Lista 2.

*1. Let $S^2 = \{(x, y, z) \in R^3; x^2 + y^2 + z^2 = 1\}$ be the unit sphere and let $A: S^2 \longrightarrow S^2$ be the (*antipodal*) map $A(x, y, z) = (-x, -y, -z)$. Prove that A is a diffeomorphism.

Dem:

Como:

$$A \circ A = id_{S^2}$$

entonces A es biyección, y $A^{-1} = A$. Así, para probar que A es difeomorfismo, basta probar que A es diferenciable. En efecto, como las derivadas parciales de A son todas funciones continuas (son -1 ó 0), entonces A es diferenciable, luego, difeomorfismo.

q.e.d.

2. Let $S \subset R^3$ be a regular surface and $\pi: S \longrightarrow R^2$ be the map which takes each $p \in S$ into its orthogonal projection over $R^2 = \{(x, y, z) \in R^3; z = 0\}$. Is π differentiable?

Sol.

Sea $p \in S$. Probaremos que π es diferenciable en p. Como S es sup. regular, $\exists\, \underline{X}: U \subseteq \mathbb{R}^2$ abierto a $S \cap V$, V vecindad de p / $\underline{X}$ cumple 1) - 3).

Entonces, si $q = \underline{X}^{-1}(p)$, la función $\pi \circ \underline{X}: U \to \mathbb{R}^3$ es diferenciable en q, pues:

$$\pi \circ \underline{X}(u, v) = (x(u,v), y(u,v), 0)$$

Cumple que x y y tienen derivadas parciales continuas. Luego $\pi \circ \underline{X}$ es diferenciable y así, π es diferenciable.

q.e.d.

3. Show that the paraboloid $z = x^2 + y^2$ is diffeomorphic to a plane.

4 Construct a diffeomorphism between the ellipsoid

Dem:

Probaremos que el paraboloide $P = \{(x,y,z) \in \mathbb{R}^3 \mid z = x^2+y^2\}$ es difeomorfo al plano $P_{xy} = \mathbb{R}^2 \times \{0\}$. En efecto: sea $f: P \to P_{xy}$ dada como:

$$\forall (x,y,z) \in P, \quad f(x,y,z) = (x,y,0).$$

f es biyectiva, pues si $f(x,y,z) = f(x',y',z') \Rightarrow x' = x$ y $y' = y \Rightarrow x^2+y^2 = x'^2+y'^2 \Rightarrow z = z'$, luego $(x,y,z) = (x',y',z')$. Además $\forall (x,y,0) \in P_{xy}$, $\exists (x,y,x^2+y^2) \in P$ m

$$f(x,y,x^2+y^2) = (x,y,0).$$

Veamos que es diferenciable. Como $f = \pi$ de la parte anterior, f es diferenciable, y su inversa:

$$f^{-1}: P_{xy} \to P$$
$$(x,y,0) \mapsto (x,y,x^2+y^2)$$

Cumple que sus funciones componentes tienen derivadas parciales continuas (Tomando dos parametrizaciones adecuadas de P y P_{xy}). Entonces f es difeomorfismo.

q.e.d.

4. Construct a diffeomorphism between the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

and the sphere $x^2 + y^2 + z^2 = 1$.

Sol.

Sea $f: S^2 \to E^2$ dado como:

$$\forall (x,y,z) \in S^2, \quad f(x,y,z) = (ax, by, cz)$$

Veamos que $f(S^2) \subseteq E^2$. En efecto: Sean $(x,y,z) \in S^2$, entonces:

$$\frac{(ax)^2}{a^2} + \frac{(by)^2}{b^2} + \frac{(cz)^2}{c^2} = x^2+y^2+z^2$$
$$= 1$$

Luego $f(S^2) \subseteq E^2$. f es biyectiva, en efecto, $\exists f^{-1}: E^2 \to S^2$ m

$$\forall (x,y,z) \in E^2 \ \text{m} \ f^{-1}(x,y,z) = \left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right)$$

Luego $f^{-1}(E^2) \subseteq S^2$. Además:

$$f \circ f^{-1} = id_{E^2}$$
$$f^{-1} \circ f = id_{S^2}$$

Así, f es biyectiva con inversa f^{-1}. Además f es difeomorfismo. Como $F: \mathbb{R}^3 \to \mathbb{R}^3$ dada como:

$$\forall (x,y,z) \in \mathbb{R}^3, \ F(x,y,z) = (ax, by, cz)$$

es diferenciable y F^{-1} también lo es, así F es difeomorfismo. Luego $f = F|_{S^2}$ es difeomorfismo.

q.e.a.

