i(\boldsymbol{\theta})t_i(x)\right)$$

show that

(a): $E\left(\sum_{i=1}^{k} \dfrac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(X)\right) = -\dfrac{\partial}{\partial \theta_j} \log c(\boldsymbol{\theta})$.

(b): $Var\left(\sum_{i=1}^{k} \dfrac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(X)\right) = -\dfrac{\partial^2}{\partial \theta_j^2} \log c(\boldsymbol{\theta}) - E\left(\sum_{i=1}^{k} \dfrac{\partial^2 w_i(\boldsymbol{\theta})}{\partial \theta_j^2} t_i(X)\right)$.

Hint:

$$\int f(x|\boldsymbol{\theta})dx = \int h(x)c(\boldsymbol{\theta}) \exp\left(\sum_{i=1}^{k} w_i(\boldsymbol{\theta}) t_i(x)\right) = 1.$$

Differentiate both sides and then rearrange terms.