American options
American options **do not have a closed-form (analytical)** solution.
The value of an American options is **always greater than or equal to its intrinsic value.**
American options **do not have a closed-form (analytical)** solution.

The **finite difference method** is one of the most widely method approximation to solve the PDE equation for American options. The Black-Scholes PDE can also be used to price American options. The main difference between European options and American options is that the latter can be executed any time prior to the expiry date.

The **European option**s under BS assumption has an **analytical solution for the fair pri**ce. In contrast, the **American option** does not have an analytical solution so the PDE has to be solved using technique like the **finite difference numerical methods.**

Optimal exercise boundary of an American option
It is not known in advance but has to be determined as part of the solution process of the pricing model.

The early exercise
American option worth more than its European counterpart because of the early exercise premium.

For American call, the older gains on the dividend yield from the asset but loses on the time value of the strike price. There is no advantage to exercise an American call prematurely when the asset received upon early exercise does not pay dividends. In this case, the American call has the same value as that of its European counterpart.

Why never optimal to early exercise American call?
The reason is that exercise requires payment of the strike price X. By holding onto X until the expiration time, the **option holder saves the interest on X.**

For an American put
It is sometimes optimal. Suppose the stock price S falls to nearly 0, then the option holder stands to gain more by exercise than by waiting. The reason is that the payout X-S cannot increase much, **but by early exercise, the option holder will get the interest on the payout.**

American put (discrete dividends)

It may or may not have jumps of discontinuity, depending on the size of the discrete dividend payments.

American call (discrete dividends)

Optimal early exercise occur only at those times immediately before the asset goes ex-dividend.

Optimal early exercise boundary

Let us define a free boundary by the value $0 < S_f < K$ such that if the current stock price S_t is smaller than S_f, then it is optimal to exercise the option, otherwise it is optimal to wait.

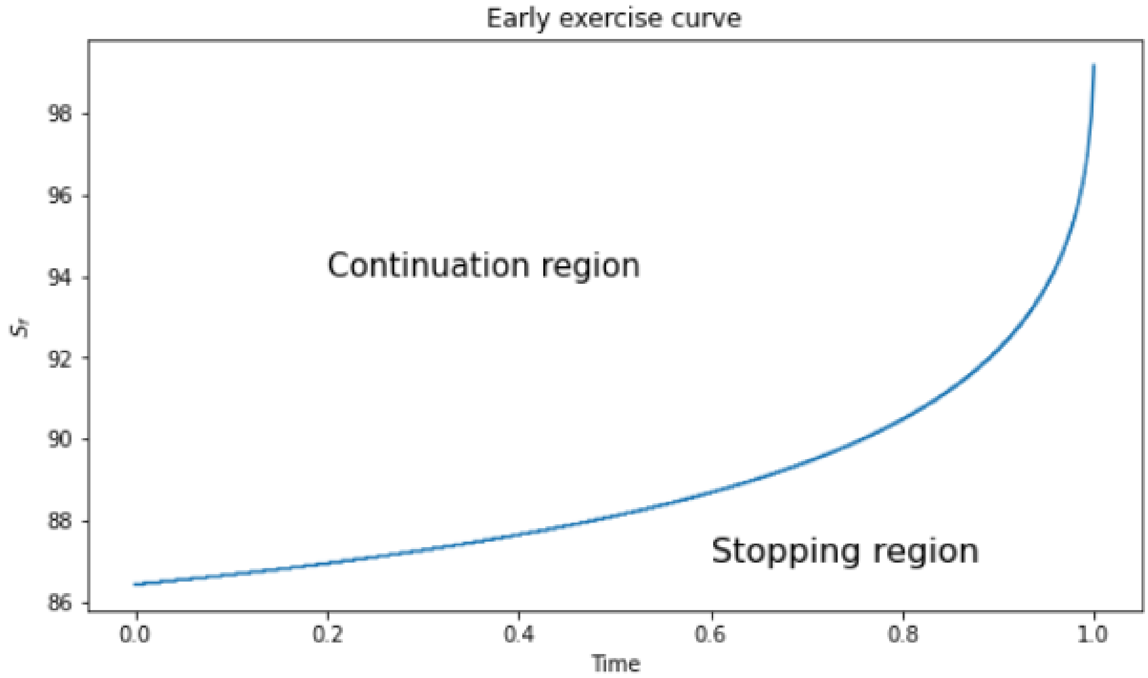
- $S_t < S_f$ called **stopping region**
- $S_t > S_f$ called **continuation region**

In the stopping region, the value of the option corresponds to its intrinsic value) i.e. $V(t, S_t) = K - S_t$.

