



COMP9020

Foundations of Computer Science
Term 3, 2024

Lecture 4: Set Theory

Outline

Recap of Key Definitions

Set Equality

Laws of Set Operations

Derived Laws

Two Useful Results

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Recap of Key Definitions

Set Equality

Laws of Set Operations

Derived Laws

Two Useful Results

Defining Sets

- ① Explicitly list elements
- ② Take a subset of an existing set by restricting the elements
- ③ Build up from existing sets using Set Operations

Set Operations

Definition

$A \cup B$ – **union** (a or b):

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

$A \cap B$ – **intersection** (a and b):

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

A^c – **complement** (with respect to a universal set \mathcal{U}):

$$A^c = \{x : x \in \mathcal{U} \text{ and } x \notin A\}.$$

We say that A, B are **disjoint** if $A \cap B = \emptyset$

Set Operations

Other set operations

Definition

$A \setminus B$ – **set difference**, relative complement (a but not b):

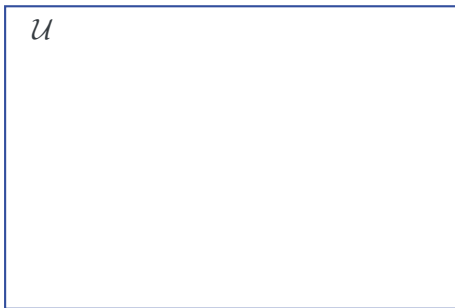
$$A \setminus B = A \cap B^c$$

$A \oplus B$ – **symmetric difference** (a and not b or b and not a ; also known as a or b exclusively; a xor b):

$$A \oplus B = (A \setminus B) \cup (B \setminus A)$$

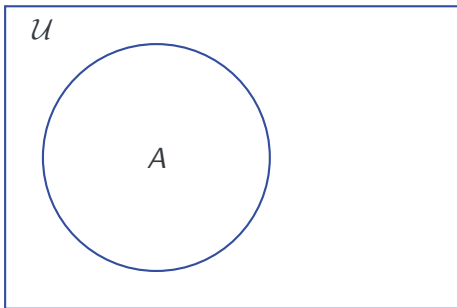
Venn Diagrams

A **Venn Diagram** is a simple graphical approach to visualize the basic set operations.



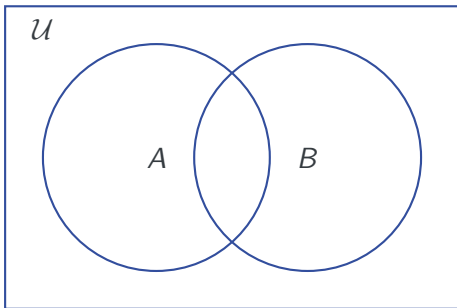
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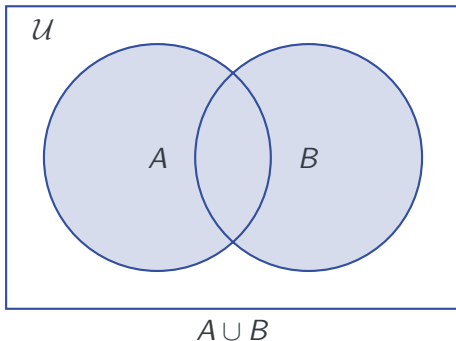
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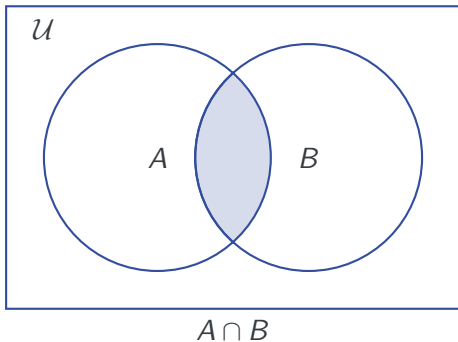
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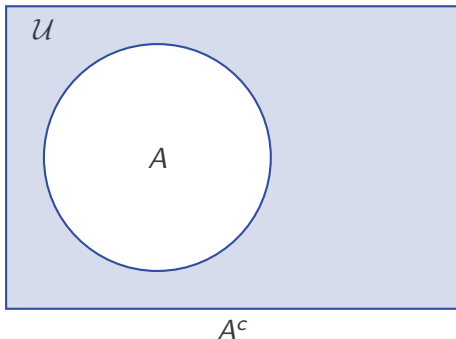
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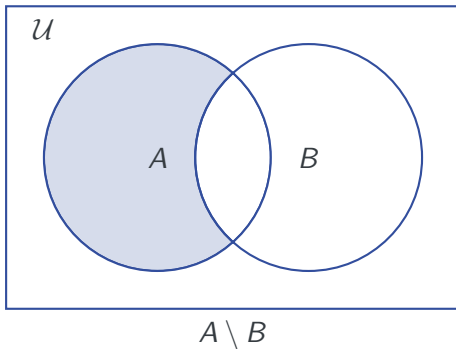
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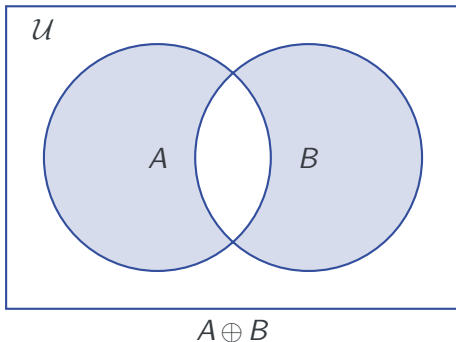
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Venn Diagrams

