ECON 7350 Applied Econometrics for Macroeconomics and Finance

Research Project II

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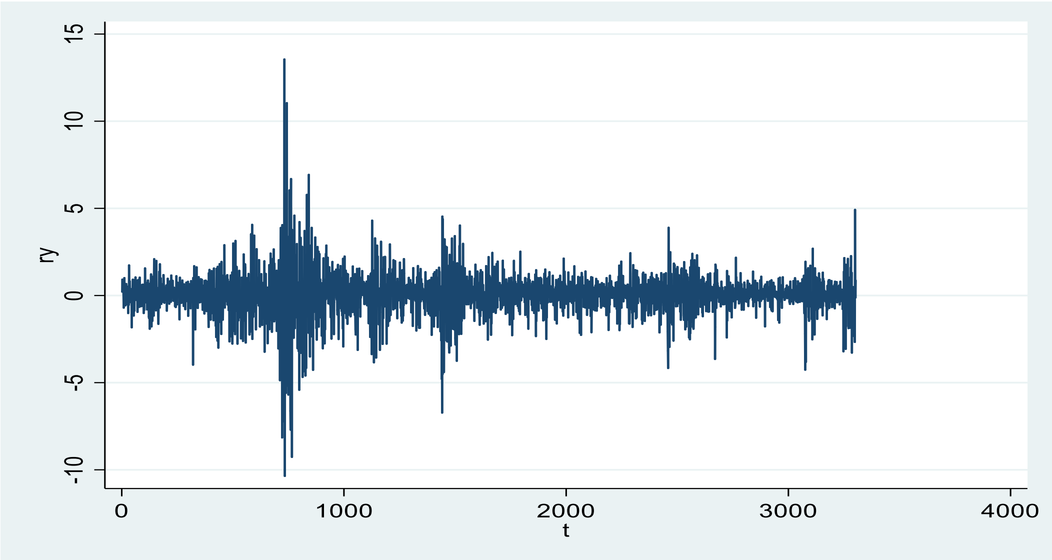
Q1

(A)

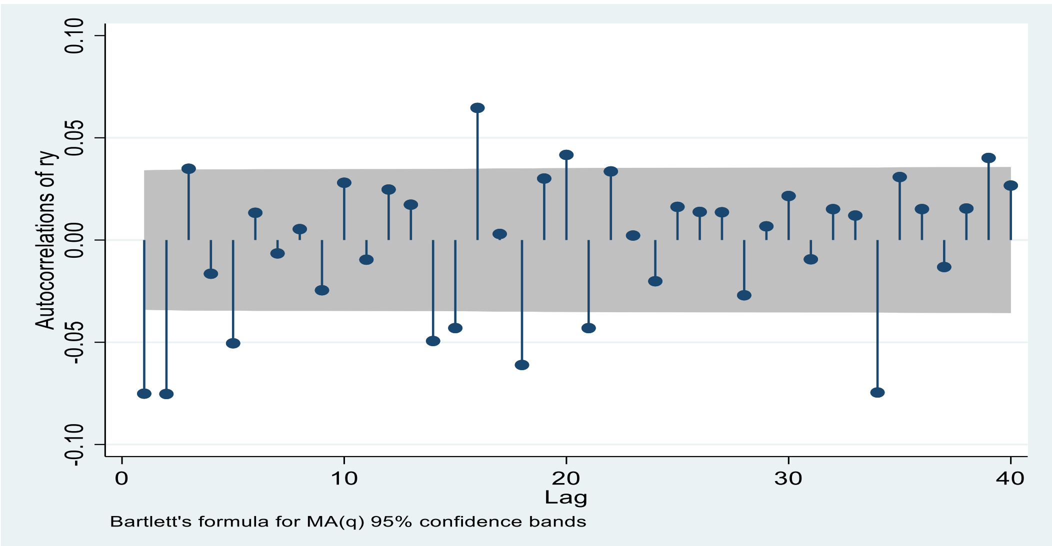
At the first glance, the time series of $r_{y,t}$ looks like it exhibits mean reversion which is the phenomenon of stationarity. However, volatility jump/fat tailed event had happened during the early period before t = 1000. That jump will make the data highly stochastic for quite a period if we conduct subsample analysis. No matter what, we still cannot conclude that it is stationary without conducting unit root test. In addition, both ACF and PACF of $r_{y,t}$ also show signs of mean reversion since they have reverted around 0 which is the signal of stationarity. Combining these information, we still cannot firmly indicate that $r_{y,t}$ is stationary. Therefore, to make sure the time series has no unit root, we then run ADF test.

From the ADF test for all three kinds of equations at lag 2, we can reject the null hypothesis that $r_{y,t}$ contains unit root at five percent significance level for all equations. In conclusion, $r_{y,t}$ is clearly stationary.

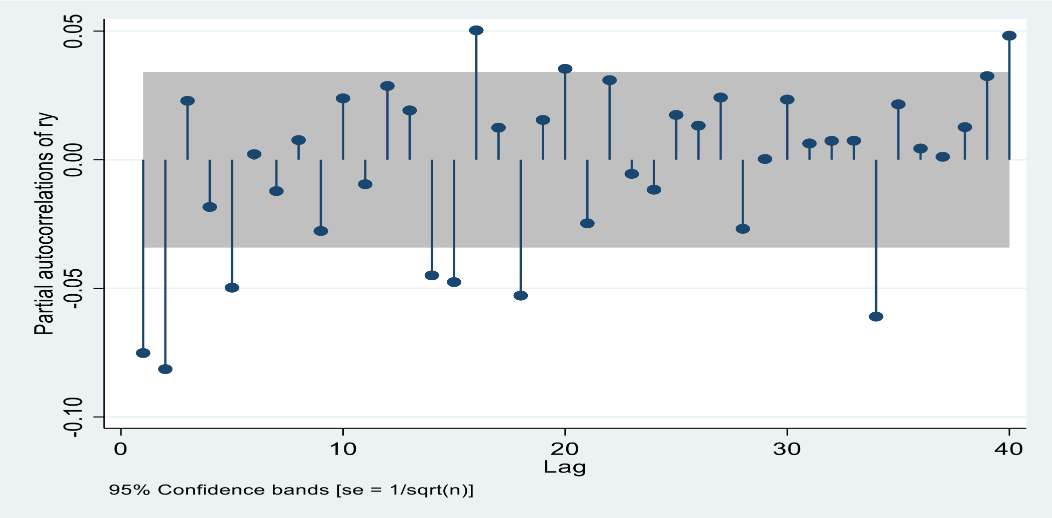
Time Series of $r_{y,t}$



ACF of $r_{y,t}$



PACF of $r_{y,t}$



ADF test of $r_{y,t}$

Ho: $r_{y,t}$ contains unit root

Ha: $r_{y,t}$ contains no unit root

The three equations of ADF test are:

Equation (1): the pure random walk model.

Equation (2): adds a drift term to the previous model, Equation (1).

Equation (3): adds a drift and a linear time trend to the random walk model.

Table of ADF test

At lag 2	Test Statistics	5% Critical Value
Equation (1)	-34.885	-1.950
Equation (2)	-34.929	-1.645
Equation (3)	-34.931	-3.410

(B)

(B-1) Model Selection

I tested the ARMA models from ARMA (1,1) to ARMA (4,4). The information criterion test indicates that ARMA(3,3) has the lowest AIC and BIC; therefore, ARMA (3,3) is the best model.