***5.** Let $S \subset R^3$ be a regular surface, and let $d: S \longrightarrow R$ be given by $d(p) = |p - p_0|$, where $p \in S$, $p_0 \in R^3$, $p_0 \notin S$; that is, d is the distance from p to a fixed point p_0 not in S. Prove that d is differentiable.

Sol.

Sea $p \in S$. Como S es sup. regular, $\exists \ \mathbf{X}: U \subseteq \mathbb{R}^2 \to V \cap S$, U abierto m $\mathbf{X}$ cumple 1)–3), y $p \in V \cap S$. Sea $q \in U$ m $\mathbf{X}(q) = p$. Probaremos que:

$$d \circ \pi : U \to \mathbb{R}$$

es diferenciable. En efecto: si $\mathbf{X}(u,v) = (x(u,v), y(u,v), z(u,v))$, entonces:

$$d \circ \pi (u,v) = \sqrt{(x(u,v) - x_0)^2 + (y(u,v) - y_0)^2 + (z(u,v) - z_0)^2}$$

Entonces:

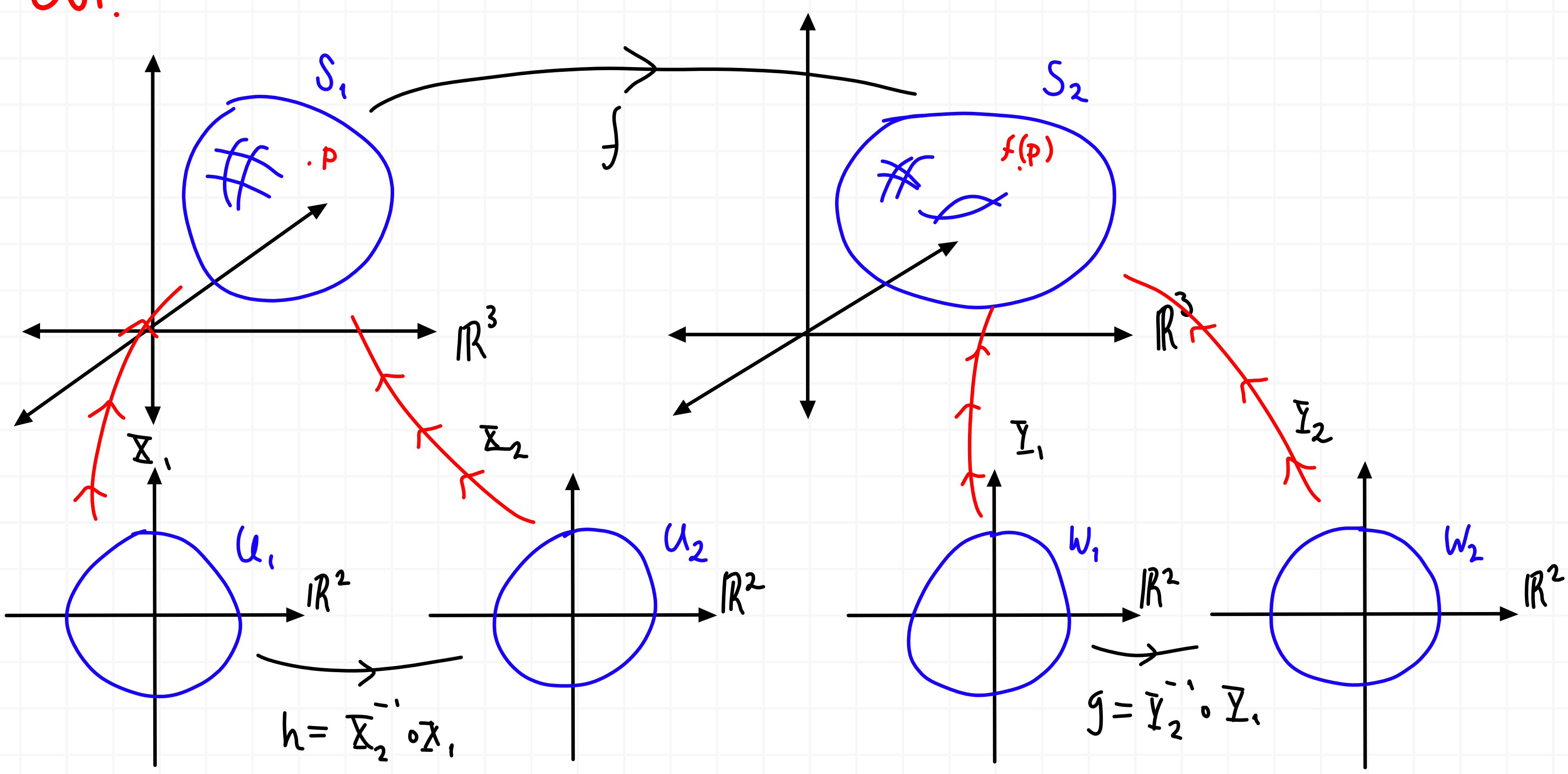
$$d(d \circ \pi)_q = \left(\frac{2 \cdot \left[(x-x_0)\frac{\partial x}{\partial u} + (y-y_0)\cdot\frac{\partial y}{\partial u} + (z-z_0)\frac{\partial z}{\partial u}\right]}{d \circ \pi} \quad \frac{2\left[(x-x_0)\cdot\frac{\partial x}{\partial v} + (y-y_0)\frac{\partial y}{\partial v} + (z-z_0)\cdot\frac{\partial z}{\partial v}\right]}{d \circ \pi} \right)\Bigg|_q$$

