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# CHAPTER 1

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## BASIC NOTIONS

Throughout this document, we work with a fixed algebraically closed field  $k$  (or sometimes denoted  $K$ ), called **ground field**.

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### 1.1 ALGEBRAIC CLOSED FIELDS

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Let  $k$  be an algebraic closed field, then for every  $f \in k[x]$  (where  $k[x]$  denotes the set of all polynomials over the field  $k$ ), we have that if  $f$  is non-constant, then  $f$  has a root on  $k$ .

#### Example 1.1.1

$\mathbb{C}$  is an algebraic closed extension of  $\mathbb{R}$ , and  $\mathbb{Q}$ .

Turns out algebraic closed fields are important due to the following fact:

*There is a one to one correspondence between geometric and algebraic objects using algebraic closed fields.*

We will first deal with simple objects and then we'll scalate to more complex in order to generalize certain notions defined in algebraic geometry.

#### Definition 1.1.1 (Affine Space)

Let  $k$  be a field. We denote by  $\mathbb{A}^n$  the  **$n$ -dimensional affine space over the field  $k$** , that is:

$$\mathbb{A}^n = \left\{ (\alpha_1, \dots, \alpha_n) \mid \alpha_i \in k, \quad \forall i \in [1, n] \right\}$$

All the geometric concepts we will be dealing with are within this space, this is due to the following fact:

#### Definition 1.1.2 (Closed Subsets of Affine Space)

Let  $X \subseteq \mathbb{A}^n$  be a subset, then  $X$  is **closed** if there exists  $f_1, \dots, f_m \in k[x_1, \dots, x_m]$  polynomials over the field  $k$  such that:

$$f_i(u) = 0, \quad \forall i \in [1, m] \iff u \in X$$

From now on, we will write  $F(T)$  to denote a polynomial in  $n$ -variables, allowing  $T$  to stand for the set of variables  $T_1, \dots, T_n$ .

#### Observation 1.1.1 (Equations of a Set)

If a closed subset  $X$  consists of all common zeros of polynomials  $F_1(T), \dots, F_m(T)$ , then we refer to:

$$F_1(T) = \dots = F_n(T) = 0$$

as the **equations of the set**  $X$ .

One really useful definition and fact that are proved in the notes of Algebra Moderna III are the following:

**Definition 1.1.3 (Noetherian Ring)**

Let  $R$  be a ring. We say that  $R$  is **noetherian** if for all  $I$  ideal of  $R$  there exists  $a_1, \dots, a_n \in R$  such that:

$$I = (a_1, \dots, a_n),$$

where  $(a_1, \dots, a_n)$  denotes the ideal generated by the set  $\{a_1, \dots, a_n\}$ .

**Observation 1.1.2 (Ideal Generated by a Set  $S$ )**

Let  $R$  be a commutative ring with identity, then the ideal generated by  $S \subseteq R$  is:

$$(S) = \left\{ \sum_{i=1}^n r_i s_i \mid r_i \in R, s_i \in S, \forall i \in [1, n]; n \in \mathbb{N} \right\}$$

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**Theorem 1.1.1 (Hilbert Basis Theorem)**

Let  $R$  be a noetherian ring, then  $R[x]$  is noetherian.

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A useful fact about the Hilbert basis theorem is that it generalizes neatly to an arbitrary number of indeterminates:

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**Corollary 1.1.1**

Let  $R$  be a noetherian ring, then  $R[T]$  is noetherian.

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This useful fact is fundamental in the proof of the following result:

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**Proposition 1.1.1**

Let  $X$  be a set defined by an infinite system of equations  $\{F_\alpha(T)\}_{\alpha \in I}$  with  $I \neq \emptyset$ . Then  $X$  is closed.