Below is the estimated model for ARMA(3,3)

$$r_t = \underset{(0.0197)}{0.0295} - \underset{(0.0643)}{0.1279} r_{t-1} - \underset{(0.0466)}{0.5386} r_{t-2} + \underset{(0.0611)}{0.5875} r_{t-3} + \underset{(0.0585)}{0.0496} \varepsilon_{t-1} + \underset{(0.0418)}{0.4708} \varepsilon_{t-2} - \underset{(0.0531)}{0.6398} \varepsilon_{t-3} + \varepsilon_t$$

(B-2) Test for existence of ARCH/GARCH effects
Breusch-Pagen LM test for existence of ARCH/GARCH effect
Ho: No ARCH effect or homoskedasticity
Ha: Existence of ARCH effect or heteroscedasticity

I then run the Breusch-Pagan type (LM) test for the errors in my ARMA(3,3) with different values of q where q=1,2,3,4. We reject the null hypothesis that variance of error term in $r_{y,t}$ has no ARCH effect in all tests as the LM test statistics are all larger than their corresponding critical values from Chi-squared distribution at five percent significance level. Hence, the ARCH effect exists in variance of error term in $r_{y,t}$.

STATA output for ARMA (3,3)

```
ARIMA regression

Sample:  2 - 3303                               Number of obs     =       3302
                                                Wald chi2(6)      =   28060.65
Log likelihood = -5259.527                      Prob > chi2       =     0.0000

------------------------------------------------------------------------------
             |                 OPG
          ry |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
ry           |
       _cons |   .0294687   .0196633     1.50   0.134    -.0090707    .0680081
-------------+----------------------------------------------------------------
ARMA         |
          ar |
         L1. |  -.1279481   .0642943    -1.99   0.047    -.2539625   -.0019336
         L2. |  -.5385931   .0465675   -11.57   0.000    -.6298638   -.4473224
         L3. |   .5875072   .0611044     9.61   0.000     .4677448    .7072696
             |
          ma |
         L1. |   .0495883   .0584833     0.85   0.396    -.0650369    .1642136
         L2. |   .4708219   .0418494    11.25   0.000     .3887986    .5528453
         L3. |  -.6398336    .053076   -12.06   0.000    -.7438606   -.5358065
-------------+----------------------------------------------------------------
      /sigma |   1.189877   .0060871   195.47   0.000     1.177946    1.201807
------------------------------------------------------------------------------
```

AIC/BIC table for all tested models

Models	AIC	BIC
ARIMA (1,0,1)	10564.76	10589.17
ARIMA (1,0,2)	10559.13	10589.65
ARIMA (1,0,3)	10560.94	10597.55
ARIMA (1,0,4)	10558.12	10600.83
ARIMA (2,0,1)	10558.22	10588.74
ARIMA (2,0,2)	10559.06	10595.67
ARIMA (2,0,3)	10542.38	10585.09
ARIMA (2,0,4)	10535.47	10584.29

ARIMA (3,0,1)	10560.09	10596.71
ARIMA (3,0,2)	10555.13	10597.85
ARIMA (3,0,3)	10535.05*	10583.87*
ARIMA (3,0,4)	10537.03	10591.95
ARIMA (4,0,1)	10556.96	10599.67
ARIMA (4,0,2)	10536.01	10584.82
ARIMA (4,0,3)	10537.04	10591.96
ARIMA (4,0,4)	10538.25	10599.27

STATA output for LM test

```
      Source |       SS           df       MS      Number of obs   =     3,301
-------------+----------------------------------   F(1, 3299)      =    128.64
       Model |  3609.48334         1  3609.48334   Prob > F        =    0.0000
    Residual |  92566.5382     3,299  28.0589689   R-squared       =    0.0375
-------------+----------------------------------   Adj R-squared   =    0.0372
       Total |  96176.0216     3,300   29.144249   Root MSE        =    5.2971

------------------------------------------------------------------------------
       ehat2 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       ehat2 |
         L1. |    .193725   .0170804    11.34   0.000     .1602357    .2272143
             |
       _cons |   1.141939   .0953159    11.98   0.000     .9550545    1.328823
------------------------------------------------------------------------------

. dis e(N)*e(r2) > invchi2(1, 0.95)
1

      Source |       SS           df       MS      Number of obs   =     3,300
-------------+----------------------------------   F(2, 3297)      =    482.40
       Model |  21772.6679         2  10886.3339   Prob > F        =    0.0000
    Residual |   74403.009     3,297  22.5668817   R-squared       =    0.2264
-------------+----------------------------------   Adj R-squared   =    0.2259
       Total |  96175.6769     3,299  29.1529787   Root MSE        =    4.7505

------------------------------------------------------------------------------
       ehat2 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       ehat2 |
         L1. |   .1079086   .0156138     6.91   0.000     .0772949    .1385224
         L2. |   .4429679   .0156138    28.37   0.000     .4123541    .4735816
             |
       _cons |   .6361496   .0873338     7.28   0.000     .4649157    .8073835
------------------------------------------------------------------------------

. dis e(N)*e(r2) > invchi2(2, 0.95)
1
```

```
      Source |       SS           df       MS      Number of obs   =     3,299
-------------+----------------------------------   F(3, 3295)      =    338.70
       Model |  22667.9235         3  7555.97452   Prob > F        =    0.0000
    Residual |  73506.2746     3,295  22.3084293   R-squared       =    0.2357
-------------+----------------------------------   Adj R-squared   =    0.2350
       Total |  96174.1981     3,298    29.16137   Root MSE        =    4.7232

------------------------------------------------------------------------------
       ehat2 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       ehat2 |
         L1. |   .0592843   .0173157     3.42   0.001     .0253337    .0932348
         L2. |   .4311167   .0156363    27.57   0.000     .4004588    .4617745
         L3. |   .1097659   .0173157     6.34   0.000     .0758153    .1437165
             |
       _cons |   .5664768   .0875422     6.47   0.000     .3948342    .7381195
------------------------------------------------------------------------------

. dis e(N)*e(r2) > invchi2(3, 0.95) //1 means reject the null
1

      Source |       SS           df       MS      Number of obs   =     3,298
-------------+----------------------------------   F(3, 3294)      =    154.71
       Model |  11877.1877         3  3959.06257   Prob > F        =    0.0000
    Residual |  84295.8052     3,294  25.5907119   R-squared       =    0.1235
-------------+----------------------------------   Adj R-squared   =    0.1227
       Total |  96172.9929     3,297  29.1698492   Root MSE        =    5.0587

------------------------------------------------------------------------------
       ehat2 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       ehat2 |
         L1. |   .0726709   .0186265     3.90   0.000     .0361503    .1091915
         L3. |   .1363213   .0185175     7.36   0.000     .1000144    .1726282
         L3. |          0  (omitted)
         L4. |   .2616511   .0168528    15.53   0.000     .2286081    .2946942
             |
       _cons |   .7517762    .093696     8.02   0.000     .5680679    .9354844
------------------------------------------------------------------------------

. dis e(N)*e(r2) > invchi2(4, 0.95) //1 means reject the null
1
```

(C)

If we also set the cap on ARCH(q) at 4 and combine with the best ARMA model from 1(B), then compare ARMA(3,3)-ARCH(1), ARMA(3,3)-ARCH(2), ARMA(3,3)-ARCH(3), and ARMA(3,3)-ARCH(4) models. We can soon find out that ARMA(3,3)-ARCH(4) model will have lowest AIC and BIC; therefore, the best q should be 4 if the cap is set at ARCH(4). Moreover, the coefficients in ARMA(3,3)-ARCH(4) are also significant. However, the above is the result with setting the cap. Now, what if we don’t set a cap?

