Lec-36, DC, 24-25, Sec A

Similarly, if 1 is Txd, then y at the MF output has a mean $+A$, & variance $N_0/2T_b$ (H.W.)

$$f_Y(y/1) = \frac{1}{\sqrt{\pi N_0/T_b}} e^{-(y-A)^2/N_0/T_b}$$

'0' Txd :- when noise is present, y 'occassionally' assumes a value $> \lambda$, in which case an error is made.

'1' Txd :- " value $< \lambda$ "

$$P_{01} = P(y < \lambda \mid \text{symbol 1 was sent}) = \int_{-\infty}^{\lambda} f_Y(y|1)\, dy$$

Since these two error events | cases are mutually exclusive, average prob. of symbol error, $P_e = P_0 P_{10} + P_1 P_{01}$

E_1 & E_2

$E_1 \cap E_2 = \Phi$

P_0 & P_1 are the apriori prob. of Tx 0 & 1, resp.

$P(E_1 \cap E_2) = 0$

$E_1 \rightarrow$ 0 Txd $\cap$ 1 Det

$E_2 \rightarrow$ 1 Txd $\cap$ 0 Det

$$P_e = P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(\underbrace{E_1 \cap E_2}_{\underbrace{\Phi}_{0}})$$

$$P_e = P(E_1) + P(E_2)$$

$$= P(\text{0 Txd} \cap \text{1 det}) + P(\text{1 Txd} \cap \text{0 Det})$$

$$P(A \cap B) = P(A) P(B|A)$$

$$= P(\text{0 Txd}) P(\text{1 det}/\text{0 Txd}) + P(\text{1 Txd}) P(\text{0 Det} | \text{1 Tx})$$

First, let us introduce a few standard special functions.

A. $$Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt \quad \text{— (1)}$$ Q-function

$X \sim \mathcal{N}(\mu, \sigma^2)$

$$Y = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy$$

$Q(x)$ = CCDF of Y or complementary error func — called interchangeably.

B. Error function – $$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-u^2}\, du$$

erf(x) is also a function.

Q. Express erfc(x) in terms of Q(x).

$\operatorname{erfc}(x) = 2Q(x\sqrt{2})$ | $Q(x) = \frac{1}{2}\operatorname{erfc}(x/\sqrt{2})$

For P_{10}, $\displaystyle\int_{\lambda}^{\infty} f_Y(\underbrace{y|0}_{\mathcal{N}\left(-A, \frac{N_0}{2T_b}\right)})\,dy = \int_{\lambda}^{\infty} \frac{1}{\sqrt{2\pi \cdot \frac{N_0}{2T_b}}} e^{-(y+A)^2/2N_0/2T_b}\,dy$ — ①

$dy = \sqrt{\frac{N_0}{T_b}}\,dz$

Subs. in ①

let $z = \dfrac{y+A}{\sqrt{N_0/T_b}}$, $\displaystyle P_{10} = \frac{1}{\sqrt{\pi}} \int_{\frac{A+\lambda}{\sqrt{N_0/T_b}}}^{\infty} e^{-z^2}\,dz$ =

$\displaystyle\int_{\frac{A+\lambda}{\sqrt{N_0/T_b}}}^{\infty} \frac{1}{\sqrt{\pi \cancel{N_0/T_b}}} e^{-z^2} \cancel{\sqrt{\frac{N_0}{T_b}}}\,dz$

$\displaystyle\int_{\frac{A+\lambda}{\sqrt{N_0/T_b}}}^{\infty} \frac{1}{\sqrt{\pi}} e^{-z^2}\,dz$

$\frac{1}{2}\operatorname{erfc}\left(\dfrac{A+\lambda}{\sqrt{N_0/T_b}}\right)$

P_{01}, $\dfrac{1}{\sqrt{2\pi \frac{N_0}{2T_b}}} \displaystyle\int_{-\infty}^{\lambda} e^{-(y-A)^2/2\cdot\frac{N_0}{2T_b}}\, dy$ (H.W.)

$= \dfrac{1}{2}\,\mathrm{erfc}\left(\dfrac{A-\lambda}{\sqrt{N_0/T_b}}\right)$

$z = \dfrac{A-y}{\sqrt{N_0/T_b}} \longrightarrow$

change of variable so that we can express P_{10} & P_{01} in terms of special function $Q(x)$/$\mathrm{erfc}(x)$

$$P_e = \frac{P_0}{2}\,\mathrm{erfc}\left(\frac{A+\lambda}{\sqrt{N_0/T_b}}\right) + \frac{P_1}{2}\,\mathrm{erfc}\left(\frac{A-\lambda}{\sqrt{N_0/T_b}}\right)$$

Here, with no control over source prob. of 0 & 1 & fixed line code (NRZ polar), P_e is a function of λ.

Also, λ is what is entirely in the hands of the receiver.

$$\lambda_{opt} = \min_{\lambda} P_e(\lambda)$$

Diff. $P_e(\lambda)$ w.r.t λ & equate it to zero to find λ_{opt}. But for this, we assume $P_e(\lambda)$ has a unique minimum & it is differentiable.

1. $$\frac{d}{du}\,\mathrm{erfc}(u) = \frac{-1}{\pi} e^{-u^2}$$ (see 4.35 - 4.36 Haykin's T.B)

2. $$\frac{dP_e}{d\lambda} = 0, \quad \lambda_{opt} = \frac{N_0}{4AT_b} \log\left(P_0 | P_1\right)$$ — (H.W.)