2. Formas Diferenciales.

VF y 1-Formas.

Exercise 2.1.i. Prove Lemma 2.1.18

Exercise 2.1.ii. Prove Lemma 2.1.11

Exercise 2.1.ii. Prove Lemma 2.1.11

Exercise 2.1.iii. Let U be an open subset of $\mathbf{R}^n$ and v_1 and v_2 vector fields on U. Show that

Exercise 2.1.iii. Let U be an open subset of $\mathbf{R}^n$ and $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ vector fields on U. Show that there is a unique vector field $\boldsymbol{w}$, on U with the property

$$L_{\boldsymbol{w}}\phi = L_{\boldsymbol{v}_1}(L_{\boldsymbol{v}_2}\phi) - L_{\boldsymbol{v}_2}(L_{\boldsymbol{v}_1}\phi)$$

for all $\phi \in C^\infty(U)$.

Exercise 2.1.iv. The vector field $\boldsymbol{w}$ in Exercise 2.1.iii is called the ***Lie bracket*** of the vector fields $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ and is denoted by $[\boldsymbol{v}_1, \boldsymbol{v}_2]$. Verify that the Lie bracket is *skew-symmetric*, i.e.,

$$[\boldsymbol{v}_1, \boldsymbol{v}_2] = -[\boldsymbol{v}_2, \boldsymbol{v}_1] ,$$

and satisfies the *Jacobi identity*

$$[\boldsymbol{v}_1, [\boldsymbol{v}_2, \boldsymbol{v}_3]] + [\boldsymbol{v}_2, [\boldsymbol{v}_3, \boldsymbol{v}_1]] + [\boldsymbol{v}_3, [\boldsymbol{v}_1, \boldsymbol{v}_2]] = 0 .$$

(And thus defines the structure of a *Lie algebra*.)

Hint: Prove analogous identities for $L_{\boldsymbol{v}_1}$, $L_{\boldsymbol{v}_2}$ and $L_{\boldsymbol{v}_3}$.

Exercise 2.1.v. Let $\boldsymbol{v}_1 = \partial/\partial x_i$ and $\boldsymbol{v}_2 = \sum_{j=1}^n g_j \partial/\partial x_j$. Show that

$$[\boldsymbol{v}_1, \boldsymbol{v}_2] = \sum_{j=1}^{n} \frac{\partial g_j}{\partial x_i} \frac{\partial}{\partial x_j} .$$

Exercise 2.1.vi. Let $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ be vector fields and f a C^∞ function. Show that

$$[\boldsymbol{v}_1, f\boldsymbol{v}_2] = L_{\boldsymbol{v}_1} f\boldsymbol{v}_2 + f[\boldsymbol{v}_1, \boldsymbol{v}_2]\,.$$

Exercise 2.1.vii. Let U be an open subset of $\mathbf{R}^n$ and let $\gamma\colon [a,b] \to U$, $t \mapsto (\gamma_1(t), \dots, \gamma_n(t))$ be a C^1 curve. Given a C^∞ one-form $\omega = \sum_{i=1}^n f_i dx_i$ on U, define the ***line integral*** of ω over γ to be the integral

$$\int_\gamma \omega := \sum_{i=1}^n \int_a^b f_i(\gamma(t)) \frac{d\gamma_i}{dt}\, dt\,.$$

Show that if $\omega = df$ for some $f \in C^\infty(U)$

$$\int_\gamma \omega = f(\gamma(b)) - f(\gamma(a))\,.$$

In particular conclude that if γ is a closed curve, i.e., $\gamma(a) = \gamma(b)$, this integral is zero.

