1.2. Espacio Cociente y Dual.

Exercise 1.2.i. Let V be an n-dimensional vector space and W a k-dimensional subspace. Show that there exists a basis, $e_1, \ldots, e_n$ of V with the property that $e_1, \ldots, e_k$ is a basis of W.

Hint: Induction on $n - k$. To start the induction suppose that $n - k = 1$. Let $e_1, \ldots, e_{n-1}$ be a basis of W and e_n any vector in $V \setminus W$.

Dem:

Sea $m = n-K$. Procedamos por inducción sobre m. Para $m = 1$ tenemos que $n = K+1$. Sea así $e_1, \ldots, e_K$ una base de W. Como $m=1$, entonces $V \setminus W \neq \emptyset$, sea así $e_{K+1} \in V/W$. Probaremos que $e_1, \ldots, e_K, e_{K+1}$ es una base de W.

En efecto: por construcción $\{e_1, \ldots, e_K, e_{K+1}\}$ son l.i. a pares. Sea ahora $x \in V$. Si $x \in W$, $\exists\, a_1, \ldots, a_K \in \mathbb{K}$ m

$$x = a_1 e_1 + \ldots + a_K e_K$$
$$= a_1 e_1 + \ldots + a_K e_K + 0 e_{K+1}$$

Luego $x \in \mathcal{L}\{e_1, \ldots, e_K, e_{K+1}\}$. Si $x \in V \setminus W$, definimos $g: \mathbb{K} \to V$ como:

Exercise 1.2.ii. In Exercise 1.2.i show that the vectors $f_i := \pi(e_{k+i})$, $i = 1, \ldots, n-k$ are a basis of V/W, where $\pi: V \to V/W$ is the quotient map.

Exercise 1.2.iii. In Exercise 1.2.i let U be the linear span of the vectors e_{k+i} for $i = 1, \ldots, n-k$.

Dem:

Sea $\mathbb{K}$ el campo sobre V. Probaremos que la aplicación

$$(c_1, c_2, \ldots, c_{n-k}) \mapsto c_1 f_1 + \ldots + c_{n-k} f_{n-k}$$

Es biyectiva. En efecto, sea $\pi(x)$ una clase de eq. en V/W con rep. $x \in V$. Como $x \in V$, $\exists\, a_1, \ldots, a_n \in \mathbb{K}$ m $x = a_1 e_1 + \ldots + a_n e_n$. Tenemos que:

$$\pi(x) = a_1 \pi(e_1) + \ldots + a_n \pi(e_n)$$

Como $e_1, \ldots, e_k \in W$, entonces $\pi(e_1) = \pi(e_2) = \ldots = \pi(e_k) = 0$, por tanto:

$$\pi(x) = a_{k+1} \pi(e_{k+1}) + \ldots + a_n \pi(e_{k+(n-k)})$$

$$= a_{k+1} f_1 + \ldots + a_n f_{n-k}$$

Por tanto, $\{f_1, \ldots, f_{n-k}\}$ genera a V/W. Veamos que son l.i. Si

$$c_1 f_1 + c_2 f_2 + \ldots + c_{n-k} f_{n-k} = 0$$

$$\Leftrightarrow \pi(c_1 e_{k+1} + \ldots + c_{n-k} e_n) = 0$$

$$\Leftrightarrow c_1 e_{k+1} + \ldots + c_{n-k} e_n \in W$$

$$\Leftrightarrow c_i = 0, \ \forall i = 1, \ldots, n-k.$$

Por tanto, son l.i y así, forman una base de V/W.

$\square$

Exercise 1.2.iii. In Exercise 1.2.i let U be the linear span of the vectors e_{k+i} for $i = 1, \ldots, n-k$. Show that the map

$$U \to V/W, \ u \mapsto \pi(u),$$

is a vector space isomorphism, i.e., show that it maps U bijectively onto V/W.

Dem: es inmediato del ejercicio anterior.

$\square$

Exercise 1.2.iv. Let U, V and W be vector spaces and let $A: V \to W$ and $B: U \to V$ be linear mappings. Show that $(AB)^* = B^* A^*$.

Exercise 1.2.v. Let $V = \mathbf{R}^2$ and let W be the x_1-axis, i.e., the one-dimensional subspace

Dem: Como $A \circ B : U \to W$, entonces $(A \circ B)^* : W^* \to U^*$. Sea $l \in W^*$, entonces:

$$\begin{aligned}(A \circ B)^*(l) &= l \circ (A \circ B) \\ &= (l \circ A) \circ B \\ &= (A^*(l)) \circ B\end{aligned}$$

donde $A^* \circ l \in V^*$. Por tanto:

$$\begin{aligned}&= B^*(A^*(l)) \\ &= B^* \circ A^*(l)\end{aligned}$$

Así, $(A \circ B)^* = B^* \circ A^*$.

$\square$

Exercise 1.2.v. Let $V = \mathbf{R}^2$ and let W be the x_1-axis, i.e., the one-dimensional subspace

$$\{(x_1, 0) \mid x_1 \in \mathbf{R}\} = W$$

of $\mathbf{R}^2$.

(1) Show that the W-cosets are the lines, $x_2 = a$, parallel to the x_1-axis.
(2) Show that the sum of the cosets, "$x_2 = a$" and "$x_2 = b$" is the coset "$x_2 = a + b$".
(3) Show that the scalar multiple of the coset, "$x_2 = c$" by the number, λ, is the coset, "$x_2 = \lambda c$".

Dem:

De (1): Sean $(u,v), (u_1,v_1) \in \mathbb{R}^2$. Tenemos que:

$$\begin{aligned}(u,v) \sim (u_1,v_1) &\iff (u-u_1, v-v_1) \in W \\ &\iff u-u_1 \in \mathbb{R} \ \text{ y } \ v-v_1 = 0 \\ &\iff u,u_1 \in \mathbb{R} \ \text{ y } \ v = v_1\end{aligned}$$

i.e, $(u_1,v_1) \in \pi(u,v) \iff v_1 = v$, i.e, si pertenece a la misma recta $x_2 = v$.

De (2): Considere dos cosets $\pi(0,a)$ y $\pi(0,b)$, entonces:

$$\begin{aligned}\pi(0,a) + \pi(0,b) &= \pi((0,a) + (0,b)) \\ &\pi(0,a+b)\end{aligned}$$

De (3): Es análogo a (2).

$\square$

Exercise 1.2.vi.

(1) Let $(V^\star)^\star$ be the dual of the vector space, $V^\star$. For every $v \in V$, let $\mathrm{ev}_v : V^\star \to \mathbf{R}$ be the ***evaluation function*** $\mathrm{ev}_v(\ell) = \ell(v)$. Show that the ev_v is a linear function on $V^\star$, i.e., an element of $(V^\star)^\star$, and show that the map

$$\mathrm{ev} = \mathrm{ev}_{(-)} : V \to (V^\star)^\star \ , \ \ v \mapsto \mathrm{ev}_v \tag{1.2.16}$$

is a linear map of V into $(V^\star)^\star$.

(2) If V is *finite dimensional*, show that the map (1.2.16) is bijective. Conclude that there is a *natural* identification of V with $(V^\star)^\star$, i.e., that V and $(V^\star)^\star$ are two descriptions of the same object.

Hint: $\dim(V^\star)^\star = \dim V^\star = \dim V$, so by equation (1.1.5) it suffices to show that (1.2.16) is injective.

Dem:

De (1): Probaremos que $ev_v : V^* \to \mathbb{R}$ es lineal. En efecto, sean $l_1, l_2 \in V^*$ y $\lambda \in \mathbb{K}$. Se cumple:

$$\begin{aligned} ev_v(l_1 + \lambda l_2) &= (l_1 + \lambda l_2)(v) \\ &= l_1(v) + \lambda l_2(v) \\ &= ev_v(l_1) + \lambda ev_v(l_2) \end{aligned}$$

Así, $ev_v : V^* \to \mathbb{R}$ es lineal. Sean ahora $v_1, v_2 \in V$ y $\lambda \in \mathbb{K}$, entonces $\forall\, l \in V^*$:

$$\begin{aligned} ev_{(v_1 + \lambda v_2)}(l) &= l(v_1 + \lambda v_2) \\ &= l(v_1) + \lambda l(v_2) \\ &= ev_{v_1}(l) + \lambda ev_{v_2}(l) \end{aligned}$$

$$\therefore\ ev_{(v_1 + \lambda v_2)} = ev_{v_1} + \lambda ev_{v_2}.$$

Por tanto, $ev : V \to (V^*)^*$ es lineal.

De (2): Suponga que V es de dimensión finita. Sea $\{e_1, \dots, e_n\}$ una base de V, como ev es lineal, basta probar que $\{ev_{e_1}, \dots, ev_{e_n}\}$ es base de $(V^*)^*$ (se sabe que $\dim(V) = \dim(V^*) = \dim((V^*)^*)$, por lo que para ver que ev es inyectivo).

Sean $v_1, v_2 \in V$ m $ev_{v_1} = ev_{v_2}$, entonces $\forall\, l \in V^*$ se cumple:

$$l(v_1) = l(v_2)$$

Sean $\pi_i : V \to \mathbb{R}$ las func. proyección, entonces:

$$\pi_i(v_1) = \pi_i(v_2),\ \forall\, i \in [\![1, n]\!],\ \text{pues } \pi_i \in V^*.$$

$$\Rightarrow v_1 = \pi_1(v_1)e_1 + \ldots + \pi_n(v_1)e_n = \pi_1(v_2)e_1 + \ldots + \pi_n(v_2)e_n = v_2$$

Así, ev es inyectivo (luego, ev es un isomorfismo).

$\square$

Exercise 1.2.vii. Let W be a vector subspace of a finite dimensional vector space V and let

$$W^{\perp} = \{\ell \in V^* \mid \ell(w) = 0 \text{ for all } w \in W\}.$$

$W^{\perp}$ is called the ***annihilator*** of W in V^*. Show that $W^{\perp}$ is a subspace of V^* of dimension $\dim V - \dim W$.

Hint: By Exercise 1.2.i we can choose a basis, $e_1, \ldots, e_n$ of V such that $e_1, \ldots e_k$ is a basis of W. Show that $e_{k+1}^*, \ldots, e_n^*$ is a basis of $W^{\perp}$.

Dem:

$W^{\perp} \neq \emptyset$, pues $0 \in W^{\perp}$. Sean $l_1, l_2 \in W^{\perp}$ y $\lambda \in \mathbb{K}$, entonces:

$$\begin{aligned}(l_1 + \lambda l_2)(w) &= l_1(w) + \lambda l_2(w)\\ &= 0, \ \forall w \in W\end{aligned}$$

Por tanto, $W^{\perp}$ es un subespacio de W. Sea $e_1, e_2, \ldots, e_n$ una base de V tal que $e_1, e_2, \ldots, e_k$ es base de W, con $k \leq n$. Sea ahora $l \in W^{\perp}$, entonces:

$$\begin{aligned}l(v) &= l(c_1e_1 + \ldots + c_ne_n)\\ &= l(c_1e_1 + \ldots + c_ke_k) + l(c_{k+1}e_{k+1} + \ldots + c_ne_n)\\ &= 0 + c_{k+1}\,l(e_{k+1}) + \ldots + c_n\,l(e_n)\\ &= l(e_{k+1})\,e_{k+1}^*(v) + \ldots + l(e_n)\,e_n^*(v), \ \forall v = c_1e_1 + \ldots + c_ne_n \in V.\end{aligned}$$

Donde $l(e_{k+1}), \ldots, l(e_n) \in \mathbb{K}$ son constantes independientes de v. Así

$$l = l(e_{k+1})e_{k+1}^* + \ldots + l(e_n)e_n^*$$

Por tanto $\{e_{k+1}^*, \ldots, e_n^*\}$ genera a $W^{\perp}$. Si

$$0 = \lambda_{k+1}e_{k+1}^* + \ldots + \lambda_n e_n^*$$

$$\Rightarrow 0 = \lambda_{k+1}e_{k+1}^*(v) + \ldots + \lambda_n e_n^*(v), \ \forall v \in V$$

En part. para $v = e_{k+1}, \ldots, e_n$:

$$\Rightarrow 0 = \lambda_{k+1} = \lambda_{k+2} = \ldots = \lambda_n$$

Luego los $\{e_{k+1}^*, \ldots, e_n^*\}$ son l.i, así y por lo ant. $\{e_{k+1}^*, \ldots, e_n^*\}$ es una base de $W^{\perp}$, con dimensión:

$$\dim(W^{\perp}) = \dim(V) - \dim(W)$$

$\square$

Exercise 1.2.viii. Let V and V' be vector spaces and $A\colon V \to V'$ a linear map. Show that if $W \subset \ker(A)$, then there exists a linear map $B\colon V/W \to V'$ with the property that $A = B \circ \pi$ (where π is the quotient map (1.2.7)). In addition show that this linear map is injective if and only if $\ker(A) = W$.

