Ejercicios (420)

1)

1-1)

1.1.1 Is $\gamma(t) = (t^2, t^4)$ a parametrization of the parabola $y = x^2$?

No, pues t^2 no toma todos los valores de $\mathbb{R}$, i.e γ solo parametriza media parábola.

1.1.2 Find parametrizations of the following level curves:

(i) $y^2 - x^2 = 1$;

(ii) $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

Sol.

De (i): Tome $y = \cosh(t)$ y $x = \operatorname{senh}(t)$, como $\cosh^2(t) - \operatorname{senh}^2(t) = 1$, $\forall t \in \mathbb{R}$, entonces $y^2 - x^2 = 1$. Así $\alpha(t) = (\operatorname{senh}(t), \cosh(t))$, $\forall t \in \mathbb{R}$ es una parametrización.

De (ii): tome $\beta(t) = (2\cos t, 3 \operatorname{sen} t)$, $\forall t \in \mathbb{R}$.

1.1.3 Find the Cartesian equations of the following parametrized curves:

(i) $\gamma(t) = (\cos^2 t, \sin^2 t)$;

(ii) $\gamma(t) = (e^t, t^2)$.

Sol.

De (i):

Claramente $x = \cos^2 t$ y $y = \operatorname{sen}^2 t$ cumplen $x + y = 1$, luego $y = 1 - x$.

De (ii):

Solutions

Chapter 1

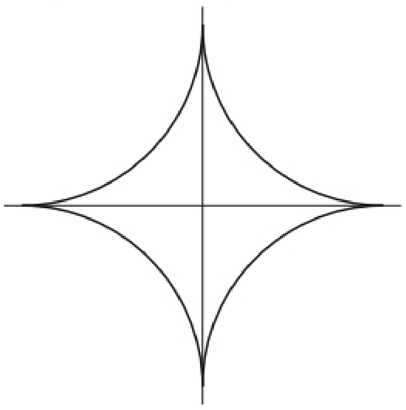
1.1.1 It is a parametrization of the part of the parabola with $x \geq 0$.

1.1.2 (i) $\boldsymbol{\gamma}(t) = (\sec t, \tan t)$ with $-\pi/2 < t < \pi/2$ and $\pi/2 < t < 3\pi/2$. Note that $\boldsymbol{\gamma}$ is defined on the union of two disjoint intervals: this corresponds to the fact that the hyperbola $y^2 - x^2 = 1$ is in two pieces, where $y \geq 1$ and where $y \leq -1$. (ii) $\boldsymbol{\gamma}(t) = (2\cos t, 3\sin t)$.

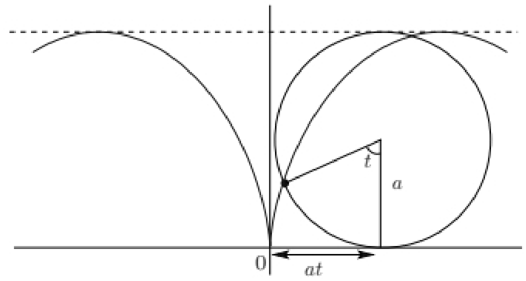
1.1.3 (i) $x + y = 1$. (ii) $y = (\ln x)^2$.

1.1.4 (i) $\dot{\boldsymbol{\gamma}}(t) = \sin 2t(-1, 1)$. (ii) $\dot{\boldsymbol{\gamma}}(t) = (e^t, 2t)$.

1.1.5 $\dot{\boldsymbol{\gamma}}(t) = 3\sin t\cos t(-\cos t, \sin t)$ vanishes where $\sin t = 0$ or $\cos t = 0$, i.e., $t = n\pi/2$ where n is any integer. These points correspond to the four cusps of the astroid (see Exercise 1.3.3).