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Set Equality

Two sets are **equal** ($A = B$) if they contain the same elements

To show equality:

- Examine all the elements
- Show $A \subseteq B$ and $B \subseteq A$
- Use the Laws of Set Operations

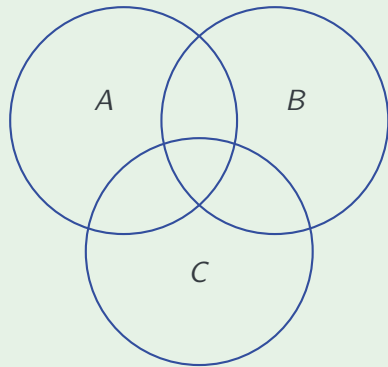
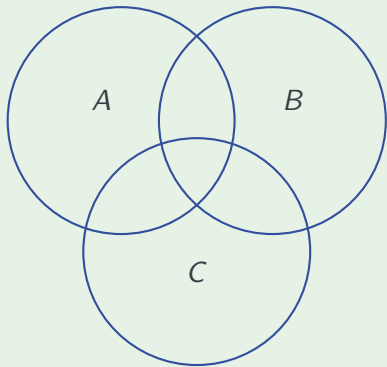
Important!

Venn diagrams can help visualize, but are **not** rigorous.

Example

Example

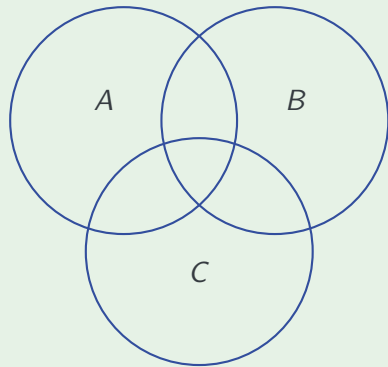
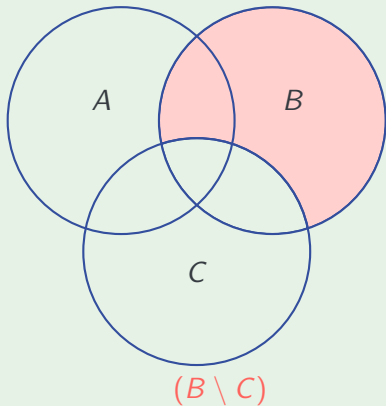
$$A \setminus (B \setminus C) \stackrel{?}{=} (A \setminus B) \setminus C$$



