Tutorial 5 Solutions: Boolean Logic and Propositional Logic

Karnaugh Maps

Concept(s)

A Karnaugh map is a diagram used to find the minimum number of minterms required for a formula as a DNF. It is used for formulas with 2, 3 or 4 variables.

	y	$\bar{y}$
x		$+$
$\bar{x}$	$+$	

$(\bar{x}y) \vee (x\bar{y})$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	$+$	$+$		$+$
$\bar{x}$	$+$		$+$	$+$

$(xy) \vee (\bar{x}\bar{y}) \vee z$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
wx	$+$	$+$		$+$
$w\bar{x}$	$+$	$+$	$+$	$+$
$\bar{w}\bar{x}$			$+$	$+$
$\bar{w}x$	$+$			$+$

$(wy) \vee (\bar{x}\bar{y}) \vee (xz)$

We want to cover the $+$ squares with the minimum number of rectangles.

They must have sides of length 1, 2 or 4. They can wrap around the map.

Exercise 1. Express $(\bar{x}y) \vee (xy)$ in DNF using the minimal number of minterms.

Answer(s)

This expression has the following Karnaugh map.

	y	$\bar{y}$
x	$+$	
$\bar{x}$	$+$	

The DNF is given by y.

Exercise 2. Express $(\bar{x}yz) \vee (xy\bar{z}) \vee (\bar{x}\bar{y}z)$ in DNF using the minimal number of minterms.

Answer(s)

This expression has the following Karnaugh map.

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x		$+$		
$\bar{x}$	$+$			$+$

The DNF is given by $(\bar{x}z) \vee (xy\bar{z})$.

Exercise 3. Express $(wx) \vee (\bar{w}x) \vee (wy\bar{z}) \vee (\bar{w}\bar{y}\bar{z})$ in DNF using the minimal number of minterms.

Answer(s)

This expression has the following Karnaugh map.

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
wx	+	+	+	+
$w\bar{x}$		+		
$\bar{w}\bar{x}$			+	
$\bar{w}x$	+	+	+	+

The DNF is given by $x \vee (wy\bar{z}) \vee (\bar{w}\bar{y}\bar{z})$.

Logical Notation

Concept(s)

A proposition is a statement that is either true or false.

Symbol	Default	Also known as
$\wedge$	and	but, ;
$\vee$	or	either . . . or . . .
$\neg$	nor	not the case
$\rightarrow$	if . . . then . . .	implies, whenever, is sufficient for
$\leftrightarrow$	. . . if and only if . . .	bi-implies, exactly when, necessary and sufficient

We will only ask you to translate the default keywords e.g. “and”, “or”, “not”.

Exercise 4. Translate the following statements into logical notation using these propositional variables:

$$p = \text{“I walk my dog”}, \quad q = \text{“It is sunny”}, \quad r = \text{“I drink coffee”}.$$

a) If it is sunny, then I walk my dog.

b) I walk my dog if and only if I drink coffee.

c) If I don’t walk my dog, then it is not sunny.

d) If it is not sunny or I walk my dog, then I drink coffee.

Answer(s)

a) $q \rightarrow p$

b) $p \leftrightarrow r$

c) $\neg p \rightarrow \neg q$

d) $(\neg q \vee p) \rightarrow r$

Well-Formed Formula

Concept(s)

Let $\mathrm{Prop} = \{p, q, r, \dots\}$ be a set of letters that represent propositions. Consider

$$\Sigma = \mathrm{Prop} \cup \{\top, \bot, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, (,)\}.$$

The well-formed formulas (wffs) over Prop is the smallest subset of Σ^* where:

- $\top, \bot$ and all elements of Prop are wffs
- If φ is a wff, then $\neg\varphi$ is a wff
- If φ and ψ are wffs, then $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$ and $(\varphi \leftrightarrow \psi)$ are wffs

Exercise 5. Which of the following terms are well-formed formulas (under the strictest definition)?

$$((p \vee p) \wedge (p \neg q)), \quad (\neg \rightarrow (q \wedge r)), \quad ((p \leftrightarrow q) \rightarrow (\neg p \rightarrow \neg q)), \quad (\neg(p \wedge q) \vee (\neg p \vee q)),$$

$$(p \vee \neg(q \rightarrow r)), \quad ((pq) \vee r), \quad ((p \rightarrow q) \rightarrow (r \vee \neg s)), \quad (((p \rightarrow\rightarrow \neg q) \vee ((p \wedge r) \rightarrow s))).$$