1.1.6 (i) The squares of the distances from $\mathbf{p}$ to the foci are $(p\cos t \pm \epsilon p)^2 + q^2\sin^2 t = (p^2 - q^2)\cos^2 t \pm 2\epsilon p^2 \cos t + p^2 = p^2(1 \pm \epsilon\cos t)^2$, so the sum of the distances is $2p$.
(ii) $\dot{\boldsymbol{\gamma}} = (-p\sin t, q\cos t)$ so if $\mathbf{n} = (q\cos t, p\sin t)$ then $\mathbf{n}\cdot\dot{\boldsymbol{\gamma}} = 0$. Hence the distances from the foci to the tangent line at $\boldsymbol{\gamma}(t)$ are $\frac{(p\cos t \mp \epsilon p, q\sin t)\cdot\mathbf{n}}{\|\mathbf{n}\|} = \frac{pq(1\mp\epsilon\cos t)}{(p^2\sin^2 t + q^2\cos^2 t)^{1/2}}$ and their product is $\frac{p^2q^2(1-\epsilon^2\cos^2 t)}{(p^2\sin^2 t + q^2\cos^2 t)} = q^2$.
(iii) It is enough to prove that $\frac{(\mathbf{p}-\mathbf{f}_1)\cdot\mathbf{n}}{\|\mathbf{p}-\mathbf{f}_1\|} = \frac{(\mathbf{p}-\mathbf{f}_2)\cdot\mathbf{n}}{\|\mathbf{p}-\mathbf{f}_2\|}$. Computation shows that both sides are equal to q.



1.1.7 When the circle has rotated through an angle t, its centre has moved to (at, a), so the point on the circle initially at the origin is now at the point $(a(t - \sin t), a(1 - \cos t))$ (see the diagram above).

1.1.8 Suppose that a point (x, y, z) lies on the cylinder if $x^2 + y^2 = 1/4$ and on the sphere if $(x+\frac{1}{2})^2 + y^2 + z^2 = 1$. From the second equation, $-1 \leq z \leq 1$ so let $z = \sin t$. Subtracting the two equations gives $x + \frac{1}{4} + \sin^2 t = \frac{3}{4}$, so $x = \frac{1}{2} - \sin^2 t = \cos^2 t - \frac{1}{2}$. From either equation we then get $y = \sin t\cos t$ (or $y = -\sin t\cos t$, but the two solutions are interchanged by $t \mapsto \pi - t$).

1.1.9 $\dot{\boldsymbol{\gamma}} = (-2\sin t + 2\sin 2t, 2\cos t - 2\cos 2t) = \sqrt{2}(\sqrt{2} - 1, 1)$ at $t = \pi/4$. So the tangent line is $y - (\frac{1}{\sqrt{2}} - 1) = (x - \sqrt{2})/(\sqrt{2} - 1)$ and the normal line is $y - (\frac{1}{\sqrt{2}} - 1) = -(x - \sqrt{2})(\sqrt{2} - 1)$.

1.2.1 $\dot{\boldsymbol{\gamma}}(t) = (1, \sinh t)$ so $\| \dot{\boldsymbol{\gamma}} \| = \cosh t$ and the arc-length is $s = \int_0^t \cosh u\, du = \sinh t$.

1.2.2 (i) $\| \dot{\boldsymbol{\gamma}} \|^2 = \frac{1}{4}(1 + t) + \frac{1}{4}(1 - t) + \frac{1}{2} = 1$.
(ii) $\| \dot{\boldsymbol{\gamma}} \|^2 = \frac{16}{25}\sin^2 t + \cos^2 t + \frac{9}{25}\sin^2 t = \cos^2 t + \sin^2 t = 1$.

1.2.3 Denoting $d/d\theta$ by a dot, $\dot{\boldsymbol{\gamma}} = (\dot{r}\cos\theta - r\sin\theta, \dot{r}\sin\theta + r\cos\theta)$ so $\| \dot{\boldsymbol{\gamma}} \|^2 = \dot{r}^2 + r^2$. Hence, $\boldsymbol{\gamma}$ is regular unless $r = \dot{r} = 0$ for some value of θ. It is unit-speed if and only if $\dot{r}^2 = 1 - r^2$, which gives $r = \pm\sin(\theta + \alpha)$ for some constant α. To see that this is the equation of a circle of radius $1/2$, see the diagram in the proof of Theorem 3.2.2.