The problem is that if we test q>4, we will find that AIC/BIC becomes smaller as q gets larger, Also, some coefficients will not be significant. This is the common phenomenon of over-parameterlization which will cause undesired outcome. The consequence is that adding more lags to q will reduce the sum of squares of the estimated residuals, but it will result in estimation of more coefficients and suffer from loss of degrees of freedom.

STATA output of ARMA(3,3)-ARCH(4)

```
ARCH family regression -- ARMA disturbances

Sample: 2 - 3303                                Number of obs     =      3,302
Distribution: Gaussian                          Wald chi2(6)      =   24771.48
Log likelihood = -4415.494                      Prob > chi2       =     0.0000
```

ry	Coef.	OPG Std. Err.	z	P>|z|	[95% Conf.	Interval]
ry						
_cons	.0831714	.0096426	8.63	0.000	.0642722	.1020706
ARMA						
ar						
L1.	.1385098	.064744	2.14	0.032	.0116138	.2654057
L2.	-.518805	.0401265	-12.93	0.000	-.5974515	-.4401584
L3.	.7468258	.0626941	11.91	0.000	.6239476	.869704
ma						
L1.	-.1959665	.0573787	-3.42	0.001	-.3084267	-.0835063
L2.	.4863421	.0351008	13.86	0.000	.4175458	.5551385
L3.	-.7948217	.0549922	-14.45	0.000	-.9026045	-.6870389
ARCH						
arch						
L1.	.1047661	.013174	7.95	0.000	.0789456	.1305867
L2.	.2478873	.0235981	10.50	0.000	.2016358	.2941388
L3.	.2235611	.0189321	11.81	0.000	.1864548	.2606674
L4.	.2715865	.0182428	14.89	0.000	.2358313	.3073417
_cons	.2796734	.0107399	26.04	0.000	.2586237	.3007232

AIC/BIC table for ARMA-ARCH(q) models

Models	AIC	BIC
ARMA(3,3)-ARCH(1)	9862.591	9917.511
ARMA(3,3)-ARCH(2)	9305.719	9366.742
ARMA(3,3)-ARCH(3)	9072.369	9139.494
ARMA(3,3)-ARCH(4)	8854.989*	8928.216*
Below shows the AIC/BIC after ARMA(3,3)-ARCH(4)		
ARMA(3,3)-ARCH(5)	8797.464	8876.794
ARMA(3,3)-ARCH(6)	8755.904	8841.336
ARMA(3,3)-ARCH(7)	8730.150	8821.684
ARMA(3,3)-ARCH(8)	8697.787	8795.424
ARMA(3,3)-ARCH(9)	8670.068	8773.807
ARMA(3,3)-ARCH(10)	8652.177	8762.018
ARMA(3,3)-ARCH(11)	8645.007	8760.951*
ARMA(3,3)-ARCH(12)	8646.814	8768.860
ARMA(3,3)-ARCH(13)	8648.752	8776.900
ARMA(3,3)-ARCH(14)	8645.582	8779.832
ARMA(3,3)-ARCH(15)	8642.157*	8782.509
ARMA(3,3)-ARCH(16)	8643.987	8790.442
ARMA(3,3)-ARCH(17)	8644.812	8797.369
ARMA(3,3)-ARCH(18)	8646.81	8805.469
ARMA(3,3)-ARCH(19)	8648.317	8813.079
ARMA(3,3)-ARCH(20)	8649.779	8820.643

(D)

I compare several candidate models: ARMA(3,3)-GARCH(1,1), ARMA(3,3)-GARCH(1,2), ARMA(3,3)-GARCH(1,3), ARMA(3,3)-GARCH(2,1), ARMA(3,3)-GARCH(2,2). Among them, ARMA(3,3)-GARCH(1,1) is preferred because it has the lowest BIC.

The estimated model for ARMA(3,3)-GARCH(1,1) can be shown as:

$$\begin{aligned} r_t = &\underset{(0.0108)}{0.0761} + \underset{(0.0779)}{0.1875} r_{t-1} - \underset{(0.0484)}{0.4894} r_{t-2} + \underset{(0.0770)}{0.8028} r_{t-3} \\ &- \underset{(0.0686)}{0.2337}\varepsilon_{t-1} + \underset{(0.0425)}{0.4597}\varepsilon_{t-2} - \underset{(0.0671)}{0.8420}\varepsilon_{t-3} + \varepsilon_t \end{aligned}$$

$$\varepsilon_t = \upsilon_t\sqrt{h_t}$$

$$h_t = \underset{(0.0024)}{0.0237} + \underset{(0.0087)}{0.1261}\varepsilon_{t-1}^2 + \underset{(0.0095)}{0.8548} h_{t-1}$$

BIC table ARMA-GARCH(p,q) models

Models	BIC
ARMA(3,3)-GARCH(1,1)	8714.184*
ARMA(3,3)-GARCH(1,2)	9007.388
ARMA(3,3)-GARCH(1,3)	8915.182
ARMA(3,3)-GARCH(2,1)	8756.743
ARMA(3,3)-GARCH(2,2)	8859.561

STATA output for ARMA(3,3)-GARCH(1,1)

```
ARCH family regression -- ARMA disturbances

Sample: 2 - 3303                                 Number of obs   =      3,302
Distribution: Gaussian                           Wald chi2(6)    =   36877.42
Log likelihood = -4316.581                       Prob > chi2     =     0.0000

------------------------------------------------------------------------------
             |                 OPG
          ry |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
ry           |
       _cons |   .0761292   .0107808     7.06   0.000     .0549991    .0972593
-------------+----------------------------------------------------------------
ARMA         |
          ar |
         L1. |   .1874695   .0778793     2.41   0.016     .0348289    .3401102
         L2. |  -.4893534   .0484142   -10.11   0.000    -.5842436   -.3944633
         L3. |   .8028334   .0770069    10.43   0.000     .6519027    .9537641
             |
          ma |
         L1. |  -.2336743    .068604    -3.41   0.001    -.3681357    -.099213
         L2. |   .4596667   .0425058    10.81   0.000     .3763569    .5429766
         L3. |  -.8419535   .0671245   -12.54   0.000    -.9735151   -.7103919
-------------+----------------------------------------------------------------
ARCH         |
        arch |
         L1. |   .1260865   .0087495    14.41   0.000     .1089379    .1432352
             |
       garch |
         L1. |   .8547735   .0094542    90.41   0.000     .8362437    .8733033
             |
       _cons |   .0237462   .0023515    10.10   0.000     .0191373    .0283551
------------------------------------------------------------------------------
```

(E)

The threshold coefficient is -0.2489 and is significantly different from 0. Hence, the t-test result provides strong evidence for the existence of leverage effect which means that volatility has tendency to decline when $r_{y,t}$ goes up and to elevate when $r_{y,t}$ goes down.