Example

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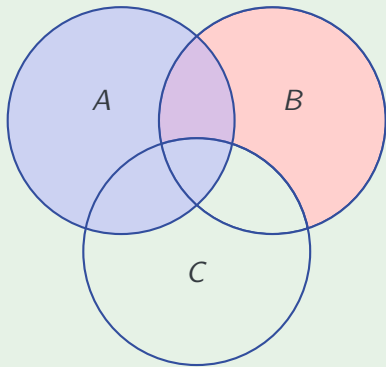
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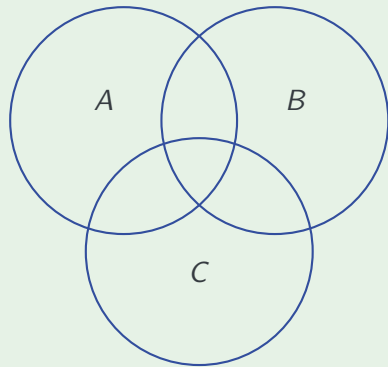
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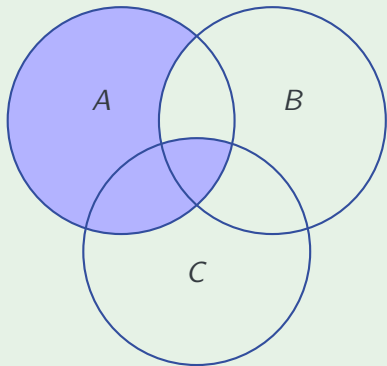
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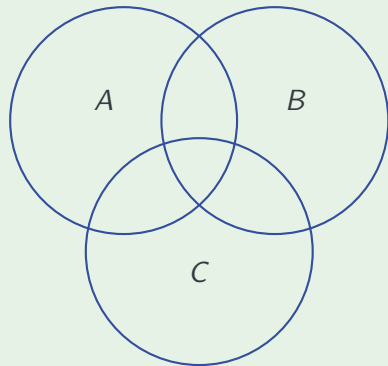
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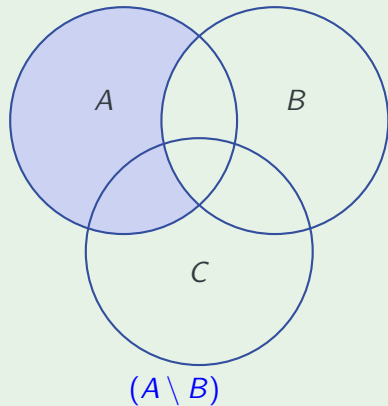
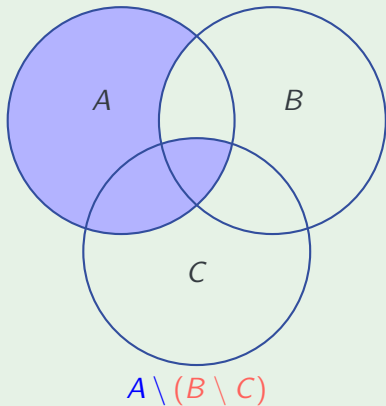
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Example

