Complemento.

4109. Calcular el volumen de $A=\{(x,y,z)\in\mathbb{R}^3 \mid (x^2+y^2+z^2)^3 \leq 3xyz\}$:

$$\phi^{-1}(A)=\{(r,\theta,\varphi)\in\Omega \mid r^6 \leq 3r^3 \operatorname{sen}\varphi\cos\theta\operatorname{sen}\varphi\operatorname{sen}\theta\cos\varphi\}$$

$$=\{(r,\theta,\varphi)\in\Omega \mid r^3 \leq 3\operatorname{sen}^2\varphi\operatorname{sen}\theta\cos\theta\cos\varphi\}$$

$$=\{(r,\theta,\varphi)\in\Omega \mid r^3 \leq \tfrac{3}{2}\operatorname{sen}2\theta\operatorname{sen}^2\varphi\cos\varphi\}$$

$$=\{(r,\theta,\varphi)\in\Omega \mid r^3 \leq \tfrac{3}{2}\operatorname{sen}2\theta(1-\cos^2\varphi)\cos\varphi\}$$

$$B=\{(r,\theta,\varphi)\in\Omega \mid r^3 \leq \underbrace{\tfrac{3}{2}\operatorname{sen}2\theta\cos\varphi-\tfrac{3}{2}\operatorname{sen}2\theta\cos^3\varphi}_{=u}\}$$

Como A es med. (por ser cerrado), ent.

$$\int_{\phi(B)} dx\,dy\,dz = \int_B |J\phi|\,dr\,d\theta\,d\varphi$$

$$=\int_B r^2\operatorname{sen}\varphi\,dr\,d\theta\,d\varphi$$

$$=\int_0^{\pi} d\varphi\int_{-\pi}^{\pi} d\theta\int_0^{u^{1/3}} r^2\operatorname{sen}\varphi\,dr$$

$$=\int_0^{\pi} d\varphi\int_{-\pi}^{\pi}\operatorname{sen}\varphi\,\tfrac{1}{3}\left[r^3\right]_0^{u^{1/3}} d\theta$$

$$=\int_0^{\pi} d\varphi\int_{-\pi}^{\pi}\tfrac{1}{2}\operatorname{sen}\varphi\left[\operatorname{sen}2\theta\cos\varphi-\operatorname{sen}2\theta\cos^3\varphi\right]d\theta$$

$$=\tfrac{1}{2}\int_0^{\pi} d\varphi\int_{-\pi}^{\pi}\left[\tfrac{1}{2}\operatorname{sen}2\theta\operatorname{sen}2\varphi-\operatorname{sen}2\theta\operatorname{sen}\varphi\cos^3\varphi\right]d\theta$$