The concept of optimality depends on what we wish to accomplish by decomposing $\bar{g}$ into two components.

→ We can think of $\bar{e}$ being the error in approximating $\bar{g}$ by $c\bar{x}$. Hence, we can now define mathematically the component (or proj.) of a vector $\bar{g}$ along vector $\bar{x}$ to be $c\bar{x}$, where c is chosen to minimize the length of error vector $\bar{e} = \bar{g} - c\bar{x}$ (mag.)

→ You can see/observe that the perpendicular has the smallest mag. or norm

& it is the unique one.

Let us see that

$$c\|x\| = \|\bar{g}\| \cos\theta \quad -(3)$$

$\because$ mag. of the component of $\bar{g}$ along $\bar{x}$ is $\|\bar{g}\| \cos\theta$

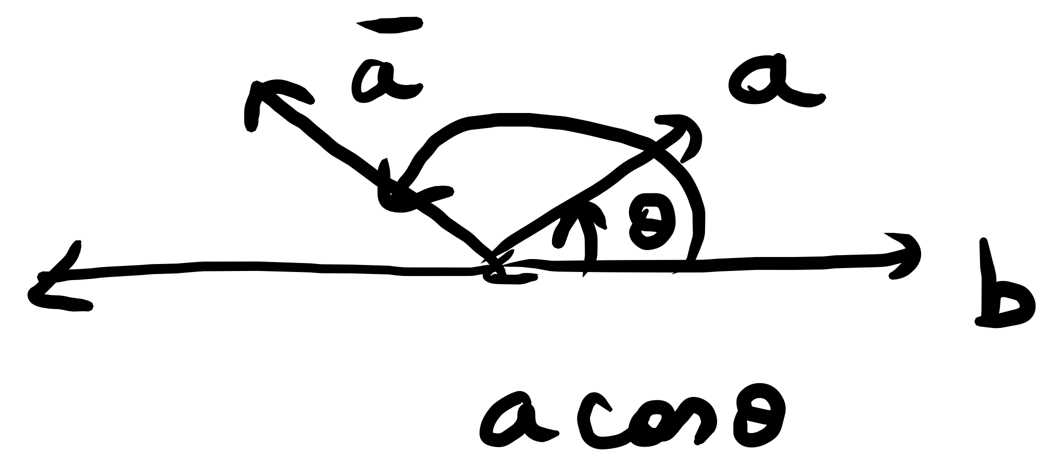
$\rightarrow$ this is for eq. (3) only.

multiply by $\|\bar{x}\|$ both sides, we get

$$c\|\bar{x}\|^2 = \|\bar{g}\|\|\bar{x}\| \cos\theta$$

$$= \langle \bar{g}, \bar{x} \rangle$$

$$\Rightarrow \quad c = \frac{\langle \bar{g}, \bar{x} \rangle}{\langle \bar{x}, \bar{x} \rangle}$$



It is aparent that when $\bar{g}$ & $\bar{x}$ are perpendicular or orthogonal then $\bar{g}$ has a zero component along $\bar{x}$, consequently, $c=0$. Hence, $\bar{g}$ & $\bar{x}$ are said to be orthogonal if the inner (scaler or dot) prod. of two vectors is zero i.e.

$$\text{if} \quad \langle \bar{g}, \bar{x} \rangle = 0$$

Gram-Schmidt procedure:-

Given M energy signals denoted as $s_1(t), s_2(t) \ldots, s_M(t)$, how do you create a complete orthonormal set of basis functions?