Dem:

Definu $B : V/W \to V'$ como:

$$B(v + W) = A(v), \ \forall\, v \in V.$$

Probaremos que B está bien definido, sean $v_1, v_2 \in V$ m $v_1 + W = v_2 + W$, entonces $v_1 - v_2 \in W \subseteq \operatorname{Ker}(A)$, luego:

$$A(v_1 - v_2) = 0 \Leftrightarrow A(v_1) = A(v_2)$$

Por ser A lineal. Luego $B(v_1 + W) = B(v_2 + W)$. Así, B está bien definido. Veamos que es lineal. Si $v_1, v_2 \in V$ y $\lambda \in \mathbb{K}$, entonces:

$$\begin{aligned} B((v_1 + W) + \lambda(v_2 + W)) &= B(v_1 + \lambda v_2 + W) \\ &= A(v_1 + \lambda v_2) \\ &= A(v_1) + \lambda A(v_2) \\ &= B(v_1 + W) + \lambda B(v_2 + W) \end{aligned}$$

Luego, B es lineal. Por def. $A = B \circ \pi$. La segunda condición se deduce de manera inmediata.

$\square$

Exercise 1.2.ix. Let W be a subspace of a finite dimensional vector space, V. From the inclusion map, $\iota\colon W^{\perp} \to V^{*}$, one gets a transpose map,

$$\iota^{*} : (V^{*})^{*} \to (W^{\perp})^{*}$$

and, by composing this with (1.2.16), a map

$$\iota^{*} \circ \mathrm{ev}\colon V \to (W^{\perp})^{*}\,.$$

Show that this map is onto and that its kernel is W. Conclude from Exercise 1.2.viii that there is a *natural* bijective linear map

$$\nu\colon V/W \to (W^{\perp})^{*}$$

with the property $\nu \circ \pi = \iota^{*} \circ \mathrm{ev}$. In other words, V/W and $(W^{\perp})^{*}$ are two descriptions of the same object. (This shows that the "quotient space" operation and the "dual space" operation are closely related.)

Dem:

Notemos que $\imath : W^{\perp} \to V^*$, $\imath(l) = l$, $\forall\, l \in W^{\perp}$. Claramente $\imath^* \circ ev : V \to (W^{\perp})^*$ es función. Luego, $\forall\, v \in V$:

$$\imath^* \circ ev(v) = \imath^*(ev_v)$$
$$= ev_v \circ \imath : W^{\perp} \to \mathbb{R} \in (W^{\perp})^*$$

Claramente $\imath^* \circ ev$ es t. lineal. Si

$$v \in V \text{ m } \imath^* \circ ev(v) = 0 \iff 0 = ev_v \circ \imath : W^{\perp} \to \mathbb{R} \text{ i.e:}$$
$$\iff \forall\, l \in W^{\perp},\ ev_v \circ \imath(l) = 0$$
$$\iff \forall\, l \in W^{\perp},\ l(v) = 0.$$
$$\iff v \in W.$$

Por tanto, $Ker(\imath^* \circ ev) = W$. Además es suprayectivo, pues $\dim(W^{\perp})^* = \dim W^{\perp} = \dim V - \dim W$. Por viii), tomando $A = \imath^* \circ ev$, como $Ker(A) = W$, $\exists\, v : V/W \to (W^{\perp})^*$ lineal inyectivo, pero:

$$\dim(V/W) = \dim V - \dim W = \dim(W^{\perp})^*$$

Luego v es isomorfismo. i.e $V/W \cong (W^{\perp})^*$.

$\square$

Exercise 1.2.x. Let V_1 and V_2 be vector spaces and $A : V_1 \to V_2$ a linear map. Verify that for the transpose map $A^* : V_2^* \to V_1^*$ we have:

$$\ker(A^*) = \operatorname{im}(A)^{\perp}$$

and

$$\operatorname{im}(A^*) = \ker(A)^{\perp}.$$

Dem:

$$l \in Ker(A^*) \subseteq V_2^* \iff A^*(l) = 0$$
$$\iff l \circ A = 0$$
$$\iff l \circ A(v_1) = 0,\ \forall\, v \in V_1.$$
$$\iff l(v_2) = 0,\ \forall\, v_2 \in im(A).$$

$$\Leftrightarrow l \in \operatorname{im}(A)^{\perp}$$

El otro es análogo. $\square$

Exercise 1.2.xi. Let V be a vector space.

(1) Let $B: V \times V \to \mathbf{R}$ be an inner product on V. For $v \in V$ let

$$\ell_v : V \to \mathbf{R}$$

be the function: $\ell_v(w) = B(v, w)$. Show that ℓ_v is linear and show that the map

$$L: V \to V^*, \quad v \mapsto \ell_v \tag{1.2.17}$$

is a linear mapping.

(2) If V is finite dimensional, prove that L bijective. Conclude that if V has an inner product one gets from it a *natural* identification of V with V^*.

Hint: Since $\dim V = \dim V^*$ it suffices by equation (1.1.5) to show that $\ker(L) = 0$. Now note that if $v \neq 0$ $\ell_v(v) = B(v, v)$ is a positive number.

Dem:

De (1): Sean $v_1, v_2 \in V$ y $\lambda \in \mathbb{K}$, entonces:

$$\begin{aligned} \forall v \in V, \ l_v(v_1 + \lambda v_2) &= B(v, v_1 + \lambda v_2) \\ &= B(v, v_1) + B(v, \lambda v_2) \\ &= l_v(v_1) + \lambda l_v(v_2) \end{aligned}$$

$\therefore l_v$ es lineal. Sean $v_1, v_2 \in V$ y $\lambda \in \mathbb{K}$, entonces:

$$\begin{aligned} L(v_1 + \lambda v_2)(u) &= l_{v_1 + \lambda v_2}(u) \\ &= B(v_1 + \lambda v_2, u) \\ &= B(v_1, u) + \lambda B(v_2, u) \\ &= l_{v_1}(u) + \lambda l_{v_2}(u) \\ &= (L(v_1) + \lambda L(v_2))(u), \ \forall u \in V. \end{aligned}$$

$\therefore L$ es lineal.

De (2): Sea $v \in \operatorname{Ker}(L) \subseteq V$, entonces:

$$\begin{aligned} & L(v) = 0 : V \to \mathbb{R} \\ \Leftrightarrow \ & l_v(u) = 0, \ \forall u \in V \\ \Leftrightarrow \ & B(v, u) = 0, \ \forall u \in V, \text{ en part. para } u = v. \end{aligned}$$

$\Leftrightarrow B(v,v)=0 \Leftrightarrow v=0.$

Así, $Ker(L)=\{0\}$. $\therefore L$ es monomorfismo $\Rightarrow L$ isomorfismo.

$\square$

Exercise 1.2.xii. Let V be an n-dimensional vector space and $B\colon V\times V \to \mathbf{R}$ an inner product on V. A basis, $e_1,\dots,e_n$ of V is ***orthonormal*** if

$$B(e_i,e_j)=\begin{cases}1, & i=j\\ 0, & i\neq j\,.\end{cases} \tag{1.2.18}$$

(1) Show that an orthonormal basis exists.

Hint: By induction let $e_1,\dots,e_k$ be vectors with the property (1.2.18) and let v be a vector which is not a linear combination of these vectors. Show that the vector

$$w=v-\sum_{i=1}^{k}B(e_i,v)e_i$$

is nonzero and is orthogonal to the e_i's. Now let $e_{k+1}=\lambda w$, where $\lambda=B(w,w)^{-\frac{1}{2}}$.

(2) Let $e_1,\dots e_n$ and $e'_1,\dots e'_n$ be two orthonormal bases of V and let

$$e'_j=\sum_{i=1}^{n}a_{i,j}e_i\,.$$

Show that

$$\sum_{i=1}^{n}a_{i,j}a_{i,k}=\begin{cases}1, & j=k\\ 0, & j\neq k\,.\end{cases} \tag{1.2.19}$$

(3) Let A be the matrix $[a_{i,j}]$. Show that equation (1.2.19) can be written more compactly as the matrix identity

$$AA^{\intercal}=\mathrm{id}_n \tag{1.2.20}$$

where id_n is the $n\times n$ identity matrix.

(4) Let $e_1,\dots,e_n$ be an orthonormal basis of V and $e_1^*,\dots,e_n^*$ the dual basis of V^*. Show that the mapping (1.2.17) is the mapping, $Le_i=e_i^*$, $i=1,\dots n$.

Dem: lo probé en A III.

$\square$

1.3. Tensores.

Exercise 1.3.i. Verify that there are exactly n^k multi-indices of length k.

Exercise 1.3.ii. Prove Proposition 1.3.18.

Dem:

Sea $n \in \mathbb{N}$. Probaremos el resultado por inducción sobre k. Para $k = 1$, hay n multi-índices. Suponga el resultado válido para $k \in \mathbb{N}$.

Probaremos el resultado para $k+1$.

Exercise 1.3.ii. Prove Proposition 1.3.18.

Proposition 1.3.18. *The map*

$$A^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)\,,\ T \mapsto A^*T\,,$$

is a linear mapping.

Dem:

Sean $T_1, T_2 \in \mathcal{L}^k(W)$ y $\lambda \in \mathbb{K}$, entonces:

$$\begin{aligned} A^*(T_1+\lambda T_2)(v_1,v_2,\dots,v_k) &= (T_1+\lambda T_2)(Av_1, Av_2,\dots, Av_k) \\ &= T_1(Av_1,Av_2,\dots,Av_k) + \lambda T_2(Av_1,\dots,Av_k) \\ &= A^*T_1(v_1,\dots,v_k) + \lambda A^*T_2(v_1,\dots,v_k) \\ &= (A^*T_1 + \lambda A^*T_2)(v_1,\dots,v_k) \end{aligned}$$

$\therefore A^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)$ es lineal.

$\square$

Exercise 1.3.iii. Verify equation (1.3.19).

Exercise 1.3.iv. Verify equation (1.3.20).

Sean $T_1, T_2 \in \mathcal{L}^k(W)$, entonces:

$$\begin{aligned} A^*(T_1 \otimes T_2)(v_1,\dots,v_{2k}) &= (T_1 \otimes T_2)(Av_1,\dots,Av_{2k}) \\ &= T_1(Av_1,\dots,Av_k)\cdot T_2(Av_{k+1},\dots,Av_{2k}) \\ &= (A^*T_1(v_1,\dots,v_k))\cdot(A^*T_2(v_{k+1},\dots,v_{2k})) \\ &= A^*T_1 \otimes A^*T_2(v_1,\dots,v_{2k}) \end{aligned}$$

$\forall (v_1,\dots,v_{2k}) \in W^{2k}$. $\therefore A^*(T_1 \otimes T_2) = A^*T_1 \otimes A^*T_2$.

$\square$

Exercise 1.3.iv. Verify equation (1.3.20).

Exercise 1.3.v. Let $A: V \to W$ be a linear map. Show that if ℓ_i, $i = 1,\dots,k$ are elements of

Sol.

Sea $A: V \to W$ y $B: U \to V$ lineal. Si $T \in \mathcal{L}^k(V)$, entonces para $(w_1,\dots,w_k) \in W^k$:

$$\begin{aligned} (A\circ B)^* T(w_1,\dots,w_k) &= T(A\circ B(w_1),\dots, A\circ B(w_k)) \\ &= A^*T(B(w_1),\dots,B(w_k)) \\ &= B^*(A^*T)(w_1,\dots,w_k) \end{aligned}$$

$$\therefore (A \circ B)^* T = B^*(A^* T)$$

$\square$

Exercise 1.3.v. Let $A: V \to W$ be a linear map. Show that if ℓ_i, $i = 1, \ldots, k$ are elements of W^*

$$A^*(\ell_1 \otimes \cdots \otimes \ell_k) = A^*(\ell_1) \otimes \cdots \otimes A^*(\ell_k).$$

Conclude that A^* maps decomposable k-tensors to decomposable k-tensors.

Dem:

Sean $w_1, \ldots, w_k \in W$, entonces:

$$A^*(\ell_1 \otimes \ldots \otimes \ell_k)(v_1, \ldots, w_k) = (\ell_1 \otimes \ldots \otimes \ell_k)(A(w_1), \ldots, A(v_k))$$
$$= \ell_1(Aw_1) \cdot \ldots \cdot \ell_k(Aw_k)$$
$$= A^*\ell_1(v_1) \cdot \ldots \cdot A^*\ell_k(v_k)$$
$$= (A^*\ell_1 \otimes \ldots \otimes A^*\ell_k)(w_1, \ldots, w_k)$$

$$\therefore A^*(\ell_1 \otimes \ldots \otimes \ell_k) = (A^*\ell_1) \otimes \ldots \otimes (A^*\ell_k)$$

$\square$

Exercise 1.3.vi. Let V be an n-dimensional vector space and ℓ_i, $i = 1, 2$, elements of V^*. Show that $\ell_1 \otimes \ell_2 = \ell_2 \otimes \ell_1$ if and only if ℓ_1 and ℓ_2 are linearly dependent.

Hint: Show that if ℓ_1 and ℓ_2 are linearly independent there exist vectors, v_i, $i =, 1, 2$ in V with property

$$\ell_i(v_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

Now compare $(\ell_1 \otimes \ell_2)(v_1, v_2)$ and $(\ell_2 \otimes \ell_1)(v_1, v_2)$. Conclude that if $\dim V \geq 2$ the tensor product operation is not commutative, i.e., it is usually not true that $\ell_1 \otimes \ell_2 = \ell_2 \otimes \ell_1$.

