Chapter 1

Basic Notions

Throughout this document, we work with a fixed algebraically closed field k (or sometimes denoted K), called **ground field**.

1.1 Algebraic Closed Fields

Let k be an algebraic closed field, then for every $f \in k[x]$ (where $k[x]$ denotes the set of all polynomials over the field k), we have that if f is non-constant, then f has a root on k.

Example 1.1.1
$\mathbb{C}$ is an algebraic closed extension of $\mathbb{R}$, and $\mathbb{Q}$.

Turns out algebraic closed fields are important due to the following fact:

There is a one to one correspondence between geometric and algebraic objects using algebraic closed fields.

We will first deal with simple objects and then we'll scalate to more complex in order to generalize certain notions defined in algebraic geometry.

Definition 1.1.1 (**Affine Space**)
Let k be a field. We denote by $\mathbb{A}^n$ the **n-dimensional affine space over the field** k, that is:

$$\mathbb{A}^n = \left\{ (\alpha_1, \dots, \alpha_n) \middle| \alpha_i \in k, \quad \forall i \in \left[\middle|1, n\middle|\right] \right\}$$

All the geometric concepts we will be dealing with are within this space, this is due to the following fact:

Definition 1.1.2 (**Closed Subsets of Affine Space**)
Let $X \subseteq \mathbb{A}^n$ be a subset, then X is **closed** if there exists $f_1, \dots, f_m \in k[x_1, \dots, x_m]$ polynomials over the field k such that:

$$f_i(u) = 0, \quad \forall i \in \left[\middle|1, m\middle|\right] \iff u \in X$$

From now on, we will write $F(T)$ to denote a polynomial in n-variables, allowing T to stand for the set of variables $T_1, \dots, T_n$.

Observation 1.1.1 (**Equations of a Set**)
If a closed subset X consists of all common zeros of polynomials $F_1(T), \dots, F_m(T)$, then we refer to:

$$F_1(T) = \cdots = F_n(T) = 0$$

as the **equations of the set** X.

One really useful definition and fact that are proved in the notes of Algebra Moderna III are the following:

Definition 1.1.3 (Noetherian Ring)
Let R be a ring. We say that R is **noetherian** if for all I ideal of R there exists $a_1, \ldots, a_n \in R$ such that:

$$I = (a_1, \ldots, a_n),$$

where $(a_1, \ldots, a_n)$ denotes the ideal generated by the set $\{a_1, \ldots, a_n\}$.

Observation 1.1.2 (Ideal Generated by a Set S)
Let R be a commutative ring with identity, then the ideal generated by $S \subseteq R$ is:

$$(S) = \left\{ \sum_{i=1}^{n} r_i s_i \middle| r_i \in R, s_i \in S, \forall i \in [|1, n|] ; n \in \mathbb{N} \right\}$$

Theorm 1.1.1 (Hilbert Basis Theorem)
Let R be a noetherian ring, then $R[x]$ is noetherian.

A useful fact about the Hilbert basis theorem is that it generalizes neatly to an arbitrary number of indeterminates:

Corollary 1.1.1
Let R be a noetherian ring, then $R[T]$ is noetherian.

This useful fact is fundamental in the proof of the following result:

Proposition 1.1.1
Let X be a set defined by an infinite system of equations $\{F_\alpha(T)\}_{\alpha\in I}$ with $I \neq \emptyset$. Then X is closed.

Proof:
Let $\mathfrak{U}$ be the ideal generated by the system of equations $\{F_\alpha(T)\}_{\alpha\in I}$. Since $k[T]$ is noetherian due to the fact that k is a field (in particular, every field is noetherian), then there exists $G_1, \ldots, G_m \in k[T]$ such that:

$$\left(\{F_\alpha(T)\}_{\alpha\in I}\right) = (G_1(T), \ldots, G_m(T))$$

We claim that $u \in X$ iff $G_i(u) = 0$, for all $i \in [|1, m|]$. If $u \in X$, then $F_\alpha(u) = 0$ for all $\alpha \in I$, so in particular by Observation (1.1.2):

$$F(u) = 0, \quad \forall F \in \left(\{F_\alpha(T)\}_{\alpha\in I}\right)$$

which is the same as:

$$F(u) = 0, \quad \forall F \in (G_1(T), \ldots, G_m(T))$$

so $G_i(u) = 0$, for all $i \in [|1, m|]$. ■

It follows from this proposition that the arbitrary intersection of closed subsets of $\mathbb{A}^n$ is closed. Also, it happens that $\emptyset$ and $\mathbb{A}^n$ are closed (using the polynomials $F = 1$ and $F = 0$).

Proposition 1.1.2
If X_1 and X_2 are closed subsets of $\mathbb{A}^n$, then $X_1 \cup X_2$ is also a closed subset of $\mathbb{A}^n$.

Proof:
Let $F_1, \dots, F_n$ and $G_1, \dots, G_m$ polinomials over the ring $k[T]$ such that:

$$F_i(u) = 0, \quad \forall u \in X_1 \text{ and } G_j(v) = 0, \quad \forall v \in X_2$$

for all $(i, j) \in [|1, n|] \times [|1, m|]$. We define $H_{i,j} \in k[T]$ as:

$$H_{i,j} = F_i G_j, \quad \forall (i, j) \in [|1, n|] \times [|1, m|]$$

Then:

$$H_{i,j}(w) = 0, \quad \forall w \in X_1 \cup X_2$$

for all $(i, j) \in [|1, n|] \times [|1, m|]$. It follows that $X_1 \cup X_2$ is closed. ∎

By all this it follows that family of all the complements of closed subsets of $\mathbb{A}^n$ are a topology over $\mathbb{A}^n$.

Example 1.1.2 (**Closed Subsets of $\mathbb{A}^1$**)
Let $X \subseteq \mathbb{A}^1$ be a closed subset of $\mathbb{A}^1$, then there exists $f_1, \dots, f_m \in k[x]$ (polynomials in one variable) such that:

$$f_i(u) = 0, \quad \forall i \in [|1, m|] \iff u \in X$$

Let $d \in [x]$ the highest degree polynomial with leading coefficient one such that:

$$f_i = u_i d, \quad \forall i \in [|1, n|]$$

where $u_i \in k[x]$. If $u_1 = 1$, then $X = \emptyset$ if one of the polynomials is non zero and $X = \mathbb{A}^1$ if all of them are equal to zero, otherwise it follows that X is the family of all the roots of d, which is finite.

If $X = \{\alpha_1, \dots, \alpha_n\}$, then X is closed because X is the family of zeros of the polynomial:

$$f(x) = (x - \alpha_1) \cdots (x - \alpha_n)$$

Example 1.1.3 (**Closed Subsets of $\mathbb{A}^2$**)

Definition 1.1.4 (**Hypersurface**)
A set $X \subseteq \mathbb{A}^n$ defined by one equation $F(T_1, \dots, T_n) = 0$ is called a **hypersurface**.

1.2 Regular Functions on Closed Subsets

Definition 1.2.1 (Nombre)
Let X be a closed subset of $\mathbb{A}^n$ over the ground field k. A function $f : X \to k$ is called **regular** if there exists a polynomial $F \in k[T]$ such that:

$$f(u) = F(u), \quad \forall u \in X$$