Answer(s)

The wffs are

$$((p \leftrightarrow q) \rightarrow (\neg p \rightarrow \neg q)), \quad (\neg(p \wedge q) \vee (\neg p \vee q)), \quad (p \vee \neg(q \rightarrow r)), \quad ((p \rightarrow q) \rightarrow (r \vee \neg s)).$$

Note(s)

For readability, we have some conventions about how we write. Check slide 29 of lecture 9-10.

Truth Assignments

Concept(s)

A truth assignment is a function v that states if a wff is true or false. We need to give all elements in Prop a truth value. The function works as follows:

$$v(\top) = 1 \qquad v(\varphi \wedge \psi) = \min\{v(\varphi), v(\psi)\}$$
$$v(\bot) = 0 \qquad v(\varphi \vee \psi) = \max\{v(\varphi), v(\psi)\}$$
$$v(\neg\varphi) = 1 - v(\varphi) \qquad v(\varphi \rightarrow \psi) = \max\{1 - v(\varphi), v(\psi)\}$$
$$v(\varphi \leftrightarrow \psi) = (1 + v(\varphi) + v(\psi)) \,\%\, 2$$

Exercise 6. Consider the truth assignments for p, q, r,

$$v(p) = 1, \quad v(q) = 0, \quad v(r) = 1.$$

Evaluate the truth value of the following propositions:

a) $\neg((p \to q) \vee \neg r)$

b) $(\neg q \leftrightarrow (p \to r))$

c) $(\neg(p \vee q) \wedge (q \leftrightarrow p))$

Answer(s)

a)

$$\begin{aligned} v(\neg((p \to q) \vee \neg r)) &= 1 - v((p \to q) \vee \neg r) \\ &= 1 - \max\{v(p \to q), v(\neg r)\} \\ &= 1 - \max\{\max\{1 - v(p), v(q)\}, 1 - v(r)\} \\ &= 1 - \max\{\max\{1 - 1, 0\}, 1 - 1\} \\ &= 1 \end{aligned}$$

b)

$$\begin{aligned} v(\neg q \leftrightarrow (p \to r)) &= (1 + v(\neg q) + v(p \to r)) \,\%\, 2 \\ &= (1 + (1 - v(q)) + \max\{1 - v(p), v(r)\}) \,\%\, 2 \\ &= (1 + (1 - 0) + \max\{1 - 1, 1\}) \,\%\, 2 \\ &= 1 \end{aligned}$$

c)

$$\begin{aligned} v(\neg(p \vee q) \wedge (q \leftrightarrow p)) &= \min\{v(\neg(p \vee q)), v(q \leftrightarrow p)\} \\ &= \min\{1 - v(p \vee q), (1 + v(q) + v(p)) \,\%\, 2\} \\ &= \min\{1 - \max\{v(p), v(q)\}, (1 + v(q) + v(p)) \,\%\, 2\} \\ &= \min\{1 - \max\{1, 0\}, (1 + 0 + 1) \,\%\, 2\} \\ &= 0 \end{aligned}$$

Concept(s)

A truth table has a row for every truth assignment (assigning each element in Prop a T/F value).

Columns are added for subformulas.

Exercise 7. Draw a truth table for the following propositions:

a) $\neg((p \wedge q) \to (p \wedge q))$

b) $(\neg p \vee q) \leftrightarrow (p \to r)$

Answer(s)

a)

p	q	$p \wedge q$	$(p \wedge q) \to (p \wedge q)$	$\neg((p \wedge q) \to (p \wedge q))$
F	F	F	T	F
F	T	F	T	F
T	F	F	T	F
T	T	T	T	F

b)

p	q	r	$\neg p$	$\neg p \vee q$	$p \to r$	$(\neg p \vee q) \leftrightarrow (p \to r)$
F	F	F	T	T	T	T
F	F	T	T	T	T	T
F	T	F	T	T	T	T
F	T	T	T	T	T	T
T	F	F	F	F	F	T
T	F	T	F	F	T	F
T	T	F	F	T	F	F
T	T	T	F	T	T	T

Exercise 8. Show that $\neg(p \wedge q) \to (p \to \neg q)$ is always true.