1.2.4 Since $\mathbf{u}$ is a unit vector, $|\dot{\boldsymbol{\gamma}}\cdot\mathbf{u}| = \| \dot{\boldsymbol{\gamma}} \| \cos\theta$, where θ is the angle between $\dot{\boldsymbol{\gamma}}$ and $\mathbf{u}$, so $\dot{\boldsymbol{\gamma}}\cdot\mathbf{u} \leq \| \dot{\boldsymbol{\gamma}} \|$. Then, $(\mathbf{q}-\mathbf{p})\cdot\mathbf{u} = (\boldsymbol{\gamma}(b) - \boldsymbol{\gamma}(a))\cdot\mathbf{u} = \int_a^b \dot{\boldsymbol{\gamma}}\cdot\mathbf{u}\, dt \leq \int_a^b \| \dot{\boldsymbol{\gamma}} \|\, dt$. Taking $\mathbf{u} = (\mathbf{q} - \mathbf{p})/\parallel \mathbf{q} - \mathbf{p} \parallel$ gives the result.

1.3.1 (i) $\dot{\boldsymbol{\gamma}} = \sin 2t(-1, 1)$ vanishes when t is an integer multiple of $\pi/2$, so $\boldsymbol{\gamma}$ is not regular. (ii) $\boldsymbol{\gamma}$ is regular since $\dot{\boldsymbol{\gamma}} \neq \mathbf{0}$ for $0 < t < \pi/2$. (iii) $\dot{\boldsymbol{\gamma}} = (1, \sinh t)$ is obviously never zero, so $\boldsymbol{\gamma}$ is regular.

1.3.2 $x = r\cos\theta = \sin^2\theta$, $y = r\sin\theta = \sin^2\theta\tan\theta$, so the parametrization in terms of θ is $\theta \mapsto (\sin^2\theta, \sin^2\theta\tan\theta)$. Since $\theta \mapsto \sin\theta$ is a bijective smooth map $(-\pi/2, \pi/2) \to (-1, 1)$, with smooth inverse $t \mapsto \sin^{-1} t$, $t = \sin\theta$ is a reparametrization map. Since $\sin^2\theta = t^2$, $\sin^2\theta\tan\theta = t^3/\sqrt{1 - t^2}$, so the reparametrized curve is as stated.

1.3.3 (i) $\dot{\boldsymbol{\gamma}} = \mathbf{0}$ at $t = 0 \iff m$ and n are both ≥ 2. If $m > 3$ the first components of $\ddot{\boldsymbol{\gamma}}$ and $\dddot{\boldsymbol{\gamma}}$ are both 0 at $t = 0$ so $\ddot{\boldsymbol{\gamma}}$ and $\dddot{\boldsymbol{\gamma}}$ are linearly dependent at $t = 0$; similarly if $n > 3$. So there are four cases: if $(m, n) = (2, 2)$ or $(3, 3)$ then either $\ddot{\boldsymbol{\gamma}}$ or $\dddot{\boldsymbol{\gamma}}$ is zero at $t = 0$, so the only possibilities for an ordinary cusp are $(m, n) = (2, 3)$ and $(3, 2)$ and then $\ddot{\boldsymbol{\gamma}}$ and $\dddot{\boldsymbol{\gamma}}$ are easily seen to be linearly independent at $t = 0$. (ii) Using the parametrization $\boldsymbol{\gamma}(t) = \left(t^2, \frac{t^3}{\sqrt{1-t^2}}\right)$, we get $\dot{\boldsymbol{\gamma}} = \mathbf{0}$, $\ddot{\boldsymbol{\gamma}} = (2, 0)$, $\dddot{\boldsymbol{\gamma}} = (0, 6)$ at $t = 0$ so the origin is an ordinary cusp. (iii) Let $\tilde{\boldsymbol{\gamma}}(\tilde{t})$ be a reparametrization of $\boldsymbol{\gamma}(t)$, and suppose $\boldsymbol{\gamma}$ has an ordinary cusp at $t = t_0$. Then, at $t = t_0$, $d\tilde{\boldsymbol{\gamma}}/d\tilde{t} = (d\boldsymbol{\gamma}/dt)(dt/d\tilde{t}) = 0$, $d^2\tilde{\boldsymbol{\gamma}}/d\tilde{t}^2 = (d^2\boldsymbol{\gamma}/dt^2)(dt/d\tilde{t})^2$, $d^3\tilde{\boldsymbol{\gamma}}/d\tilde{t}^3 = (d^3\boldsymbol{\gamma}/dt^3)(dt/d\tilde{t})^3 + 3(d^2\boldsymbol{\gamma}/dt^2)(dt/d\tilde{t})(d^2t/d\tilde{t}^2)$. Using the fact that $dt/d\tilde{t} \neq 0$, it is easy to see that $d^2\tilde{\boldsymbol{\gamma}}/d\tilde{t}^2$ and $d^3\tilde{\boldsymbol{\gamma}}/d\tilde{t}^3$ are linearly independent when $t = t_0$.