In order to find S_f we have to find the maximum value s such that $V(t, s) - (K - s) = 0$:

```
payoff = BS.payoff_f(BS.S_vec).reshape(len(BS.S_vec),1)    # Transform the payoff in a column vector
mesh = (BS.mesh - payoff)[1:-1,:-1]                         # I remove the boundary terms
optimal_indeces = np.argmax( np.abs(mesh)>1e-10, axis=0  ) # I introduce the error 1e-10
T_vec = np.linspace(0, BS.T, N_time)                        # Time vector
```

```
fig = plt.figure(figsize=(9,5))
plt.plot(T_vec[:-1], BS.S_vec[optimal_indeces])
plt.text(0.2, 94, "Continuation region", fontsize=15)
plt.text(0.6, 87, "Stopping region", fontsize=16)
plt.xlabel("Time"); plt.ylabel("$S_f$"); plt.title("Early exercise curve");
plt.show()
```



Binomial tree

The binomial tree method was pioneered by Cox, Ross, and Robinstein in 1979. The binomial model allows the stock to move up or down a specific amount over the next time step. Given the initial stock price S, we allow it to either go up or down by a factor u and v resulting in the value uS and vS after the next time step.

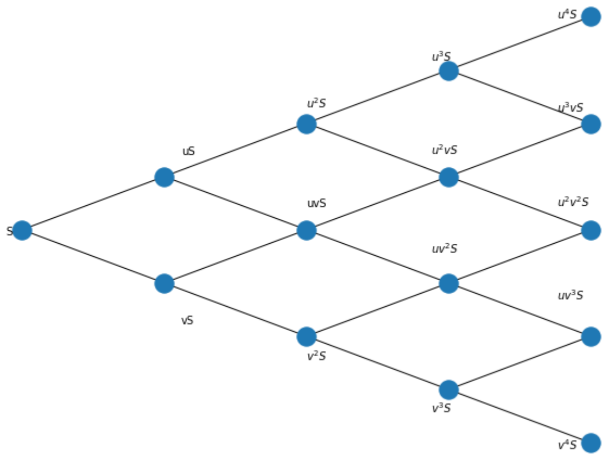
```python
# Import math functions from NumPy
from numpy import *

# Import plotting functions from helper
from helper import plot_asset_path, plot_probability, plot_binomial_tree
```

