Tutorial 1: Number Theory

Floor, Ceiling and Absolute Value

Concept(s)

Floor Function: $\lfloor x \rfloor$ gives the largest integer less than or equal to x.

Ceiling Function: $\lceil x \rceil$ gives the smallest integer greater than or equal to x.

Absolute Value: $|x|$ is the non-negative value of x without regard to its sign.

Exercise 1. Calculate the following:

1. $\lfloor \sqrt{10} \rfloor + \lceil \pi \rceil$
2. $\lfloor -3.01 \rfloor + \lceil 2.99 \rceil$
3. $|\lfloor -5.6 \rfloor| + \lfloor |-5.6| \rfloor$

Exercise 2. Explain why $\lfloor x \rfloor = \lceil x \rceil$ means that x is an integer.

Exercise 3. For which numbers n is the statement $\lfloor \sqrt{n} \rfloor = \lceil \sqrt{n} \rceil$ true?

Exercise 4. Find all integer x such that the following equation is true:

$$\left\lfloor \frac{x}{2} \right\rfloor + \left\lceil \frac{x}{3} \right\rceil = 5.$$

Divisibility and GCD/LCM

Concept(s)

For integers m and n, we say m divides n when $n = k \cdot m$ for some integer k. We write $m \mid n$.

Exercise 5. Are the following statements true?

$$5 \mid 35, \ 8 \mid 35, \ 2 \mid -14, \ -2 \mid 14.$$

Exercise 6. How many numbers between 1 and 653 are divisible by 3 or 5?

Exercise 7. For positive integers a, b, c where $ab \mid bc$, show that $a \mid c$.

Concept(s)

Consider two integers m and n.

The largest integer d such that $d \mid m$ and $d \mid n$ is called the $\gcd(m, n)$.

The smallest integer k such that $m \mid k$ and $n \mid k$ is called the $\mathrm{lcm}(m, n)$.

$$\gcd(m, n) \cdot \mathrm{lcm}(m, n) = |m|\,|n|\,.$$

We say that m and n are coprime if $\gcd(m, n) = 1$.

Exercise 8. Suppose that n is an integer. Explain why n and $n + 1$ are coprime.

Modular Arithmetic

Concept(s)

For integers m and n, we define the following operations:

$$m \text{ div } n = \left\lfloor \frac{m}{n} \right\rfloor \text{ and } m \% n = m - n \cdot \left\lfloor \frac{m}{n} \right\rfloor.$$

This gives us

$$m = q \cdot n + r, \text{ where } q = m \text{ div } n \text{ and } r = m \% n.$$

Exercise 9. Find the last two digits of 7^{7^7}.

Exercise 10. Find the least positive integer n for which $5^n \% 17 = 16$. Hence, evaluate $5^{200} \% 17$.

Concept(s)

We denote $m =_{(n)} p$ to mean that $(m \% n) = (p \% n)$.

Exercise 11. Suppose that $k \mid n$ and $a =_{(n)} b$ for positive integers a, b, k, n. Show that $a =_{(k)} b$.

Euclidean Algorithm

Concept(s)

The Euclidean algorithm provides us a way to calculate the gcd:

$$\gcd(m, n) = \begin{cases} m & \text{if } n = 0 \\ n & \text{if } m = 0 \\ \gcd(m \% n, n) & \text{if } m > n > 0 \\ \gcd(m, n \% m) & \text{if } n > m > 0 \end{cases}$$

Exercise 12. Calculate $\gcd(a, b)$ and $\text{lcm}(a, b)$ for the following pairs (a, b):

1. $(44, 17)$
2. $(56, 72)$
3. $(123, 321)$

Exercise 13. Find $\gcd(615, 220)$. Are there integers x and y such that $4 = 615x + 220y$? Explain why.

Extra Practice Problems

Note(s)

These practice questions are designed to help deepen your understanding. No answers will be provided, as the goal is to encourage independent problem-solving and reinforce key concepts.

1. Find the following values:

   $$\lfloor 17.73 \rfloor, \quad \lceil 73.17 \rceil, \quad \lceil \lfloor 1 \rfloor \rceil, \quad \left| -\frac{1}{2} \right|, \quad \left\lfloor \sqrt{59} \right\rfloor, \quad \left| \left\lceil -\frac{222}{10} \right\rceil \right|, \quad \left\lceil \left| -\frac{222}{10} \right| \right\rceil.$$

2. a) Provide an example of numbers x and y such that $\lceil x \rceil + \lceil y \rceil = \lceil x + y \rceil$.
   b) Provide an example of numbers x and y such that $\lceil x \rceil + \lceil y \rceil > \lceil x + y \rceil$.
   c) Give an argument that $\lceil x \rceil + \lceil y \rceil \geq \lceil x + y \rceil$ for all numbers x and y.

3. If t is an integer, then $\lceil x + t \rceil = \lceil x \rceil + t$ for every number x.
   a) State a similar fact about the floor function $\lfloor \cdot \rfloor$.
   b) Explain why this fact is true.

4. Is there an example of numbers x and y such that $|x| + |y| < |x + y|$?

5. Which of the following are true?

   $$7 \mid 161, \quad 7 \mid 162, \quad 8 \mid 4, \quad 17 \mid 68, \quad 3 \mid 10^{400}, \quad 11 \mid 1001.$$

6. Find the following values:

   $$100 \text{ div } 13, \quad 100 \ \% \ 13, \quad 67 \text{ div } -22, \quad 67 \ \% \ -22, \quad (-238) \text{ div } 11, \quad (-238) \ \% \ 11.$$

7. Suppose that $a \mid c$ and $b \mid d$. Explain why $ab \mid cd$.

8. Find the least positive integer n for which $3^n \ \% \ 7 = 1$. Evaluate $3^{100} \ \% \ 7$.

9. Find the following gcd values and use them to find the corresponding lcm:
   a) $\gcd(12, 18)$ and $\text{lcm}(12, 18)$
   b) $\gcd(83, 36)$ and $\text{lcm}(83, 36)$
   c) $\gcd(533, 182)$ and $\text{lcm}(533, 182)$
   d) $\gcd(112, 629)$ and $\text{lcm}(112, 629)$

10. Which pairs of numbers from the previous question are coprime?

11. The amount of integers between integers m and n, where $n > m$ is $n - m + 1$. How many integers are there between two real numbers x, y where $x > y$?

12. Suppose that $a =_{(n)} b$ for positive integers a, b, n. Show that $\gcd(a, n) = \gcd(b, n)$.