Ejercicio 1.1.1
Let A be a square matrix that is filled with all zeros except for the coordinates where the row number equals the column number. In those cells, the numbers from 1 to n appear in alphabetical order based on each number's English spelling. For example if n=3 then the order would be 1-3-2. Find the trace of A**2

(A) n^2

(B) $n(n+1)/2$

(C) $n(n+1)(2n+1)/6$

(D) n^3

Solución:
First, lets compute A^2. We have for all $i,j=1,...,n$;

$$(A^2)_{i,j}=\sum_{k=1}^{n}(A)_{i,k}(A)_{k,j}$$

because A is filled with zeros except for the coordinates where the row number equals the column number, when $i\neq j$ we have that:

$$(A)_{i,k}(A)_{k,j}=0,\quad \forall k=1,...,n$$

wich implies that $(A^2)_{i,j}=0$ when $i\neq j$. When $i=j$ the sum becomes:

$$\begin{aligned}(A^2)_{i,i}&=\sum_{k=1}^{n}(A)_{i,k}(A)_{k,i}\\&=\sum_{k=1}^{n}(A)_{i,k}^2\\&=(A)_{i,i}^2\end{aligned}$$

so now, the trace of A^2 would be:

$$\begin{aligned}\mathrm{Trace}(A)&=\sum_{i=1}^{n}(A^2)_{i,i}\\&=\sum_{i=1}^{n}(A)_{i,i}^2\end{aligned}$$

because all the numbers from 1 to n appear in the diagonal of A, then we are just making the sum of all squared numbers from 1 to n, so rearranging all the terrms, the sum becomes:

$$\mathrm{Trace}(A)=\sum_{i=1}^{n}i^2=\frac{n(n+1)(2n+1)}{6}$$

so the answer is (C). $\square$

Ejercicio 1.1.2
Let a be a prime number bigger than 3 and b an integer coprime to a. What is the smallest prime number that divides both a^4b^4 and a^2+ab?

(A) The smallest prime divisor of b

(B) The smallest prime divisor of ab

(C) a

(D) b

Solución:

It can't be (D) because b not necessarly is a prime number. Also, it can't be (A) because the smallest prime divisor of b not necessarly divides $a^2 + ab = a(a + b)$.

If p is prime such that $p \mid a^4 b^4$ then because a and b are coprime we must only one of these: $p \mid a$ or $p \mid b$.

In the second part, we have that $p \mid a(a + b)$. If $p \mid b$ then p cant divide a, so $p \mid a + b$ which by linearity implies that $p \mid a$, a contradiction.

So, $p \mid a$, whichi implies that $p = a$. Therefore the answer is (C).

□

Ejercicio 1.1.3
Let m, n be the 11th and 12-th Fibonacci numbers where the first and second Fibonacci numbers are both 1. How many subgroups of Z_{m*n} are there?

(A) 20

(B) 25

(C) 30

(D) 35

Solución:

We compute the Fibonacci numbers up to 11 and 12 position:

$$1, 1, 2, 3, 5, 8, 13, 21, 34{,}55, 89, 144, ...$$

so we must compute all subgroups of $Z_{89 \cdot 144}$. Recall that:

$$89 \cdot 144 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 89$$

we note first that $Z_{89 \cdot 144}$ is isomorphic to $Z/89 \cdot 144Z$. Let $n = 89 \cdot 144$. Now, by correspondence theorem, all subgroups of Z/nZ (namely rZ/nZ) are in correspondence with the subgroups of Z such that:

$$nZ \subseteq rZ \subseteq Z$$

the condition $nZ \subseteq rZ$ implies that $r \mid n$, so the set of all subgroups of Z/nZ is:

$$\left\{ rZ/nZ \middle| r \mid n \right\}$$

so, we must compute all divisors of n, with the prime decomposition of $89 \cdot 144$ its seen that there are 30 divisors, so the answer is (C). □

Ejercicio 1.1.4
Let v= [1,2] be a vector in the plane and let $A = 2[[1/sqrt(2), -1/sqrt(2)], [1/sqrt(2), 1/sqrt(2)]]$. What is $(A^8)v$?