Price Path

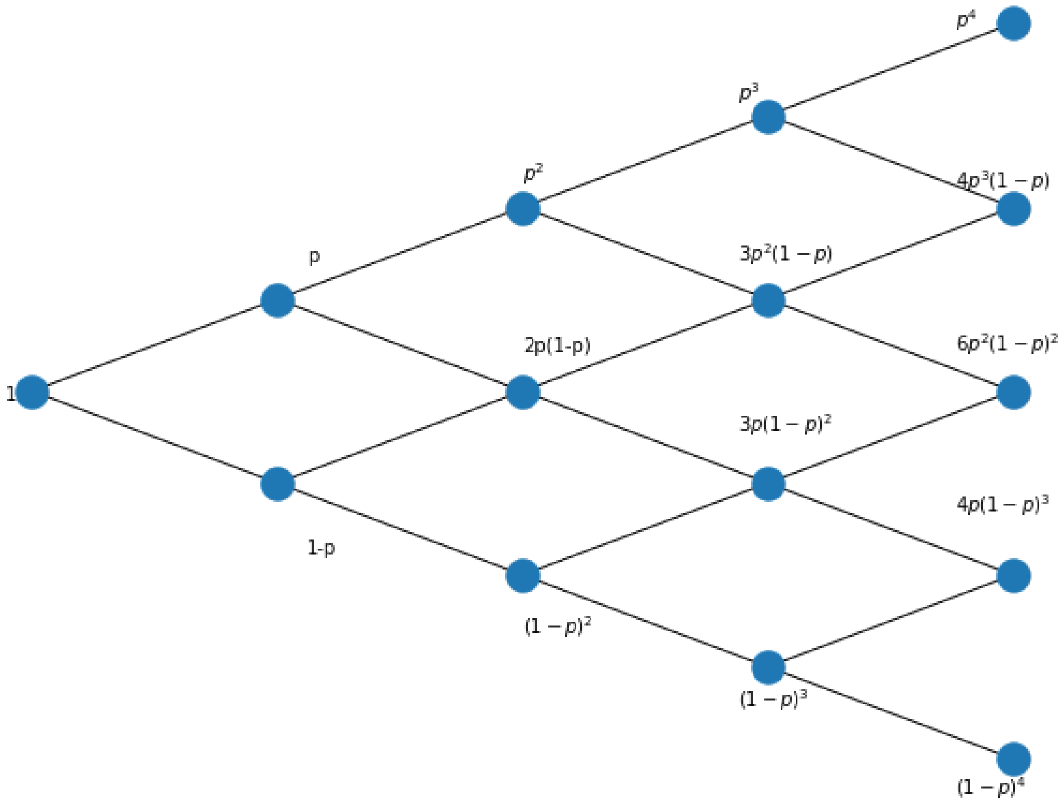
The probability of reaching a particular node in the binomial tree depends on the numbers of distinct paths to that node and the probabilities of the up and down moves. The following figures shows the number of paths to each node and the probability of reaching to that node.

```python
[2]: # Plot asset price path
plot_asset_path()
```



Path Probability

```
# Plot node probability
plot_probability()
```



Risk Neutral Probability

Risk-neutral measure is a probability measure such that each share price today is the discounted expectations of the share price. From Paul's lecture, we know the formula for u, v, p' and V are as follows,

$u = 1 + \sigma\sqrt{\delta t}$

$v = 1 - \sigma\sqrt{\delta t}$

The underlying instrument will move up or down by a specific factor u or v per step of the tree where $u \geq 1$ and $0 < v \leq 1$.

$p' = \frac{1}{2} + \frac{r\sqrt{\delta t}}{2\sigma}$

where, p' the risk-neutral probability.

$V = \frac{1}{1+r\delta t}\left(p'V^{+} + (1-p')V^{-}\right)$

where, V is the option value which is present value of some expectation : sum probabilities multiplied by events.

Building Binomial Tree

Next, we will build a binomial tree using the risk neutral probability. Building a tree is a multi step process which involves.

Step 1: Draw a n-step tree

Step 2: At the end of n-step, estimate terminal prices

Step 3: Calculate the option value at each node based on the terminal price, exercise price and type

Step 4: Discount it back one step, that is, from n to n-1, according to the risk neutral probability

Step 5: Repeat the previous step until we find the final value at step 0

Finite Difference Method

FDM are similar in approach to the (binomial) tress. However, instead of discretizing asset prices and the passage of time in a tree structure, it discretizes in a grid - with **time** and **price steps** - by **calculating the value at every possible grid points.**

Differentiation Using The Grid

The Binomial method contains the diffusion - the volatility - in the tree structure whereas in FDM, the *'tree'* is fixed and we change the parameters to reflect the changing diffusion. We will now define the grid by specifying the time step δt and asset step δs and discretize S and t as

$$S = i\delta s$$

and times

$$t = T - k\delta t$$

where 0 $\leq$ i $\leq$ I and 0 $\leq$ k $\leq$ K

Here i and k are respective steps in the grid and we can write the value of the option at each grid points as

$$V_i^k = (i\delta S, T - k\delta t)$$

Approximating Greeks

The greeks terms, the Black–Scholes equation can be written as

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma + rS\Delta - rV = 0$$

Assume that we know the option value at each grid points, we can extract the derivatives of the option using Taylor series expansion.