1.3.4 (i) If $\tilde{\boldsymbol{\gamma}}(t) = \boldsymbol{\gamma}(\varphi(t))$, let ψ be the inverse of the reparametrization map φ. Then $\tilde{\boldsymbol{\gamma}}(\psi(t)) = \boldsymbol{\gamma}(\varphi(\psi(t))) = \boldsymbol{\gamma}(t)$. (ii) If $\tilde{\boldsymbol{\gamma}}(t) = \boldsymbol{\gamma}(\varphi(t))$ and $\hat{\boldsymbol{\gamma}}(t) = \tilde{\boldsymbol{\gamma}}(\psi(t))$, where φ and ψ are reparametrization maps, then $\hat{\boldsymbol{\gamma}}(t) = \boldsymbol{\gamma}((\varphi\circ\psi)(t))$ and $\varphi\circ\psi$ is a reparametrization map because it is smooth and $\frac{d}{dt}(\varphi(\psi(t)) = \dot{\varphi}(\psi(t))\dot{\psi}(t) \neq 0$ as $\dot{\varphi}$ and $\dot{\psi}$ are both $\neq 0$.

1.4.1 It is closed because $\boldsymbol{\gamma}(t + 2\pi) = \boldsymbol{\gamma}(t)$ for all t. Suppose that $\boldsymbol{\gamma}(t) = \boldsymbol{\gamma}(u)$. Then $\cos^3 t(\cos 3t, \sin 3t) = \cos^3 u(\cos 3u, \sin 3u)$. Taking lengths gives $\cos^3 t = \pm\cos^3 u$ so $\cos t = \pm\cos u$, so $u = t, \pi - t, \pi + t$ or $2\pi - t$ (up to adding multiples of 2π). The second possibility forces $t = n\pi/3$ for some integer n and the third possibility is true for all t. Hence, the period is π and for the self-intersections we need only consider $t = \pi/3, 2\pi/3$, giving $u = 2\pi/3, \pi/3$, respectively. Hence, there is a unique self-intersection at $\boldsymbol{\gamma}(\pi/3) = (-1/8, 0)$.

1.4.2 The curve $\tilde{\boldsymbol{\gamma}}(t) = (\cos(t^3 + t), \sin(t^3 + t))$ is a reparametrization of the circle $\boldsymbol{\gamma}(t) = (\cos t, \sin t)$ but it is not closed.