Como $p_0 \notin S$, entonces $d(p) \neq 0, \forall \ p \in S$. Luego $d \circ \pi (q) \neq 0$, así $d(d \circ \pi)_q$ está bien definida, así $d \circ \pi$ es diferenciable $\Rightarrow d$ es diferenciable.

q.e.a.

6. Prove that the definition of a differentiable map between surfaces does not depend on the parametrizations chosen.

Sol.



Sea $p \in S_1$, y $\bar{X}_1, \bar{X}_2$ parametrizaciones de S_1 $\ni$ $p \in \bar{X}_1(U_1), \bar{X}_2(U_2)$. Además $\bar{Y}_1, \bar{Y}_2$ son dos parametrizaciones de S_2 $\ni$ $f(p) \in \bar{Y}_1(W_1), \bar{Y}_2(W_2)$, donde $f: S_1 \to S_2$ es una función.

Veamos que:

$$\bar{X}_1 = \bar{X}_2 \circ h \quad y \quad \bar{Y}_1 = \bar{Y}_2 \circ h$$

Por tanto:

$$\begin{aligned} \bar{Y}_1^{-1} \circ f \circ \bar{X}_1 &= (\bar{Y}_2 \circ h)^{-1} \circ f \circ (\bar{X}_2 \circ h) \\ &= h^{-1} \circ \bar{Y}_2^{-1} \circ f \circ \bar{X}_2 \circ h \; \ldots (1) \end{aligned}$$

$$\bar{Y}_2^{-1} \circ f \circ \bar{X}_2 = h \circ \bar{Y}_1^{-1} \circ f \circ \bar{X}_1 \circ h^{-1} \ldots (2)$$

Como h es difeomorfismo (local), si $\bar{Y}_1^{-1} \circ f \circ \bar{X}_1$ es diferenciable, entonces $\bar{Y}_2^{-1} \circ f \circ \bar{X}_2$ también lo es (por (2)), y viceversa (por (1)). Así la diferenciabilidad de f no depende de las parametrizaciones elegidas.

q.e.d.

7. Prove that the relation "S_1 is diffeomorphic to S_2" is an equivalence relation in the set of regular surfaces.

Dem:

Sea $\mathcal{S} = \{S \in P(\mathbb{R}^3) \mid S$ es sup. regular en $\mathbb{R}^3\}$, y defina la relación:

$\forall\, S_1, S_2 \in \mathcal{S},\ S_1 \sim S_2 \iff S_1$ es difeomorfo a S_2.

