Lec-42, DC, 24-25, SecA

16 Nov. 2024 - (9 - 11 AM)
Saturday

18 Nov. 2024 - (9-10 AM)

No lecture on Thursday & Friday

$H_i : y(t) = s_i(t) + n(t)$

$i = 0, 1, 2 .., M-1$

$0 \le t \le T_s$

$n(t)$ :- Gaussian Random process (GRP)

Can be called as sample function of a GRP.

If our observation has finite enteries, we can easily use the derived ML/MAP rules. here $y(t)$ has infinite values b/w $[0, T_s]$.

GS procedure

$s_0(t), s_1(t), \ldots, s_{M-1}(t)$

$\varphi_0(t), \varphi_1(t) .., \varphi_{N-1}(t)$

$N \le M$

$$s_i(t) = \sum_{j=0}^{N-1} s_{ij}\, \varphi_j(t) \quad , \quad i = \{0, 1, 2, \ldots, M-1\}$$

$$s_i(t) \equiv \begin{bmatrix} s_{i0} \\ s_{i1} \\ \vdots \\ s_{iN-1} \end{bmatrix} \rightarrow \bar{s}_i \quad , \quad i = \{0, 1, 2, \ldots, M-1\}$$

Transmitting $s_i(t)$ implies, we are Tx a vector in the space generated by $\{\varphi_0(t), \varphi_1(t), \ldots, \varphi_{N-1}(t)\}$

From this space, which has infinite vectors, we choose 1 out of M vectors :- $\{\tilde{s}_0, \tilde{s}_1, \ldots, \tilde{s}_{M-1}\}$

$n(t)$ adds to $s_i(t)$, where $i$ is not known at the

Receiver. Q. Does $n(t)$ belong to the space generated by $\{\varphi_0(t), \varphi_1(t), .., \varphi_{N-1}(t)\}$?

$$n(t) \overset{?}{\underset{\times}{=}} \sum_{j=0}^{N-1} n_j \varphi_j(t)$$

$n(t)$ is a GRP & is infinite dimensional.

$n(t)$ need to projected on the signal space or space generated by $\{\varphi_0(t), \varphi_1(t), .., \varphi_{N-1}(t)\}$ to access its impact on the vector transmitted.

$$n_0 = \int_0^{T_s} n(t)\varphi_0(t)\,dt \quad .... \quad n_{N-1} = \int_0^{T_s} n(t)\varphi_{N-1}(t)\,dt$$

$$\bar{n}(t) = \sum_{j=0}^{N-1} n_j \varphi_j(t) \quad \Big| \quad n(t) = \bar{n}(t) + e(t)$$

Working with $\bar{n}(t)$ & discarding $e(t)$, will it lead to any loss?

$$e(t) = n(t) - \bar{n}(t)$$

✓

$e(t) \perp$ to the Space generated by $\{\varphi_0(t), \varphi_1(t), \ldots \varphi_{N-1}(t)\}$?

as per the principle of projection.

$$y(t) = s_i(t) + n(t)$$

project $y(t)$ on the signal space.

$$y_0 = \int_0^{T_s} y(t)\varphi_0(t)\,dt = s_{i0} + n_0$$

$\vdots$

$$y_{N-1} = \int_0^{T_s} y(t)\varphi_{N-1}(t)\,dt = s_{iN-1} + n_{N-1}$$

$$\bar{y} = \bar{s}_i + \bar{n}$$

$$\bar{y} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{bmatrix} \quad \bar{s}_i = \begin{bmatrix} s_{i0} \\ s_{i1} \\ \vdots \\ s_{iN-1} \end{bmatrix}$$

$$\bar{n} = \begin{bmatrix} n_0 \\ n_1 \\ \vdots \\ n_{N-1} \end{bmatrix}$$

Another important issue:- $n(t)$ is a GRP, what about $\bar{n}$?

$n(t)$: mean 0, psd $= \frac{N_0}{2}$ → also called WGN

$\bar{n}$ → Random Vector (R.V.)

What is the statistic of the R.V. $\bar{n}$ given the statistics of GRP $n(t)$?

Theorem 6.2.1

$$z_1 = \int_{-\infty}^{\infty} n(t)\, \psi_0(t)\, dt$$

$$z_2 = \int_{-\infty}^{\infty} n(t)\, \psi_1(t)\, dt$$

where $n(t)$ is WGN, mean $= 0$, psd $= N_0/2$

The RVs. $z_1$ & $z_2$ are zero mean, jointly Gaussian, with

$$\mathrm{cov}(z_1, z_2) = \sigma^2 \langle \varphi_0(t), \varphi_1(t) \rangle$$

Thus, we obtain that $\underline{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \sim N(\underline{0}, C)$ with

$$\underline{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad C = \begin{pmatrix} \sigma^2 \|\varphi_0(t)\|^2 & \sigma^2 \langle \varphi_0(t), \varphi_1(t) \rangle \\ \sigma^2 \langle \varphi_0(t), \varphi_1(t) \rangle & \sigma^2 \|\varphi_1(t)\|^2 \end{pmatrix}$$

$$C = E\left[ \begin{pmatrix} z_1 - u_{z_1} \\ z_2 - u_{z_2} \end{pmatrix} \begin{pmatrix} z_1 - u_{z_1} & z_2 - u_{z_2} \end{pmatrix} \right] = \begin{bmatrix} E(z_1 - u_{z_1})^2 & E[(z_1 - u_{z_1})(z_2 - u_{z_2})] \\ E[(z_2 - u_{z_2})(z_1 - u_{z_1})] & E(z_2 - u_{z_2})^2 \end{bmatrix}$$

$$E(z_1 - u_{z_1})^2 = \mathrm{cov}(z_1, z_1) \qquad C = \begin{pmatrix} \sigma_{z_1}^2 & \mathrm{cov}(z_1, z_2) \\ \mathrm{cov}(z_2, z_1) & \sigma_{z_2}^2 \end{pmatrix}$$

Using the fact, $\underbrace{\langle \varphi_0(t), \varphi_0(t) \rangle}_{\|\varphi_0(t)\|^2} = 1$ & $\underbrace{\langle \varphi_1(t), \varphi_1(t) \rangle}_{\|\varphi_1(t)\|^2} = 1$

& $\langle \varphi_0(t), \varphi_1(t) \rangle = 0$, $\langle \varphi_1(t), \varphi_0(t) \rangle = 0$.

$$Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \sim \mathcal{N}\left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} \right)$$

The above theorem implies that $\vec{n} = \begin{bmatrix} n_0 \\ n_1 \\ n_2 \\ \vdots \\ n_{N-1} \end{bmatrix}$ is $\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ mean & cov. $\begin{pmatrix} \sigma^2 & 0 & 0 & \cdots & 0 \\ 0 & \sigma^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \cdots & \sigma^2 \end{pmatrix}$ i.e $n_j$s are iid Gaussian R.Vs with each having mean of 0 & var. $\sigma^2$.

$$H_i: \ \bar{y} = \bar{s}_i + \bar{n} \quad , i = 0, 1, \ldots, M-1$$