(A) v

(B) 256 v

(C) $[128, 0]$

(D) -v

Solución:

Recall the matrix A is:

$$A = 2\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

we remember the form of the rotation matrix of angle θ in the euclidean plane:

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

so, $A = 2R_\theta$ where $\theta = \frac{\pi}{4}$. We observe that:

$$A^8 = (2R_\theta)^8 = 2^8 R_\theta^8 = 256 R_\theta^8$$

but rotation matrix has the property that:

$$R_\alpha R_\beta = R_{\alpha+\beta}$$

so,

$$R_\theta^8 = R_{8\theta} = R_{2\pi} = I$$

We conclude that $A^8 = 256I$, which implies that $(A^8)v = 256Iv = 256v$, so the answer is (B). $\square$

Ejercicio 1.1.5

Consider the subset of the real line A = (-inf, 0]. Which of the following are open sets (there may be more than 1 correct answer)?

(A) $A \cap [0, 1]$

(B) $A \cap (-inf, -1)$

(C) $A \cup \{1/2\}$

(D) $A \cup (-1, 1)$

(E) $A \cup (0, 1, 1)$

Solución:

(A) cannot be, because closed sets are closed under intersection, also with (C) but now with union of sets. (E) es not even a subset of the real line.

Now, $A \subseteq (-inf, -1)$, so $A \cap (-inf, -1) = A$, it can't be open because A is closed, which discards (B)

Finally, $A \cup (-1, 1) = (-inf, 1)$, which is open. So the answer is (D). $\square$

1. Hint (1): The square of a diagonal matrix is just the squares of its elements. It's trace is just the sum of the squared numbers from 1 to n, regardless of the order. Use the formula of the sum of squares.

2. Hint (2): If p is a prime number that divides $a^4 b^4$ then it must divide only a or b because both of them are coprime. Proof that if we suppose p divides b

3. Hint (3): Use the fact that Z_{m*n} is isomorphic to $Z/m*nZ$. By correspondence theorem all subgroups of $Z/m*nZ$ are in correspondence with the subgroups of Z such that those contain $m*nZ$.

   Subgroups of $Z/m*nZ$ are of the form $rZ/m*nZ$ with $m*nZ \subseteq rZ$. Proof this implies $r \mid n$. Then, all subgroups of $Z/m*nZ$ are of the form: $\left\{ rZ/m*nZ \middle| r \mid m*n \right\}$ find all positive integer divisors of $m*n$, then use the latter fact to count all subgroups of Z_{m*n}.

Ejercicio 1.2.6

Solución:

$$n^2 - n + 1 \mod 3 \equiv n^2 \mod 3 - n \mod 3 + 1 \mod 3$$

1	1	0
2	1	0
3	0	1
4	1	1
5	1	0
6	0	0

$$(3n-1)^2 - (3n-1) + 1 \mod 3 \equiv$$

$$\begin{aligned} n^2 - n + 1 = 3^k &\Rightarrow n^2 - n + 1 - 3^k = 0 \\ &\Rightarrow n(n-1) + 1 - 3^k = 0 \\ &\Rightarrow n(n-1) = 3^k - 1 \end{aligned}$$

el producto de dos números consecutivos debe ser tal que sucede eso, para algún k, uno de los dos debe ser par.

$$(3n-1)^2 - (3n-1) + 1 = 9n^2 - 6n + 1 - 3n + 1 + 1 = 9n^2 - 9n + 3 = 3(3n^2 - 3n + 1)$$

□

Ejercicio 1.2.7

$$x^4 - 2x^3 - 35x^2 + 36x + 180 = (x-a_1)(x-a_2)(x-a_3)(x-a_4)$$

Solución:

Se tiene que:

$$(-a_1)(-a_2)(-a_3) + (-a_1)(-a_2)(-a_4) + (-a_1)(-a_4)(-a_3) + (-a_4)(-a_2)(-a_3) = -2$$

tiene como raíces:

$$b_1 - b_3 = -5 - 3 = -8$$

Factorize 180 in its prime decomposition and substitute

1. Find roots of the polinomial. 2. Order roots from least to greatest. 3. Compute $b_1 - b_3$. 4. Convert from decimal to binary the result of $b_1 - b_3$.

□

Solución:

$[-5,5] \cap (1,\infty] \cap [2,6] \setminus \{2,3\} = (2,5] \setminus \{3\}$.

$$[-5,5] \cap (1,\infty] \cap [2,6] = [2,5]$$

□

First, compute the domain of each of the function summands in $f(x)$, then for each function we compute it's domain. Next, find the intersection of all domains to find the domain of f. Finally count all prime numbers in the domain of f.

