SC301 - Numerical Linear Algebra (3-0-0-3)

Aditya Tatu

Lecture 1

Why $Ax = b$?

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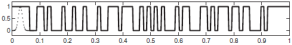

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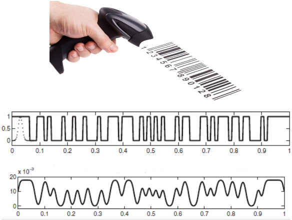

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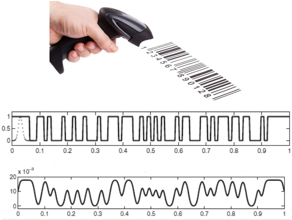

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$$0 \leq s \leq 1, \quad g(s) = \int_0^1 \exp\left(-\frac{(s-t)^2}{\sigma^2}\right) f(t)\, dt.$$

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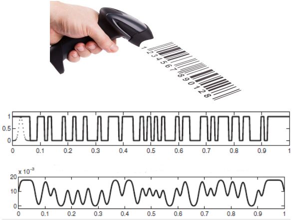

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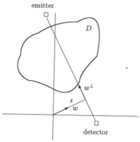
  - Sampling gives $Ax = b$.

- **X-ray tomography:**



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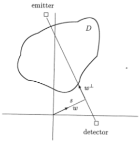

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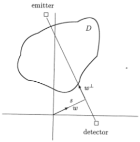

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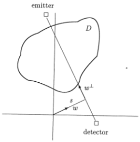

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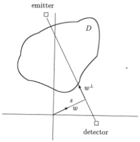

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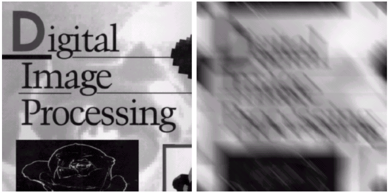
● **Image deblurring:** ● Image blurred by camera being out of focus, object motion.



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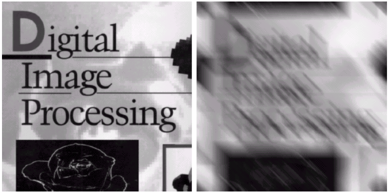

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● Blur is assumed to be linear position invariant, modeled by a point spread function h; relation between original and blurred image is

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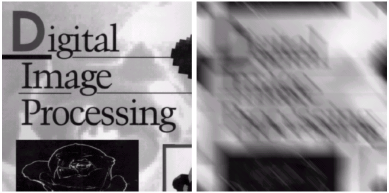

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● Compute f, given g.

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$$\forall(i,j) \in [1, N_x] \times [1, N_y], g(x_i, y_j)$$
$$= \frac{f(x_i + h, y_j) + f(x_i, y_j + h) + f(x_i - h, y_j) + f(x_i, y_j - h) - 4f(x_i, y_j)}{h^2}.$$

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- Compute f such that $Lf = g$.

- **Principal Component Analysis:**

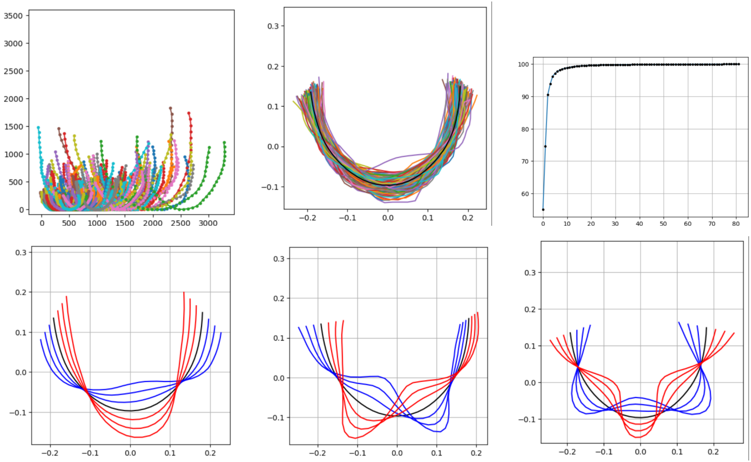

Figure: PCA of jawlines of 300 faces

- **Nonlinear dimensionality reduction: ISOMAP**

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  - Compute the inter-point distance matrix D.

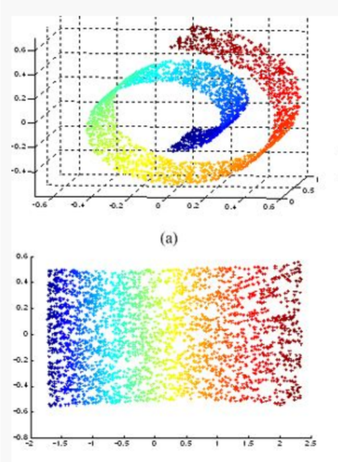

Figure: The swissroll and its embedding using two largest eigenvectors of D.

- **Fourier Analysis on meshes:**

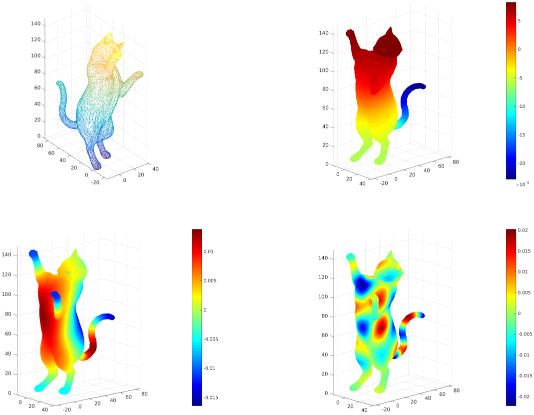

Figure: (top-left) Cat mesh, (top-right) Fundamental harmonic, (bottom-left) 10^{th} harmonic, (bottom-right) 50^{th} harmonic.

Singular Value Decomposition: $A = U\Sigma V^T$

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- **Image compression:**

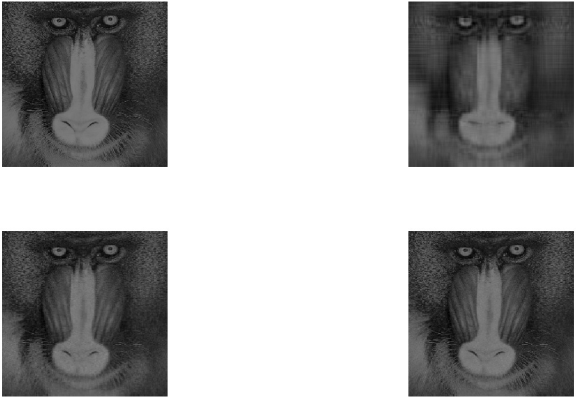

Figure: (top-left) Original image, (top-right) Rank-10 approx. (bottom-left) Rank-50 approx., (bottom-right) Rank-100 approx.

Contents

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- **SVD:** Applications and algorithms. Ill-posed Inverse problems.

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- Evaluation:
- In Sem-1 and 2: 25% each.
- End Sem: 35%
- Quiz: 15%.

Issues in $Ax = b$

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  - How about $A \in \mathbb{R}^{n\times n}$, $n \sim 10000$?