Sample Finals

1. a) Prove that
$$2 \mid (k^2 + k) \text{ for all integers } k.$$
[6 marks]

   b) Find the greatest positive integer n such that
$$n \mid (k^3 - k) \text{ for all integers } k.$$
[6 marks]

2. Prove, using the laws of set operations, for all sets A, B, C:

   a)
$$A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$$
[6 marks]

   b)
$$(A \oplus B) \oplus (B \oplus C) = A \oplus C$$
[6 marks]

   Partial marks are available for proofs that do not use the laws of set operations.

3. Let $\Sigma = \{a, b\}$ and let $W = \Sigma^{=3}$. Define the relation $R \subseteq W \times W$ where:
$$w_1 \; R \; w_2 \text{ if and only if for each position } i \text{, either:}$$
   - $w_1[i] = w_2[i]$, or
   - $w_1[i] = a$ and $w_2[i] = b$

   a) Prove R is a partially ordered relation [8 marks]

   b) Draw the Hasse Diagram for R [7 marks]

4. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined as:
$$f(x) = \begin{cases} \lfloor \frac{x}{2} \rfloor & \text{if } x \geq 0, \\ 1 - 2x & \text{if } x < 0. \end{cases}$$

   a) Suppose that integer $n \geq 3$. Find $f^n(7)$, where f^n is f composed with itself n times.
[4 marks]

   b) Prove or disprove that f is injective. [5 marks]

   c) Let $g(x) = f(x+1) - f(x)$. Find the range of g. [5 marks]

5. Let $M = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   a) Find M^4. [6 marks]

   b) Prove, by induction (or otherwise), that for all $n \in \mathbb{Z}$ with $n \geq 1$,
$$M^{4n} = I.$$

[6 marks]

6. A social media platform is testing a new system for recommending content. Each user is randomly assigned to one of two groups:

   - **Group A**: users see the new content recommendation system.
   - **Group B**: users see the old content recommendation system.

   The platform checks whether users interact with the content by liking, sharing, or commenting. Here is what we know:

   - 60% of users are assigned to Group A, and 40% are assigned to Group B.
   - 40% of Group A users will engage with the content.
   - 30% of Group B users will engage with the content.

   a) What is the probability that the randomly selected user who interacted with the content is from Group A? [5 marks]

   b) The platform earns \$2 for each user who interacts with the content but pays \$0.50 for each recommendation. For a randomly selected user, what is the expected profit and the variance of the profit? [7 marks]

7. Let $\Sigma = \{a, b\}$. We define $\mathrm{swap}(w)$ as the word obtained by swapping each a with b and each b with a. For example, $\mathrm{swap}(aab) = bba$.

   Define the language L recursively:

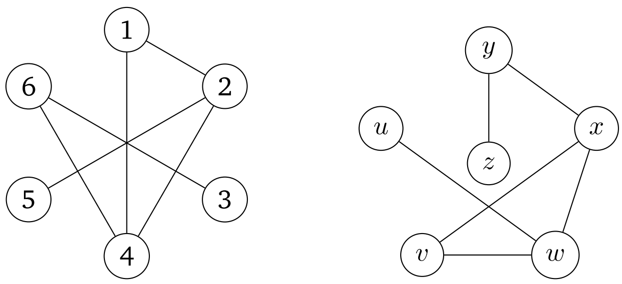
   - $\lambda \in L$
   - If $w \in L$, then $awa \in L$
   - If $w \in L$, then $\mathrm{swap}(w) \in L$

   a) List all words in L with a length less than 3. [3 marks]

   b) Define $f(n)$ as the number of words of length n in L. Find a recurrence relation for $f(n)$.

   [6 marks]

8. Consider the graphs $G = (V_G, E_G)$ (on the left) and $H = (V_H, E_H)$ (on the right) drawn as



   a) Prove that G has a Hamiltonian path. [2 marks]

   b) Prove that H does not have an Euler trail. [2 marks]

   c) Provide a function $\phi : V_G \to V_H$ to prove that G and H are isomorphic. [4 marks]

   d) Prove that $\deg(v) = \deg(\phi(v))$ for all $v \in V_G$. [4 marks]

9. In a room with three logicians Alice, Bob and Charlie:

   - Alice says: "Bob and Charlie are lying"
   - Bob says: "If Charlie is telling the truth, then Alice is telling the truth"

- Charlie says: "Alice and Bob are lying"

a) Express each statement using propositional logic. [4 marks]

b) Find all truth assignments where every statement is satisfied. [6 marks]

c) You know Bob is telling the truth. Is Alice lying? Is Charlie lying? Explain. [5 marks]

10. Consider this algorithm for finding the maximum sum of any subarray:

```
max_subarray(A, n):
    if n == 1:
        return A[0]
    mid = ⌊n/2⌋
    left_max = max_subarray(A[0:mid], mid)
    right_max = max_subarray(A[mid+1:n], n-mid)
    cross_max = find_crossing_max(A, mid)
    return max(left_max, right_max, cross_max)
```

where find_crossing_max runs in $\Theta(n)$ time.

a) Give a recurrence relation for the running time of `max_subarray`. [4 marks]

b) Give, with justification, an asymptotic upper bound for the running time of `max_subarray` in Big-O notation. [4 marks]