Approximating Θ

We know that the first derivative of option as,

$$\frac{\partial V}{\partial t} = \lim_{h \to 0} \frac{V(S, t+h) - V(S, t)}{h}$$

Approximating Δ

From the lecture, we know that the central difference has much lower error when compared to forward and backward differences. Accordingly, we can approximate the first derivative of option with respect to the underlying as

$$\frac{\partial V}{\partial S}(S,t) \approx \frac{V_{i+1}^k - V_{i-1}^k}{2\delta S}$$

Approximating Γ

The gamma of the option is the second derivative of option with respective to the underlying and approximating it we have,

$$\frac{\partial V^2}{\partial S^2}(S,t) \approx \frac{V_{i+1}^k - 2V_i^k + V_{i-1}^k}{\delta S^2}$$

FDM are similar to binomial trees. However, instead of discretizing asset prices and passage of time in a tree structure, it discretizes in a grid-with **time** and **price steps** by calculating the value at every possible grid points.

Step 1: Generate the grid by specifying grid points.
Step 2: Specify the final or initial conditions.
Step 3: Use boundary conditions to calculate option values and step back down the grid to fill it.

The Monte Carlo Method for American options
Options prices are **first determined at expiration** and then **worked backwards to the starting date**. Therefore at every point where an early exercise is possible, the choice is easy to make because the future value of the option is known if the option is not exercised.

There is a region in which if the option were **sufficiently deep in the money**, depending on the time to expiry, then it should be **exercised early.**

The great virtues of Monte Carlo are:
Convergence is order $t^{-1/2}$ in all dimensions
Path dependence is easy

Local volatility model
LVM has one factor: only the stock price is stochastic and so most of the standard BSM

schem for perfect replication in terms of a riskless bond and stock still works.
With LVM, we can use **risk neutral valuation methods** to obtain unique arbitrage free options values, just as for BSM.

Evolution of stock price in LVM:

$\frac{dS}{S}$=u(S,t)dt + $\sigma(S,t)$dZ

Note that $\sigma(S,t)$ is a deterministic function of a stochastic variable S.

Advantage of LVM:
Once calibrated, the LVM provides **arbitrage free option values** and hedge ratios for standard and exotic options. It has **closeness to the original BSM** model and its dynamics.

Disadvantage of LVM:
One general objection to LVM is that it needs to be frequently calibrated. As time passes and the underlying stock price or index level changes, the implied volatility surface changes, and a new local volatility surface must be extracted from the data.

New hedge ratios and exotic option values must then be calculated from this updated surface.
The parameters of the model are not stationary.
LVM tend to have difficulty matching the future short term skew.

Constant Elasticity of Variance (CEV)
CEV is one of the LVM.

Evolution of stock price in CEV:

$\frac{dS}{S}$=u(S,t)dt + $\sigma S^{\beta-1}$dZ

In this model, the volatility is proportional to $S^{\beta-1}$, where β is a constant to be determined by calibration. If β =1, then the CEVM reduces to standard lognormal geometric Brownian motion. When β =0, returns are normally distributed. In order to account for the observed **skew** in equity markets, $\boldsymbol{\beta}$ **needs to be negative and large in magnitude**.

Stochastic Volatility Models

In SVM, there are two random processes, **one for stock** itself, and **another for the volatility** or variance of the stock. These two random processes may be correlated.

European options

Interpretation of d_1 *and* d_2

$d_2 = d_1 - \sigma\sqrt{T}$

d_1 is the upper envelope for the projected amount by which a call is ITM normalized by standard deviation.

d_2 is the lower envelope.

Probability the options ends ITM

$\mathrm{P}(call\ finishes\ ITM) = \Phi(d_2)$

$\mathrm{P}(put\ finishes\ ITM) = \Phi(-d_2)$

$\Phi(\mathrm{X})$: CDF of standard normal distribution

N(d_2) is the probability that the call will be exercised

N(d_1) does not lend itself to a simple probability interpretation. Stock price *N(d_1)is the present value using the risk free interest rate of the expected asset price at expiration.

N(d_1) is the factor by which the present value of contingent receipt of the stock exceeds the current stock price.