Ejercicio 1.1.1
Let A be a square matrix that is filled with all zeros except for the coordinates where the row number equals the column number. In those cells, the numbers from 1 to n appear in alphabetical order based on each number's English spelling. For example if n=3 then the order would be 1-3-2. Find the trace of A**2

(A) n^2

(B) $n(n+1)/2$

(C) $n(n+1)(2n+1)/6$

(D) n^3

Solución:
First, lets compute A^2. We have for all $i, j = 1, ..., n$;

$$(A^2)_{i,j} = \sum_{k=1}^{n} (A)_{i,k}(A)_{k,j}$$

because A is filled with zeros except for the coordinates where the row number equals the column number, when $i \neq j$ we have that:

$$(A)_{i,k}(A)_{k,j} = 0, \quad \forall k = 1, ..., n$$

wich implies that $(A^2)_{i,j} = 0$ when $i \neq j$. When $i = j$ the sum becomes:

$$\begin{aligned}(A^2)_{i,i} &= \sum_{k=1}^{n} (A)_{i,k}(A)_{k,i} \\ &= \sum_{k=1}^{n} (A)_{i,k}^2 \\ &= (A)_{i,i}^2\end{aligned}$$

so now, the trace of A^2 would be:

$$\begin{aligned}\text{Trace}(A) &= \sum_{i=1}^{n} (A^2)_{i,i} \\ &= \sum_{i=1}^{n} (A)_{i,i}^2\end{aligned}$$

because all the numbers from 1 to n appear in the diagonal of A, then we are just making the sum of all squared numbers from 1 to n, so rearranging all the terrms, the sum becomes:

$$\text{Trace}(A) = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

so the answer is (C). $\square$

Ejercicio 1.1.2
Let a be a prime number bigger than 3 and b an integer coprime to a. What is the smallest prime number that divides both a^4b^4 and $a^2 + ab$?

(A) The smallest prime divisor of b

(B) The smallest prime divisor of ab

(*C*) a

(*D*) b

Solución:

It can't be (D) because b not necessarly is a prime number. Also, it can't be (A) because the smallest prime divisor of b not necessarly divides $a^2 + ab = a(a + b)$.

If p is prime such that $p \mid a^4b^4$ then because a and b are coprime we must only one of these: $p \mid a$ or $p \mid b$.

In the second part, we have that $p \mid a(a + b)$. If $p \mid b$ then p cant divide a, so $p \mid a + b$ which by linearity implies that $p \mid a$, a contradiction.

So, $p \mid a$, whichi implies that $p = a$. Therefore the answer is (C).

□

Ejercicio 1.1.3
Let m, n be the 11th and 12-th Fibonacci numbers where the first and second Fibonacci numbers are both 1. How many subgroups of Z_{m*n} are there?

(*A*) 20

(*B*) 25

(*C*) 30

(*D*) 35

Solución:

We compute the Fibonacci numbers up to 11 and 12 position:

$$1, 1, 2, 3, 5, 8, 13, 21, 34{,}55, 89, 144, ...$$

so we must compute all subgroups of $Z_{89\cdot 144}$. Recall that:

$$89 \cdot 144 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 89$$

we note first that $Z_{89\cdot 144}$ is isomorphic to $Z/89 \cdot 144Z$. Let $n = 89 \cdot 144$. Now, by correspondence theorem, all subgroups of Z/nZ (namely rZ/nZ) are in correspondence with the subgroups of Z such that:

$$nZ \subseteq rZ \subseteq Z$$

the condition $nZ \subseteq rZ$ implies that $r \mid n$, so the set of all subgroups of Z/nZ is:

$$\left\{ rZ/nZ \middle| r \mid n \right\}$$

so, we must compute all divisors of n, with the prime decomposition of $89 \cdot 144$ its seen that there are 30 divisors, so the answer is (C). □

Ejercicio 1.1.4
Let v= [1,2] be a vector in the plane and let $A = 2[[1/sqrt(2), -1/sqrt(2)], [1/sqrt(2), 1/sqrt(2)]]$. What is $(A^8)v$?

(*A*) v

(*B*) 256 v

(*C*) $[128, 0]$

(D) -v

Solución:

Recall the matrix A is:

$$A = 2\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

we remember the form of the rotation matrix of angle θ in the euclidean plane:

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

so, $A = 2R_\theta$ where $\theta = \frac{\pi}{4}$. We observe that:

$$A^8 = (2R_\theta)^8 = 2^8 R_\theta^8 = 256 R_\theta^8$$

but rotation matrix has the property that:

$$R_\alpha R_\beta = R_{\alpha+\beta}$$

so,

$$R_\theta^8 = R_{8\theta} = R_{2\pi} = I$$

We conclude that $A^8 = 256I$, which implies that $(A^8)v = 256Iv = 256v$, so the answer is (B). $\square$

Ejercicio 1.1.5

Consider the subset of the real line A = (-inf, 0]. Which of the following are open sets (there may be more than 1 correct answer)?

(A) $A \cap [0, 1]$

(B) $A \cap (-inf, -1)$

(C) $A \cup \{1/2\}$

(D) $A \cup (-1, 1)$

(E) $A \cup (0, 1, 1)$

Solución:

(A) cannot be, because closed sets are closed under intersection, also with (C) but now with union of sets. (E) es not even a subset of the real line.

Now, $A \subseteq (-inf, -1)$, so $A \cap (-inf, -1) = A$, it can't be open because A is closed, which discards (B)

Finally, $A \cup (-1, 1) = (-inf, 1)$, which is open. So the answer is (D). $\square$