1.4.3 If $\boldsymbol{\gamma}$ is T-periodic then it is kT-periodic for all $k \neq 0$ (this can be proved by induction on k if $k > 0$, or on $-k$ if $k < 0$). If $\boldsymbol{\gamma}$ is T_1-periodic and T_2-periodic then it is k_1T_1- and k_2T_2-periodic for all non-zero integers k_1, k_2, so $\boldsymbol{\gamma}(t + k_1T_1 + k_2T_2) = \boldsymbol{\gamma}(t + k_1T_1)$ as $\boldsymbol{\gamma}$ is k_2T_2-periodic, which $= \boldsymbol{\gamma}(t)$ as $\boldsymbol{\gamma}$ is k_1T_1-periodic.

1.4.4 If $\boldsymbol{\gamma}$ is T-periodic write $T = kT_0 + T_1$ where k is an integer and $0 \leq T_1 < T_0$. By Exercise 1.4.3 $\boldsymbol{\gamma}$ is T_1-periodic; if $T_1 > 0$ this contradicts the definition of T_0.

1.4.5 (i) Choose $T_1 > 0$ such that $\boldsymbol{\gamma}$ is T_1-periodic; then T_1 is not the smallest positive number with this property, so there is a $T_2 > 0$ such that $\boldsymbol{\gamma}$ is T_2-periodic. Iterating this argument gives the desired sequence. (ii) The sequence $\{T_r\}_{r\geq 1}$ in (i) is decreasing and bounded below, so must converge to some $T_\infty \geq 0$. Then $\boldsymbol{\gamma}$ is T_∞-periodic because (using continuity of $\boldsymbol{\gamma}$) $\boldsymbol{\gamma}(t + T_\infty) = \lim_{r\to\infty}\boldsymbol{\gamma}(t + T_r) = \lim_{r\to\infty}\boldsymbol{\gamma}(t) = \boldsymbol{\gamma}(t)$. By Exercise 1.4.3, $\boldsymbol{\gamma}$ is $(T_r - T_\infty)$-periodic for all $r \geq 1$, and this sequence of positive numbers converges to 0. (iii) If $\{T_r\}$ is as in (i) and $T_r \to 0$ as $r \to \infty$, then by the mean value theorem $0 = (f(t + T_r) - f(t))/T_r = \dot{f}(t + \lambda T_r)$ for some $0 < \lambda < 1$. Letting $r \to \infty$ gives $\dot{f}(t) = 0$ for all t, so f is constant.

1.4.6 Following the hint, since $T_0 = (k_i/k)T_i$ is an integer multiple of T_i, each γ_i is T_0-periodic. Let $\mathcal{T}$ be the union of the finite sets of real numbers $\{T_i, 2T_i, \ldots, k_iT_i\}$ over all i such that γ_i is not constant, and let $\mathcal{P} = \{T' \in \mathcal{T} \,|\, \gamma \text{ is } T'\text{-periodic}\}$. Then $\mathcal{P}$ is finite (because $\mathcal{T}$ is) and non-empty (because $T \in \mathcal{P}$). The smallest element of $\mathcal{P}$ is the smallest positive number T_0' such that γ is T_0'-periodic (since if γ is T'-periodic either $T' > T$ or $T' \in \mathcal{P}$). By Exercise 1.4.4, $T_0 = k'T_0'$ for some integer k' and then there are integers k_i' such that $T_0' = k_i'T_i$ for all i such that γ_i is not constant. Then, $k_iT_i/k = k'k_i'T_i$ so kk' divides each k_i. As k is the largest such divisor, $k' = 1$, so $T_0 = T_0'$.

1.5.1 $x(1 - x^2) \geq 0 \iff x \leq -1$ or $0 \leq x \leq 1$ so the curve is in (at least) two pieces. The parametrization is defined for $t \leq -1$ and $0 \leq t \leq 1$ and it covers the part of the curve with $y \geq 0$.