ARMA-TARCH leverage effect (λ) test

Ho: $\lambda = 0$

Ha: $\lambda > 0$

The estimated model for this ARMA-TARCH can be shown as:

$$r_t = 0\ .0351 - 1.0067r_{t-1} - 0.7401r_{t-2} - 0.6879r_{t-3}$$

(0.0132) (0.2035) (0.2400) (0.1361)

$$+ 0.9724\varepsilon_{t-1} + 0.6968\varepsilon_{t-2} + 0.6899\varepsilon_{t-3} + \varepsilon_t$$

(0.2029) (0.2321) (0.1380)

$$\varepsilon_t = \upsilon_t\sqrt{h_t}$$

$$h_t = 0.0257 + 0.2267\varepsilon_{t-1}^2 - 0.2489d_{t-1}\varepsilon_{t-1}^2 + 0.8714h_{t-1}$$

(0.0021) (0.0163) (0.0173) (0.0091)

STATA output for ARMA(3,3)-TARCH model

```
ARCH family regression -- ARMA disturbances

Sample: 2 - 3303                                Number of obs     =      3,302
Distribution: Gaussian                          Wald chi2(6)      =    2471.86
Log likelihood = -4241.674                      Prob > chi2       =     0.0000

------------------------------------------------------------------------------
             |                 OPG
          ry |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
ry           |
       _cons |   .0351308   .0132038     2.66   0.008     .0092518    .0610097
-------------+----------------------------------------------------------------
ARMA         |
          ar |
         L1. |   -1.00669   .2034746    -4.95   0.000    -1.405493   -.6078873
         L2. |  -.7401432   .2400422    -3.08   0.002    -1.210617   -.2696692
         L3. |  -.6879237   .1360943    -5.05   0.000    -.9546636   -.4211838
             |
          ma |
         L1. |   .9723864   .2029096     4.79   0.000     .5746908    1.370082
         L2. |   .6968229   .2320916     3.00   0.003     .2419317    1.151714
         L3. |   .6899381   .1379915     5.00   0.000     .4194796    .9603965
-------------+----------------------------------------------------------------
ARCH         |
        arch |
         L1. |   .2266948   .0163396    13.87   0.000     .1946697    .2587198
             |
      tarch  |
         L1. |  -.2488891   .0173478   -14.35   0.000    -.2828902    -.214888
             |
       garch |
         L1. |   .8713512   .0090755    96.01   0.000     .8535636    .8891389
             |
       _cons |   .0256514   .0021439    11.96   0.000     .0214495    .0298534
------------------------------------------------------------------------------
```

(F)

This GARCH-M model with a GARCH(2,1) parameterization is not favourable to the existence of a time-varying risk premium as the coefficient on h_t in the conditional mean model is not significant at five percent significance level. Therefore, we cannot conclude that the increased expected return is caused by the conditional variance.

GARCH-M time-varying risk premium (λ) test
Ho: λ = 0
Ha: λ > 0

The estimated model for this GARCH-M with GARCH(2,1) can be shown as:

$$r_t = \underset{(0.0166)}{0.0581} + \underset{(0.0156)}{0.0278}h_t + \varepsilon_t$$

$$\varepsilon_t = \upsilon_t\sqrt{h_t}$$

$$h_t = \underset{(0.0025)}{0.0197} + \underset{(0.0111)}{0.0979}\varepsilon_{t-1}^2 + \underset{(0.0981)}{1.1927}h_{t-1} - \underset{(0.0872)}{0.3064}h_{t-2}$$

STATA output for GARCH-M model

```
ARCH family regression

Sample: 2 - 3303                              Number of obs    =      3,302
Distribution: Gaussian                        Wald chi2(1)     =       3.18
Log likelihood = -4327.446                    Prob > chi2      =     0.0747
```

ry	Coef.	OPG Std. Err.	z	P>|z|	[95% Conf.	Interval]
ry						
_cons	.0581438	.0165553	3.51	0.000	.0256961	.0905915
ARCHM						
sigma2	.0277773	.0155832	1.78	0.075	-.0027652	.0583198
ARCH						
arch						
L1.	.0979044	.0110715	8.84	0.000	.0762046	.1196043
garch						
L1.	1.192714	.0980694	12.16	0.000	1.000502	1.384927
L2.	-.3063752	.0871774	-3.51	0.000	-.4772398	-.1355105
_cons	.01971	.0024847	7.93	0.000	.0148402	.0245799

Q2

(A)

AIC suggests using p = 5. BIC and LR test are the alternatives that can also help selecting p but may give conflicting results. The major difference is that AIC is better when the sample size is small and BIC tend to select the more parsimonious model. The LR test is Chi-square distributed and has degrees of freedom equal to the number of max lags we choose to test. The suggestion of LR test in this case may result in over-parameterlization.

STATA output for optimal lag selection table

```
Selection-order criteria
Sample:  1070 - 3303                          Number of obs      =      2234
```

lag	LL	LR	df	p	FPE	AIC	HQIC	SBIC
0	-5146.35				.020168	4.60998	4.61279	4.61765
1	-5056.81	179.08	9	0.000	.018765	4.53788	4.54908	4.56856*
2	-5034.45	44.734	9	0.000	.018542	4.52592	4.54552*	4.5796
3	-5026.02	16.851	9	0.051	.018552	4.52643	4.55443	4.60313
4	-5018.3	15.435	9	0.080	.018573	4.52758	4.56398	4.62729
5	-5003.16	30.294	9	0.000	.018471*	4.52207*	4.56688	4.64479
6	-4999.37	7.571	9	0.578	.018557	4.52674	4.57995	4.67247
7	-4991.6	15.544	9	0.077	.018578	4.52784	4.58945	4.69658
8	-4983.08	17.033	9	0.048	.018586	4.52828	4.59829	4.72002
9	-4969.44	27.29	9	0.001	.018509	4.52412	4.60253	4.73888
10	-4959.87	19.136*	9	0.024	.018499	4.52361	4.61042	4.76138

(B)

Yes, the system R_t satisfies the stationary condition because there is no root or eigenvalues lies outside the unit circle. Therefore, spurious regression can be prevented. After computing the residual autocorrelation from the LM test, I find that we cannot reject the null hypothesis that there is no autocorrelation for p = 5, which is consistent with suggestion by AIC, at five percent significance level. We need to eliminate autocorrelation that results in inaccurate computed variance and standard error of forecast. Since there is no autocorrelation in the residuals and the system is stationary at p=5, this model is adequate.