Example

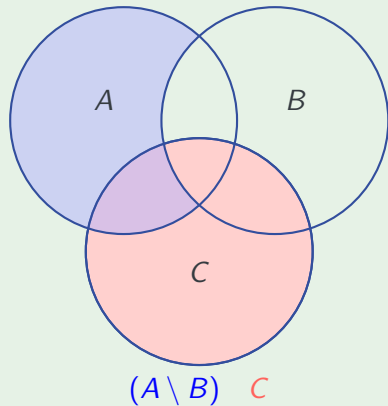
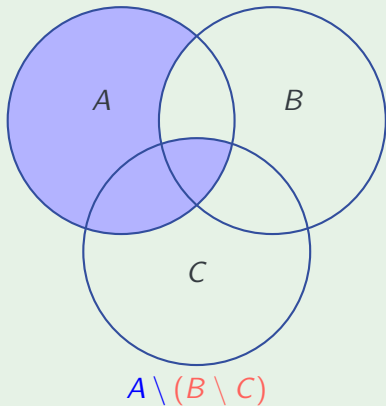
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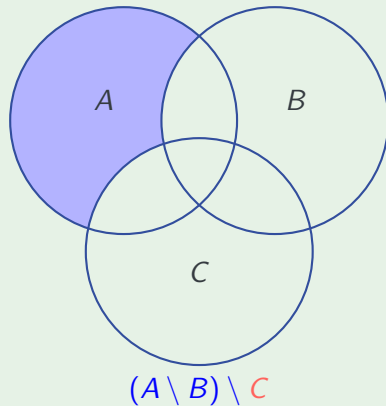
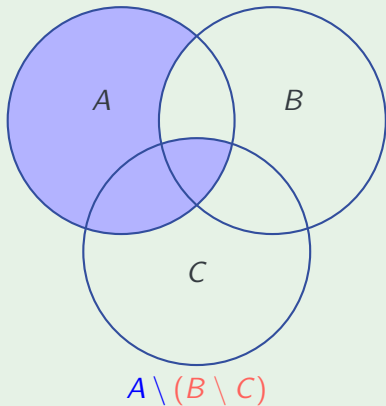
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Example

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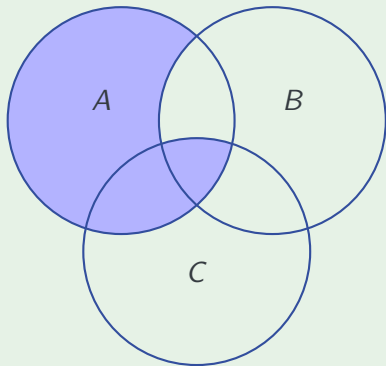
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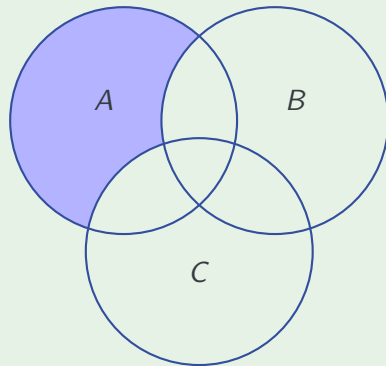
Example

Example

$$A \setminus (B \setminus C) \neq (A \setminus B) \setminus C$$



$A \setminus (B \setminus C)$

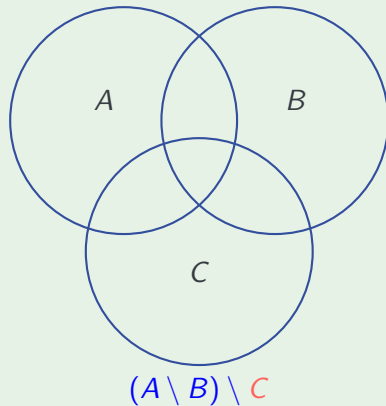
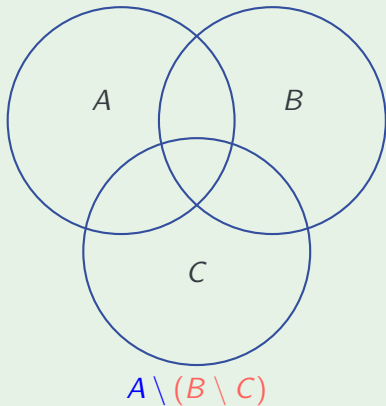


$(A \setminus B) \setminus C$

Example

Example

$$A \setminus (B \setminus C) \neq (A \setminus B) \setminus C$$



Examples

Example

Show $\{3, 2, 1\} = (0, 4)$.

Examples

Example

Show $\{3, 2, 1\} = (0, 4)$.

$(0, 4) = \{1, 2, 3\} = \{3, 2, 1\}$.