Veamos qué es de equivalencia. En efecto:

a) Sea $S_1 \in \mathcal{S}$, $S_1 \sim S_1$, pues $\exists\ id : S_1 \to S_1$ biyección, la cual es diferenciable (y por ende, difeomorfismo, pues $id^{-1} = id$). En efecto, sea $p \in S_1$, y sean X, Y dos parametrizaciones de S_1 $\ni$ $p \in X(U), Y(W)$. Entonces como $h = Y^{-1} \circ X$ (mapeo de transición el cuál es difeomorfismo), tenemos:

$$\begin{aligned} Y^{-1} \circ id \circ X &= Y^{-1} \circ X \\ &= h \end{aligned}$$

Luego $Y^{-1} \circ id \circ X$ es difeomorfismo, en particular diferenciable.

b) Sean $S_1, S_2 \in \mathcal{S}$ $\ni$ $S_1 \sim S_2$, entonces $\exists\ f : S_1 \to S_2$ difeomorfismo, luego $f^{-1} : S_2 \to S_1$ es difeomorfismo, así $S_2 \sim S_1$.

c) Sean $S_1, S_2, S_3 \in \mathcal{S}$ $\ni$ $S_1 \sim S_2$ y $S_2 \sim S_3$, entonces $\exists\ f : S_1 \to S_2$ y $g : S_2 \to S_3$ difeomorfismos. Considere $g \circ f : S_1 \to S_3$, veamos que es difeomorfismo.

Sea $p \in S_1$, $\exists\ X : U_1 \to S_1$, $Y : U_2 \to S_2$ y $Z : U_3 \to S_3$ $\ni$ $p \in X(U_1)$, $f(p) \in Y(U_2)$, $g \circ f(p) \in Z(U_3)$. Veamos que:

$$Z^{-1} \circ (g \circ f) \circ X = (Z^{-1} \circ g \circ Y) \circ (Y^{-1} \circ f \circ X)$$

Como f y g son diferenciables, $Z^{-1} \circ g \circ Y$ y $Y^{-1} \circ f \circ X$ lo son, así $g \circ f$ es diferenciable. De forma similar $f^{-1} \circ g^{-1}$ es diferenciable. Así $g \circ f$ es difeomorfismo, luego $S_1 \sim S_3$.

q.e.d.

***8.** Let $S^2 = \{(x, y, z) \in R^3;\ x^2 + y^2 + z^2 = 1\}$ and $H = \{(x, y, z) \in R^3;\ x^2 + y^2 - z^2 = 1\}$. Denote by $N = (0, 0, 1)$ and $S = (0, 0, -1)$ the north and south poles of S^2, respectively, and let $F: S^2 - \{N\} \cup \{S\} \longrightarrow H$ be defined as follows: For each $p \in S^2 - \{N\} \cup \{S\}$ let the perpendicular from p to the z axis meet $0z$ at q. Consider the half-line l starting at q and containing p. Then $F(p) = l \cap H$ (Fig. 2-20). Prove that F is differentiable.

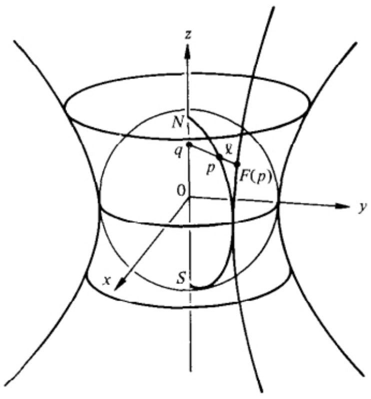


Figure 2-20

Dem:

Primero, veámos cómo está dada F. Sea $p \in S^2 \setminus \{N, S\}$, digamos $p = (x, y, z)$. La recta $l_p : \mathbb{R} \to \mathbb{R}^3$ que pasa por p e intersecta a s está dada como:

$$l_p(t) := (0, 0, z) + (tx, ty, 0)$$
$$= (tx, ty, z)$$

l_p intersecta a H cuando $l_p(t) \in H$, i.e:

$$(tx)^2 + (ty)^2 - z^2 = 1$$
$$\Leftrightarrow t^2x^2 + t^2y^2 - z^2 = 1$$
$$\Leftrightarrow t^2(x^2 + y^2) = 1 + z^2$$
$$\Rightarrow t_f = \frac{\sqrt{1+z^2}}{\sqrt{x^2+y^2}}, \text{ con } t > 0$$