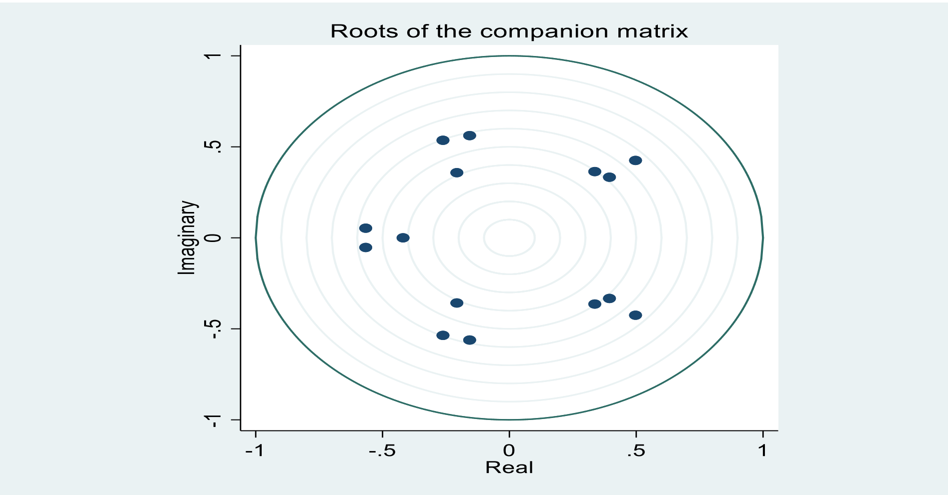
Lagrange-multiplier (LM) test
Ho: No autocorrelation in the residuals at lag p
Ha: Autocorrelation in the residuals at lag p

STATA output of LM test

Lagrange-multiplier test

lag	chi2	df	Prob > chi2
1	6.8290	9	0.65492
2	11.8910	9	0.21953
3	9.0051	9	0.43681
4	16.0774	9	0.06528
5	8.0389	9	0.53023
6	5.8186	9	0.75792
7	18.2391	9	0.03250
8	18.6395	9	0.02844
9	17.8694	9	0.03672
10	23.7967	9	0.00463
11	5.0263	9	0.83201
12	13.3923	9	0.14564

STATA unit circle graph for checking VAR stability



(C)

The advantage of companion representation is that it is useful in analysing VAR(p) model as the stationarity of the dynamic system can be assessed by checking the eigenvalues of the matrix.

The VAR(5) can be expressed as:

$R_t = a_0 + \sum_{j=1}^{5} A_j\, R_{t-j} + e_t$

The companion form of this model is expressed as:

$$\begin{pmatrix} R_t \\ R_{t-1} \\ R_{t-2} \\ R_{t-3} \\ R_{t-4} \end{pmatrix} = \begin{bmatrix} a_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ I_3 & 0 & 0 & 0 & 0 \\ 0 & I_3 & 0 & 0 & 0 \\ 0 & 0 & I_3 & 0 & 0 \\ 0 & 0 & 0 & I_3 & 0 \end{pmatrix} \begin{pmatrix} R_{t-1} \\ R_{t-2} \\ R_{t-3} \\ R_{t-4} \\ R_{t-5} \end{pmatrix} + \begin{pmatrix} e_t \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Specifically, VAR(p) is stable if the eigenvalues of the companion matrix have modules less than 1.

$$\tilde{A}_1 = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ I_3 & 0 & 0 & 0 & 0 \\ 0 & I_3 & 0 & 0 & 0 \\ 0 & 0 & I_3 & 0 & 0 \\ 0 & 0 & 0 & I_3 & 0 \end{pmatrix}$$

(D)

The structural form of VAR model is shown below as:

$\Theta R_t = \Omega_0 + \sum_{i=1}^{n} \Theta_i R_{t-i} + U_t$

Where Θ is coefficient matrix of R_t and its diagonal elements are all one. Ω_0 is a parameter vector. U_t represents the structural disturbance.

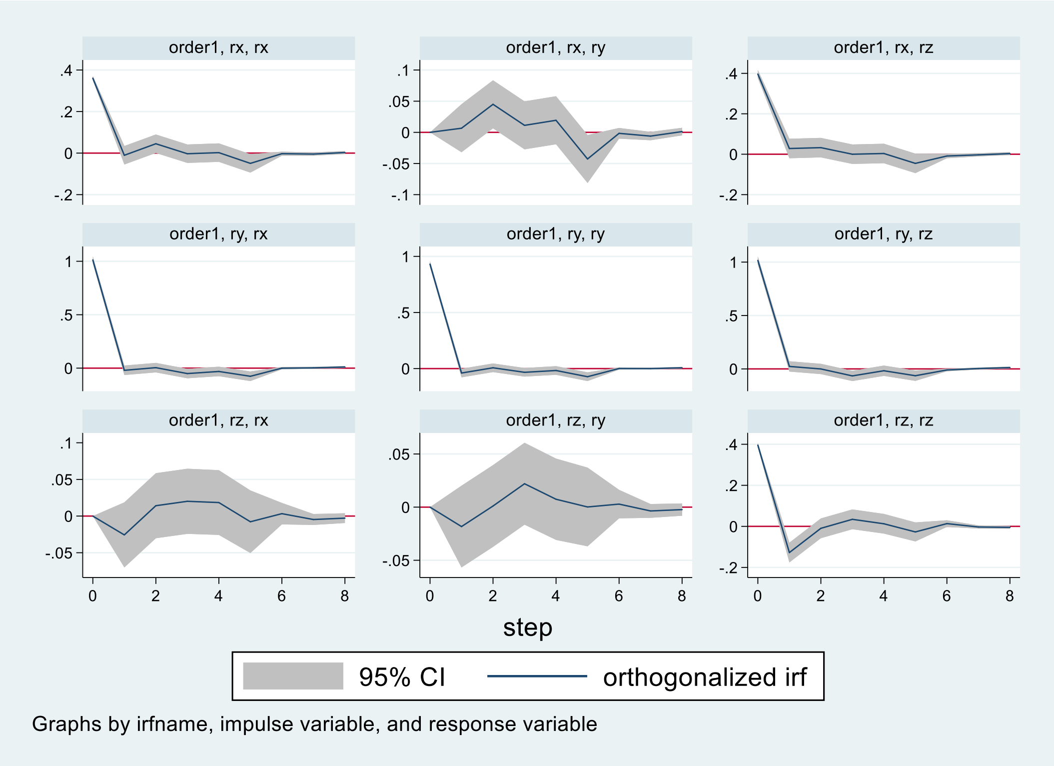
Panel of R_t that exhibits the relationship between the reduced form errors and the structural disturbances is shown below: (Note: X is the coefficient to be estimated)

$$\begin{bmatrix} \mathrm{Ury} \\ \mathrm{Urx} \\ \mathrm{Urz} \end{bmatrix} \begin{pmatrix} 1 & \square & \square \\ X & 1 & \square \\ X & X & 1 \end{pmatrix} = \begin{pmatrix} 1 & \square & \square \\ \square & 1 & \square \\ \square & \square & 1 \end{pmatrix} \begin{bmatrix} \varepsilon_{\mathrm{ry}} \\ \varepsilon_{rx} \\ \varepsilon_{rz} \end{bmatrix}$$