Examples

Example

Show $\{n : n \in \mathbb{Z} \text{ and } n^2 < 5\} = \{n : n \in \mathbb{Z} \text{ and } |n| \leq 2\}$

Examples

Example

Show $\{n : n \in \mathbb{Z} \text{ and } n^2 < 5\} = \{n : n \in \mathbb{Z} \text{ and } |n| \leq 2\}$

$$\begin{aligned}\{n : n \in \mathbb{Z} \text{ and } n^2 < 5\} &= \{-2, -1, 0, 1, 2\} \\ &= \{n : n \in \mathbb{Z} \text{ and } |n| \leq 2\}\end{aligned}$$

Examples

Example

Show $\{n : n \in \mathbb{Z} \text{ and } n^2 > 5\} = \{n : n \in \mathbb{Z} \text{ and } |n| > 2\}$

Examples

Example

Show $\{n : n \in \mathbb{Z} \text{ and } n^2 > 5\} = \{n : n \in \mathbb{Z} \text{ and } |n| > 2\}$

Show:

- For all $n \in \mathbb{Z}$, if $n^2 > 5$ then $|n| > 2$; and
- For all $n \in \mathbb{Z}$, if $|n| > 2$ then $n^2 > 5$.

That is, show:

For all $n \in \mathbb{Z}$: $n^2 > 5$ if, and only if $|n| > 2$

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Laws of Set Operations

For all sets A, B, C :

Commutativity $A \cup B = B \cup A$
 $A \cap B = B \cap A$

Associativity $(A \cup B) \cup C = A \cup (B \cup C)$
 $(A \cap B) \cap C = A \cap (B \cap C)$

Distribution $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
 $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Identity $A \cup \emptyset = A$
 $A \cap \mathcal{U} = A$

Complementation $A \cup (A^c) = \mathcal{U}$
 $A \cap (A^c) = \emptyset$

Substitution

Because the laws hold for all sets, we can substitute complex expressions for each set symbol.

Example

Commutativity

$$A \cup B = B \cup A$$

Substitution

Because the laws hold for all sets, we can substitute complex expressions for each set symbol.

Example

Commutativity

$$A \cup B = B \cup A$$

Therefore: $(C \cap D) \cup (D \oplus E) = (D \oplus E) \cup (C \cap D)$

Example

Example

Show that for all sets $A \cap (B \cap C) = C \cap (B \cap A)$:

Example

Example

Show that for all sets $A \cap (B \cap C) = C \cap (B \cap A)$:

$$\begin{aligned} A \cap (B \cap C) &= (A \cap B) \cap C && [\text{Associativity}] \\ &= C \cap (A \cap B) && [\text{Commutativity}] \\ &= C \cap (B \cap A) && [\text{Commutativity}] \end{aligned}$$

Important!

(Aim to) limit each step to a non-overlapping applications of a single rule

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Other useful set laws

The following are all derivable from the previous 10 laws.

Idempotence $A \cap A = A$
 $A \cup A = A$

Double complementation $(A^c)^c = A$

Annihilation $A \cap \emptyset = \emptyset$
 $A \cup \mathcal{U} = \mathcal{U}$

de Morgan's Laws $(A \cap B)^c = A^c \cup B^c$
 $(A \cup B)^c = A^c \cap B^c$

Example (Idempotence of \cup)

$$A = A \cup \emptyset$$

(Identity)

Example (Idempotence of \cup)

$$\begin{aligned} A &= A \cup \emptyset && \text{(Identity)} \\ &= A \cup (A \cap A^c) && \text{(Complementation)} \end{aligned}$$

Example (Idempotence of \cup)

$$\begin{aligned} A &= A \cup \emptyset && \text{(Identity)} \\ &= A \cup (A \cap A^c) && \text{(Complementation)} \\ &= (A \cup A) \cap (A \cup A^c) && \text{(Distributivity)} \end{aligned}$$

Example (Idempotence of \cup)

$$\begin{aligned} A &= A \cup \emptyset && \text{(Identity)} \\ &= A \cup (A \cap A^c) && \text{(Complementation)} \\ &= (A \cup A) \cap (A \cup A^c) && \text{(Distributivity)} \\ &= (A \cup A) \cap \mathcal{U} && \text{(Complementation)} \end{aligned}$$