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**Proof:**

Let  $\mathfrak{U}$  be the ideal generated by the system of equations  $\{F_\alpha(T)\}_{\alpha \in I}$ . Since  $k[T]$  is noetherian due to the fact that  $k$  is a field (in particular, every field is noetherian), then there exists  $G_1, \dots, G_m \in k[T]$  such that:

$$(\{F_\alpha(T)\}_{\alpha \in I}) = (G_1(T), \dots, G_m(T))$$

We claim that  $u \in X$  iff  $G_i(u) = 0$ , for all  $i \in [1, m]$ . If  $u \in X$ , then  $F_\alpha(u) = 0$  for all  $\alpha \in I$ , so in particular by Observation (1.1.2):

$$F(u) = 0, \quad \forall F \in (\{F_\alpha(T)\}_{\alpha \in I})$$

which is the same as:

$$F(u) = 0, \quad \forall F \in (G_1(T), \dots, G_m(T))$$

so  $G_i(u) = 0$ , for all  $i \in [1, m]$ . ■

It follows from this proposition that the arbitrary intersection of closed subsets of  $\mathbb{A}^n$  is closed. Also, it happens that  $\emptyset$  and  $\mathbb{A}^n$  are closed (using the polynomials  $F = 1$  and  $F = 0$ ).

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**Proposition 1.1.2**

If  $X_1$  and  $X_2$  are closed subsets of  $\mathbb{A}^n$ , then  $X_1 \cup X_2$  is also a closed subset of  $\mathbb{A}^n$ .

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**Proof:**

Let  $F_1, \dots, F_n$  and  $G_1, \dots, G_m$  polynomials over the ring  $k[T]$  such that:

$$F_i(u) = 0, \quad \forall u \in X_1 \text{ and } G_j(v) = 0, \quad \forall v \in X_2$$

for all  $(i, j) \in [1, n] \times [1, m]$ . We define  $H_{i,j} \in k[T]$  as:

$$H_{i,j} = F_i G_j, \quad \forall (i, j) \in [1, n] \times [1, m]$$

Then:

$$H_{i,j}(w) = 0, \quad \forall w \in X_1 \cup X_2$$

for all  $(i, j) \in [1, n] \times [1, m]$ . It follows that  $X_1 \cup X_2$  is closed. ■

By all this it follows that family of all the complements of closed subsets of  $\mathbb{A}^n$  are a topology over  $\mathbb{A}^n$ .

**Example 1.1.2 (Closed Subsets of  $\mathbb{A}^1$ )**

Let  $X \subseteq \mathbb{A}^1$  be a closed subset of  $\mathbb{A}^1$ , then there exists  $f_1, \dots, f_m \in k[x]$  (polynomials in one variable) such that:

$$f_i(u) = 0, \quad \forall i \in [1, m] \iff u \in X$$

Let  $d \in [x]$  the highest degree polynomial with leading coefficient one such that:

$$f_i = u_i d, \quad \forall i \in [1, m]$$

where  $u_i \in k[x]$ . If  $u_1 = 1$ , then  $X = \emptyset$  if one of the polynomials is non zero and  $X = \mathbb{A}^1$  if all of them are equal to zero, otherwise it follows that  $X$  is the family of all the roots of  $d$ , which is finite.

If  $X = \{\alpha_1, \dots, \alpha_n\}$ , then  $X$  is closed because  $X$  is the family of zeros of the polynomial:

$$f(x) = (x - \alpha_1) \cdots (x - \alpha_n)$$

**Example 1.1.3 (Closed Subsets of  $\mathbb{A}^2$ )****Definition 1.1.4 (Hypersurface)**

A set  $X \subseteq \mathbb{A}^n$  defined by one equation  $F(T_1, \dots, T_n) = 0$  is called a **hypersurface**.

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## 1.2 REGULAR FUNCTIONS ON CLOSED SUBSETS

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**Definition 1.2.1 (Regular Functions)**

Let  $X$  be a closed subset of  $\mathbb{A}^n$  over the ground field  $k$ . A function  $f : X \rightarrow k$  is called **regular** if there exists a polynomial  $F \in k[T]$  such that:

$$f(u) = F(u), \quad \forall u \in X$$

In general, there's not a unique polynomial that defines a regular function.

**Observation 1.2.1**

We can add to  $F$  any polynomial entering in the system of equations that defines the set  $X$ , this doesn't always result in an alteration of  $F$ .