1.5.2 If $\boldsymbol{\gamma}(t) = (x(t), y(t), z(t))$ is a curve in the surface $f(x, y, z) = 0$, differentiating $f(x(t), y(t), z(t)) = 0$ with respect to t gives $\dot{x}f_x + \dot{y}f_y + \dot{z}f_z = 0$, so $\dot{\boldsymbol{\gamma}}$ is perpendicular to $\boldsymbol{\nabla}f = (f_x, f_y, f_z)$. Since this holds for every curve in the surface, $\boldsymbol{\nabla}f$ is perpendicular to the surface. The surfaces $f = 0$ and $g = 0$ should intersect in a curve if the vectors $\boldsymbol{\nabla}f$ and $\boldsymbol{\nabla}g$ are not parallel at any point of the intersection.

1.5.3 Let $\boldsymbol{\gamma}(t) = (u(t), v(t), w(t))$ be a regular curve in $\mathbb{R}^3$. At least one of $\dot{u}, \dot{v}, \dot{w}$ is non-zero at each value of t. Suppose that $\dot{u}(t_0) \neq 0$ and $x_0 = u(t_0)$. As in the 'proof' of Theorem 1.5.2, there is a smooth function $h(x)$ defined for x near x_0 such that $t = h(x)$ is the unique solution of $x = u(t)$ for each t near t_0. Then, for t near t_0, $\boldsymbol{\gamma}(t)$ is contained in the level curve $f(x,y,z) = g(x,y,z) = 0$, where $f(x,y,z) = y - v(h(x))$ and $g(x,y,z) = z - w(h(x))$. The functions f and g satisfy the conditions in the previous exercise, since $\nabla f = (-\dot{v}h', 1, 0)$, $\nabla g = (-\dot{w}h', 0, 1)$, a dash denoting d/dx.

Chapter 2

2.1.1 (i) $\boldsymbol{\gamma}$ is unit-speed (Exercise 1.2.2(i)) so $\kappa = \parallel \ddot{\boldsymbol{\gamma}} \parallel = \parallel (\frac{1}{4}(1+t)^{-1/2}, \frac{1}{4}(1-t)^{-1/2}, 0) \parallel = \frac{1}{\sqrt{8(1-t^2)}}$. (ii) $\boldsymbol{\gamma}$ is unit-speed (Exercise 1.2.2(ii)) so $\kappa = \parallel \ddot{\boldsymbol{\gamma}} \parallel = \parallel (-\frac{4}{5}\cos t, \sin t, \frac{3}{5}\cos t) \parallel = 1$.
(iii) $\kappa = \frac{\|(1,\sinh t,0)\times(0,\cosh t,0)\|}{\|(1,\sinh t,0)\|^3} = \frac{\cosh t}{\cosh^3 t} = \mathrm{sech}^2 t$ using Proposition 2.1.2.
(iv) $(-3\cos^2 t\sin t, 3\sin^2 t\cos t, 0)\times(-3\cos^3 t + 6\cos t\sin^2 t, 6\sin t\cos^2 t - 3\sin^3 t, 0) = (0, 0, -9\sin^2 t\cos^2 t)$, so $\kappa = \frac{\|(0,0,-9\sin^2 t\cos^2 t)\|}{\|(-3\cos^2 t\sin t, 3\sin^2 t\cos t, 0)\|^3} = \frac{1}{3|\sin t\cos t|}$. This becomes infinite when t is an integer multiple of $\pi/2$, i.e., at the four cusps $(\pm 1, 0)$ and $(0, \pm 1)$ of the astroid.

2.1.2 The proof of Proposition 1.3.5 shows that, if $\mathbf{v}(t)$ is a smooth (vector) function of t, then $\parallel \mathbf{v}(t) \parallel$ is a smooth (scalar) function of t provided $\mathbf{v}(t)$ is non-zero for all t. The result now follows from the formula in Proposition 2.1.2. The curvature of the regular curve $\boldsymbol{\gamma}(t) = (t, t^3)$ is $\kappa(t) = 6|t|/(1+9t^4)^{3/2}$, which is not differentiable at $t = 0$.