Example (Idempotence of \cup)

$$\begin{aligned} A &= A \cup \emptyset && \text{(Identity)} \\ &= A \cup (A \cap A^c) && \text{(Complementation)} \\ &= (A \cup A) \cap (A \cup A^c) && \text{(Distributivity)} \\ &= (A \cup A) \cap \mathcal{U} && \text{(Complementation)} \\ &= (A \cup A) && \text{(Identity)} \end{aligned}$$

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Two useful results

Definition

If A is a set defined using \cap , \cup , \emptyset and \mathcal{U} , then $\text{dual}(A)$ is the expression obtained by replacing \cap with \cup (and vice-versa) and \emptyset with \mathcal{U} (and vice-versa).

Theorem (Principle of Duality)

If you can prove $A_1 = A_2$ using the Laws of Set Operations then you can prove $\text{dual}(A_1) = \text{dual}(A_2)$

Example

Absorption law: $A \cup (A \cap B) = A$

Dual: $A \cap (A \cup B) = A$

Application (Idempotence of \cap)

Recall Idempotence of \cup :

$$\begin{aligned} A &= A \cup \emptyset && \text{(Identity)} \\ &= A \cup (A \cap A^c) && \text{(Complementation)} \\ &= (A \cup A) \cap (A \cup A^c) && \text{(Distributivity)} \\ &= (A \cup A) \cap \mathcal{U} && \text{(Complementation)} \\ &= (A \cup A) && \text{(Identity)} \end{aligned}$$

Application (Idempotence of \cap)

Invoke the dual laws!

$$\begin{aligned} A &= A \cap \mathcal{U} && \text{(Identity)} \\ &= A \cap (A \cup A^c) && \text{(Complementation)} \\ &= (A \cap A) \cup (A \cap A^c) && \text{(Distributivity)} \\ &= (A \cap A) \cup \emptyset && \text{(Complementation)} \\ &= (A \cap A) && \text{(Identity)} \end{aligned}$$

Two useful results

Theorem (Uniqueness of complement)

$A \cap B = \emptyset$ and $A \cup B = \mathcal{U}$ if, and only if, $B = A^c$.

Two useful results

Theorem (Uniqueness of complement)

$A \cap B = \emptyset$ and $A \cup B = \mathcal{U}$ if, and only if, $B = A^c$.

Proof (Only if).

$$\begin{aligned} B &= B \cap \mathcal{U} && \text{(Identity)} \\ &= B \cap (A \cup A^c) && \text{(Complement)} \\ &= (B \cap A) \cup (B \cap A^c) && \text{(Distributivity)} \\ &= (A \cap B) \cup (A^c \cap B) && \text{(Commutativity)} \\ &= \emptyset \cup (A^c \cap B) && \text{(Given)} \\ &= (A \cap A^c) \cup (A^c \cap B) && \text{(Complement)} \\ &= (A^c \cap A) \cup (A^c \cap B) && \text{(Commutativity)} \\ &= A^c \cap (A \cup B) && \text{(Distributivity)} \\ &= A^c \cap \mathcal{U} && \text{(Given)} \\ &= A^c && \text{(Identity)} \end{aligned}$$



Application (Double complement)

Take $A = X^c$ and $B = X$:

$$\begin{aligned} X^c \cap X &= X \cap X^c && \text{(Commutativity)} \\ &= \emptyset && \text{(Complement)} \end{aligned}$$

$$X^c \cup X = \mathcal{U} \quad \text{(Principle of duality)}$$

By the uniqueness of complement, $(X^c)^c = X$.

Exercises

Exercises

Show the following for all sets A , B , C :

- $B \cup (A \cap \emptyset) = B$
- $(C \cup A) \cap (B \cup A) = A \cup (B \cap C)$
- $(A \cap B) \cup (A \cup B^c)^c = B$

Exercises

Give counterexamples to show the following do not hold for all sets:

- $A \setminus (B \setminus C) = (A \setminus B) \setminus C$
- $(A \cup B) \setminus C = A \cup (B \setminus C)$
- $(A \setminus B) \cup B = A$