For the next proposition, we need to remember the definition of an algebra:

**Definition 1.2.2 ( $k$ -Algebra)**

A  $k$ -algebra is a cuatruplet  $(A, k, +, \cdot)$  such that:

- (1)  $(A, +, \cdot)$  is a ring.
- (2)  $(A, +)$  is a vector space over the field  $k$ .
- (3) The ring multiplication is  $k$ -bilinear, that is:

$$\alpha(ab) = (\alpha a)b = a(\alpha b), \quad \forall \alpha \in k, a, b \in A$$

We will usually denote a  $k$ -algebra simply by  $A$ .

**Proposition 1.2.1 (Algebra of Regular Functions)**

Let  $X$  be a closed subset of  $\mathbb{A}^n$  over the ground field  $k$ . The set of all regular functions over  $X$  is a  $k$ -algebra with the usual addition and multiplication of functions.

The ring obtained is denoted by  $k[X]$  and is called the **ring of regular functions over  $X$** .

**Proof:**

Let  $k[X]$  be the set of all regular functions over  $X$ . We will prove that  $k[X]$  is a  $k$ -algebra by checking the three conditions of the definition:

- (1)  **$(k[X], +, \cdot)$  is a ring.** This is obvious since the sum and product of polynomials is a polynomial.
- (2)  **$(k[X], +)$  is a vector space over the field  $k$ .** This is also obvious since the sum of polynomials and the multiplication of a polynomial by a scalar is a polynomial.
- (3) **The ring multiplication is  $k$ -bilinear.** This is also immediate from the fact that the multiplication of elements in the field  $k$  is commutative.

It follows that  $k[X]$  is a  $k$ -algebra. ■

**Definition 1.2.3 (Coordinate Ring)**

Let  $X$  be a closed subset of  $\mathbb{A}^n$  over the ground field  $k$ . The  $k$ -algebra  $k[X]$  is called the **coordinate ring of  $X$** .

We will only deal with the ring structure of the algebra of regular functions, leaving the vector space structure aside (for the moment). We write  $k[T]$  for the polynomial ring in  $T_1, \dots, T_n$ -variables over the field  $k$ .

**Observation 1.2.2**

Let  $ap : k[T] \rightarrow k[X]$  be the function defined as:

$$ap(F) = f, \quad \forall F \in k[T]$$

where  $f : X \rightarrow k$  is defined as  $f(u) = F(u)$  for all  $u \in X$ . It's immediate that  $ap$  is an epimorphism of rings.

Due to the latter observation and using the first isomorphism theorem, it follows that:

$$k[X] \cong k[T] / \ker(ap)$$

where:

$$\ker(ap) = \left\{ F \in k[T] \mid F(u) = 0, \quad \forall u \in X \right\}$$

is an ideal of  $k[T]$ .

**Definition 1.2.4 (Ideal of a Closed Set)**

Let  $X$  be a closed subset of  $\mathbb{A}^n$  over the ground field  $k$ . The ideal  $\ker(ap)$  of  $k[T]$  is called the **ideal of the closed set  $X$**  and is denoted by  $\mathfrak{I}_X$ .

From the above, it follows that:

$$k[X] \cong k[T] / \mathfrak{I}_X$$

Thus, the ring is fully determined by the ideal  $\mathfrak{I}_X$ . We will now focus on studying this ideal in order to understand better the structure of the ring of regular functions over a closed set.

**Example 1.2.1**

If  $X \subseteq \mathbb{A}^n$  is such that  $X = \{(x_1, \dots, x_n)\}$ , then:

$$k[X] \cong k[T] / \mathfrak{I}_X$$

where  $\mathfrak{I}_X$  is the ideal of all polynomials vanishing at the point  $x = (x_1, \dots, x_n)$ , which is given by:

$$\mathfrak{I}_X = \left\{ F \mid \text{where } F(T) = \alpha(T - x); \alpha \in k \right\}$$

So,  $k[T] / \mathfrak{I}_X \cong k$ . It follows that  $k[X] \cong k$ .

**Example 1.2.2**

If  $X = \mathbb{A}^n$ , then  $\mathfrak{I}_X = \{0\}$ , where 0 is the 0 polynomial, so  $k[X] \cong k[T]$ .

