Statistical Inference Assignment 9

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PROBLEM 1.

Suppose $X_1, \ldots, X_{20} \overset{iid}{\sim} B(p)$. We want to test the hypothesis

$$H_0 : p = 0.2 \qquad vs \qquad H_1 : p \neq 0.2. \tag{1}$$

Suppose the decision rule is $\phi(\boldsymbol{x})$ and $\phi(\boldsymbol{x}) = 1$ if and only if $\sum_{i=1}^{20} x_i \geq 7$ or $\sum_{i=1}^{20} x_i \leq 1$.

(a) Find the power of the decision rule when $p = 0, 0.1, 0.2, \ldots, 0.9$. Draw a graph of the power as a function of p.

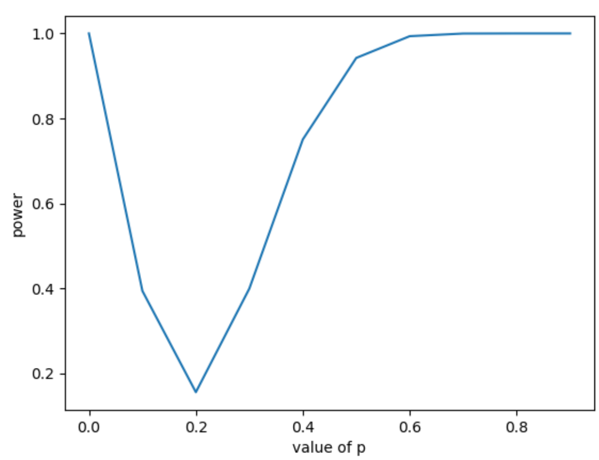
(b) Find the type I error probability of the rule ϕ. What is the type II error probability when $p = 0.05$?

SOLUTION.

(a) The power of the decision rule can be computed by:

$$\beta(p) = (1-p)^{20} + 20p(1-p)^{19} + \sum_{k=7}^{20} p^k (1-p)^{20-k}.$$

And the result is shown below:



(b) By the calculation results in problem (a), we can know that the type I error probability is $\beta(0.2) \approx 0.156$.

When $p = 0.05$, by (1), we have $\beta(0.05) \approx 0.736$. Therefore, the type II error probability is $1 - \beta(0.05) \approx 0.264$.

PROBLEM 2.

Suppose $X_1, \ldots, X_{10} \overset{iid}{\sim} U(0, \theta)$. Construct a test for testing

$$H_0 : \theta \leq 1 \qquad vs \qquad H_a : \theta \geq 1 \tag{2}$$

at a level α (Construct a test means finding a rejection region).

SOLUTION.

Intuitively, we can use $X_{(n)}$ to estimate θ, and $X_{(n)}/\theta \sim Beta(n, 1)$. Under H_0, we have $X_{(n)} \sim Beta(n, 1)$. Suppose there exists a c such that

$$P(X_{(n)} > c) = \alpha,$$

then a level α rejection region is $(\, c, +\infty\,)$. Since $P(X_{(n)} > c) \overset{H_0}{=} 1 - c^n$, we have $c = (1-\alpha)^{1/n}$, which means the rejection region with level α is $(\,(1-\alpha)^{1/n}, +\infty)$. In this problem, it's $(\,(1-\alpha)^{1/10}, +\infty)$. Therefore, when $X_{(10)}$ is grater than $(1-\alpha)^{1/n}$, we can reject the null hypothesis. Otherwise, we can not reject it.

PROBLEM 3.

Suppose $X_1, \ldots, X_n \overset{iid}{\sim} U(0, \theta)$. Use $\max\{X_i\}_{i=1}^n$ to find a $(1-\beta, 1-\gamma)$ tolerance interval.

PROBLEM 3.

Suppose there exists a λ such that

$$P_\theta(P_\theta^*(\lambda X_{(n)} \leq X \leq X_{(n)}) \geq 1 - \beta) \geq 1 - \gamma. \tag{3}$$

Since

$$P_\theta^*(\lambda X_{(n)} \leq X \leq X_{(n)}) = (1-\lambda)\frac{X_{(n)}}{\theta},$$

where $X_{(n)}/\theta \sim Beta(n, 1)$, we can rewrite (3) as

$$P_\theta\left(\frac{X_{(n)}}{\theta} \geq \frac{1-\beta}{1-\lambda}\right) \geq 1 - \gamma.$$

Let $Y = X_{(n)}/\theta \sim Beta(n, 1)$ with cdf $F(y) = y^n$.Then we have

$$P_\theta\left(\frac{X_{(n)}}{\theta} \geq \frac{1-\beta}{1-\lambda}\right) = 1 - F\left(\frac{1-\beta}{1-\lambda}\right) \geq 1 - \gamma.$$

We can let

$$1 - F\left(\frac{1-\beta}{1-\lambda}\right) = 1 - \left(\frac{1-\beta}{1-\lambda}\right)^n = 1 - \gamma,$$

which gives us a reasonable value of λ:

$$\lambda_0 = 1 - \frac{1-\beta}{\gamma^{1/n}}.$$

Therefore, one $(1-\beta, 1-\gamma)$ tolerance interval is

$$[\,\lambda_0 X_{(n)},\ X_{(n)}\,].$$