MANA130353.01: Statistic Inference	**1st Semester, 2022-2023**

Homework 7

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Note: This note is a reference answer for the homework.

Disclaimer: *This note is only used as a reference solution for the homework, and the solution to each question is not unique. If you have any questions, you can add my WeChat ID statchaij, or you can find my lecture video here. You can also come to discuss with me an hour before class every week.*

Problem 1 $(5')$**.** *Suppose* $X_1, \ldots, X_n \overset{iid}{\sim} \Gamma(\alpha, \lambda)$, *where* α *is known and* $\lambda > 0$. *Prove that* $\bar{X}/\alpha$ *is the best unbiased estimator of* $1/\lambda$.

Pf: The jpdf of $(X_1, \cdots, X_n)$ is

$$f(X \mid \alpha, \lambda) = \prod_{i=1}^{n} \frac{\lambda^{\alpha} x_i^{\alpha-1} e^{-\lambda x_i}}{\Gamma(\alpha)} = \frac{\lambda^{n\alpha}}{\Gamma(\alpha)^n} \left(\prod_{i=1}^{n} x_i \right)^{\alpha-1} e^{-\lambda \sum x_i},$$

which belongs to the exponential family, and the natural parameter space has an open set thus $\sum X_i$ is complete sufficient statistics.

It is easy to obtain $E(T) = \frac{n\alpha}{\lambda}$. Therefore, $\frac{\bar{X}}{\alpha}$ is an unbiaeds estimator of $\frac{1}{\lambda}$ and by the Lehmann-Scheffe Theorem, $\frac{\bar{X}}{\alpha}$ is the best unbiased estimator of $\frac{1}{\lambda}$. $\square$

Problem 2 $(8')$**.** *Suppose* $X_1, \ldots, X_n \overset{iid}{\sim} N\left(0, \sigma^2\right)$.

(a). Find the UMVUE of σ.

(b). Find the UMVUE of σ^4.

Pf: By the property of the exponential family[1], we have $T = \sum_{i=1}^{n} X_i^2$ is complete sufficient statistics. Note that $\frac{T}{\sigma^2} \sim \chi_n^2 = Ga(\frac{n}{2}, \frac{1}{2})$ and $E(T^k) = 2^k \frac{\Gamma(\frac{n}{2}+k)}{\Gamma(\frac{n}{2})} \sigma^{2k}$ by the definition of gamma function.

(a). From above we discuss, we have $E(\frac{\Gamma(\frac{n}{2})}{\sqrt{2}\Gamma(\frac{n}{2}+\frac{1}{2})} T^{1/2}) = \sigma$, by the Lehmann-Scheffe Theorem, $\frac{\Gamma(\frac{n}{2})}{\sqrt{2}\Gamma(\frac{n}{2}+\frac{1}{2})} T^{1/2}$ is the UMVUE of σ.

[1]The detatils are omitted here.

(b). Simalar to above, we have $E(\frac{\Gamma(\frac{n}{2})}{4\Gamma(\frac{n}{2}+2)}T^2) = E(\frac{1}{n^2+2n}T^2) = \sigma^4$, by the Lehmann-Scheffe Theorem, $\frac{1}{n^2+2n}T^2$ is the UMVUE of σ^4.

$\square$

Problem 3 (12′)**.** *Suppose $X_1, \ldots, X_n \overset{iid}{\sim} \mathrm{Geom}(\theta)$, where the pmf of $\mathrm{Geom}(\theta)$ is*

$$P(X = i) = \theta(1-\theta)^{i-1}, \quad i = 1, 2, \ldots, \quad 0 < \theta < 1.$$

Geometric distribution can be interpreted as number of Bernoulli trials $(\mathrm{Ber}(\theta))$ *needed to get one success.*

(a). Show that $T = \sum_{i=1}^n X_i$ is a sufficient statistic for θ, and has pmf

$$P_\theta(T = t) = \begin{pmatrix} t-1 \\ n-1 \end{pmatrix} \theta^n (1-\theta)^{t-n}, t = n, n+1, n+2, \cdots$$

(b). Compute $E_\theta(T)$ and use it to find the UMVUE of θ^{-1}. (Hint: $\sum_{i=0}^\infty z^i = \frac{1}{1-z}$ for $|z| < 1$.)

(c). Show that $\psi(X_1) = I(X_1 = 1)$ is an unbiased estimator of θ, use this fact to find the UMVUE of θ.