Dem:

Si $\dim(V) = 0^{2)}$ el resultado es inmediato. Ahora si $\dim V \geq 2$ y alguno de los ℓ_i es trivial (i.e $\ell_i(v) = 0$, $\forall\ v \in V$) el resultado tambien es inmediato ya que si $\ell_i = \ell_1$:

$$\Rightarrow \ell_1 = \alpha \ell_2 \quad \alpha = 0$$

i.e $\ell_1 \otimes \ell_2(v_1, v_2) = 0 = \ell_2 \otimes \ell_1(v_1, v_2)$, $\forall\ v_1, v_2 \in V \Leftrightarrow \ell_1 = \alpha \ell_2$ o $\ell_2 = \alpha \ell_1$ (con $\alpha = 0$),

i.e ℓ_1 y ℓ_2 son l.d.

Suponga entonces que $\dim(V) \geq 1$ y tanto ℓ_1 como ℓ_2 no son triviales.

$\Leftarrow$) Suponga que ℓ_1 y ℓ_2 son l.d. entonces $\exists\ \alpha \in \mathbb{R}$ m $\ell_1 = \alpha \ell_2$. Por tanto, $\forall\ v_1, v_2 \in V$ se cumple que:

$$\ell_1 \otimes \ell_2(v_1, v_2) = \ell_1(v_1) \cdot \ell_2(v_2)$$
$$= \alpha \ell_2(v_1) \cdot \ell_2(v_2)$$

$$= l_2(v_1)\cdot \alpha l_2(v_2)$$

$$= l_2(v_1)\, l_1(v_2)$$

$$= l_2 \otimes l_1(v_1, v_2)$$

$$\therefore\ l_1 \otimes l_2 = l_2 \otimes l_1$$

$\Rightarrow$) Suponga que l_1 y l_2 son l.i. Probaremos que $\exists\ v_1, v_2 \in V$ ∋

$$l_i(v_j) = \begin{cases} 1 & \text{si } i = j. \\ 0 & \text{si } i \neq j. \end{cases}$$

(basta con probar la existencia del v_1, para el v_2 el caso es análogo). Basta probar que $\exists\ v_1 \in V$ ∋ $l_1(v_1) \neq 0$ y $l_2(v_1) = 0$. Suponga que $\forall\ v \in V$ ∋ $l_1(v) \neq 0$, $l_2(v) \neq 0$. Por el. T. rango nulidad:

$$\dim(V) = \dim(\mathrm{Ker}(l_1)) + \dim \mathbb{R}$$

$$\Rightarrow \dim(\mathrm{Ker}\, l_1) = n - 1 = \dim(\mathrm{Ker}\, l_2)$$

Por tanto, como $\forall\ v \notin \mathrm{Ker}\, l_1 \Rightarrow v \notin \mathrm{Ker}\, l_2$ y sus dimensiones son iguales, se tiene que $\mathrm{Ker}\, l_1 = \mathrm{Ker}\, l_2$. Sea $\{e_1, \dots, e_n\}$ una base de V ∋ $\{e_1, \dots, e_{n-1}\}$ es base del Kernel de l_1 (por ende, de l_2), ent. $l_1(e_n) \neq 0$, $l_2(e_n) \neq 0$ y se cumple que $l_1(e_n) = \frac{l_1(e_n)}{l_2(e_n)} l_2(e_n)$. Tome $\alpha = \frac{l_1(e_n)}{l_2(e_n)}$, ent. $\forall\ v \in V$:

$$l_1(v) = l_1(c_1 e_1 + \dots + c_n e_n)$$

$$= c_n l_1(e_n)$$

$$= c_n \alpha l_2(e_n)$$

$$= \alpha l_2(c_n e_n)$$

$$= \alpha l_2(v)$$

$\therefore\ l_1$ y l_2 son l.d. $\cancel{*}$c. Así, $\exists\ v \in V$ ∋ $l_1(v) \neq 0$ y $l_2(v) = 0$. Tome $v_1 = \frac{v}{l_1(v)}$, ent.

$$l_1(v_1) = 1 \text{ y } l_2(v_1) = 0.$$

Exercise 1.3.vii. Let T be a k-tensor and v a vector. Define $T_v : V^{k-1} \to \mathbf{R}$ to be the map

$$T_v(v_1, \ldots, v_{k-1}) := T(v, v_1, \ldots, v_{k-1}) . \tag{1.3.21}$$

Show that T_v is a $(k-1)$-tensor. (Si $K=1$, el resultado es inmediato).

Dem:

Basta probar que T_v ($v \in V$ arbitrario, fijo) es lineal en su i-ésima entrada. Sea $i \in [1, K-1]$ y $v, w \in V$, $\alpha \in \mathbb{R}$. Entonces:

$$T_v(v_1, \ldots, v_{i-1}, v+\alpha w, v_{i+1}, \ldots, v_{k-1}) = T(v, v_1, \ldots, v_{i-1}, v+\alpha w, \ldots, v_{k-1})$$

$$= T(v, v_1, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_{k-1}) + \alpha T(v, \ldots, v_{i-1}, w, v_{i+1}, \ldots, v_{k-1})$$

$$= T_v(v_1, \ldots, v, \ldots, v_{k-1}) + \alpha T_v(v_1, \ldots, w, \ldots, v_{k-1})$$

Luego T_v es lineal en su i-ésima entrada. Como el i fue arbitrario, $T_v \in \mathcal{L}^{k-1}(V)$.

$\square$

Exercise 1.3.viii. Show that if T_1 is an r-tensor and T_2 is an s-tensor, then if $r > 0$,

$$(T_1 \otimes T_2)_v = (T_1)_v \otimes T_2 .$$

Dem:

Sea $v \in V$. Ya se sabe que T_v es un $r-1$-tensor y $(T_1 \otimes T_2)_v$ es un $(r+s-1)$-tensor. Sean $v_1, \ldots, v_{r+s-1} \in V$, entonces:

$$(T_1 \otimes T_2)_v(v_1, \ldots, v_{r+s-1}) = T_1 \otimes T_2(v, v_1, \ldots, v_{r+s-1})$$

$$= T_1(v, v_1, \ldots, v_{r-1}) T_2(v_r, v_{r+1}, \ldots, v_{r+s-1})$$

$$= (T_1)_v(v_1, \ldots, v_{r-1}) \cdot T_2(v_r, \ldots, v_{r+s-1})$$

$$= (T_1)_v \otimes T_2(v_1, \ldots, v_{r+s-1})$$

$$\therefore (T_1 \otimes T_2)_v = (T_1)_v \otimes T_2$$

$\square$

Exercise 1.3.ix. Let $A : V \to W$ be a linear map mapping, and $v \in V$. Write $w := Av$. Show that for $T \in \mathcal{L}^k(W)$, $A^*(T_w) = (A^*T)_v$.

Dem:

Sean $v_1, \ldots, v_{k-1} \in V$. Entonces:

$$A^*(T_w)(v_1, \ldots, v_{k-1}) = T_w(A(v_1), \ldots, A(v_{k-1}))$$

$$= T(Av, Av_1, \ldots, Av_{k-1})$$

$$= A^* T(v, v_1, \dots, v_{k-1})$$

$$= (A^* T)_v (v_1, \dots, v_{k-1})$$

$$\therefore \quad A^* (T_w) = (A^* T)_v$$

$$\square$$

1.4 K-Tensores Alternantes.

Exercise 1.4.i. Show that there are exactly $k!$ permutations of order k.

Hint: Induction on k: Let $\sigma \in S_k$, and let $\sigma(k) = i$, $1 \leq i \leq k$. Show that $\tau_{i,k}\sigma$ leaves k fixed and hence is, in effect, a permutation of Σ_{k-1}.

Dem:

Lo probé en AM II.

$\square$

Exercise 1.4.ii. Prove that if $\tau \in S_k$ is a transposition, $(-1)^\tau = -1$ and deduce from this Proposition 1.4.11.

Exercise 1.4.iii. Prove assertion 2 in Proposition 1.4.14.

Dem:

La primera parte la hice en AM II. Como $\tau = \tau$, entonces τ es el producto de un número impar de transposiciones. Por 1.4.11: $(-1)^\tau = -1$.

$\square$

Exercise 1.4.iii. Prove assertion 2 in Proposition 1.4.14.

Exercise 1.4.iv. Prove that $\dim \mathcal{A}^k(V)$ is given by (1.4.27).

Dem:

Ya lo probé en las notas del curso.

$\square$

Exercise 1.4.iv. Prove that $\dim \mathcal{A}^k(V)$ is given by (1.4.27).

Exercise 1.4.v. Verify that for $i < j - 1$

$$\tau_{i,j} = \tau_{j-1,i}\tau_{j,j-1}, \tau_{j-1,i}.$$

Dem:

Probaremos que, si $1 \leq k \leq n$:

$$\dim(\mathcal{A}^k(V)) = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Procederemos por inducción sobre n. Si $n=1$ el resultado es inmediato, pues $k=1$, y:

$$\dim(\mathcal{A}^k(V)) = |\{I \mid I \text{ es } k\text{-multi-índice de } 1 \text{ est. creciente}\}|$$
$$= |\{(1)\}|$$

$$= 1$$
$$= \binom{1}{k}$$

Suponga el resultado válido para $1, 2, \dots, m$. Probaremos que se cumple para $n = m+1$. En efecto: si $k = m+1$ el resultado se cumple, ya que:

$$\begin{aligned} \dim \Lambda^k(V) &= |\{ I \mid I \text{ es } k\text{-multi-índice de } m+1 \text{ est. creciente} \}| \\ &= |\{ (1\ 2\ \dots\ m+1) \}| \\ &= 1 \\ &= \binom{m+1}{k} \end{aligned}$$

Suponga $1 \leq k \leq m$. Veamos que:

$$\begin{aligned} \dim \Lambda^k(V) &= |\{ I \mid I \text{ es un } k \text{ multi-índice de } m+1 \text{ est. creciente} \}| \\ &= |A_1 \cup A_2 \cup \dots \cup A_{m-k+1}| \end{aligned}$$

Donde:

$$A_i = \{ I \mid I = (i\ i_1\ \dots\ i_{k-1}) \text{ m } i_j \in [\![1, m+1]\!] \text{ y } i_1 < i_2 < \dots < i_{k-1} \}, \ \forall\ i \in [\![1, m-k+2]\!]$$

Claramente los A_i's son todos disjuntos, y se cumple por tanto:

$$= |A_1| + |A_2| + \dots + |A_{m-k+2}|$$

Por construcción, $|A_i| = \binom{m+1-i}{k-1}$ ya que el A_i contiene k-multi-índices crecientes que van de $i+1$ a $m+1$. Ent.

$$\begin{aligned} \dim \Lambda^k(V) &= \sum_{i=1}^{m-k+2} \binom{m+1-i}{k-1} \\ &= \binom{m}{k-1} + \binom{m-1}{k-1} + \dots + \binom{m+1-(m-k+2)}{k-1} \\ &= \binom{m}{k-1} + \binom{m-1}{k-1} + \dots + \binom{k-1}{k-1} \\ &= \binom{m+1}{k}^{3)} \end{aligned}$$

Luego el resultado se cumple para $n = m+1$. Por inducción se cumple $\forall\ n \in \mathbb{N}$. Por tanto si $1 \leq k \leq n$:

$$\dim(\Lambda^k(V)) = \binom{n}{k}$$

$\square$

Exercise 1.4.v. Verify that for $i < j - 1$

$$\tau_{i,j} = \tau_{j-1,j}\tau_{i,j-1}, \tau_{j-1,j} \, .$$

Sea $K \in \mathbb{N}$ m $1 \leq i < j-1 < j \leq K$. Probaremos que:

$$\tau_{i,j} = \tau_{j-1,j} \circ \tau_{i,j-1} \circ \tau_{j-1,j}$$

Como en las permutaciones sólo hay transposiciones donde aparecen i, j y $j-1$, se sigue la igualdad, $\forall\, l \in [\![1, K]\!] \setminus \{i, j, j-1\}$. i.e $\tau_{i,j}(l) = \tau_{j-1,j} \circ \tau_{i,j-1} \circ \tau_{j-1,j}(l)$.

1) $l = i$. entonces:

$$\begin{aligned} \tau_{j-1,j} \cdot \tau_{i,j-1} \cdot \tau_{j-1,j}(l) &= \tau_{j-1,j}(j-1) \\ &= j \\ &= \tau_{i,j}(l) \end{aligned}$$

2) $l = j-1$, entonces:

$$\begin{aligned} \tau_{j-1,j} \circ \tau_{i,j-1} \cdot \tau_{j-1,j}(l) &= j-1 \\ &= \tau_{i,j}(l) \end{aligned}$$

3) $l = j$, entonces:

$$\begin{aligned} \tau_{j-1,j} \circ \tau_{i,j-1} \circ \tau_{j-1,j}(l) &= i \\ &= \tau_{i,j}(l) \end{aligned}$$

Por 1)–3), se cumple la igualdad.