Exercise 2.1.viii. Let ω be the C^∞ one-form on $\mathbf{R}^2 \smallsetminus \{0\}$ defined by

$$\omega = \frac{x_1 dx_2 - x_2 dx_1}{x_1^2 + x_2^2},$$

and let $\gamma\colon [0, 2\pi] \to \mathbf{R}^2 \smallsetminus \{0\}$ be the closed curve $t \mapsto (\cos t, \sin t)$. Compute the line integral $\int_\gamma \omega$, and note that $\int_\gamma \omega \neq 0$. Conclude that ω is not of the form df for $f \in C^\infty(\mathbf{R}^2 \smallsetminus \{0\})$.

Exercise 2.1.ix. Let f be the function

$$f(x_1, x_2) = \begin{cases} \arctan \frac{x_2}{x_1} & x_1 > 0 \\ \frac{\pi}{2}, & x_2 > 0 \text{ or } x_1 = 0 \\ \arctan \frac{x_2}{x_1} + \pi & x_1 < 0 \,. \end{cases}$$

Recall that $-\frac{\pi}{2} < \arctan(t) < \frac{\pi}{2}$. Show that f is C^∞ and that df is the 1-form ω in Exercise 2.1.viii. Why does not this contradict what you proved in Exercise 2.1.viii?

2. Curvas Int. y VF.

Exercise 2.2.i. Prove the reparameterization result Theorem 2.2.20.

Exercise 2.2.ii. Let U be an open subset of $\mathbf{R}^n$, V an open subset of $\mathbf{R}^n$ and $f : U \to V$ a C^k

Exercise 2.2.ii. Let U be an open subset of $\mathbf{R}^n$, V an open subset of $\mathbf{R}^n$ and $f : U \to V$ a C^k map.

(1) Show that for $\phi \in C^\infty(V)$ (2.2) can be rewritten

$$f^\star d\phi = d f^\star \phi \, .$$

(2) Let μ be the one-form

$$\mu = \sum_{i=1}^{m} \phi_i dx_i \, , \quad \phi_i \in C^\infty(V)$$

on V. Show that if $f = (f_1, \dots, f_m)$ then

$$f^\star \mu = \sum_{i=1}^{m} f^\star \phi_i \, d f_i \, .$$

(3) Show that if μ is C^∞ and f is C^∞, $f^\star \mu$ is C^∞.

Exercise 2.2.iii. Let $\boldsymbol{v}$ be a complete vector field on U and $f_t : U \to U$, the one parameter group of diffeomorphisms generated by $\boldsymbol{v}$. Show that if $\phi \in C^1(U)$

$$L_v \phi = \left. \frac{d}{dt} f_t^\star \phi \right|_{t=0} .$$

Exercise 2.2.iv.

(1) Let $U = \mathbf{R}^2$ and let $\boldsymbol{v}$ be the vector field, $x_1\partial/\partial x_2 - x_2\partial/\partial x_1$. Show that the curve

$$t \mapsto (r\cos(t+\theta), r\sin(t+\theta)),$$

for $t \in \mathbf{R}$, is the unique integral curve of $\boldsymbol{v}$ passing through the point, $(r\cos\theta, r\sin\theta)$, at $t = 0$.

(2) Let $U = \mathbf{R}^n$ and let $\boldsymbol{v}$ be the constant vector field: $\sum_{i=1}^n c_i\partial/\partial x_i$. Show that the curve

$$t \mapsto a + t(c_1, \ldots, c_n),$$

for $t \in \mathbf{R}$, is the unique integral curve of $\boldsymbol{v}$ passing through $a \in \mathbf{R}^n$ at $t = 0$.

(3) Let $U = \mathbf{R}^n$ and let $\boldsymbol{v}$ be the vector field, $\sum_{i=1}^n x_i\partial/\partial x_i$. Show that the curve

$$t \mapsto e^t(a_1, \ldots, a_n),$$

for $t \in \mathbf{R}$, is the unique integral curve of $\boldsymbol{v}$ passing through a at $t = 0$.