**Example 1.2.3**

Let  $X \subseteq \mathbb{A}^2$  be given by the equation:

$$X = \left\{ (x_1, x_2) \in \mathbb{A}^2 \mid x_1 x_2 - 1 = 0 \right\}$$

Then, the ideal  $\mathfrak{I}_X$  is given by:

$$\mathfrak{I}_X = \left\{ F \in k[T_1, T_2] \mid F(T_1, T_2) = (T_1 T_2 - 1)G(T_1, T_2); G \in k[T_1, T_2] \right\}$$

It is not so difficult to show that there is a ring isomorphism between  $k[X]$  and  $k[T_1, T_1^{-1}]$ , the ring of Laurent polynomials in one variable over the field  $k$ .

**Observation 1.2.3**

Let  $R$  be a ring and  $I$  an ideal of  $R$ . We know that there is a correspondence between the ideals of  $R/I$  and the ideals of  $R$  that contain  $I$ . This correspondence is given by:

$$J \mapsto J/I, \text{ where } J \text{ is an ideal of } R \text{ such that } I \subseteq J$$

Since  $k[T]$  is Noetherian, it follows that  $k[X]$  is also Noetherian, no matter the closed subset  $X$  of  $\mathbb{A}^n$ .

And furthermore, we have that  $k[X]$  satisfies the following analogue of the Nullstellensatz Theorem. Before that, we shall give the following definition:

**Definition 1.2.5 (Radical Ideal)**

Let  $R$  be a ring and  $I$  an ideal of  $R$ . The radical of the ideal  $I$ , denoted by  $\text{rad}(I)$  or  $\sqrt{I}$ , is given by:

$$\text{rad}(I) = \left\{ a \in R \mid a^n \in I, \text{ for some } n \in \mathbb{N} \right\}$$

**Proposition 1.2.2**

Let  $X$  be a closed subset of  $\mathbb{A}^n$  over the ground field  $k$ . Then, if  $f, g_1, \dots, g_m \in k[X]$  are functions such that  $f$  vanishes at all points where  $g_1, \dots, g_m$  vanish, then there exists  $r \in \mathbb{N}$  such that:

$$f^r \in (g_1, \dots, g_m)$$

where  $(g_1, \dots, g_m)$  denotes the ideal of  $k[X]$  generated by the set  $\{g_1, \dots, g_m\}$ .

**Proof:**

Let  $F$  and  $G_1, \dots, G_m \in k[T]$  be polynomials such that:

$$i(F) = f \text{ and } i(G_j) = g_j, \quad \forall j \in [1, m]$$

where  $i : k[T]/\mathfrak{I}_X \rightarrow k[X]$  is given by:

$$i(H + \mathfrak{I}_X) = h, \quad \forall H \in k[T]$$

being  $h : X \rightarrow k$  the function defined as  $h(u) = H(u)$  for all  $u \in X$ .

Let  $F_1, \dots, F_l$  be the equations of  $X$ , that is:

$$F_i(u) = 0, \quad \forall u \in X, \quad \forall i \in [1, l]$$

If  $u \in X$  is such that  $G_j(u) = 0$  for all  $j \in [1, m]$ , then  $F_i(u) = 0$  for all  $i \in [1, l]$  and  $F(u) = 0$ . It follows from the Nullstellensatz Theorem that there exists  $r \in \mathbb{N}$  such that:

$$F^r \in (G_1, \dots, G_m, F_1, \dots, F_l)$$

Therefore, since  $F_1, \dots, F_l \in \mathfrak{I}_X$ , it follows that  $f^r \in (g_1, \dots, g_m)$ . ■

**Question 1.2.1**

What's the relation between the ideal  $\mathfrak{I}_X$  of a closed set  $X$  and a system  $F_1 = \dots = F_m = 0$  of defining equations of  $X$ ?

**Solution:**

Clearly, by definition we have that  $F_i \in \mathfrak{I}_X$ , for all  $i \in [1, m]$ . Therefore, the ideal generated by the set  $\{F_1, \dots, F_m\}$  is contained in  $\mathfrak{I}_X$ .

However, it is not always true that  $\mathfrak{I}_X$  is equal to the ideal generated by the set  $\{F_1, \dots, F_m\}$ .  $\square$

**Example 1.2.4**

Let  $X \subseteq \mathbb{A}^2$  be given by the equation:

$$T^2 = 0$$

Then,  $X = \{0\}$ , so  $\mathfrak{I}_X$  consists of all the polynomials vanishing at 0, that is, all polynomials without constant term. However, the ideal generated by the polynomial  $T^2$  doesn't contain the polynomial  $T$ , so they are not equal.