$\square$

Exercise 1.4.vi. For $k = 3$ show that every one of the six elements of S_3 is either a transposition or can be written as a product of two transpositions.

Exercise 1.4.vii. Let $\sigma \in S_k$ be the "cyclic" permutation

Dem:

Es inmediato de la def. de S_3.

$\square$

Exercise 1.4.vii. Let $\sigma \in S_k$ be the "cyclic" permutation

$$\sigma(i) := i + 1 \, , \quad i = 1, \ldots, k - 1$$

and $\sigma(k) := 1$. Show explicitly how to write σ as a product of transpositions and compute $(-1)^\sigma$.

Hint: Same hint as in Exercise 1.4.i.

Dem:

$$\sigma = \tau_{1,2}\,\tau_{1,3}\,\tau_{1,4}\cdots\tau_{1,K-1}\,\tau_{1,K}$$

y $(-1)^{\sigma} = (-1)^{K-1}$.

$\square$

Exercise 1.4.viii. In Exercise 1.3.vii show that if T is in $\mathcal{A}^k(V)$, T_v is in $\mathcal{A}^{k-1}(V)$. Show in addition that for $v, w \in V$ and $T \in \mathcal{A}^k(V)$ we have $(T_v)_w = -(T_w)_v$.

Exercise 1.4.ix. Let $A: V \to W$ be a linear mapping. Show that if T is in $\mathcal{A}^k(W)$, A^*T is in

Dem.

Sea $v \in V$ y $T \in \mathcal{A}^k(V)$, ent. $\forall\, \sigma \in S_k$:

$$T^{\sigma} = (-1)^{\sigma} T$$

En otras palabras, si $v_1, \dots, v_k \in V$, ent.

$$T(v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(k)}) = (-1)^{\sigma} T(v_1, \dots, v_k)$$

Sea $\theta \in S_{k-1}$. Entonces:

$$\begin{aligned}
T_{v_K}(v_1, \dots, v_{K-1})^{\theta} &= T_{v_k}(v_{\theta^{-1}(1)}, \dots, v_{\theta^{-1}(K-1)}) \\
&= T(v_k\, v_{\theta^{-1}(1)}, \dots, v_{\theta^{-1}(K-1)}) \\
&= T(v_{\sigma^{-1}(k)}, v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(K-1)}) \\
&= (T(v_k, v_1, \dots, v_k))^{\sigma} \\
&= (-1)^{\sigma} T(v_k, v_1, \dots, v_{k-1}) \\
&= (-1)^{\theta} T_{v_k}(v_1, \dots, v_{k-1})
\end{aligned}$$

donde $\sigma \in S_k$, $\sigma(i) = \theta(i)$ $\forall\, i \in [1, K-1]$ y $\theta(K) = K$. Claramente θ y σ se factorizan en el mismo número de transposiciones. Por tanto $(-1)^{\sigma} = (-1)^{\theta}$. Así:

$$T_{v_k}^{\theta} = (-1)^{\theta} T_{v_K}$$

$\therefore T_{v_k} \in \mathcal{A}^{K-1}(V)$, $\forall\, v_K \in V$.

Para la otra parte. Supongo que $K \geq 2$. Sean $v, w \in V$ y $v_1, \dots, v_{K-2} \in V$. Ent.

$$(T_v)_w(v_1, \dots, v_{K-2}) = T_v(w, v_1, \dots, v_{K-2})$$

$$= T(v, w, v_1, \dots, v_{k-2})$$
$$= (-1)\, T(w, v, v_1, \dots, v_{k-2})$$
$$= - T_w(v, v_1, \dots, v_{k-2})$$
$$= -(T_w)_v (v_1, \dots, v_{k-2})$$

$$\therefore (T_v)_w = -(T_w)_v$$

$\square$

Exercise 1.4.ix. Let $A\colon V \to W$ be a linear mapping. Show that if T is in $\mathcal{A}^k(W)$, A^*T is in $\mathcal{A}^k(V)$.

Exercise 1.4.x. In Exercise 1.4.ix show that if T is in $\mathcal{L}^k(W)$ then $\mathrm{Alt}(A^*T) = A^*(\mathrm{Alt}(T))$, i.e., show that the Alt operation commutes with the pullback operation.

Dem:

Sea $\sigma \in S_k$ y $v_1, \dots, v_k \in V$. Entonces si $T \in \mathcal{A}^k(W)$:

$$(A^*T)^\sigma (v_1, \dots, v_k) = A^*T(v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(k)})$$
$$= T(Av_{\sigma^{-1}(1)}, \dots, Av_{\sigma^{-1}(k)})$$
$$= T^\sigma (Av_1, \dots, Av_k)$$
$$= (-1)^\sigma T(Av_1, \dots, Av_k)$$
$$= (-1)^\sigma A^*T(v_1, \dots, v_k)$$

Por tanto $A^*T \in \mathcal{A}^k(V)$.

$\square$

$\mathcal{A}^k(V)$.

Exercise 1.4.x. In Exercise 1.4.ix show that if T is in $\mathcal{L}^k(W)$ then $\mathrm{Alt}(A^*T) = A^*(\mathrm{Alt}(T))$, i.e., show that the Alt operation commutes with the pullback operation.

Dem:

Como $T \in \mathcal{L}^k(W)$, $A^*T \in \mathcal{L}^k(V)$. Sean $v_1, \dots, v_k \in V$, ent.

$$A^*(\mathrm{Alt}(T))(v_1, \dots, v_k) = \mathrm{Alt}(T)(Av_1, \dots, Av_k)$$
$$= \sum_{\tau \in S_k} (-1)^\tau T^\tau (Av_1, \dots, Av_k)$$
$$= \sum_{\tau \in S_k} (-1)^\tau T(Av_{\tau^{-1}(1)}, \dots, Av_{\tau^{-1}(k)})$$
$$= \sum_{\tau \in S_k} (-1)^\tau A^*T(v_{\tau^{-1}(1)}, \dots, v_{\tau^{-1}(k)})$$

$$= \sum_{\tau \in S_k} (-1)^{\tau} (A^* T)^{\tau} (v_1, \ldots, v_k)$$

$$= \mathrm{Alt}(A^* T)(v_1, \ldots, v_k)$$

$$\therefore A^* \mathrm{Alt}(T) = \mathrm{Alt}(A^* T)$$

$\square$

El Espacio $\Lambda^k(V^*)$.

Exercise 1.5.i. A k-tensor $T \in \mathcal{L}^k(V)$ is ***symmetric*** if $T^\sigma = T$ for all $\sigma \in S_k$. Show that the set $\mathcal{S}^k(V)$ of symmetric k tensors is a vector subspace of $\mathcal{L}^k(V)$.

Exercise 1.5.ii. Let $e_1, \dots, e_n$ be a basis of V. Show that every symmetric 2-tensor is of the form

Sol.

Claramente $\mathcal{S}^k(V) \neq \emptyset$, pues $I : V^k \to \mathbb{R}$, $(v_1, \dots, v_k) \mapsto 0$ es un k-tensor, y: $\forall v_1, \dots, v_k \in V$:

$$\begin{aligned} I^\sigma(v_1, \dots, v_k) &= I(v_{\sigma^{-1}(v_1)}, \dots, \sigma^{-1}(v_k)) \\ &= 0 \\ &= I(v_1, \dots, v_k) \end{aligned}$$

Luego $I \in \mathcal{S}^k(V)$. Sean $T_1, T_2 \in \mathcal{S}^k(V)$, $\lambda \in \mathbb{R}$, ent. $\forall \sigma \in S_k$:

$$\begin{aligned} (T_1 + \lambda T_2)^\sigma &= T_1^\sigma + \lambda T_2^\sigma \\ &= T_1 + \lambda T_2 \end{aligned}$$

Así, $T_1 + \lambda T_2 \in \mathcal{S}^k(V)$. Por tanto $\mathcal{S}^k(V)$ es subespacio de $\mathcal{L}^k(V)$.

$\square$

Exercise 1.5.ii. Let $e_1, \dots, e_n$ be a basis of V. Show that every symmetric 2-tensor is of the form

$$\sum_{1 \le i,j \le n} a_{i,j} e_i^\star \otimes e_j^\star$$

where $a_{i,j} = a_{j,i}$ and $e_1^\star, \dots, e_n^\star$ are the dual basis vectors of $V^\star$.

Sol.

Sea $T \in \mathcal{S}^2(V)$. Como $S_2 = \{e, \tau\}$, donde $e(i) = i$ y $\tau(1) = 2$, $\tau(2) = 1$, $\forall i \in [\![1, 2]\!]$,

Por tanto:

$$\begin{aligned} T^\tau(e_i, e_j) &= T(e_j, e_i) \\ &= T(e_i, e_j) \end{aligned}$$

$$\therefore T(e_i, e_j) = T(e_j, e_i), \forall i, j \in [\![1, n]\!]. \quad \dots (1)$$

Ahora, como $T \in \mathcal{L}^2(V)$, $\exists\ T_{11}, T_{12}, \dots, T_{1n}, T_{21}, \dots, T_{nn} \in \mathbb{R}$ m

$$T = \sum_{1 \leq i,j \leq n} T_{ij}\, e_i^* \otimes e_j^*$$

(Pues $\{e_i^* \otimes e_j^*\}$ es base de $\mathcal{L}^2(V)$). En part, por (1):

$$T_{ij} = T_{ji}, \; \forall\, i,j \in [\![1,n]\!]$$

Si $a_{ij} = T_{ij}$, se tiene el resultado.

$\square$

Exercise 1.5.iii. Show that if T is a symmetric k-tensor, then for $k \geq 2$, then T is in $\mathcal{I}^k(V)$.

Hint: Let σ be a transposition and deduce from the identity, $T^\sigma = T$, that T has to be in the kernel of Alt.

Exercise 1.5.iv (a warning). In general $\mathcal{S}^k(V) \neq \mathcal{I}^k(V)$. Show, however, that if $k = 2$ these

Dem:

Sea $T \in \mathcal{S}^k(V)$ (con $k \geq 2$). Si $\tau \in S_k$ es una transposición cualquiera, entonces:

$$T^\tau = T$$

$$\begin{aligned} \Rightarrow \mathrm{Alt}(T) &= \mathrm{Alt}(T^\tau) \\ &= \mathrm{Alt}(T)^\tau \\ &= (-1)^\tau \mathrm{Alt}(T) \\ &= -\mathrm{Alt}(T) \end{aligned}$$

$$\therefore \mathrm{Alt}(T) = 0$$

así, $T \in \mathrm{Ker}(\mathrm{Alt}) = \mathcal{I}^k(V)$. $\Rightarrow \mathcal{S}^k(V) \subseteq \mathcal{I}^k(V)$.

$\square$

Exercise 1.5.iv (a warning). In general $\mathcal{S}^k(V) \neq \mathcal{I}^k(V)$. Show, however, that if $k = 2$ these two spaces are equal.

Exercise 1.5.v. Show that if $\ell \in V^*$ and $T \in \mathcal{L}^{k-2}(V)$, then $\ell \otimes T \otimes \ell$ is in $\mathcal{I}^k(V)$.

Dem:

Por el ejercicio ant, basta probar que $\mathcal{I}^2(V) \subseteq \mathcal{S}^2(V)$. Para ello, basta probar el resultado para $T \in \mathcal{I}^2(V)$ 2-tensor descomponible y redundante. i.e $T = \ell_1 \otimes \ell_2$, donde $\ell_1 = \ell_2 = \ell$. Ent.

$$T = \ell \otimes \ell$$

Ent $T^\tau = T$ y $T^e = T$, donde $S_2 = \{e, \tau\}$. Por tanto $T \in \mathcal{S}^2(V)$.

$\square$

Exercise 1.5.v. Show that if $\ell \in V^*$ and $T \in \mathcal{L}^{k-2}(V)$, then $\ell \otimes T \otimes \ell$ is in $\mathcal{I}^k(V)$.

Exercise 1.5.vi. Show that if ℓ_1 and ℓ_2 are in V^* and $T \in \mathcal{L}^{k-2}(V)$, then $\ell_1 \otimes T \otimes \ell_2 + \ell_2 \otimes T \otimes \ell_1$ is in $\mathcal{I}^k(V)$.

Dem:

Observemos que:

$$(T \otimes \ell + \ell \otimes T) \otimes \ell = T \otimes \ell \otimes \ell + \ell \otimes T \otimes \ell$$

$$\Rightarrow \ell \otimes T \otimes \ell = (T \otimes \ell + \ell \otimes T) \otimes \ell - T \otimes \ell \otimes \ell$$

donde $T \otimes \ell \otimes \ell \in \mathcal{I}^k(V)$ y $T \otimes \ell + \ell \otimes T \in \mathcal{I}^{k-1}(V)$, pues si T es descomponible, con $T = \ell_1 \otimes \dots \otimes \ell_{k-2}$ $(m = k-2)$, ent.

$$T \otimes \ell + \ell \otimes T = \ell_1 \otimes \dots \otimes \ell_m \otimes \ell + \ell \otimes \ell_1 \dots \otimes \ell_m$$

Exercise 1.5.vi. Show that if ℓ_1 and ℓ_2 are in V^* and $T \in \mathcal{L}^{k-2}(V)$, then $\ell_1 \otimes T \otimes \ell_2 + \ell_2 \otimes T \otimes \ell_1$ is in $\mathcal{I}^k(V)$.

