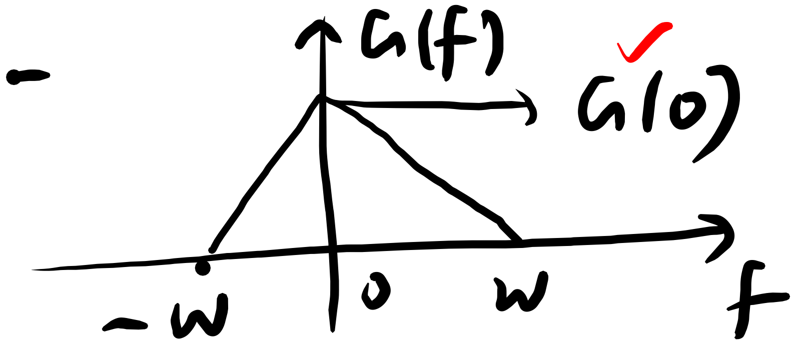
Lec-6, DC, 24-25, Sec A

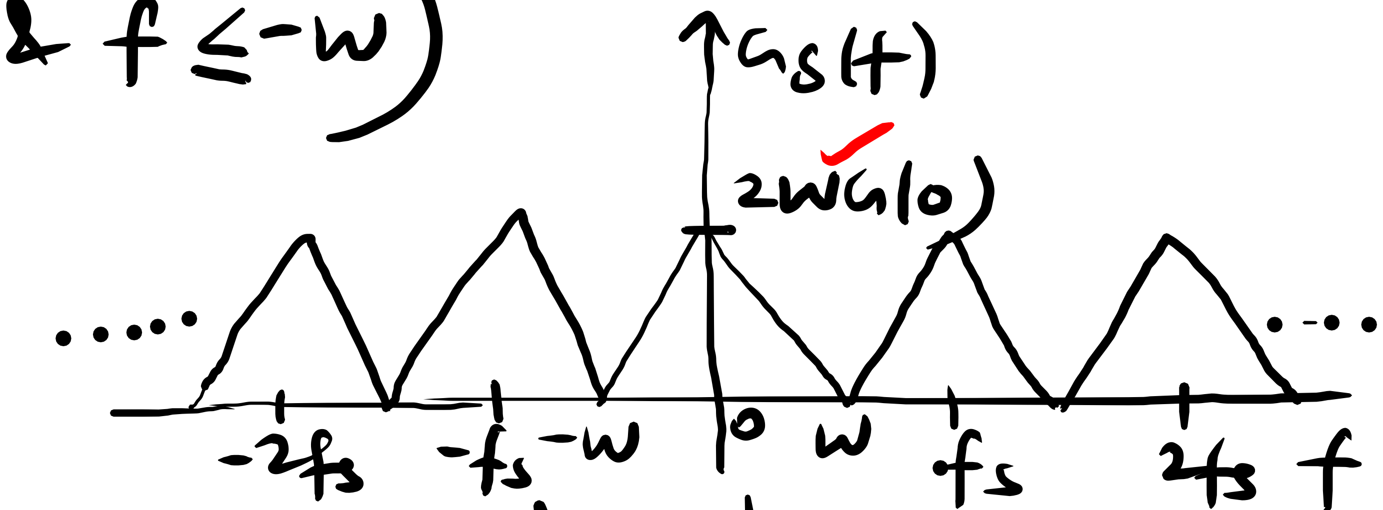
We assumed $g(t)$ to be of finite energy & infinite duration. Let us also assume $g(t)$ to be band-limited (strictly) to W Hz.

$\Rightarrow \quad G(f) = 0, \quad |f| \geq W \quad (f \geq W \; \& \; f \leq -W)$

a typical ex-



$\rightarrow$



Now, choose the sampling rate as 2W samples/sec. or

$T_s = \frac{1}{2W}$, then

$$G_\delta(f) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) e^{-j\frac{\pi n f}{W}} \quad -(3)$$

& also,

$$G_\delta(f) = f_s G(f) + f_s \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} G(f - m f_s) \quad -(2)$$

Hence, under 2 conditions, (1) $G(f)=0$, $|f|\geq W$ & (2) $f_s=2W$, from eq(2)

$$G(f)=\begin{cases}\frac{1}{2W}G_s(f) & , -W<f<W \\ 0 & , \text{otherwise}\end{cases} \quad -(4)$$

Now, put eq(3) in eq(4).

$$G(f)=\begin{cases}\frac{1}{2W}\sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right)e^{-j\frac{\pi n f}{W}} & , -W<f<W \\ 0 & , \text{otherwise}\end{cases} \quad -(5)$$

Therefore, if the sample values $g\left(\frac{n}{2W}\right)$ of a signal $g(t)$ are specified for all n, then FT $G(f)$ of the signal is uniquely determined by using DTFT in eq. (5). You can check from

5. that $G(f+2W) = G(f)$. From the FS theory, we know that the signal constructed using $g\left(\frac{n}{2W}\right)$ alongwith complex exponentials $e^{-j\pi n f/W}$ will lead to $G(f)$ uniquely.

$\left[\right.$ $g(t)$ & $G(f)$ are related through inverse FT, hence $g(t)$ is itself uniquely determined by the sample values $g\left(\frac{n}{2W}\right)$ for $-\infty < n < \infty$

$\boxed{\text{seq. } g(n/2W) \text{ has all the information contained in } g(t)}$

Recovery | Reconstruction :- We know $g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t} df$

Inverse FT

$$g(t) = \int_{-W}^{W} \left(\frac{1}{2W} \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) e^{-j\pi n f/W} \right) e^{j2\pi f t}\, df$$

Interchange order of summation & integration

$$g(t) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \frac{1}{2W} \int_{-W}^{W} e^{j2\pi f\left(t-\frac{n}{2W}\right)}\, df$$

$$= \frac{e^{j2\pi W\left(t-\frac{n}{2W}\right)} - e^{-j2\pi W\left(t-\frac{n}{2W}\right)}}{j2\pi\left(t-\frac{n}{2W}\right)} \Leftarrow \left. \frac{e^{j2\pi f\left(t-\frac{n}{2W}\right)}}{j2\pi\left(t-\frac{m}{2W}\right)} \right|_{-W}^{W}$$

$$\frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin\theta$$

$$\left(\frac{1}{2W}\right) \frac{\sin\left(2\pi W\left(t-\frac{n}{2W}\right)\right)}{\pi\left(t-n/2W\right)} = \frac{\sin(2\pi W t - n\pi)}{2\pi W t - n\pi}$$

We get, $g(t) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \text{sinc}(2Wt - n)$

$-\infty < t < \infty$

$\text{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}$

This is the interpolation formula for reconstructing the original signal $g(t)$ from 'seq. of sample values' $\{g(\frac{n}{2W})\}$ with sinc function, $\text{sinc}(2Wt)$ as the interpolation function.

delay sinc $\longrightarrow$ multiply $\longrightarrow$ add