Exercise 2.2.v. Show that the following are one-parameter groups of diffeomorphisms:

(1) $f_t : \mathbf{R} \to \mathbf{R}$, $f_t(x) = x + t$
(2) $f_t : \mathbf{R} \to \mathbf{R}$, $f_t(x) = e^t x$
(3) $f_t : \mathbf{R}^2 \to \mathbf{R}^2$, $f_t(x, y) = (x\cos(t) - y\sin(t), x\sin(t) + y\cos(t))$.

Exercise 2.2.vi. Let $A\colon \mathbf{R}^n \to \mathbf{R}^n$ be a linear mapping. Show that the series

$$\exp(tA) := \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} = \mathrm{id}_n + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

converges and defines a one-parameter group of diffeomorphisms of $\mathbf{R}^n$.

Exercise 2.2.vii.

(1) What are the infinitesimal generators of the one-parameter groups in Exercise 2.2.v?
(2) Show that the infinitesimal generator of the one-parameter group in Exercise 2.2.vi is the vector field

$$\sum_{1 \le i,j \le n} a_{i,j} x_j \frac{\partial}{\partial x_i}$$

where $(a_{i,j})$ is the defining matrix of A.

Exercise 2.2.viii. Let $\boldsymbol{v}$ be the vector field on $\mathbf{R}$ given by $x^2 \frac{d}{dx}$. Show that the curve

$$x(t) = \frac{a}{a - at}$$

is an integral curve of $\boldsymbol{v}$ with initial point $x(0) = a$. Conclude that for $a > 0$ the curve

$$x(t) = \frac{a}{1 - at}, \quad 0 < t < \frac{1}{a}$$

is a maximal integral curve. (In particular, conclude that $\boldsymbol{v}$ is not complete.)

Exercise 2.2.ix. Let U and V be open subsets of $\mathbf{R}^n$ and $f : U \xrightarrow{\sim} V$ a diffeomorphism. If $\boldsymbol{w}$ is a vector field on V, define the ***pullback*** of $\boldsymbol{w}$ to U to be the vector field

$$f^\star \boldsymbol{w} := (f_\star^{-1} \boldsymbol{w}) .$$

Show that if ϕ is a C^∞ function on V

$$f^\star L_{\boldsymbol{w}} \phi = L_{f^\star \boldsymbol{w}} f^\star \phi .$$

Hint: Equation (2.2.25).

Exercise 2.2.x. Let U be an open subset of $\mathbf{R}^n$ and $\boldsymbol{v}$ and $\boldsymbol{w}$ vector fields on U. Suppose $\boldsymbol{v}$ is the infinitesimal generator of a one-parameter group of diffeomorphisms

$$f_t : U \rightrightarrows U\,, \quad -\infty < t < \infty\,.$$

Let $\boldsymbol{w}_t = f_t^\star \boldsymbol{w}$. Show that for $\phi \in C^\infty(U)$ we have

$$L_{[\boldsymbol{v},\boldsymbol{w}]}\phi = L_{\dot{\boldsymbol{w}}}\phi\,,$$

where

$$\dot{\boldsymbol{w}} = \frac{d}{dt} f_t^\star \boldsymbol{w}\Big|_{t=0}\,.$$

Hint: Differentiate the identity

$$f_t^\star L_{\boldsymbol{w}}\phi = L_{\boldsymbol{w}_t} f_t^\star \phi$$

with respect to t and show that at $t = 0$ the derivative of the left hand side is $L_{\boldsymbol{v}} L_{\boldsymbol{w}}\phi$ by Exercise 2.2.iii, and the derivative of the right hand side is

$$L_{\dot{\boldsymbol{w}}} + L_{\boldsymbol{w}}(L_{\boldsymbol{v}}\phi)\,.$$

Exercise 2.2.xi. Conclude from Exercise 2.2.x that

$$[\boldsymbol{v}, \boldsymbol{w}] = \frac{d}{dt} f_t^{\star} \boldsymbol{w} \bigg|_{t=0} . \tag{2.2.25}$$

K-FORMAS.

Notas.