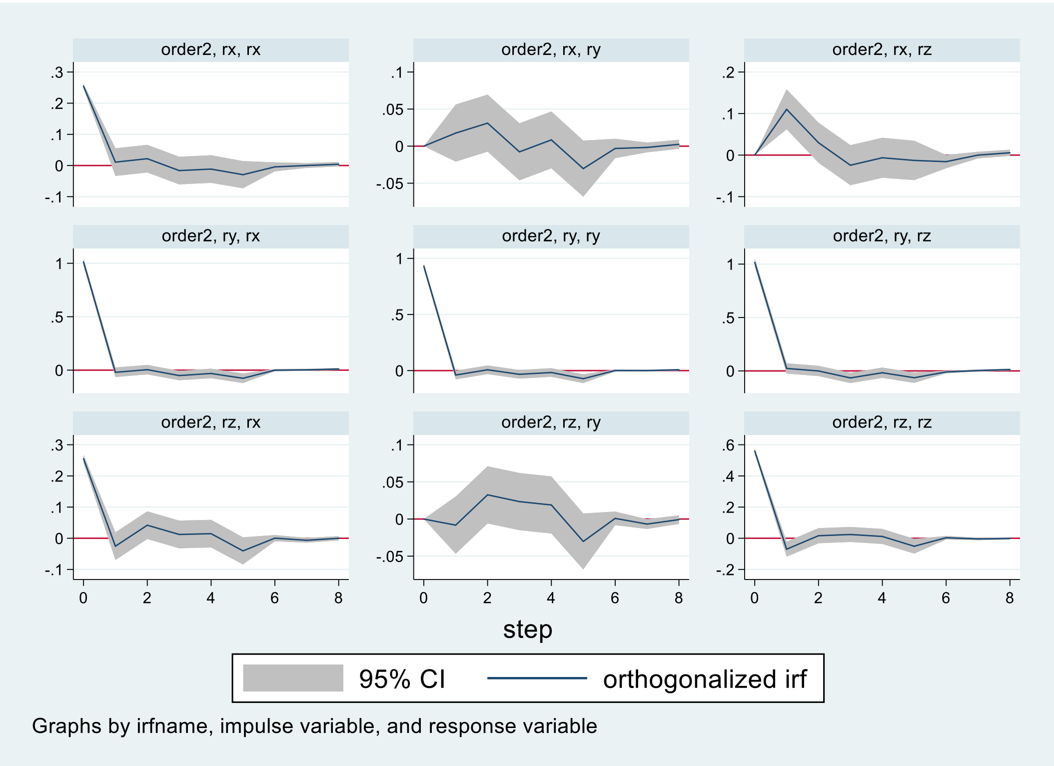
(D-1)

Below is the STATA output of all the ordering where ry denotes $r_{y,t}$, rx denotes $r_{x,t}$, and rz denotes $r_{z,t}$. The system appears to be sensitive to ordering because direction, shape, and persistence of impulse response to the same shock can be quite diverse under different ordering.

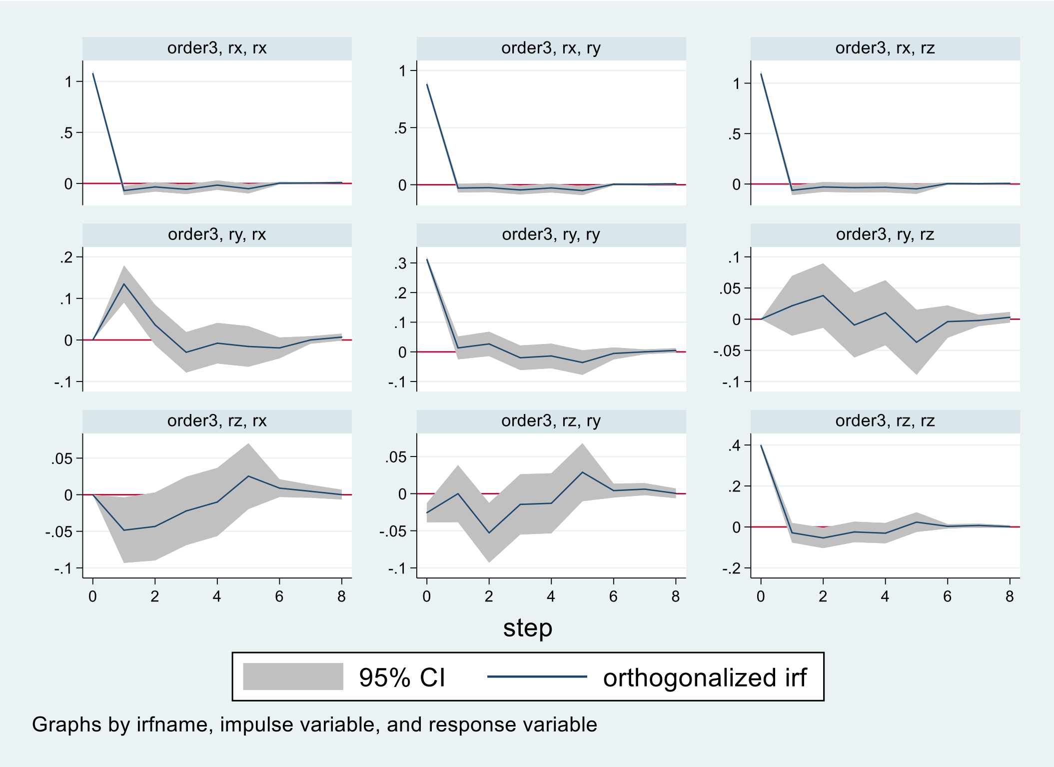
Order 1(ry rx rz)



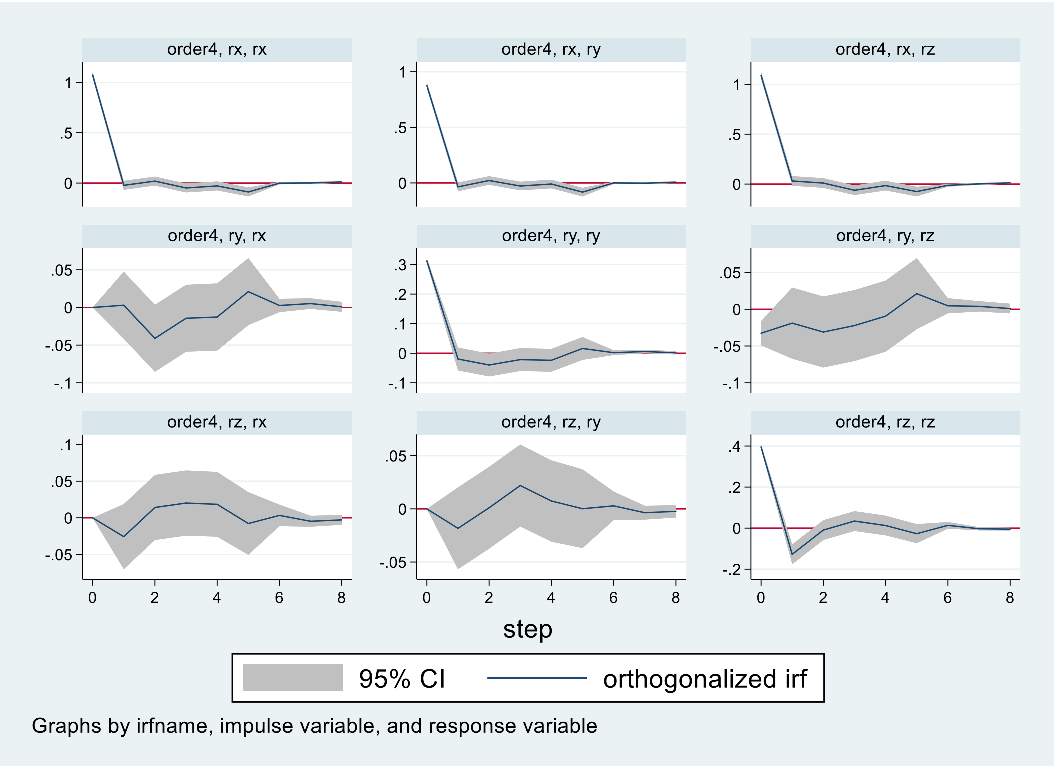
Order 2(ry rz rx)