2.2.1 Differentiate $\mathbf{t}\cdot\mathbf{n}_s = 0$ and use $\dot{\mathbf{t}} = \kappa_s\mathbf{n}_s$.

2.2.2 If $\boldsymbol{\gamma}$ is smooth, $\mathbf{t} = \dot{\boldsymbol{\gamma}}$ is smooth and hence so are $\dot{\mathbf{t}}$ and $\mathbf{n}_s$ (since $\mathbf{n}_s$ is obtained by applying a rotation to $\mathbf{t}$). So $\kappa_s = \dot{\mathbf{t}}\cdot\mathbf{n}_s$ is smooth.

2.2.3 For the first part, from the results in Appendix 2 it suffices to show that $\tilde{\kappa}_s = -\kappa_s$ if M is the reflection in a straight line l. But this is clear: if we take the fixed angle φ_0 in Proposition 2.2.1 to be the angle between l and the positive x-axis, then (in the obvious notation) $\tilde{\varphi} = -\varphi$. Conversely, if $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ have the same non-zero curvature, their signed curvatures are either the same or differ in sign. In the first case the curves differ by a direct isometry by Theorem 2.2.5; in the latter case, applying a reflection to one curve gives two curves with the same signed curvature, and these curves then differ by a direct isometry, so the original curves differ by an opposite isometry.

2.2.4 The first part is obvious as the effect of the dilation is to multiply s by a and leave φ unchanged. For the second part, consider the small piece of the chain between the points with arc-length s and $s + \delta s$. The net horizontal force on this piece is (in the obvious notation) $\delta(T\cos\varphi)$, and as this must vanish $T\cos\varphi$ must be a constant, say λ. The net vertical force is $\delta(T\sin\varphi)$, and this must balance the weight of the piece of chain, which is a constant multiple of δs. This shows that $T\sin\varphi = \mu s + \nu$ for some constants μ, ν, and ν must be zero because $\varphi = s = 0$ at the lowest point of $\mathcal{C}$. From $T\cos\varphi = \lambda$, $T\sin\varphi = \mu s$, we get $\tan\varphi = s/a$ where $a = \lambda/\mu$. Hence, $\sec^2\varphi\frac{d\varphi}{ds} = 1/a$, so the signed curvature is $\kappa_s = d\varphi/ds = 1/a\sec^2\varphi = 1/a(1+\tan^2\varphi) = \frac{1}{a}(1+s^2/a^2)^{-1}$. Using the first part and Example 2.2.4 gives the result.

2.2.5 We have $d\boldsymbol{\gamma}^\lambda/dt = d\boldsymbol{\gamma}/dt + \lambda d\mathbf{n}_s/dt = (1-\lambda\kappa_s)ds/dt\,\mathbf{t}$, so the arc-length s^λ of $\boldsymbol{\gamma}^\lambda$ satisfies $ds^\lambda/dt = |1-\lambda\kappa_s|ds/dt$. The unit tangent vector of $\boldsymbol{\gamma}^\lambda$ is $\mathbf{t}^\lambda = (d\boldsymbol{\gamma}^\lambda/dt)/(ds^\lambda/dt) = \epsilon\mathbf{t}$, hence the signed unit normal of $\boldsymbol{\gamma}^\lambda$ is $\mathbf{n}_s^\lambda = \epsilon\mathbf{n}_s$. Then, the signed curvature κ_s^λ of $\boldsymbol{\gamma}^\lambda$ is given by $\kappa_s^\lambda\mathbf{n}_s^\lambda = d\mathbf{t}^\lambda/ds^\lambda = (d\mathbf{t}^\lambda/dt)/|1-\lambda\kappa_s|(ds/dt) = \epsilon|1-\lambda\kappa_s|^{-1}d\mathbf{t}/ds = \kappa_s(1-\lambda\kappa_s)^{-1}\mathbf{n}_s = \epsilon\kappa_s(1-\lambda\kappa_s)^{-1}\mathbf{n}_s^\lambda = \kappa_s|1-\lambda\kappa_s|^{-1}\mathbf{n}_s^\lambda$.