1. Start with $s_1(t)$, chosen from the set arbitrarily, the first basis function is defined by.

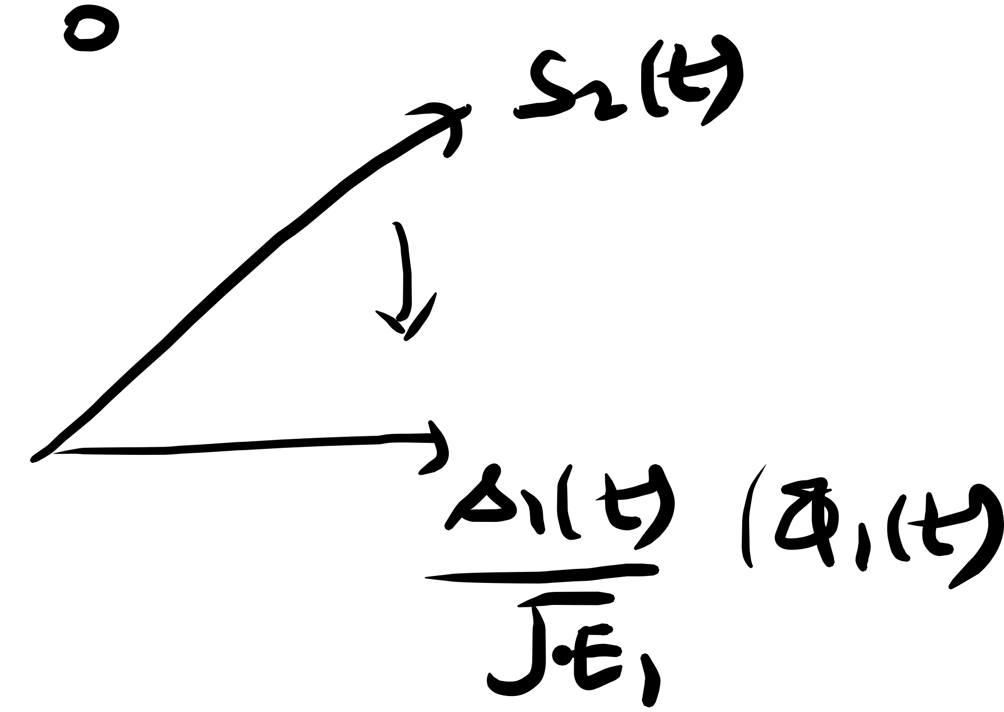
$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$$, where $E_1 \rightarrow$ energy of $s_1(t)$

$$\int_0^T s_1^2(t)\,dt$$

Then, clearly $s_1(t) = \sqrt{E_1}\,\phi_1(t)$

or $s_{11}\,\phi_1(t)$

2. Define
$$s_{21} = \frac{\int_0^T s_2(t)\,\phi_1(t)\,dt}{\underbrace{\int_0^T \phi_1^2(t)\,dt}_{\rightarrow 1}}$$



let $g_2(t) = s_2(t) - s_{21}\,\phi_1(t)$

↳ this is orthogonal to $\phi_1(t)$ over the interval $0 \le t \le T$.

second basis function $\phi_2(t) = \dfrac{s_2(t) - s_{21}\phi_1(t)}{\sqrt{E_2 - s_{21}^2}}$

$$E_{num} = \int_0^T \left[s_2(t) - s_{21}\phi_1(t)\right]^2 dt \qquad E_2 = \int_0^T s_2^2(t)\,dt$$

$$= E_2 + s_{21}^2 - 2\int_0^T s_2(t)\, s_{21}\,\phi_1(t)\,dt$$

$$= E_2 + s_{21}^2 - 2\, s_{21} \times s_{21} \qquad (1, -1, 2)$$

$$= E_2 - s_{21}^2$$

3. Continuing in this fashion, we have

$$g_i(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij}\,\phi_j(t) \quad \text{—} \quad (4)$$

where, $s_{ij} = \int_0^T s_i(t)\,\phi_j(t)\,dt$, $j = 1, 2, \ldots, i-1$

We have discussed special cases of eq. (4) with $i=2$ & $i=1$ earlier.

Given the $g_i(t)$, we may define the set of basis functions

$$\phi_i(t) = \frac{g_i(t)}{\sqrt{\int_0^T g_i^2(t)\,dt}} \quad , \quad i = 1, 2, .., N$$

which form an orthonormal set.

$N \leq M$:-

1. The signals $s_1(t), s_2(t), \ldots, s_M(t)$ form a linearly independent set, in which case $N = M$.

2. Signals $s_1(t), s_2(t), \ldots s_M(t)$ are not L.I., in which case $N < M$ & function $g_i(t)$ is zero for $i > N$

H.W. Why not $N > M$ possible?

Two examples:- Conventional Fourier series of a periodic signal

&

B.L. signal's expansion in terms of its samples taken at the Nyquist rate.

Two distinctions:-

1. Form of basis functions $\phi_1(t), \phi_2(t) \ldots \phi_N(t)$

has not been specified.

i.e., not sinusoidal functions (as in FS) or sinc " (as in reconstruction formula)

2. Here (GS) using finite no. of terms is not an approximation, wherein only the first N terms are significant rather an exact exp. where N & only N terms are significant.

H.W. Do ex 7.1.1 from comm. system engg. by J. G. Proakis, 2nd edition