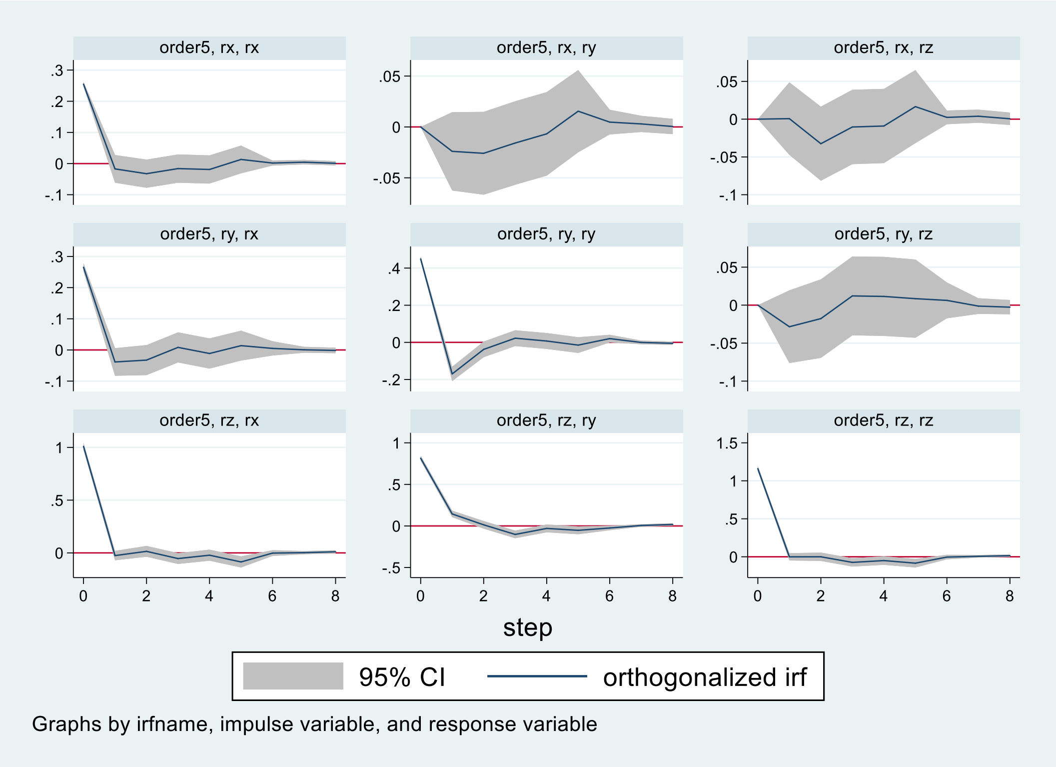
Order 3(rx rz ry)



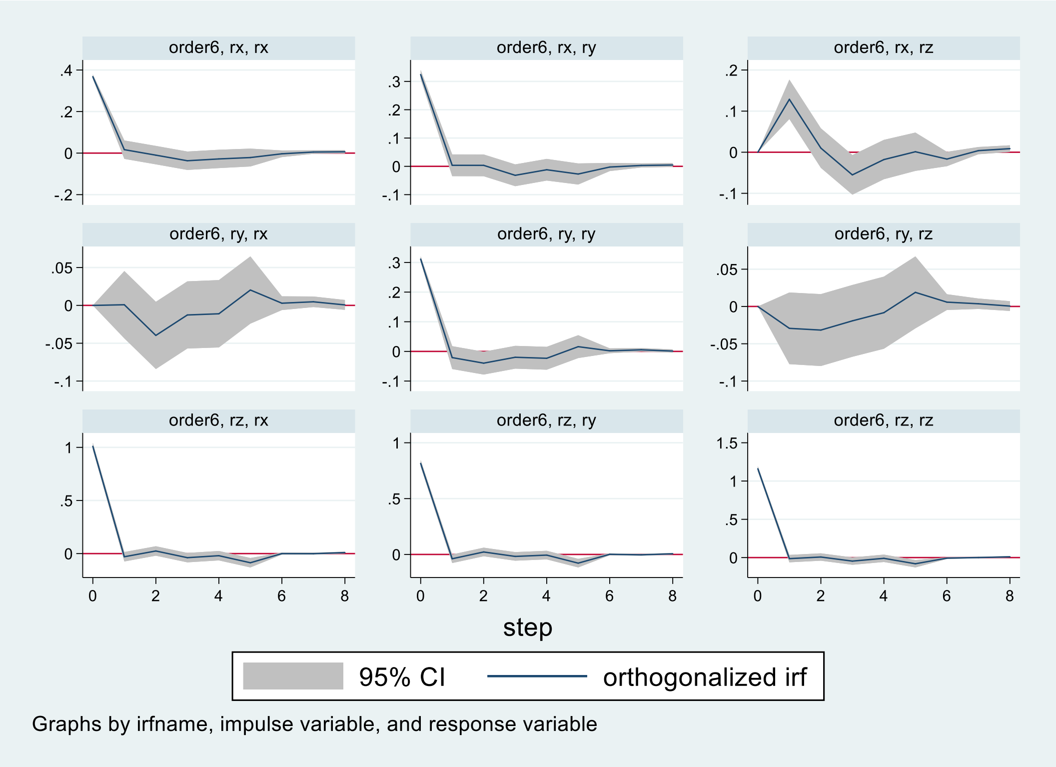
Order 4(rx ry rz)



Order 5(rz ry rx)



Order 6(rz rx ry)



(D-2)

Round I

ry rx rz vs ry rz rx
Patterns are all very similar. Nevertheless, the response of rz to exogenous shock in rx is larger when the order is (ry rx rz) but response of rx to unexpected shock in rz is greater when the order is (ry rz rx).The response of rx to exogenous shock in itself is more significant when the order is (ry rx rz) while the response of rz to surprise change in itself is larger when the order is (ry rz rx). At this point, they are tied. Since rz is more sensitive to exogenous shock in rx when rx comes before rz, the order (ry rx rz) makes more sense. Overall, the order (ry rx rz) is a better model.

rx rz ry vs rx ry rz
The order (rx rz ry) is clearly better. The response of rz to exogenous shock in ry is stronger four periods after the initial shock when the order is (rx rz ry). The response of rx to shock in ry is stronger when the order is (rx rz ry). The response of ry to shock in rz is stronger when the order is (rx rz ry). The response of rx to unexpected change in rz is greater when the order is (rx rz ry). Others are either tied or order (rx ry rz) only has two sets that have stronger actions. In sum, all of the above conditions indicate that the model with order (rx rz ry) is more sound.

rz ry rx vs rz rx ry
At the first glance, their patterns look quite even. However, the response of ry to exogenous shock in rx is much more significant under the order (rz rx ry). The response of ry to unexpected change in rz is also slightly more significant under the order (rz rx ry). The response of rz to sudden and unexpected change in rx is also greater under the order (rz rx ry). The response of rx to surprise change in itself is slightly larger under the order (rz rx ry). In contrast, only the response of rx to shock in ry and the response of ry to unexpected change in itself are clearly more significant under order (rz ry rx). Apparently, the better model is with the order (rz rx ry).