2.2.6 $\boldsymbol{\epsilon}(s_0)$ lies on the perpendicular bisector of the line joining $\boldsymbol{\gamma}(s_0)$ and $\boldsymbol{\gamma}(s_0 + \delta s)$. So

$$(\boldsymbol{\epsilon}(s_0) - \frac{1}{2}(\boldsymbol{\gamma}(s_0) + \boldsymbol{\gamma}(s_0 + \delta s)))\cdot(\boldsymbol{\gamma}(s_0 + \delta s) - \boldsymbol{\gamma}(s_0)) = 0.$$

Using Taylor's theorem, and discarding terms involving powers of δs higher than the second, this gives (with all quantities evaluated at s_0) $(\boldsymbol{\epsilon}-\boldsymbol{\gamma})\cdot\dot{\boldsymbol{\gamma}}\delta s + \frac{1}{2}(\boldsymbol{\epsilon}\cdot\ddot{\boldsymbol{\gamma}} - 1 - \boldsymbol{\gamma}\cdot\ddot{\boldsymbol{\gamma}})(\delta s)^2 = 0$. This must also hold when δs is replaced by $-\delta s$; adding and subtracting the two equations give $(\boldsymbol{\epsilon}-\boldsymbol{\gamma})\cdot\dot{\boldsymbol{\gamma}} = 0$ and $(\boldsymbol{\epsilon}-\boldsymbol{\gamma})\cdot\ddot{\boldsymbol{\gamma}} = 1$. The first equation gives $\boldsymbol{\epsilon} = \boldsymbol{\gamma} + \lambda\mathbf{n}_s$ for some scalar λ, and since $\ddot{\boldsymbol{\gamma}} = \kappa_s\mathbf{n}_s$ the second gives $\lambda = 1/\kappa_s$.

2.2.7 The tangent vector of $\boldsymbol{\epsilon}$ is $\mathbf{t} + \frac{1}{\kappa_s}(-\kappa_s\mathbf{t}) - \frac{\dot{\kappa}_s}{\kappa_s^2}\mathbf{n}_s = -\frac{\dot{\kappa}_s}{\kappa_s^2}\mathbf{n}_s$ so its arc-length is $u = \int \parallel \dot{\boldsymbol{\epsilon}} \parallel\, ds = \int \frac{\dot{\kappa}_s}{\kappa_s^2}ds = u_0 - \frac{1}{\kappa_s}$, where u_0 is a constant. Hence, the unit tangent vector of $\boldsymbol{\epsilon}$ is $-\mathbf{n}_s$ and its signed unit normal is $\mathbf{t}$. Since $-d\mathbf{n}_s/du = \kappa_s\mathbf{t}/(du/ds) = \frac{\kappa_s^3}{\dot{\kappa}_s}\mathbf{t}$, the signed curvature of $\boldsymbol{\epsilon}$ is $\kappa_s^3/\dot{\kappa}_s$.

Any point on the normal line to $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(s)$ is $\boldsymbol{\gamma}(s) + \lambda\mathbf{n}_s(s)$ for some λ. Hence, the normal line intersects $\boldsymbol{\epsilon}$ at the point $\boldsymbol{\epsilon}(s)$, where $\lambda = 1/\kappa_s(s)$, and since the tangent vector of $\boldsymbol{\epsilon}$ there is parallel to $\mathbf{n}_s(s)$ by the first part, the normal line is tangent to $\boldsymbol{\epsilon}$ at $\boldsymbol{\epsilon}(s)$.

Denoting d/dt by a dash, $\boldsymbol{\gamma}' = a(1-\cos t, \sin t)$ so the arc-length s of $\boldsymbol{\gamma}$ is given by $ds/dt = 2a\sin(t/2)$ and $\mathbf{t} = d\boldsymbol{\gamma}/ds = (\sin(t/2), \cos(t/2))$. So $\mathbf{n}_s = (-\cos(t/2), \sin(t/2))$ and $\dot{\mathbf{t}} = (d\mathbf{t}/dt)/(ds/dt) = \frac{1}{4a\sin(t/2)}(\cos(t/2),$