However, we can always define the same closed set  $X$  with a system of equations  $G_1 = \dots = G_l = 0$  whose generated ideal is equal to  $\mathfrak{I}_X$ . This follows from the Hilbert's Basis Theorem, since  $\mathfrak{I}_X$  is finitely generated.

**Observation 1.2.4 (Generating System of Equations)**

Let  $X$  be a closed subset of  $\mathbb{A}^n$ , and let  $G_1, \dots, G_m \in k[T]$  be polynomials such that:

$$\mathfrak{I}_X = (G_1, \dots, G_m)$$

then, for all  $F \in \mathfrak{I}_X$  we have that there exists  $H_1, \dots, H_m \in k[T]$  such that:

$$F = H_1 G_1 + \dots + H_m G_m$$

Since  $G_i \in \mathfrak{I}_X$  it follows that  $X$  is defined by the system of equations  $G_1 = \dots = G_m = 0$ .

Therefore, we can always find a system of equations defining  $X$  whose generated ideal is equal to  $\mathfrak{I}_X$ .

**Idea 1.2.1**

It is sometimes even convenient to consider a closed set as defined by the infinite system of equations  $F = 0$  for all polynomials  $F \in \mathfrak{I}_X$ . Indeed, if  $(F_1, \dots, F_m) = \mathfrak{I}_X$  then these equations are all consequences of  $F_1 = \dots = F_m = 0$ .

It turns out that relations between closed sets can be translated to relations between their corresponding ideals.

**Example 1.2.5**

Let  $X$  and  $Y$  be closed subsets of  $\mathbb{A}^n$ . Then,  $Y \subseteq X$  if and only if  $\mathfrak{I}_X \subseteq \mathfrak{I}_Y$ .

This follows directly from the definitions. The latter example allows us to associate to every closed subset of a closed set  $X$  of the affine space  $\mathbb{A}^n$  an ideal of the coordinate ring  $k[X]$ .

Indeed, if  $Y \subseteq X$  is a closed subset of  $X$ , then we can associate to  $Y$  the ideal  $\mathfrak{a}_Y = \mathfrak{I}_Y / \mathfrak{I}_X$  of the ring  $k[X] = k[T] / \mathfrak{I}_X$  (using correspondence theorem).

Conversely, if  $\mathfrak{a}$  is an ideal of  $k[X]$ , then we can associate to  $\mathfrak{a}$  the closed subset  $Y$  of  $X$  defined by the equations  $F = 0$  for all polynomials  $F \in \mathfrak{a}$ .

**Observation 1.2.5**

$Y = \emptyset$  if and only if  $\mathfrak{I}_Y = k[X]$

Something interesting happens when we consider isolated points of  $X$ . Indeed, if  $x \in X$  is an isolated point of  $X$ , then the ideal  $\mathfrak{m}_x = \mathfrak{I}_{\{x\}}/\mathfrak{I}_X$  has to be a maximal ideal of  $k[X]$ .

**Observation 1.2.6**

By definition, this ideal is the kernel of the homomorphism  $ap : k[T] \rightarrow k[X]$ , so:

$$\mathfrak{m}_x = \left\{ F \in k[T] \mid F(x) = 0 \right\}$$

Since  $k$  is an algebraically closed field, then:

$$\mathfrak{m}_x = \left\{ \alpha(x - T) \mid \alpha \in k \right\}$$

So,  $k[T]/\mathfrak{m}_x = k$ . It follows that  $\mathfrak{m}_x$  has to be a maximal ideal of  $k[X]$ .

We can do the converse too, and to each maximal ideal of  $k[X]$  there is an isolated point of  $X$  associated.

**Definition 1.2.6 (Hypersurface)**

Let  $X$  be a closed subset of  $\mathbb{A}^n$  over the ground field  $k$ . We say that  $X$  is an **hypersurface** if there exists  $F \in k[T]$  such that:

$$X = \left\{ u \in \mathbb{A}^n \mid F(u) = 0 \right\}$$

## 1.3 REGULAR MAPS

From now on,  $X$  will denote a closed subset of  $\mathbb{A}^n$  and  $Y$  a closed subset of  $\mathbb{A}^m$ .