Exercise 1.5.vii. Given a permutation $\sigma \in S_k$ and $T \in \mathcal{I}^k(V)$, show that $T^\sigma \in \mathcal{I}^k(V)$.

Dem:

Observemos que:

$$(l_1 + l_2) \otimes T \otimes (l_1 + l_2) = l_1 \otimes T \otimes l_1 + l_1 \otimes T \otimes l_2 + l_2 \otimes T \otimes l_1 + l_2 \otimes T \otimes l_2$$

$$\Rightarrow l_1 \otimes T \otimes l_2 + l_2 \otimes T \otimes l_1 = l_1 \otimes T \otimes l_1 + l_2 \otimes T \otimes l_2 - (l_1 + l_2) \otimes T \otimes (l_1 + l_2)$$

donde $l_i \otimes T \otimes l_i$, $(l_1+l_2) \otimes T \otimes (l_1 + l_2) \in \mathcal{I}^k(V)$ (por el ejercicio anterior). Luego se tiene el resultado. $\square$

Exercise 1.5.vii. Given a permutation $\sigma \in S_k$ and $T \in \mathcal{I}^k(V)$, show that $T^\sigma \in \mathcal{I}^k(V)$.

Exercise 1.5.viii. Let $\mathcal{W}(V)$ be a subspace of $\mathcal{L}^k(V)$ having the following two properties.
(1) For $S \in \mathcal{S}^2(V)$ and $T \in \mathcal{L}^{k-2}(V)$, $S \otimes T$ is in $\mathcal{W}(V)$.

Dem:

Como $T \in \mathcal{I}^k(V) \Rightarrow \mathrm{Alt}(T) = 0$. Pero:

$$\begin{aligned} \mathrm{Alt}(T^\sigma) &= \mathrm{Alt}(T)^\sigma \\ &= (-1)^\sigma \mathrm{Alt}(T) \\ &= 0 \end{aligned}$$

$\therefore T^\sigma \in \mathrm{Ker}(\mathrm{Alt}) = \mathcal{I}^k(V)$. $\square$

Exercise 1.5.viii. Let $\mathcal{W}(V)$ be a subspace of $\mathcal{L}^k(V)$ having the following two properties.
(1) For $S \in \mathcal{S}^2(V)$ and $T \in \mathcal{L}^{k-2}(V)$, $S \otimes T$ is in $\mathcal{W}(V)$.

(2) For T in $\mathcal{W}(V)$ and $\sigma \in S_k$, T^σ is in $\mathcal{W}(V)$.

Show that $\mathcal{W}(V)$ has to contain $\mathcal{I}^k(V)$ and conclude that $\mathcal{I}^k(V)$ is the smallest subspace of $\mathcal{L}^k(V)$ having properties (1) and (2).

Exercise 1.5.ix. Show that there is a bijective linear map

Dem:

Por un ejercicio ant. $\mathcal{S}^2(V) = \mathcal{I}^2(V)$. Luego (1) es equivalente a:

(3) Para $S \in \mathcal{I}^2(V)$, $T \in \mathcal{I}^{k-2}(V)$, $S \otimes T \in \mathcal{W}(V)$.

Por el ejercicio ant. $\mathcal{I}^k(V)$ cumple (2)

Exercise 1.5.ix. Show that there is a bijective linear map

$$\alpha : \Lambda^k(V^\star) \xrightarrow{\sim} \mathcal{A}^k(V)$$

with the property

$$\alpha\pi(T) = \frac{1}{k!}\operatorname{Alt}(T) \tag{1.5.15}$$

for all $T \in \mathcal{L}^k(V)$, and show that α is the inverse of the map of $\mathcal{A}^k(V)$ onto $\Lambda^k(V^\star)$ described in Theorem 1.5.13.

Hint: Exercise 1.2.viii.

Dem:

Considere $\operatorname{Alt}' : \mathcal{L}^k(V) \to \mathcal{A}^k(V)$, Alt' es supreyectivo y $\operatorname{Ker}(\operatorname{Alt}) = \mathcal{I}^k(V)$. Por el ejercicio 1.2.viii) $\exists\, \alpha : \mathcal{L}^k(V)/\mathcal{I}^k(V) = \Lambda^k(V^*) \to \mathcal{A}^k(V)$ inyectivo $\ni$ $\alpha \circ \pi = \operatorname{Alt}$, es decir:

$$\alpha \circ \pi(T) = \operatorname{Alt}' T = \frac{1}{k!}\operatorname{Alt} T$$

donde $\operatorname{Alt}' T = \frac{1}{k!}\operatorname{Alt} T$, $\forall\, T \in \mathcal{L}^k(V)$. En part. $\forall\, T \in \mathcal{A}^k(V)$:

$$\alpha \circ \pi(T) = \frac{1}{k!}\operatorname{Alt} T = T = id_{\mathcal{A}^k(V)}(T)$$

$$\therefore \alpha \circ \pi\big|_{\mathcal{A}^k(V)} = id_{\mathcal{A}^k(V)}$$

y $\forall\, T + \mathcal{I}^k(V) \in \Lambda^k(V^*)$, donde $T \in \mathcal{A}^k(V)$, entonces:

$$\pi\big|_{\mathcal{A}^k(V)}\, \alpha(T + \mathcal{I}^k(V)) = \pi\big|_{\mathcal{A}^k(V)}\left(\frac{1}{k!}\operatorname{Alt} T\right)$$

$$= T + \mathcal{I}^k(V)$$

$$= id_{\Lambda^k(V^*)}(T)$$

$$\therefore \alpha = \pi\big|_{\mathcal{A}^k(V)}^{-1}.$$

$\square$

Exercise 1.5.x. Let V be an n-dimensional vector space. Compute the dimension of $\mathcal{S}^k(V)$.

Hints:

(1) Introduce the following symmetrization operation on tensors $T \in \mathcal{L}^k(V)$:

$$\mathrm{Sym}(T) = \sum_{\tau \in S_k} T^\tau \,.$$

Prove that this operation has properties (2)–(4) of Proposition 1.4.17 and, as a substitute for (1), has the property: $\mathrm{Sym}(T)^\sigma = \mathrm{Sym}(T)$.

(2) Let $\phi_I = \mathrm{Sym}(e_I^\star)$, $e_I^\star = e_{i_1}^\star \otimes \cdots \otimes e_{i_n}^\star$. Prove that $\{\, \phi_I \mid I \text{ is non-decreasing} \,\}$ form a basis of $S^k(V)$.

(3) Conclude that $\dim(\mathcal{S}^k(V))$ is equal to the number of non-decreasing multi-indices of length k: $1 \le i_1 \le i_2 \le \cdots \le \ell_k \le n$.

(4) Compute this number by noticing that the assignment

$$(i_1, \dots, i_k) \mapsto (i_1 + 0, i_2 + 1, \dots, i_k + k - 1)$$

is a bijection between the set of these non-decreasing multi-indicesand the set of increasing multi-indices $1 \le j_1 < \cdots < j_k \le n + k - 1$.

1.6. The wedge product

The tensor algebra operations on the spaces $\mathcal{L}^k(V)$ which we discussed in §§1.2 and 1.3, i.e., the "tensor product operation" and the "pullback" operation, give rise to similar operations on the spaces, $\Lambda^k(V^\star)$. We will discuss in this section the analogue of the tensor product operation.

El Producto Wedge.

Exercise 1.6.i. Prove the assertions (1.6.4), (1.6.5), and (1.6.6).

Exercise 1.6.ii. Verify the multiplication law in equation (1.6.11) for the wedge product.

Dem:

(1.6.4) $\lambda(\omega_1 \wedge \omega_2) = (\lambda\omega_1)\wedge\omega_2 = \omega_1 \wedge (\lambda\omega_2)$.

(1.6.5) $(\omega_1+\omega_2)\wedge\omega_3 = \omega_1\wedge\omega_3 + \omega_2\wedge\omega_3$.

(1.6.6) $\omega_1\wedge(\omega_2+\omega_3) = \omega_1\wedge\omega_2 + \omega_1\wedge\omega_3$.

Se probará (1.6.4). Sean $\omega_1 \in \Lambda^{k_1}(V^*)$, $\omega_2 \in \Lambda^{k_2}(V^*)$ y $\lambda \in \mathbb{R}$, ent. $\exists\ T_1 \in \mathcal{L}^{k_1}(V)$ y $T_2 \in \mathcal{L}^{k_2}(V)$ tales que $\omega_1 = \pi(T_1)$ y $\omega_2 = \pi(T_2)$. Entonces:

$$\begin{aligned}\lambda(\omega_1\wedge\omega_2) &= \lambda\pi(T_1\otimes T_2)\\ &= \pi(\lambda(T_1\otimes T_2))\\ &= \pi((\lambda T_1)\otimes T_2) = (\lambda\omega_1)\wedge\omega_2\\ &= \pi(T_1\otimes(\lambda T_2)) = \omega_1\wedge(\lambda\omega_2)\end{aligned}$$

De (1.6.5). Sean $\omega_1, \omega_2 \in \Lambda^r(V^*)$ y $\omega_3 \in \Lambda^s(V^*)$, entonces $\exists\ T_1, T_2 \in \mathcal{L}^r(V^*)$ y $T_3 \in \mathcal{L}^s(V)$

π

$$\begin{aligned}(\omega_1+\omega_2)\wedge\omega_3 &= \pi((T_1+T_2)\otimes T_3)\\ &= \pi(T_1\otimes T_3 + T_2\otimes T_3)\\ &= \pi(T_1\otimes T_3) + \pi(T_2\otimes T_3)\\ &= \omega_1\wedge\omega_3 + \omega_2\wedge\omega_3\end{aligned}$$

De (1.6.6). Es análogo a (1.6.5).

$\square$

Exercise 1.6.ii. Verify the multiplication law in equation (1.6.11) for the wedge product.

Exercise 1.6.iii. Given $\omega \in \Lambda^r(V^*)$ let ω^k be the k-fold wedge product of ω with itself, i.e.

Dem:

Se hizo en las notas.

$\square$

Exercise 1.6.iii. Given $\omega \in \Lambda^r(V^*)$ let ω^k be the k-fold wedge product of ω with itself, i.e., let $\omega^2 = \omega \wedge \omega$, $\omega^3 = \omega \wedge \omega \wedge \omega$, etc.

(1) Show that if r is odd then for $k > 1$, $\omega^k = 0$.
(2) Show that if ω is decomposable, then for $k > 1$, $\omega^k = 0$.

Dem.

De (1): Como r es impar, r^2 también lo es, luego:

$$\omega \wedge \omega = (-1)^{r^2} \omega \wedge \omega$$

$$\Rightarrow \omega^2 = -\omega^2$$

$$\Rightarrow \omega^2 = 0$$

Si $k > 2$, ent. r^{k-2} es impar, así:

$$\begin{aligned} \omega^k &= \omega^{k-2} \wedge \omega^2 \\ &= (-1)^{r^2 \cdot r^{k-2}} \omega^k \\ &= (-1)^{rk} \omega^k \\ &= -\omega^k \end{aligned}$$

$$\therefore \omega^k = 0$$

De (ii): Suponga que ω es descomponible, entonces $\exists\, l_1^*, \ldots, l_r^* \in \Lambda^1(V^*) = V^*$ m

$$\omega = l_1^* \wedge l_2^* \wedge \ldots \wedge l_r^*$$

Procederemos por inducción sobre K. Para $K = 2$: $\omega = l_1^* \wedge l_2^* \wedge \ldots \wedge l_r^*$, ent.

$$\omega^2 = l_1^* \wedge \ldots \wedge l_r^* \wedge l_1^* \wedge \ldots \wedge l_r^*$$

Exercise 1.6.iv. If ω and μ are in $\Lambda^r(V^*)$ prove:

$$(\omega+\mu)^k = \sum_{\ell=0}^{k} \binom{k}{\ell} \omega^\ell \wedge \mu^{k-\ell}.$$

Hint: As in freshman calculus, prove this binomial theorem by induction using the identity: $\binom{k}{\ell} = \binom{k-1}{\ell-1} + \binom{k-1}{\ell}$.

Dem.

Si r es impar, el resultado es inmediato. Suponga r par, entonces r^u es par, $\forall\, u \in \mathbb{N}$.