Demostración:

$$\mathbb{Q}\times\mathbb{Q}=\bigcup_{a\in\mathbb{Q}}\left\{(a,b)\Big|b\in\mathbb{Q}\right\}$$

Sea $a\in\mathbb{Q}$. Entonces el conjunto:

$$\left\{(a,b)\Big|a<b,b\in\mathbb{Q}\right\}\subseteq\mathcal{L}$$

es numerable.

Si $\mathcal{L}$ fuese finito, entonces:

$$a=\min\left\{a_i\right\}$$

$(a-1,a)$. ■

Demostración:

Recordemos:

$$\overline{A}=A\cup A'$$

y la otra equivalencia es que:

$$x\in\overline{A}\text{ sii }\exists\left\{x_n\right\}\text{ en }A\text{ que converge a }x$$

$$x\in\overline{A}\text{ sii }\forall r>0,B_d(x,r)\cap A\neq\emptyset$$

Un conjunto U es abierto si para todo $x\in U$ existe $r>0$ tal que $B_d(x,r)\subseteq U$.

$\Rightarrow$) : Suponga que $x\in\overline{A}$, entonces

- $x\in A$, entonces:

$$0\leq d(x,A)=\inf\left\{d(x,a)\Big|a\in A\right\}\leq d(x,x)=0\Rightarrow d(x,A)=0$$

- $x\in A'$, si para toda vecindad (para todo $r>0$) se tiene que

$$(B_d(x,r)\setminus\{x\})\cap A\neq\emptyset$$

entonces existe $a_x\in(B_d(x,r)\setminus\{x\})\cap A$, por lo que:

$$0\leq\inf\left\{d(x,a)\Big|a\in A\right\}\leq d(x,a_x)<r$$

donde el $r>0$ fue arbitrario.

Por tanto:

$$d(x,A)=\inf\left\{d(x,a)\Big|a\in A\right\}=0$$

$\Leftarrow$): ■

Demostración:

Sea (X, d), como es separable existe un conjunto denso D a lo sumo numerable.

Sea A el conjunto de puntos aislados de X.

- Si A es finito ya hemos terminado.
- Suponga que A es infinito.

$$x \in A \text{ si y sólo si } \exists r > 0 \text{ abierto es tal que } B_d(x,r) = B_d(x,r) \cap X = \{x\}$$

Ahora, como D es denso entonces:
$$\overline{D} = X$$
lo que quiere decir que
$$\forall x \in X, \forall r > 0 \quad B_d(x,r) \cap D \neq \emptyset$$
Entonces, $A \subseteq D$ lo cual implica que A es numerable.

■

Demostración:

$$A = \left\{ f \in \mathcal{C}([0,1]) \middle| f(1/2) = 1 \right\}$$

el complemento de A es:

$$\begin{aligned} \mathcal{C}A &= \left\{ f \in \mathcal{C}([0,1]) \middle| f(1/2) \neq 1 \right\} \\ &= \left\{ f \in \mathcal{C}([0,1]) \middle| f(1/2) < 1 \right\} \cup \left\{ f \in \mathcal{C}([0,1]) \middle| f(1/2) > 1 \right\} \end{aligned}$$

objetivo: probar que
$$B = \left\{ f \in \mathcal{C}([0,1]) \middle| f(1/2) < 1 \right\}$$
es abierto. Sea $f \in B$, se tiene que:
$$f(1/2) < 1$$
Recordemos que:
$$\mathcal{N}_\infty(g) = \sup \left\{ |g(x)| \middle| x \in [0,1] \right\}, g \in \mathcal{C}([0,1]) \tag{1.1}$$
tomemos:
$$0 < 1 - f(1/2) = r$$

$x \mapsto d(x, A)$ es continua. Sea $f(x) = d(x, A)$.