Answer(s)

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg q$	$p \to \neg q$	$\neg(p \wedge q) \to (p \to \neg q)$
F	F	F	T	T	T	T
F	T	F	T	F	T	T
T	F	F	T	T	T	T
T	T	T	F	F	F	T

Logical Equivalence

Concept(s)

Two wffs φ and ψ are logically equivalent, $\varphi \equiv \psi$, if $v(\varphi) \equiv v(\psi)$ for all truth assignments v.

We can show that $\varphi \equiv \psi$ by:

- Comparing all rows of a truth table
- Show that $(\varphi \leftrightarrow \psi)$ is always true
- Use logical equivalence laws

Exercise 9. Prove or disprove the following logical equivalences with truth tables.

a) $\neg p \to (q \vee r) \equiv q \to (\neg p \to r)$

b) $\neg p \to (q \vee r) \equiv \neg q \to (\neg p \to r)$

Answer(s)

p	q	r	$\neg p$	$\neg q$	$q \vee r$	$\neg p \to (q \vee r)$	$\neg p \to r$	$q \to (\neg p \to r)$	$\neg q \to (\neg p \to r)$
F	F	F	T	T	F	F	F	T	F
F	F	T	T	T	T	T	T	T	T
F	T	F	T	F	T	T	F	F	T
F	T	T	T	F	T	T	T	T	T
T	F	F	F	T	F	T	T	T	T
T	F	T	F	T	T	T	T	T	T
T	T	F	F	F	T	T	T	T	T
T	T	T	F	F	T	T	T	T	T

a) This equivalence is not true as $\neg p \to (q \vee r)$ and $q \to (\neg p \to r)$ are different when $p = q = r = F$, from the truth table.

b) The equivalence is true as their columns are the same so they have the same truth value for all assignments of p, q, r.

Concept(s)

Commutativity	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
Associativity	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Distribution	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity	$p \vee \bot \equiv p$	$p \wedge \top \equiv p$
Complement	$p \vee (\neg p) \equiv \top$	$p \wedge (\neg p) \equiv \bot$
Idempotence	$p \vee p \equiv p$	$p \wedge p \equiv p$
Implication	$p \to q \equiv \neg p \vee q$	
Double Negation	$\neg\neg p \equiv p$	
Contrapositive	$p \to q \equiv \neg q \to \neg p$	
De Morgan's	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$

Exercise 10. Prove this logical equivalence by using the equivalence laws:

$$(p \wedge q) \to r \equiv p \to (\neg q \vee r)$$

Answer(s)

$$\begin{aligned}
(p \wedge q) \to r &\equiv \neg(p \wedge q) \vee r && \text{(Implication)} \\
&\equiv (\neg p \vee \neg q) \vee r && \text{(De Morgan's)} \\
&\equiv \neg p \vee (\neg q \vee r) && \text{(Associativity)} \\
&\equiv p \to (\neg q \vee r) && \text{(Implication)}
\end{aligned}$$

Entailment and Validity

Concept(s)

Let $T = \{\varphi_1, \ldots, \varphi_n\}$ be a set of wffs. Truth assignment v satisfies T if $v(\varphi) = \text{true}$ for all $\varphi \in T$.

We say T entails a formula ψ, $\varphi_1, \ldots, \varphi_n \models \psi$, if $v(\psi) = \text{true}$ for all v that satisfy T. We can:

- Draw a truth table for $\varphi_1, \ldots, \varphi_n, \psi$. Check ψ is true when $\varphi_1, \ldots, \varphi_n$ are all true.
- Show that $((\varphi_1 \wedge \varphi_2) \ldots \varphi_n) \to \psi$ is always true.
- Show that $\varphi_1 \to (\varphi_1 \to (\cdots \to (\varphi_n \to \psi) \ldots))$ is always true.

Exercise 11. Show that $p \wedge \neg p \models q$.