Procederemos por inducción sobre K. Para $K=2$:

$$\begin{aligned}(\omega+\mu)^2 &= (\omega+\mu)\wedge(\omega+\mu)\\ &= (\omega+\mu)\wedge\omega + (\omega+\mu)\wedge\mu\\ &= \omega^2 + \mu\wedge\omega + \omega\wedge\mu + \mu^2\end{aligned}$$

Pero $\mu\wedge\omega = (-1)^{r^2}\omega\wedge\mu = \omega\wedge\mu$, luego:

$$\begin{aligned}&= \omega^2 + 2\omega\wedge\mu + \mu^2\\ &= \sum_{\ell=0}^{2}\binom{2}{\ell}\omega^\ell\wedge\mu^{2-\ell}\end{aligned}$$

Luego, el resultado se cumple para $K=2$. Suponga se cumple para algún $K\in\mathbb{N}$. Ent.

$$\begin{aligned}(\omega+\mu)^{K+1} &= (\omega+\mu)\wedge(\omega+\mu)^K\\ &= (\omega+\mu)\wedge\sum_{\ell=0}^{K}\binom{K}{\ell}\omega^\ell\wedge\mu^{K-\ell}\\ &= \sum_{\ell=0}^{K}\binom{K}{\ell}\omega^{\ell+1}\wedge\mu^{K-\ell} + \sum_{\ell=0}^{K}\binom{K}{\ell}\mu\wedge\omega^\ell\wedge\mu^{K-\ell}\end{aligned}$$

Pero $\mu\wedge\omega^\ell = (-1)^{r(\ell r)}\omega^\ell\wedge\mu = (-1)^{r^2\ell}\omega^\ell\wedge\mu = \omega^\ell\wedge\mu$, luego:

$$\begin{aligned}&= \sum_{\ell=0}^{K}\binom{K}{\ell}\omega^{\ell+1}\wedge\mu^{K-\ell} + \sum_{\ell=0}^{K}\binom{K}{\ell}\omega^\ell\wedge\mu^{K+1-\ell}\\ &= \binom{K}{K}\omega^{K+1}\wedge\mu^0 + \sum_{\ell=0}^{K-1}\binom{K}{\ell}\omega^{\ell+1}\wedge\mu^{K-\ell} + \sum_{\ell=1}^{K}\binom{K}{\ell}\omega^\ell\wedge\mu^{K+1-\ell} + \binom{K}{0}\omega^0\wedge\mu^{K+1}\\ &= \binom{K+1}{K+1}\omega^{K+1}\wedge\mu^{(K+1)-(K+1)} + \sum_{\ell=1}^{K}\binom{K}{\ell-1}\omega^\ell\wedge\mu^{K+1-\ell} + \sum_{\ell=1}^{K}\binom{K}{\ell}\omega^\ell\wedge\mu^{K+1-\ell} + \binom{K+1}{0}\omega^0\wedge\mu^{K+1-0}\\ &= \binom{K+1}{K+1}\omega^{K+1}\wedge\mu^0 + \sum_{\ell=1}^{K}\left[\binom{K}{\ell-1}+\binom{K}{\ell}\right]\omega^\ell\wedge\mu^{K+1-\ell} + \binom{K+1}{0}\omega^0\wedge\mu^{K+1}\\ &= \sum_{\ell=0}^{K+1}\binom{K+1}{\ell}\omega^\ell\wedge\mu^{K+1-\ell}\end{aligned}$$

Luego se cumple para $K+1$. Aplicando inducción, se cumple $\forall\, n\in\mathbb{N}$. $\square$

Exercise 1.6.v. Let ω be an element of $\Lambda^2(V^*)$. By definition the ***rank*** of ω is k if $\omega^k \neq 0$ and $\omega^{k+1} = 0$. Show that if

$$\omega = e_1 \wedge f_1 + \cdots + e_k \wedge f_k$$

with $e_i, f_i \in V^*$, then ω is of rank $\leq k$.

Hint: Show that

$$\omega^k = k! e_1 \wedge f_1 \wedge \cdots \wedge e_k \wedge f_k .$$

Dem:

Procederemos por inducción sobre K (para probar la pista). Si $K=1$, es claro que

$$\omega^1 = 1!\, e_1 \wedge f_1$$

Suponga se cumple para algún $K \in \mathbb{N}$. Sea $\omega = e_1\wedge f_1 + \dots + e_{K+1}\wedge f_{K+1}$. Llamemos $\omega_1 = e_1 \wedge f_1 + \dots + e_K \wedge f_K$.

Entonces:

$$\begin{aligned}
\omega^{K+1} &= (\omega_1 + \omega_2)^{K+1}, \quad \omega_2 = e_{K+1}\wedge f_{K+1} \\
&= (\omega_1+\omega_2)\wedge(\omega_1+\omega_2)^K \\
&= \omega_1 \wedge (\omega_1+\omega_2)^K + \omega_2\wedge(\omega_1+\omega_2)^K
\end{aligned}$$

donde $(\omega_1+\omega_2)^K = \sum_{l=0}^{K} \binom{K}{l} \omega_1^l \wedge \omega_2^{K-l}$. Ent.

$$= \omega_1 \wedge \sum_{l=0}^{K} \omega_1^l \wedge \omega_2^{K-l} + \omega_2 \wedge \sum_{l=0}^{K} \omega_1^l \wedge \omega_2^{K-l}$$

Pero $\omega_1 \wedge \sum_{l=0}^{K} \omega_1^l \wedge \omega_2^{K-l} = \sum_{l=0}^{K} \omega_1^{l+1} \wedge \omega_2^{K-l}$ y $\omega_2 \wedge \sum_{l=0}^{K} \omega_1^l \wedge \omega_2^{K-l} = \sum_{l=0}^{K} \omega_2 \wedge \omega_1^l \wedge \omega_2^{K-l}$, y $\omega_2 \wedge \omega_1^l = (-1)^{2\cdot 2l}$

$\omega_1^l \wedge \omega_2 = \omega_1^l \wedge \omega_2$. Por ende $= \sum_{l=0}^{K} \omega_1^l \wedge \omega_2^{K+1-l}$.

Ahora, $\omega_2^2 = (e_1\wedge f_1)\wedge(e_1\wedge f_1) = e_1\wedge f_1 \wedge e_1 \wedge f_1 = 0$. Por tanto:

$$\sum_{l=0}^{K} \omega_1^l \wedge \omega_2^{K+1-l} = \omega_1^K \wedge \omega_2$$

(ya que en los demás, $\omega_2^{K+1-l} = 0$, $\forall\, l \in [0, K-1]$). También:

$$\sum_{l=0}^{K} \binom{K}{l} \omega_1^{l+1} \wedge \omega_2^{K-l} = \omega_1^{K+1} + K\, \omega^K \wedge \omega_1$$

Pero $\omega_1^{K+1} = 0$, así:

$$\begin{aligned}
\omega^K &= K\, \omega_1^K \wedge \omega_2 + \omega_1^K \wedge \omega_2 \\
&= (K+1)\, \omega_1^K \wedge \omega_2
\end{aligned}$$

donde por hip. de ind. $\omega_1^K = K!\, e_1 \wedge f_1 \wedge \dots \wedge e_K \wedge f_K$, así:

$$= (K+1)!\, e_1 \wedge f_1 \wedge \dots \wedge e_K \wedge f_K \wedge e_{K+1} \wedge f_{K+1}.$$

Aplicando inducción se cumple el resultado $\forall K \in \mathbb{N}$. Ahora:

$$\begin{aligned}
\omega^{k+1} &= \omega \wedge \omega^{k} \\
&= k!\left(e_1\wedge f_1 + \dots + e_k \wedge f_k\right)\wedge\left(e_1\wedge f_1 \wedge \dots \wedge e_k \wedge f_k\right) \\
&= k!\left(e_1\wedge f_1 \wedge e_1 \wedge f_1 \wedge \dots \wedge e_k \wedge f_k + \dots + e_k \wedge f_k \wedge e_1 \wedge f_1 \wedge \dots \wedge e_k \wedge f_k\right) \\
&= k! \cdot 0 = 0
\end{aligned}$$

(pues en cada sumando hay términos repetidos). Por tanto $\operatorname{rank}(\omega) \leq k$.

$\square$

Exercise 1.6.vi. Given $e_i \in V^*$, $i = 1, \dots, k$ show that $e_1 \wedge \dots \wedge e_k \neq 0$ if and only if the e_i's are linearly independent.

Hint: Induction on k.

$\Rightarrow$) Suponga que los e_i no son l.i, entonces $e_i = e_j$, para algun $i,j \in [\![1,K]\!]$, $i \neq j$. Luego:

$$e_1 \wedge \dots \wedge e_k = \pi\left(e_1 \otimes \dots \otimes e_k\right)$$

donde $e_1 \otimes \dots \otimes e_k \in \mathcal{L}^k(V)$ es redundante. Así $e_1 \wedge \dots \wedge e_k = 0$.

$\Leftarrow$) Procederemos por inducción sobre K. Para $K=2$, suponga que e_1 y e_2 son l.i. Para probar que $e_1 \wedge e_2 \neq 0$, basta ver que $e_1 \otimes e_2 \in \mathcal{A}^2(V) \setminus \{0\}$. Claramente $e_1 \otimes e_2 \neq 0$ pues

$$\begin{aligned}
e_1 \otimes e_2 (1,1) &= e_1(1,1) \cdot e_2(1,1) \\
&= 1 \cdot 1 \\
&= 1 \neq 0
\end{aligned}$$

Ahora, sea

1.7. El Producto Interior

Exercise 1.7.i. Prove Lemma 1.7.4.

Exercise 1.7.ii. Prove Lemma 1.7.6.

Dem.

La prueba se hizo en las notas.

$\square$

Exercise 1.7.ii. Prove Lemma 1.7.6.

Exercise 1.7.iii. Show that if $T \in \mathcal{A}^k$, $\iota_v T = kT_v$ where T_v is the tensor (1.3.21). In particular

Dem.

La prueba se hizo en las notas.

$\square$

Exercise 1.7.iii. Show that if $T \in \mathcal{A}^k$, $\iota_v T = kT_v$ where T_v is the tensor (1.3.21). In particular
conclude that $\iota_v T \in \mathcal{A}^{k-1}(V)$. (See Exercise 1.4.viii.)

Dem.

Sea $T \in \mathcal{A}^k(V)$. Se tiene para $v_1, \dots, v_{k-1} \in V$ que: $\forall\, v \in V$:

$$i_v T(v_1, \dots, v_{k-1}) = \sum_{r=1}^{k} (-1)^{r-1} T(v_1, \dots, v_{r-1}, v, v_r, \dots, v_{k-1})$$

Sea $r \in [\![1, k]\!]$. Tenemos 3 casos:

i) $r = 1$, ent. $(-1)^{r-1} T(v_1, \dots, v_{r-1}, v, v_r, \dots, v_{k-1}) = (-1)^0 T(v, v_1, \dots, v_{k-1}) = T_v(v_1, \dots, v_{k-1})$.

ii) $r = k$. ent $(-1)^{r-1} T(v_1, \dots, v_{r-1}, v, v_r, \dots, v_{k-1}) = (-1)^{k-1} T(v_1, \dots, v_{k-1}, v)$. Por ser T alternante, se tiene que:

$$\begin{aligned}(-1)^{k-1} T(v_1, \dots, v_{k-1}, v) &= (-1)^{2k-2} T(v, v_1, \dots, v_{k-1}) \\ &= T_v(v_1, \dots, v_{k-1})\end{aligned}$$

iii) $1 < r < k$, ent.

$$\begin{aligned}(-1)^{r-1} T(v_1, \dots, v_{r-1}, v, v_r, \dots, v_{k-1}) &= (-1)^{2r-2} T(v, v_1, \dots, v_{k-1}) \\ &= T_v(v_1, \dots, v_{k-1})\end{aligned}$$

Por (i)-(iii), se tiene que:

$$i_v T(v_1,\dots,v_{k-1}) = \sum_{r=1}^{k} (-1)^{r-1} T(v_1,\dots,v_{r-1},v,v_r,v_{r-1})$$
$$= \sum_{r=1}^{k} T_v(v_1,\dots,v_{k-1})$$
$$= k\,T_v(v_1,\dots,v_{k-1})$$
$$\therefore i_v T = k T_v$$

$\square$

Exercise 1.7.iv. Assume the dimension of V is n and let Ω be a nonzero element of the one dimensional vector space $\Lambda^n(V^\star)$. Show that the map

(1.7.17) $$\phi: V \to \Lambda^{n-1}(V^\star),\ v \mapsto \iota_v\Omega,$$

is a bijective linear map. *Hint:* One can assume $\Omega = e_1^\star \wedge \cdots \wedge e_n^\star$ where $e_1,\dots,e_n$ is a basis of V. Now use equation (1.7.16) to compute this map on basis elements.

Dem.