Luego:

$$l_p(t_f) = \left(\left(\frac{1+z^2}{x^2+y^2}\right)^{1/2} x, \left(\frac{1+z^2}{x^2+y^2}\right)^{1/2} y, z \right) \in H$$

Sea entonces $F : S^2 \setminus \{N, S\} \to H$, dada como:

$$(x, y, z) \mapsto \left(\left(\frac{1+z^2}{x^2+y^2}\right)^{1/2} x, \left(\frac{1+z^2}{x^2+y^2}\right)^{1/2} y, z \right)$$

Vemos que F es diferenciable en $S^2 \setminus \{N, S\}$, por tener derivadas parciales continuas.

q.e.d.

9. a. Define the notion of differentiable function on a regular curve. What does one need to prove for the definition to make sense? Do not prove it now. If you have not omitted the proofs in this section, you will be asked to do it in Exercise 15.

b. Show that the map $E: R \longrightarrow S^1 = \{(x, y) \in R^2; x^2 + y^2 = 1\}$ given by

$$E(t) = (\cos t, \sin t), \qquad t \in R,$$

is differentiable (geometrically, E "wraps" R around S^1).

Sol.

De a): Sea C una curva regular y $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^2$, con I intervalo abierto de extremos $a < b$. Sea $f : C \to \mathbb{R}$, entonces f es diferenciable.

De b): ¿?

10. Let C be a plane regular curve which lies in one side of a straight line r of the plane and meets r at the points p, q (Fig. 2-21). What conditions should C satisfy to ensure that the rotation of C about r generates an extended (regular) surface of revolution?

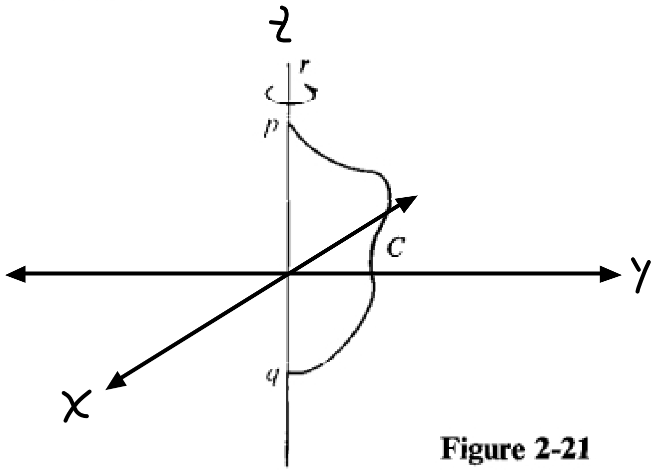


Figure 2-21

Para ver que la sup. de rev. generada sea regular, primero, no debe tener auto intersecciones. Para evitarlas, hagamos que $C \subseteq \bar{Y}^{+} = \{(x,y,z) \in \mathbb{R}^3 \mid y > 0\}$.

Además, C debe ser curva regular (para que sea suave.

$\square$

11. Prove that the rotations of a surface of revolution S about its axis are diffeomorphisms of S.

12. Parametrized surfaces are often useful to describe sets Σ which are regular sur-

Suponga (sin pérdida de generalidad) que el eje de la sup. S es el z, considere la función $F_\theta : \mathbb{R}^3 \to \mathbb{R}^3$ de rotación:

$$\forall \theta \in [0, 2\pi[, \; F_\theta = \begin{pmatrix} \cos\theta & -\operatorname{sen}\theta & 0 \\ \operatorname{sen}\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Entonces, $\forall p \in \mathbb{R}^3$, $p = (x,y,z)$:

$$F_\theta(p) = (x\cos\theta - y\operatorname{sen}\theta, \; x\operatorname{sen}\theta + y\cos\theta, \; z)$$

Claramente F_θ es diferenciable. Luego $F_\theta|_S$ es diferenciable y, como S es sup. de revolución, $F_\theta|_S : S \to S$, $\forall \theta \in [0, 2\pi[$ arbitrario.

q.e.d.