Veamos que:

$$\begin{aligned} f^{-1}(]-\infty, \delta[) &= \left\{ x \in X \middle| f(x) \in]-\infty, \delta[\right\} \\ &= \left\{ x \in X \middle| -\infty < d(x, A) < \delta \right\} \\ &= \left\{ x \in X \middle| d(x, A) < \delta \right\} \\ &= G_\delta \end{aligned}$$

■

Demostración:

Considere la función:

$$h : X \to \mathbb{R}_{\geq 0}$$

tal que $x \mapsto d(x, f(x))$.

$$d(x, f(x)) > 0, \quad \forall x \in X \iff x \neq f(x), \quad \forall x \in X$$

Objetivo: ver que h vale cero en algún punto.

Veamos que h es continua. En efecto, pues $d : X \times X \to \mathbb{R}_{\geq 0}$ y $f : X \to X$ es continua, luego la composición:

$$x \mapsto (x, f(x))$$

es continua, luego la composición de esta función con d es continua. Así que h es continua.

Como X es compacto, entonces $h(X) \subseteq \mathbb{R}_{\geq 0}$ es compacto.

Si $0 \notin h(X)$, entonces denotemos por $k \in \mathbb{R}_{\geq 0}$ al elemento mínimo de $h(X)$.

Entonces, existe $x' \in X$ tal que:

$$h(x') = d(x', f(x')) = k$$

Luego:

$$h(f(x')) = d(f(x'), f^2(x')) < h(x') = k$$

$\#_c$. Por tanto, $0 \in h(X)$ luego existe $x \in X$ tal que $h(x) = 0 \Rightarrow d(x, f(x)) = 0 \Rightarrow x = f(x)$.

Supongamos que existe $y \in X$ tal que:

$$h(y) = 0 \Rightarrow y = f(y)$$

queremos probar que $x = y$. En efecto:

$$d(f(x), f(y)) = d(x, y) \Rightarrow x = y$$

Por ende, X tiene un único punto fijo. ■

Demostración:

Sea $r > 0$. Considere la función $f : E \to E$:

$$x \mapsto \frac{1}{r} \cdot x$$

de esta forma $B'(0, r)$ es mapeada a $B'(0, 1)$. f es continua y es una aplicación lineal acotada. Tiene inversa continua.

f es isomorfismo continuo. Se tiene que

$$B'(0, r) \text{ es compacta} \iff B'(0, 1) \text{ es compacta}$$

Aplicando el Corolario al teorema de Riez:

$$B'(0, r) \text{ es compacta} \iff E \text{ tiene dimensión finita}$$

para todo $r > 0$.

Se tiene que: W_r será compacto si E es de dimensión finita es condición suficiente.

Probaremos ahora que:

$$W_r = \left\{ x \in X \middle| d(x, C) \leq r \right\} = C + B'(0, r)$$

Como C es compacto, es cerrado. Luego:

$$x \in \overline{C} = C \text{ si y sólo si } d(x, C) = 0$$

- Suponga que $x\in W_r$, entonces $d(x,C)\leq r$. Se tienen dos casos:
    - $0=d(x,C)$ por lo anterior se tiene que $x\in C$. Tomamos $c=x$ y $b=0$, se tiene que:

      $$x=c+b$$

      con $c\in C$ y $b=0\in B'(0,r)$. Por ende, $x\in C+B'(0,r)$.
    - $0<d(x,C)\leq r$. Entonces $x\notin C$, por lo que existe $\epsilon>0$ tal que:

      $$B(x,\varepsilon)\subseteq X\setminus C$$

      La función $c\mapsto d(x,c)$ es continua. Entonces como C es compacto, luego alcanza su mínimo:

      $$\inf\left\{d(x,c)\Big| c\in C\right\}=d(x,C)$$

      por lo que, existe $c\in C$ tal que:

      $$d(x,c)=d(x,C)$$

      Tomemos:

      $$b=x-c$$

      afirmamos que $b\in B'(0,r)$. En efecto, veamos que:

      $$d(0,b)=\|0-b\|=\|-x+c\|=\|x-c\|=d(x,c)=d(x,C)\leq r$$

      por tanto, $b\in B'(0,r)$. Así que $x=c+b\in C+B'(0,r)$.
- Si $c+b\in C+B'(0,r)$:

  $$d(c+b,C)=\inf\left\{\|c+b-c'\|\Big| c'\in C\right\}\leq\|c-c+b\|=\|b\|=d(0,b)\leq r$$

  pues, $c\in C$. Por tanto:

  $$d(c+b,C)\leq r$$

  es decir, que $c+b\in W_r$.

De ambas contenciones se sigue la igualdad. ■