Pf: (a). The jpdf of $(X_1, \cdots, X_n)$ is

$$f(X \mid \theta) = \prod_{i=1}^n \theta(1-\theta)^{x_i - 1} = \left(\frac{\theta}{1-\theta}\right)^n (1-\theta)^T = \left(\frac{1 - e^{\log(1-\theta)}}{e^{\log(1-\theta)}}\right)^n e^{\log(1-\theta)T}$$

By the factorization theorem, T is sufficient.

The event of T equals to t is equivalent to the number of Bernoulli trials needed to get n success, and one of the success is the t_{th} trial. Therefore,

$$P_\theta(T = t) = \begin{pmatrix} t-1 \\ n-1 \end{pmatrix} \theta^n (1-\theta)^{t-n}, t = n, n+1, n+2, \ldots$$

(b). Since the sample distribution belongs to the exponential family and the natural parameter space contains an open set, T is complete.

$$E(X) = \sum_{k=1}^\infty k\theta(1-\theta)^{k-1} = \theta \sum_{k=1}^\infty k(1-\theta)^{k-1} = \theta \sum_{k=1}^\infty \frac{\mathrm{d}(1-\theta)^k}{\mathrm{d}(1-\theta)}$$
$$= \theta \frac{\mathrm{d}}{\mathrm{d}(1-\theta)} \left(\sum_{k=0}^\infty (1-\theta)^k\right) = \theta \frac{\mathrm{d}}{\mathrm{d}(1-\theta)} \left(\frac{1}{\theta}\right) = \frac{\theta}{\theta^2} = \frac{1}{\theta}.$$

Thus $E(T) = \frac{n}{\theta}$, and $E(\frac{T}{n}) = \frac{1}{\theta}$. By the Lehmann-Scheffe Theorem, $\frac{T}{n}$ is the UMVUE of θ^{-1}.

(c). Since $E\left(\psi\left(X_1\right)\right) = P\left(X_1 = 1\right) = \theta$, ψ is an unbiased estimator of θ. By the Lehmann-Scheffe Theorem $E\left(\psi\left(X_1\right) \mid T\right)$ is UMVUE of θ, where

$$E\left(\psi\left(X_1\right) \mid T = t\right) = \frac{P\left(X_1 = 1, \sum_{i=2}^{n} X_i = t-1\right)}{P\left(\sum_{i=1}^{n} X_i = t\right)} = \frac{\theta \cdot \begin{pmatrix} t-2 \\ n-2 \end{pmatrix} \theta^{n-1}(1-\theta)^{t-n}}{\begin{pmatrix} t-1 \\ n-1 \end{pmatrix} \theta^{n}(1-\theta)^{t-n}} = \frac{n-1}{t-1},$$

for $t = n, n+1, \ldots$, that is $E\left(\psi\left(X_1\right) \mid T\right) = \frac{n-1}{T-1}$.

□

Problem 4 (5′). *Suppose* $X_1, \ldots, X_n \overset{iid}{\sim} f_a(x)$ *where*

$$f_a(x) = e^{-(x-a)} \mathbb{I}_{(a,\infty)}(x), \quad -\infty < a\infty.$$

Find the UMVUE of a.

Sol: The jpdf of $(X_1, \cdots, X_n)$ is

$$f(X \mid a) = e^{na} \mathbb{I}\left(x_{(1)} > a\right) \cdot e^{-\sum x_i}.$$

By the factorization theorem, $T = X_{(1)}$ is the sufficient statistics. Then,we show T is also complete. The pdf of T is

$$f(t) = n f_a(t) \left(1 - F_a(t)\right)^{n-1} = n e^{-n(t-a)} \mathbb{I}_{(a,\infty)}(t).$$

Suppose $\phi(T)$ is any real value function satisfying $E(\phi(T)) = 0$, then we have

$$\int_a^\infty \phi(t) n e^{-n(t-a)} dt = 0 \Rightarrow \int_a^\infty \phi(t) e^{-nt} dt = 0.$$

Take derivative with respect to a on both sides, then we have[2]

$$0 = \phi(a) e^{-na}, \quad \forall a \in \mathbb{R}.$$

Thus T is also complete. Since $E(T) = \int_a^\infty n t e^{-n(t-a)} dt = a + \frac{1}{n}$, $T - \frac{1}{n}$ is an unbiased estimator of a, by the Lehmann-Scheffe Theorem, $X_{(1)} - \frac{1}{n}$ is the UMVUE of a. □

[2]In fact, this is true only for function which is Riemann integrable.