Round II

ry rx rz vs rx rz ry vs rz rx ry
I then compare the winners which are orders (ry rx rz), (rx rz ry), and (rz rx ry) from the round I comparison. The order (rx rz ry) is eliminated because it is clearly the worst out of three orders due to lower responses in most cases except for the responses to exogenous shock in rx.

Round III

ry rx rz vs rz rx ry

The response of rx to change to itself is greater under the order (ry rx rz). The response of rz to unexpected change in rx is more significant under the order (ry rx rz). The response of rx to unexpected shock in ry is larger under the order (ry rx rz). The response of rz to sudden change in ry is larger under the order (ry rx rz). The response of ry to unexpected change in itself is larger under the order (ry rx rz).

In contrast, the response of ry to unexpected change in rx is clearly greater under the order (rz rx ry). The response of rx to exogenous shock in rz is stronger under the order (rz rx ry). The response of ry to surprise change in rz is larger under the order (rz rx ry). The response of rz to surprise shock in itself is greater under the order (rz rx ry).

Putting everything together, the order (ry rx rz) is a slightly better model. Therefore, the large cap index (ry) should be the strongest mover, followed by mid cap (rx), and small cap (rz).

<u>Conclusion</u>

Stocks or companies belong to S&P 500 large cap or SPY tend to have much larger liquidity and is being tracked by more institutional investors including traders of proprietary trading desk at investment bank, commercial bank, long-only fund, and quantitative trading fund. In addition, traders at large banks are blocked from trading small cap stocks by middle office risk managers because large banks usually set very high limit regarding liquidity risk unless he/she belongs to market making desk for mid/small equties. Therefore, the large cap should be the first in the order and the order (ry rx rz) is inevitably the best model of all.

(D-3)

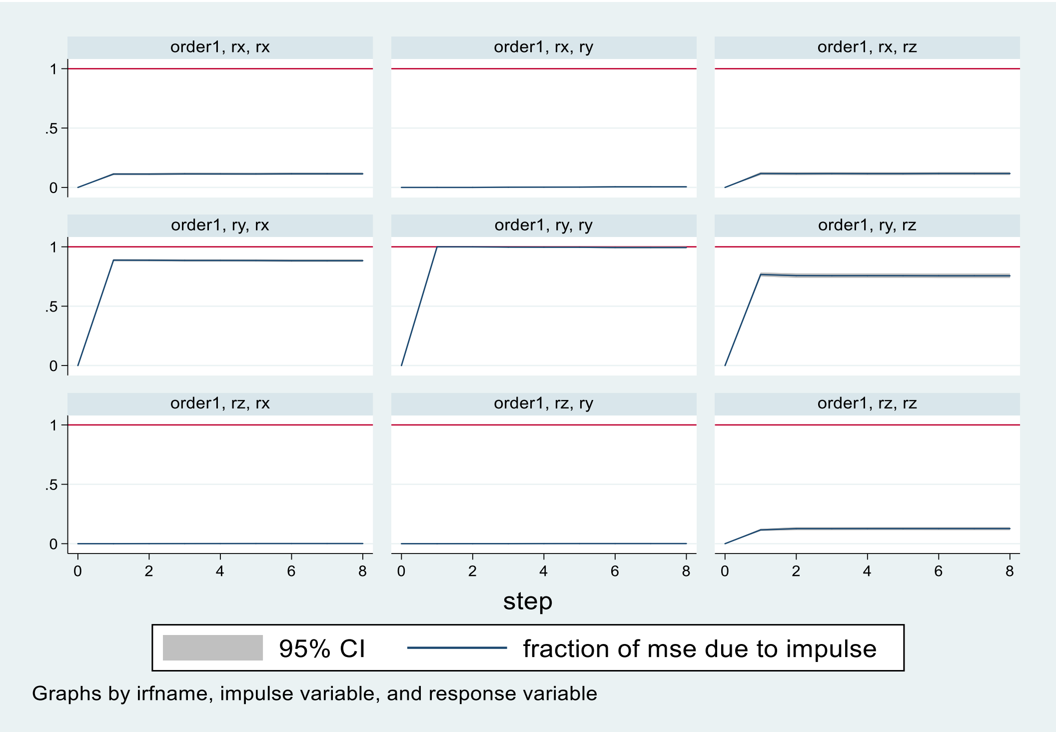
The forecast error variance decomposition (FEVD):

1. Most of the forecast error variances in the return of rx are not explained by changes in itself but by changes in ry. Thus, rx is not exogenous.

2. The entire forecast error of variance in the return of ry is due to change in changes in itself. Thus, ry is exogenous.

3. Most of the forecast error of variances in the return of rz are not due to changes in itself but by changes in ry followed by changes in rx. Hence, rz is not exogenous because it can be explained by other two variables.

STATA graph of forecast error variance decomposition



(E)

We can use Granger causality Wald test to determine whether lags of one variable Granger cause other variables. From the Granger causality Wald test, we fail to reject most of them at the five percent significance level excepts “MDY Granger cause SLY” at five percent significance level with five lags. Besides, the test also shows that all Granger cause SLY at five percent significance level with ten lags. To interpret, we can say that lags of MDY Granger cause lags of SLY in the system. MDY is said to Granger-cause a variable SLY if, given the past values of SLY, past values of MDY are useful for predicting SLY.

A three-variable causal model can be expressed as:

X_t = a_1(U) X_t + b_1(U) Y_t + c_1(U) Z_t + $\varepsilon_{1,t}$

Y_t = a_2(U) X_t + b_2(U) Y_t + c_2(U) Z_t + $\varepsilon_{2,t}$

Z_t = a_3(U) X_t + b_3(U) Y_t + c_3(U) Z_t + $\varepsilon_{3,t}$

Where a_i(U) are polynomials in U, the shift operator. $\varepsilon_{i,t}$ are uncorrelated, white-noise series and denote the variance $\varepsilon_{i,t} = \sigma_i^2$.

Granger causality Wald test
Ho: No Granger causality
Ha: Granger causality exists

Table of Granger causality Wald test

Equation	Excluded	χ^2	df	Prob. > χ^2	Decision
ry	rx	6.4427	5	0.265	MDY not Granger cause SPY at 5% significance level
ry	rz	2.3076	5	0.805	SLY not Granger cause SPY at 5% significance level
rx	ry	4.6154	5	0.465	SPY not Granger cause MDY at 5% significance level
rx	rz	3.1617	5	0.675	SLY not Granger cause MDY at 5% significance level
rz	ry	5.4575	5	0.363	SPY not Granger cause SLY at 5% significance level
rz	rx	24.644	5	0.000	MDY Granger cause SLY at 5% significance level

STATE output of Granger causality Wald test

Granger causality Wald tests

Equation	Excluded	chi2	df	Prob > chi2
ry	rx	6.4427	5	0.265
ry	rz	2.3076	5	0.805
ry	ALL	13.488	10	0.198
rx	ry	4.6154	5	0.465
rx	rz	3.1617	5	0.675
rx	ALL	8.0142	10	0.627
rz	ry	5.4575	5	0.363
rz	rx	24.644	5	0.000
rz	ALL	38.027	10	0.000