Se sabe que $\dim(\Lambda^n(V^*)) = \binom{n}{n} = 1$. Por tanto, todos los elementos de $\Lambda^n(V^*)$ están caracterizados (son de la forma $c e_1^* \wedge \dots \wedge e_n^*$, donde $c \in \mathbb{R}$). Por tanto, podemos asumir a Ω de la forma:

$$\Omega = c e_1^* \wedge \dots \wedge e_n^*,\quad c \neq 0.$$

entonces, Sea $\phi: V \to \Lambda^{n-1}(V^*)$, $v \mapsto \phi(v) = i_v\Omega$. Claramente ϕ está bien definido. Veamos que es lineal. Sean $w_1, w_2 \in V$ y $\alpha \in \mathbb{R}$. Ent.

$$\phi(w_1+\alpha w_2) = i_{w_1+\alpha w_2}\Omega$$
$$= i_{w_1}\Omega + i_{\alpha w_2}\Omega$$
$$= i_{w_1}\Omega + \alpha\, i_{w_2}\Omega$$
$$= \phi(w_1) + \alpha\,\phi(w_2)$$

pues:

$$i_{\alpha w_2}\Omega(v_1,\dots,v_{k-1}) = \sum_{r=1}^{k} (-1)^{r-1}\Omega(v_1,\dots v_{r-1},\alpha w_2, v_r,\dots,v_{k-1})$$
$$= \sum_{r=1}^{k} (-1)^{r-1}\alpha\,\Omega(v_1,\dots v_{k-1}, w_2, v_r,\dots,v_{k-1})$$
$$= \alpha\, i_{w_2}\Omega$$

Así, ϕ es lineal. Veamos que es biyectiva.

i) ϕ es monomorfismo. Sea $v \in \operatorname{Ker}(\phi)$. Ent. como Ω es descomponible:

$$i_v\Omega(v_1,\dots,v_{k-1}) = \sum_{r=1}^{k}(-1)^{r-1}e_r^*(v)\,(e_1^*\wedge\dots\wedge\hat{e}_r^*\wedge\dots\wedge e_k^*)(v_1,\dots,v_{k-1})$$
$$=0,\ \forall\, v_1,\dots,v_{k-1}\in V.$$

Consideremos $\forall\, l\in[\![1,n]\!]$ y tomemos $w=(e_1,\dots,e_{l-1},\hat{e}_l,e_l,\dots,e_k)\in V^{k-1}$. Ent.

$$i_v\Omega(w) = \sum_{r=1}^{k}(-1)^{r-1}e_r^*(v)\,(e_1^*\wedge\dots\wedge\hat{e}_r^*\wedge\dots\wedge e_k^*)(w)$$

donde $(e_1^*\wedge\dots\wedge\hat{e}_r^*\wedge\dots\wedge e_k^*)(w)=0$ si $r\neq l$, pues $e_r^*(e_l)=\delta_{rl}$. Si $r=l$ en cambio:

$$(e_1^*\wedge\dots\wedge\hat{e}_l^*\wedge\dots\wedge e_k^*)(w)=1$$

por tanto:

$$i_v\Omega(w) = (-1)^{l-1}e_r^*(v)$$
$$=0$$
$$\Rightarrow e_r^*(v)=0,\ \forall\, r\in[\![1,k]\!]$$
$$\Rightarrow v=0$$

Así $\operatorname{Ker}\phi=\{0\}$.

ii) Por el t. rango-nulidad:

$$\dim(\Lambda^{n-1}(V^*)) = \dim(\operatorname{Ker}\phi)+\dim(\operatorname{im}(\phi))$$
$$\Rightarrow \dim(\Lambda^{n-1}(V^*)) = \dim(\operatorname{im}\phi)$$
$$\Rightarrow n=\dim(\operatorname{im}\phi)$$

donde $\dim(V)=n \Rightarrow \dim(\operatorname{im}\phi)=\dim(V)<\infty \Rightarrow \phi$ es epimorfismo.

Por (i) y (ii), ϕ es isomorfismo. $\square$

Exercise 1.7.v (cross-product). Let V be a 3-dimensional vector space, B an inner product on V and Ω a nonzero element of $\Lambda^3(V^*)$. Define a map

$$-\times- : V\times V\to V$$

by setting

$$v_1\times v_2 := \rho^{-1}(Lv_1\wedge Lv_2)$$

where ρ is the map (1.7.17) and $L\colon V\to V^*$ the map (1.2.17). Show that this map is linear in v_1, with v_2 fixed and linear in v_2 with v_1 fixed, and show that $v_1\times v_2=-v_2\times v_1$.

Dem.

Claramente $\times : V\times V\to V$ está bien definido, pues ρ lo está y L también. Veamos que es lineal entrada por entrada. Como $\dim(V)=3$, ent. L es isomorfismo.

Basta probar que es lineal por la entrada izquierda, pues: $\forall\, v_1, v_2 \in V$:

$$\begin{aligned} v_1 \times v_2 &= \rho^{-1}(Lv_1 \wedge Lv_2) \\ &= \rho^{-1}(l_{v_1} \wedge l_{v_2}) \\ &= \rho^{-1}(-l_{v_2} \wedge l_{v_1}) \\ &= -\rho^{-1}(l_{v_2} \wedge l_{v_1}) \\ &= -\rho^{-1}(Lv_2 \wedge Lv_1) \\ &= -v_2 \times v_1 \end{aligned}$$

Sea $v_1 \in V$ fijo. Si $w_1, w_2 \in V$, $\alpha \in \mathbb{R}$, ent.

$$\begin{aligned} v_1 \times (w_1 + \alpha w_2) &= \rho^{-1}(Lv_1 \wedge L(w_1 + \alpha w_2)) \\ &= \rho^{-1}(Lv_1 \wedge (Lw_1 + \alpha L w_2)) \\ &= \rho^{-1}(Lv_1 \wedge Lw_1 + \alpha Lv_1 \wedge Lw_2) \\ &= \rho^{-1}(Lv_1 \wedge Lw_1) + \alpha\, \rho^{-1}(Lv_1 \wedge w_2) \\ &= v_1 \times w_1 + \alpha\, v_1 \times w_2 \end{aligned}$$

$\therefore$ es lineal en la 2^{da} ent.

$\square$

Exercise 1.7.vi. For $V = \mathbf{R}^3$ let e_1, e_2 and e_3 be the standard basis vectors and B the standard inner product. (See § 1.1.) Show that if $\Omega = e_1^\star \wedge e_2^\star \wedge e_3^\star$ the cross-product above is the standard cross-product:

$$e_1 \times e_2 = e_3$$
$$e_2 \times e_3 = e_1$$
$$e_3 \times e_1 = e_2 .$$

Hint: If B is the standard inner product, then $Le_i = e_i^\star$.

Dem.

Se probará para $e_1, e_2 \in V$. En efecto:

$$\begin{aligned} e_1 \times e_2 &= \rho^{-1}(Le_1 \wedge Le_2) \\ &= \rho^{-1}(l_{e_1} \wedge l_{e_2}) \end{aligned}$$

donde: $l_{e_i}(v) = B(e_i, v) = v_i \Rightarrow l_{e_i} = e_i^*$. Por tanto:

$$= \mathcal{D}^{-1}(e_1^* \wedge e_2^*)$$

Notemos que:

$$\mathcal{D}(e_3) = i_{e_3}\Omega$$

$$= i_{e_3}(e_1^* \wedge e_2^* \wedge e_3^*)$$

$$= \cancelto{0}{e_1^*(e_3)}\, e_2^* \wedge e_3^* - \cancelto{0}{e_2^*(e_3)}\, e_1^* \wedge e_3^* + \cancelto{1}{e_3^*(e_3)}\, e_1^* \wedge e_2^*$$

$$= e_1^* \wedge e_2^*$$

$$\therefore\ \mathcal{D}^{-1}(e_1^* \wedge e_2^*) = e_3$$

$$\therefore\ e_1 \times e_2 = e_3$$

los demás se prueban de forma análoga.

$\square$

1.8 Pullback.

Exercise 1.8.i. Verify the three assertions of Proposition 1.8.4.

Exercise 1.8.ii. Deduce from Proposition 1.8.9 a well-known fact about determinants of $n \times n$ matrices: If two columns are equal, the determinant is zero.

Dem.

Se hizo en las notas.

$\square$

Exercise 1.8.ii. Deduce from Proposition 1.8.9 a well-known fact about determinants of $n \times n$ matrices: If two columns are equal, the determinant is zero. 5)

Exercise 1.8.iii. Deduce from Proposition 1.8.7 another well-known fact about determinants of $n \times n$ matrices: If one interchanges two columns, then one changes the sign of the

Sol.

Sea V esp. vect. de dim. n y $A: V \to V$ una aplicación lineal, $\{e_i\}_{i=1}^n$ una base de V. Como 2 columnas de la rep. mat. de A son iguales, por 5) se tiene que:

$$A(e_i) = A(e_j)$$

para algunos $i,j \in [\![1,n]\!]$, $i \neq j$. Luego $e_i - e_j \in \operatorname{Ker} A \Rightarrow \dim(\operatorname{Ker} A) \geq 1$. Por tanto A no es suprayectiva, luego $\operatorname{Det} A = 0$.

$\square$

Exercise 1.8.iii. Deduce from Proposition 1.8.7 another well-known fact about determinants of $n \times n$ matrices: If one interchanges two columns, then one changes the sign of the determinant.

Hint: Let $e_1, \ldots, e_n$ be a basis of V and let $B: V \to V$ be the linear mapping: $Be_i = e_j$, $Be_j = e_i$ and $Be_\ell = e_\ell$, $\ell \neq i, j$. What is $B^*(e_1^* \wedge \cdots \wedge e_n^*)$?

Sol.

Sea V esp. vect. de dim. n, $\{e_i\}_{i=1}^n$ una base de V y $A: V \to V$ un operador lineal. Considere el operador lineal $B: V \to V$, dado como sigue (dados $i,j \in [\![1,n]\!]$ fijos, $i \neq j$):

$$B(e_\ell) := \begin{cases} e_\ell, & \ell \in [\![1,n]\!] \setminus \{i,j\} \\ e_j, & \ell = i. \\ e_i, & \ell = j. \end{cases}$$

Ahora, la t. lineal $A \circ B$ intercambia la i-ésima columna de A, con la j-ésima. En efecto:

$$A\circ B(e_l)=\begin{cases} A(e_l), & l\in [\![1,n]\!]\setminus\{i,j\}. \\ A(e_j), & l=i. \\ A(e_i), & l=j. \end{cases}$$

Sea $A' = A\circ B$. Ent.

$$\begin{aligned}\mathrm{Det}(A') &= \mathrm{Det}(A\circ B)\\ &= \mathrm{Det}\,A\cdot \mathrm{Det}\,B\end{aligned}$$

Computaremos $\mathrm{Det}\,B$. Claramente $B\circ B = id_V$, por ende

$$(\mathrm{Det}\,B)^2 = 1$$

Veamos que

$$B^*(e_1^*\wedge\ldots\wedge e_n^*) = (B^*e_1^*)\wedge\ldots\wedge(B^*e_n^*)$$

donde $B^*(e_k^*) = e_k^*\circ B$. Si $l\in[\![1,n]\!]$ y $l\neq i,j$, ent.

$$\begin{aligned}B^*(e_l^*)(v) &= e_l^*\circ B(v)\\ &= e_l^*(B(v_1e_1+\ldots+v_ne_n))\\ &= e_l^*(v_1e_1+\ldots+v_ie_j+\ldots+v_je_i+\ldots+v_le_l+\ldots+v_ne_n)\\ &= v_l\\ &= e_l^*(v)\end{aligned}$$

Por otro lado,

$$\begin{aligned}B^*e_i^*(v) &= e_i^*\circ B(v)\\ &= e_i^*(B(v_1e_1+\ldots+v_ne_n))\\ &= e_i^*(v_1e_1+\ldots+v_ie_j+\ldots+v_je_i+\ldots+v_ne_n)\\ &= v_j\\ &= e_j^*(v)\end{aligned}$$

Por lo tanto, $B^*e_l^* = e_l^*$ y $B^*e_i^* = e_j^*$, $B^*e_j^* = e_i^*$ (procediendo de forma similar). Luego (siendo $i<j$):

$$B^*(e_1^* \wedge \dots \wedge e_i^* \wedge \dots \wedge e_j^* \wedge \dots \wedge e_n^*) = e_1^* \wedge \dots \wedge e_j^* \wedge \dots \wedge e_i^* \wedge \dots \wedge e_n^*$$

$$= (-1)^{\tau} e_1^* \wedge \dots \wedge e_i^* \wedge \dots \wedge e_j^* \wedge \dots \wedge e_n^*$$

donde τ es la transposición que intercambia a i con j, y $(-1)^{\tau} = -1$, luego:

$$= - e_1^* \wedge \dots \wedge e_n^*$$

$$\therefore \operatorname{Det} B = -1$$

$$\Rightarrow \operatorname{Det} A' = -\operatorname{Det} A$$

por lo que, intercambiar dos columnas sólo cambia el signo del determinante.

$\square$

Exercise 1.8.iv. Deduce from Propositions 1.8.3 and 1.8.4 another well-known fact about determinants of $n \times n$ matrix: If $(b_{i,j})$ is the inverse of $[a_{i,j}]$, its determinant is the inverse of the determinant of $[a_{i,j}]$.

Dem.

Sean A y B las t. lineales asociadas a las matrices $[a_{ij}]$ y (b_{ij}). Ent.

$$A \circ B = id_V$$

$$\Rightarrow \operatorname{Det} A \cdot \operatorname{Det} B = 1$$

$$\Rightarrow \operatorname{Det} B = \frac{1}{\operatorname{Det} A}.$$

$\square$

Exercise 1.8.v. Extract from (1.8.11) a well-known formula for determinants of 2×2 matrices:

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21}, & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21} .$$

Sol.