12. Parametrized surfaces are often useful to describe sets Σ which are regular surfaces except for a finite number of points and a finite number of lines. For instance, let C be the trace of a regular parametrized curve $\alpha: (a, b) \longrightarrow R^3$ which does not pass through the origin $O = (0, 0, 0)$. Let Σ be the set generated by the

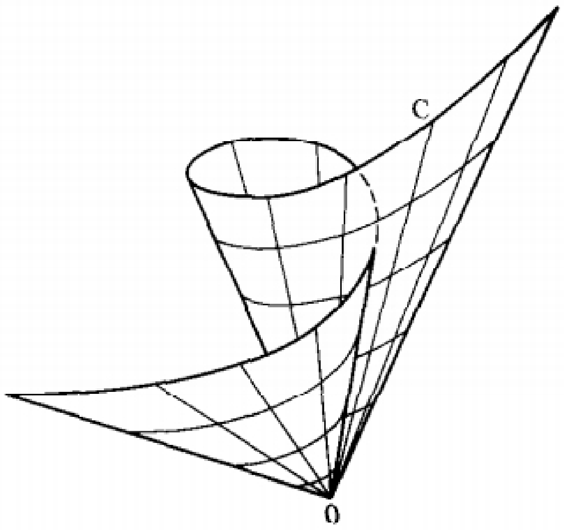


Figure 2-22

displacement of a straight line l passing through a moving point $p \in C$ and the fixed point 0 (a cone with vertex 0; see Fig. 2-22).

a. Find a parametrized surface $\mathbf{x}$ whose trace is Σ.

b. Find the points where $\mathbf{x}$ is not regular.

c. What should be removed from Σ so that the remaining set is a regular surface?

***13.** Show that the definition of differentiability of a function $f\colon V \subset S \to R$ given in the text (Def. 1) is equivalent to the following: f is differentiable in $p \in V$ if it is the restriction to V of a differentiable function defined in an open set of R^3 containing p. (Had we started with this definition of differentiability, we could have defined a surface as a set which is locally diffeomorphic to R^2; see Remark 3.)

14. Let $A \subset S$ be a subset of a regular surface S. Prove that A is itself a regular surface if and only if A is open in S; that is, $A = U \cap S$, where U is an open set in R^3.

15. Let C be a regular curve and let $\alpha : I \subset R \longrightarrow C$, $\beta : J \subset R \longrightarrow C$ be two parametrizations of C in a neighborhood of $p \in \alpha(I) \cap \beta(I) = W$. Let

$$h = \alpha^{-1} \circ \beta : \beta^{-1}(W) \longrightarrow \alpha^{-1}(W)$$

be the change of parameters. Prove that

a. h is a diffeomorphism.

b. The absolute value of the arc length of C in W does not depend on which parametrization is chosen to define it, that is,

$$\left| \int_{t_0}^{t} |\alpha'(t)|\, dt \right| = \left| \int_{\tau_0}^{\tau} |\beta'(\tau)|\, d\tau \right|, \qquad t = h(\tau),\ t \in I,\ \tau \in J.$$

***16.** Let $R^2 = \{(x, y, z) \in R^3; z = -1\}$ be identified with the complex plane $\mathbb{C}$ by setting $(x, y, -1) = x + iy = \zeta \in \mathbb{C}$. Let $P: \mathbb{C} \longrightarrow \mathbb{C}$ be the complex polynomial

$$P(\zeta) = a_0\zeta^n + a_1\zeta^{n-1} + \cdots + a_n, \qquad a_0 \neq 0,\ a_i \in \mathbb{C},\ i = 1, \ldots, n.$$

Denote by π_N the stereographic projection of $S^2 = \{(x, y, z) \in R^3; x^2 + y^2 + z^2 = 1\}$ from the north pole $N = (0, 0, 1)$ onto R^2. Prove that the map $F: S^2 \longrightarrow S^2$ given by

$$F(p) = \pi_N^{-1} \circ P \circ \pi_N(p), \qquad \text{if } p \in S^2 - \{N\},$$
$$F(N) = N$$

is differentiable.