Answer(s)

p	q	$\neg p$	$p \wedge \neg p$
F	F	T	F
F	T	T	F
T	F	F	F
T	T	F	F

When $p \wedge \neg p$ is true, we find that q is true so $p \wedge \neg p \models q$. When there is no scenario where the premises can be true, the entailment is always true.

Concept(s)

An argument has a set of propositions called premises and a proposition called the conclusion.

Argument Validity means "If the premises are true, then the conclusion is true"

We can show an argument is valid by showing the premises entail the conclusion.

Exercise 12. Consider the following argument

Premises	Today is not hot
	If we eat ice cream then today is hot
	If we don't ice cream then we save money
Conclusion	We save money

a) Translate the premises and conclusion into logical notation.

b) Determine if the premises entail the conclusion.

c) Is the argument valid? Why or why not?

Answer(s)

a) We assign the following variables:

$$p = \text{“Today is hot”}, \quad q = \text{“We eat ice cream”}, \quad r = \text{“We save money”}.$$

This gets us the following argument

$$\begin{array}{l} \neg p \\ q \to p \\ \neg q \to r \\ \hline r \end{array}$$

b) We draw the following truth table to find that

p	q	r	$\neg p$	$\neg q$	$q \to p$	$\neg q \to r$
F	F	F	T	T		
F	F	T	T	T	T	T
F	T	F	T	F		
F	T	T	T	F		
T	F	F				
T	F	T				
T	T	F				
T	T	T				

The only row where all premises are true is when $p = F, q = F, r = T$. We see that $r = T$ so the premises entail the conclusion.

c) Since the premises entail the conclusion, the argument is valid.

Extra Practice Problems

Note(s)

These practice questions are designed to help deepen your understanding. No answers will be provided, as the goal is to encourage independent problem-solving and reinforce key concepts.

1. Express the following formulae in DNF using the minimal number of minterms.
   a) $(yz) \vee (\bar{y}z) \vee (x\bar{z})$
   b) $(\bar{x}yz) \vee (x\bar{y}z) \vee (zy\bar{z}) \vee (xyz)$
   c) $(\bar{w}xy) \vee (\bar{w}\bar{x}\bar{z}) \vee (w\bar{x}\bar{z}) \vee (wxy) \vee (\bar{w}x\bar{y}z) \vee (wx\bar{y}z)$
2. Determine whether the following are propositions or not:
   a) This is a microphone.
   b) The sun is on fire.
   c) $x^2 - 4 = 0$.
   d) Two is not a prime number.
   e) Is two a prime number?
3. Translate the following statements into logical notation using these propositional variables:

$$p = \text{“Cats can fly”}, \quad q = \text{“Birds are liquid”}, \quad r = \text{“Cats are birds”}.$$

   a) If cats can fly or birds are liquid then cats are birds.
   b) Birds are not liquid and cats are not birds.
   c) Birds are liquid, if cats are birds.
   d) Cats are birds exactly when cats can fly.
4. Prove or disprove the following:
   a) $((p \vee q) \to r) \to (p \to r)$ is a tautology
   b) $(\neg(p \vee \neg q) \to (q \to r)) \leftrightarrow (\neg(q \to p) \to (\neg q \vee r))$ is a contingency
5. Prove or, using a counterexample, disprove the following logical equivalences:
   a) $\neg(p \vee q) \equiv \neg p \vee \neg q$
   b) $p \vee (p \wedge q) \equiv p$
   c) $p \to (q \to r) \equiv q \to (p \to r)$
   d) $(q \to (r \wedge p)) \equiv r \to (p \wedge q)$
6. Use logical equivalence laws to prove that $q \to (p \vee r) \equiv (q \to p) \vee (q \to r)$.
7. Prove that $p \models (p \to q) \to q$ is true.

8. Determine whether the following argument is valid or not:

$$\begin{array}{l} W \to X \\ Y \to Z \\ \hline W \vee Y \to X \vee Z \end{array}$$

9. Determine whether the following argument is valid or not:

$$\begin{array}{l} \text{If cats can fly or birds are liquid then cats are birds.} \\ \text{Birds are not liquid and cats are not birds.} \\ \text{Birds are liquid, if cats are birds.} \\ \text{Cats are birds exactly when cats can fly.} \\ \hline \text{Whenever birds are liquid, cats are birds.} \end{array}$$