Como $\Sigma_2 = \{1,2\} \Rightarrow S_2 = \{id, \tau\}$ dónde $\tau(1) = 2$ y $\tau(2) = 1$. Por tanto:

$$\det([a_{ij}]) = \sum_{\sigma \in S_2} a_{1\sigma(1)} \cdot a_{2\sigma(2)} (-1)^{\sigma}$$

$$= a_{1e(1)} a_{2e(2)} (-1)^{e} + a_{1\tau(1)} a_{2\tau(2)} (-1)^{\tau}$$

$$= a_{11} a_{22} - a_{12} a_{21}.$$

$\square$

Exercise 1.8.vi. Show that if $A = [a_{i,j}]$ is an $n \times n$ matrix and $A^\intercal = [a_{j,i}]$ is its transpose $\det A = \det A^\intercal$.

Hint: You are required to show that the sums

$$\sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$$

and

$$\sum_{\sigma \in S_n} (-1)^\sigma a_{\sigma(1),1} \cdots a_{\sigma(n),n}$$

are the same. Show that the second sum is identical with

$$\sum_{\sigma \in S_n} (-1)^{\sigma^{-1}} a_{\sigma^{-1}(1),1} \cdots a_{\sigma^{-1}(n),n} \cdot$$

Sol.

Observemos que:

$$\begin{aligned}
\det([a_{ij}]) &= \sum_{\sigma \in S_n} (-1)^\sigma a_{1\sigma(1)} \cdots a_{n\sigma(n)} \\
&= \sum_{\sigma \in S_n} (-1)^{\sigma^{-1}} \cdot (-1)^{\sigma \circ \sigma^{-1}} a_{1\sigma(1)} \cdots a_{n\sigma(n)} \\
&= \sum_{\sigma \in S_n} (-1)^{\sigma^{-1}} a_{1\sigma(1)} \cdots a_{n\sigma(n)} \\
&= \sum_{\sigma \in S_n} (-1)^{\sigma^{-1}} a_{\sigma^{-1}(1)1} \cdots a_{\sigma^{-1}(n)n} \\
&= \sum_{\tau \in S_n} (-1)^{\tau} a_{\tau(1)1} \cdots a_{\tau(n)n} \\
&= \det([a_{ji}])
\end{aligned}$$

$\therefore \operatorname{Det} A = \operatorname{Det} A^T$

$\square$

Exercise 1.8.vii. Let A be an $n \times n$ matrix of the form

$$A = \begin{pmatrix} B & * \\ 0 & C \end{pmatrix},$$

where B is a $k \times k$ matrix and C the $\ell \times \ell$ matrix and the bottom $\ell \times k$ block is zero. Show that

$$\det(A) = \det(B)\det(C).$$

Hint: Show that in equation (1.8.11) every nonzero term is of the form

$$(-1)^{\sigma\tau} b_{1,\sigma(1)} \cdots b_{k,\sigma(k)} c_{1,\tau(1)} \cdots c_{\ell,\tau(\ell)}$$

where $\sigma \in S_k$ and $\tau \in S_\ell$.

Exercise 1.8.viii. Let V and W be vector spaces and let $A: V \to W$ be a linear map. Show that if $Av = w$ then for $\omega \in \Lambda^p(W^\star)$,

$$A^\star \iota_w \omega = \iota_v A^\star \omega \, .$$

Hint: By equation (1.7.14) and Proposition 1.8.4 it suffices to prove this for $\omega \in \Lambda^1(W^\star)$, i.e., for $\omega \in W^\star$.

1.9. Orientation.

Exercise 1.9.i. Prove Corollary 1.9.8.

Exercise 1.9.ii. Show that the argument in the proof of Theorem 1.9.9 can be modified to prove that if V and W are oriented then these orientations induce a natural orientation on V/W.

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Exercise 1.9.iii. Similarly show that if W and V/W are oriented these orientations induce

Exercise 1.9.iii. Similarly show that if W and V/W are oriented these orientations induce a natural orientation on V.

Exercise 1.9.iv. Let V be an n-dimensional vector space and $W \subset V$ a k-dimensional subspace. Let $U = V/W$ and let $\iota\colon W \to V$ and $\pi\colon V \to U$ be the inclusion and projection

Exercise 1.9.iv. Let V be an n-dimensional vector space and $W \subset V$ a k-dimensional subspace. Let $U = V/W$ and let $\iota\colon W \to V$ and $\pi\colon V \to U$ be the inclusion and projection maps. Suppose V and U are oriented. Let μ be in $\Lambda^{n-k}(U^\star)_+$ and let ω be in $\Lambda^n(V^\star)_+$. Show that there exists a ν in $\Lambda^k(V^\star)$ such that $\pi^\star\mu \wedge \nu = \omega$. Moreover show that $\iota^\star\nu$ is *intrinsically* defined (i.e., does not depend on how we choose ν) and sits in the positive part $\Lambda^k(W^\star)_+$ of $\Lambda^k(W^\star)$.

Exercise 1.9.v. Let $e_1, \ldots, e_n$ be the standard basis vectors of $\mathbf{R}^n$. The ***standard*** orientation of $\mathbf{R}^n$ is, by definition, the orientation associated with this basis. Show that if W is the subspace of $\mathbf{R}^n$ defined by the equation $x_1 = 0$, and $v = e_1 \notin W$ then the natural orientation of W associated with v and the standard orientation of $\mathbf{R}^n$ coincide with the orientation given by the basis vectors $e_2, \ldots, e_n$ of W.

Exercise 1.9.vi. Let V be an oriented n-dimensional vector space and W an $n-1$-dimensional subspace. Show that if v and v' are in $V \smallsetminus W$ then $v' = \lambda v + w$, where w is in W and $\lambda \in \mathbf{R} \smallsetminus \{0\}$. Show that v and v' give rise to the same orientation of W if and only if λ is positive.

Exercise 1.9.vii. Prove Proposition 1.9.14.

Exercise 1.9.viii. A key step in the proof of Theorem 1.9.9 was the assertion that the matrix A expressing the vectors, e_i as linear combinations of the vectors f_J, had to have the form (1.9.11). Why is this the case?

Exercise 1.9.viii. A key step in the proof of Theorem 1.9.9 was the assertion that the matrix A expressing the vectors, e_i as linear combinations of the vectors f_J, had to have the form (1.9.11). Why is this the case?

Exercise 1.9.ix.

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(1) Let V be a vector space, W a subspace of V and $A\colon V \to V$ a bijective linear map which maps W onto W. Show that one gets from A a bijective linear map

$$B\colon V/W \to V/W$$

with the property

$$\pi A = B\pi\,,$$

where $\pi\colon V \to V/W$ is the quotient map.

(2) Assume that V, W and V/W are compatibly oriented. Show that if A is orientation-preserving and its restriction to W is orientation preserving then B is orientation preserving.

serving.

Exercise 1.9.x. Let V be a oriented n-dimensional vector space, W an $(n-1)$-dimensional subspace of V and $i\colon W \to V$ the inclusion map. Given $\omega \in \Lambda^n(V^\star)_+$ and $v \in V \setminus W$ show that for the orientation of W described in Exercise 1.9.v, $i^\star(\iota_v\omega) \in \Lambda^{n-1}(W^\star)_+$.

Exercise 1.9.xi. Let V be an n-dimensional vector space, $B\colon V \times V \to \mathbf{R}$ an inner product and $e_1, \dots, e_n$ a basis of V which is positively oriented and orthonormal. Show that the ***volume element***

$$\mathrm{vol} := e_1^\star \wedge \cdots \wedge e_n^\star \in \Lambda^n(V^\star)$$

is intrinsically defined, independent of the choice of this basis.

Hint: Equations (1.2.20) and (1.8.10).

Exercise 1.9.xii.

(1) Let V be an oriented n-dimensional vector space and B an inner product on V. Fix an oriented orthonormal basis, $e_1, \dots, e_n$, of V and let $A \colon V \to V$ be a linear map. Show that if

$$Ae_i = v_i = \sum_{j=1}^{n} a_{j,i} e_j$$

and $b_{i,j} = B(v_i, v_j)$, the matrices $M_A = [a_{i,j}]$ and $M_B = (b_{i,j})$ are related by: $M_B = M_A^{\intercal} M_A$.

(2) Show that if vol is the volume form $e_1^\star \wedge \cdots \wedge e_n^\star$, and A is orientation preserving

$$A^\star \operatorname{vol} = \det(M_B)^{\frac{1}{2}} \operatorname{vol} .$$

(3) ByTheorem 1.5.13 one has a bijective map

$$\Lambda^n(V^\star) \cong \mathcal{A}^n(V) .$$

Show that the element, Ω, of $A^n(V)$ corresponding to the form vol has the property

$$|\Omega(v_1, \dots, v_n)|^2 = \det((b_{i,j}))$$

where $v_1, \dots, v_n$ are any n-tuple of vectors in V and $b_{i,j} = B(v_i, v_j)$.

Notas:

2) Si $\dim(V) = 1$, entonces $V^* \setminus \{l_i\} = \{l, -l\}$, donde l_i es la t. lineal trivial y $l : V \to \mathbb{R}$ es un isomorfismo (de esp. vect.) de $\mathbb{R}$ y V. En este caso $l_1 \otimes l_2 = l_2 \otimes l_1$, $\forall\, l_1, l_2 \in V^*$ (pues l y $-l$ son l.d).

3) Lo probaremos por inducción sobre m. Si $m=1$ el resultado es inmediato, pues con $1 \leq k \leq 1$, $k-1=0$, así:

$$\binom{1}{0} + \binom{0}{0} = 1 + 1$$
$$= 2$$
$$= \binom{2}{1}$$

Suponga el resultado válido para $n=m$. Probaremos que se cumple para $m+1$. En efecto, por hip. se tiene que:

$$\binom{m}{k-1} + \binom{m-1}{k-1} + \dots + \binom{k-1}{k-1} = \binom{m+1}{k}$$

Veamos que si $1 \leq k \leq m$:

$$\binom{m+1}{k-1} + \binom{m}{k-1} + \dots + \binom{k-1}{k-1} = \binom{m+1}{k-1} + \binom{m+1}{k}$$
$$= \frac{(m+1)!}{(k-1)!\,(m+1-k+1)!} + \frac{(m+1)!}{k!\,(m+1-k)!}$$
$$= \frac{(m+1)!\,k + (m+1)!\,(m-k+2)}{k!\,(m+2-k)!}$$
$$= \frac{(m+2)!}{k!\,(m+2-k)!}$$
$$= \binom{m+2}{k}$$

Si $k = m+2$, el resultado es inmediato, pues:

$$\binom{m+1}{k-1} + \binom{m}{k-1} + \dots + \binom{k-1}{k-1} = \binom{m+1}{m+1}$$
$$= 1$$
$$= \binom{m+2}{m+2}$$
$$= \binom{m+2}{k}$$

$\square$

4) $\sum_{l=0}^{k} \binom{k}{l} w_1^{l} \wedge w_2^{k+1-l} = w_1^k \wedge w_2$. En efecto, observemos que

$$\sum_{l=0}^{k} \binom{k}{l} w_1^{l} \wedge w_2^{k+1-l} = w_2^{k+1} + k\, w_1 \wedge w_2^{k} + \dots + k\, w_1^{k-1} \wedge w_2^{2} + w_1^{k} \wedge w_2$$

Veamos que todos los demás términos son cero. Como $w_2^2 = 0$, ent. $\exists\, T \in \mathcal{I}^2(V)$ m $T \otimes T \in \mathcal{I}^4(V)$.

Ent. $T^u \in \mathcal{I}^{2u}(V)$, $\forall u \in \mathbb{N} \setminus \{1\}$, así si $L \in \mathcal{L}^2(V)$ es tal que $w_1 = \pi(L)$, ent.

$$w_1^{l} \wedge w_2^{k+1-l} = \pi\left(L^{l} \otimes T^{k+1-l}\right)$$

$$= 0 \quad \text{si} \quad k+1-l > 1 \Rightarrow k > l$$

Luego todos los términos son cero (salvo posiblemente el último). Así $\sum_{l=0}^{k} \dots = w_1^k \wedge w_2$.

Proposición.

Sean $w_1 \in \Lambda^r(V^*)$ y $w_2 \in \Lambda^s(V^*)$. Si $w_1 = 0$ ent $w_1 \wedge w_2 = 0$.

Dem:

Como $w_1 \in \Lambda^r(V^*)$ y $w_2 \in \Lambda^s(V^*)$ ent. $\exists\, T_1 \in \mathcal{L}^r(V)$ y $T_2 \in \mathcal{L}^s(V)$ m $w_1 = \pi(T_1)$ y $w_2 = \pi(T_2)$.

Siendo que $w_1 = 0 \Rightarrow \pi(T_1) = 0 \Rightarrow T_1 \in \mathcal{I}^r(V) \Rightarrow T_1 \otimes T_2 \in \mathcal{I}^{r+s}(V)$, así:

$$w_1 \wedge w_2 = \pi(T_1 \otimes T_2) = 0$$

$\square$

5) ¿Qué es una columna? Si $A: V \to V$ es lineal, ent.

$$A(e_i) = \sum_{j=1}^{n} a_{ij}\, e_j$$

donde $[a_{ij}]_{1 \le i,j \le n}$ es la rep. matricial de A. Sus columnas son la rep. ant. i.e la j-ésima columna será:

$$\begin{bmatrix} a_{j1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix}$$