**Definition 1.3.1 (Regular Maps)**

A map  $f : X \rightarrow Y$  is called **regular** if there exists  $f_1, \dots, f_m \in k[X]$  regular functions on  $X$  such that:

$$f(x) = (f_1(x), \dots, f_m(x)), \quad \forall x \in X$$

Any regular map  $f : X \rightarrow \mathbb{A}^m$  is always given by  $m$ -functions  $f_1, \dots, f_m \in k[X]$ .

**Observation 1.3.1**

In order to know that this maps into the closed subset  $Y \subseteq \mathbb{A}^n$ , we need to check that:

$$G(f_1(x), \dots, f_m(x)) = 0, \quad \forall x \in X$$

for all  $G \in \mathfrak{I}_Y$ .

**Example 1.3.1**

A regular function is the same thing as a regular map  $f : X \rightarrow \mathbb{A}^1 = k$ .

**Example 1.3.2**

A linear map  $L : \mathbb{A}^n \rightarrow \mathbb{A}^m$  is a regular map.

**Proof:**

It suffices to show that the components of  $L$ , let say  $L = (L_1, \dots, L_m)$  are regular functions. Indeed, since  $L$  is linear, it follows that  $L_j$  is  $k$ -linear for all  $j \in [1, m]$ , so there exists  $\alpha_1, \dots, \alpha_m \in \mathbb{A}^n$  such that:

$$L_j(x) = \alpha_j \cdot x, \quad \forall x \in \mathbb{A}^n$$

where  $\alpha_j \cdot x$  denotes the usual dot product. Since each  $L_j$  is a polynomial of degree 1, it follows that  $L_j$  is a polynomial function, so it's regular. ■

**Example 1.3.3**

The projection map  $(x, y) \mapsto x$  defines a regular map of the curve defined by  $xy = 1$  to  $\mathbb{A}^1$ .

**Example 1.3.4**

The map  $f(t) = (t^2, t^3)$  is a regular map of the line  $\mathbb{A}^1$  to the curve defined by  $x^3 - y^2 = 0$  in  $\mathbb{A}^2$ .

**1.3.1 THE ZETA FUNCTION OF A VARIETY OVER  $\mathbb{F}_p$** 

This example is very important to number theorists, since it relates algebraic geometry with number theory in a very deep way.

Let  $p$  be a prime number and  $\mathbb{F}_p$  the finite field with  $p$  elements. Let  $X$  be a closed subset of  $\mathbb{A}^n$  defined over  $\mathbb{F}_p$ , that is, the polynomials defining  $X$  have coefficients in  $\mathbb{F}_p$ .

**Observation 1.3.2**

Let's consider the closed set  $X$  and  $F_1 = \dots = F_m$  the polynomials defining  $X$ . If  $x \in X$  then:

$$F_i(x) = 0, \quad \forall i \in [1, m]$$

So it follows that:

Let's consider the map  $\varphi : \mathbb{A}^n \rightarrow \mathbb{A}^n$  given by:

$$\varphi(x_1, \dots, x_n) = (x_1^p, \dots, x_n^p)$$

It's immediate to see that  $\varphi$  is a regular map. Furthermore, if  $x \in X$ , then  $\varphi(x) \in X$ . Let  $x \in X$ , so:

$$F_i(x) = 0, \quad \forall i \in [1, m]$$

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## 1.4 EXCERSIES

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### Excercise 1.4.1

If  $X$  and  $Y$  are closed subsets of  $\mathbb{A}^n$  and  $\mathbb{A}^m$ , then  $k[X \times Y]$  is isomorphic to  $k[X] \otimes_k k[Y]$ , where:

$$k[X] \otimes_k k[Y] = \left\{ f \otimes_k g \mid f \in k[X] \text{ and } g \in k[Y] \right\}$$

where  $f \otimes_k g : X \times Y \rightarrow k$  is defined as:

$$f \otimes_k g(x, y) = f(x)g(y), \quad \forall (x, y) \in X \times Y$$