Ecuaciones De Hamilton.

Recordemos a la Lagrangiana.

$$L(q_j, \dot{q}_j, t), \quad j=1,\dots,n.$$

Hamilton define otra cantidad llamada momento canónico:

$$p_j = \frac{\partial L}{\partial \dot{q}_j}, \quad j = 1,\dots, n. \quad \dots (0)$$

De esta forma, los momentos canónicos determinarán las velocidades:

$$\dot{q}_j = \dot{q}_j(q_j, p_j, t), \quad j=1,\dots,n.$$

q_j y p_j son llamados variables conjugadas. Si $\frac{\partial L}{\partial q_j} = 0$, entonces:

$$\frac{\partial L}{\partial q_j} = 0 \Rightarrow p_j = cte.$$

Función De Hamilton.

$$\frac{dL}{dt} = \sum_{j=1}^{n} \frac{\partial L}{\partial q_j}\dot{q}_j + \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j + \frac{\partial L}{\partial t}$$

Considerando las ecs. de Lagrange:

$$\frac{dL}{dt} = \sum_{j=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j + \frac{\partial L}{\partial t}$$

$$= \sum_{j=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j\right) + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{d}{dt}\left(\sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j - L\right) = -\frac{\partial L}{\partial t}$$

Si t no aparece explícitamente en L, entonces:

$$\sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j - L = cte.$$

$$\Rightarrow \sum_{j=1}^{n} p_j \dot{q}_j - L = cte. \quad \dots (1)$$

(1) es llamada función de Hamilton, o Hamiltoneana, denotada por:

$$H = H(p_j, q_j, t)$$

$$= \sum_{j=1}^{n} p_j \dot{q}_j - L(q_j, \dot{q}_j, t) \dots (2)$$

De (0) se despeja $\dot{q}_j$ y se sustituye en (1) y (2)

$$\Rightarrow dH = \sum_{j=1}^{n} \frac{\partial H}{\partial q_j} dq_j + \sum_{j=1}^{n} \frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial t} dt \quad \dots (4)$$

de igual manera, de (2):

$$dH = \sum_{j=1}^{n} \dot{q}_j\, dp_j + \sum_{j=1}^{n} d\dot{q}_j\, p_j - dL(q_j, \dot{q}_j, t) \quad \ldots (5)$$

Ahora:

$$dL = \sum_{j=1}^{n} \frac{\partial L}{\partial q_j} dq_j + \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j + \frac{\partial L}{\partial t} dt \quad \ldots (6)$$

Sustituyendo (5) en (6):

$$\begin{aligned} dH &= \sum_{j=1}^{n} \dot{q}_j\, dp_j + \sum_{j=1}^{n} p_j\, d\dot{q}_j - \sum_{j=1}^{n} \frac{\partial L}{\partial q_j} dq_j - \sum_{j=1}^{n} p_j\, d\dot{q}_j - \frac{\partial L}{\partial t} dt \\ &= \sum_{j=1}^{n} \dot{q}_j\, dp_j - \sum_{j=1}^{n} \frac{\partial L}{\partial q_j} dq_j - \frac{\partial L}{\partial t} dt \quad \ldots (7) \\ &= \sum_{j=1}^{n} \dot{q}_j\, dp_j - \sum_{j=1}^{n} \dot{p}_j\, dq_j - \frac{\partial L}{\partial t} dt \end{aligned}$$

Comparando (4) y (7):

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \frac{\partial H}{\partial q_j} = -\dot{p}_j, \quad -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \quad \ldots (8)$$

(8) son llamadas Ecuaciones de mov. de Hamilton.

SIGNIFICADO DE H.

Recordando:

$$H = H(q_j, p_j, t), \quad j = 1, \ldots, n.$$

$$\frac{dH}{dt} = \sum_{j=1}^{n} \frac{\partial H}{\partial q_j} \dot{q}_j + \sum_{j=1}^{n} \frac{\partial H}{\partial p_j} \dot{p}_j + \frac{\partial H}{\partial t}$$

Considerando las ecs. de Hamilton:

$$= \frac{\partial H}{\partial t}$$

Si t no aparece explícitamente en H, entonces $\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0$, $H = cte$. Supongamos que el campo es conservativo $V(q_i)$ donde las reestricciones son holonómicas y escleronómicas, entonces $H = E$.

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\vec{r}}_i^{\,2}, \quad r_i = r_i(q_j), \quad V = v(q_j), \quad j = 1, \ldots, n.$$

Entonces:

$$\dot{\vec{r}}_i = \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j, \quad i = 1, \ldots, N$$

$$\therefore T = \frac{1}{2}\sum_{i=1}^{N} m_i \left(\sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j\right)\cdot\left(\sum_{k=1}^{n}\frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k\right)$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n}\dot{q}_j\dot{q}_k \underbrace{\left(\frac{1}{2}\sum_{i=1}^{n}\frac{\partial \vec{r}_i}{\partial q_j}\cdot\frac{\partial \vec{r}_i}{\partial q_k} m_i\right)}_{a_{jk}}$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}\,\dot{q}_j\dot{q}_k$$

Ahora consideremos $V = V(q_j)$

$$\Rightarrow \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} = P_j$$

$$\Rightarrow H = \sum_{j=1}^{n}\dot{q}_j P_j - L$$

$$= \sum_{j=1}^{n}\dot{q}_j\frac{\partial T}{\partial \dot{q}_j} - L$$

Calculemos primero:

$$\sum_{j=1}^{n}\dot{q}_j\frac{\partial T}{\partial \dot{q}_j} = \sum_{j=1}^{n}\dot{q}_j\frac{\partial}{\partial \dot{q}_j}\left(\sum_{k=1}^{n}\sum_{l=1}^{n} a_{kl}\dot{q}_k\dot{q}_l\right)$$

$$= \sum_{j=1}^{n}\dot{q}_j\sum_{k=1}^{n}\sum_{l=1}^{n} a_{kl}\frac{\partial}{\partial \dot{q}_j}(\dot{q}_k\dot{q}_l)$$

$$= \sum_{j,k,l=1}^{n}\dot{q}_j a_{kl}\left(\delta_{kj}\dot{q}_l + \dot{q}_k\delta_{lj}\right)$$

$$= \sum_{k,l=1}^{n} a_{kl}\sum_{j=1}^{n}\left(\dot{q}_j\dot{q}_l\delta_{kj} + \dot{q}_j\dot{q}_k\delta_{lj}\right)$$

$$= \sum_{k,l=1}^{n} a_{kl}\left(\dot{q}_k\dot{q}_l + \dot{q}_l\dot{q}_k\right) = 2T$$

$$\therefore H = 2T - L$$

$$= T + V = E$$

y.q.u.

Paréntesis de Poisson

$f = f(q_j, p_j, t)$ se llama integral de movimiento de las ecs. de Hamilton, si para cualquier movimiento, dicha función es igual a una cte. C.

$$f(q_j, p_j, t) = C$$

Ejemplo:

1) $mr^2\dot{\theta} = \text{cte.}$ en el problema de los dos cuerpos. Donde:

$$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$\Rightarrow \dot{\theta} = \frac{P_\theta}{mr^2}$$

Supongamos un sistema compuesto con $2n$ ctes