COMP9444: Neural Networks and Deep Learning

Week 1d. Backpropagation

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June 4, 2025

Outline

➢ Multi-Layer Neural Networks

➢ Continuous Activation Functions

➢ Gradient Descent

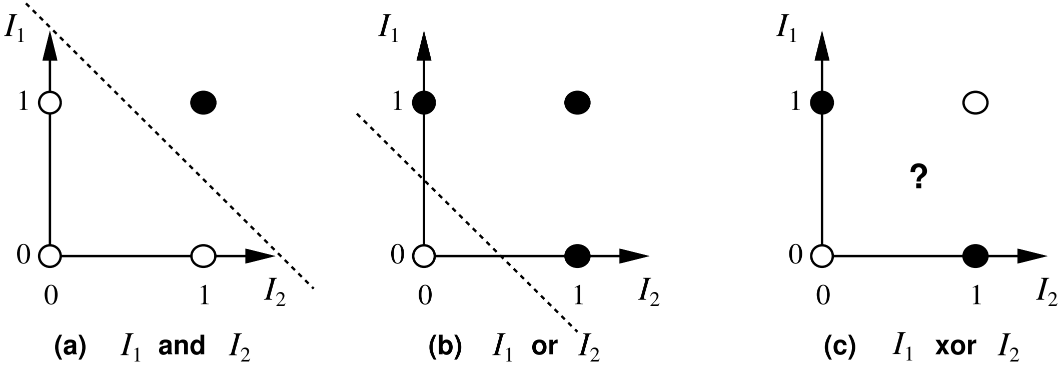
➢ Backpropagation

➢ Examples

➢ Momentum and Adam

Recall: Limitations of Perceptrons

Problem: many useful functions are not linearly separable (e.g. XOR)

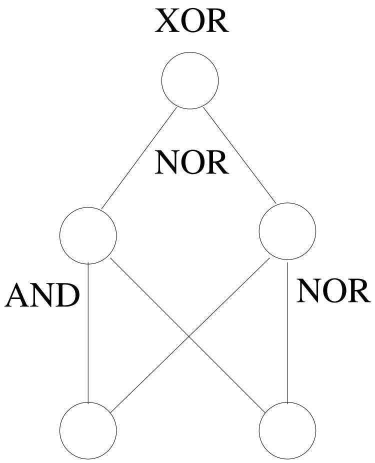


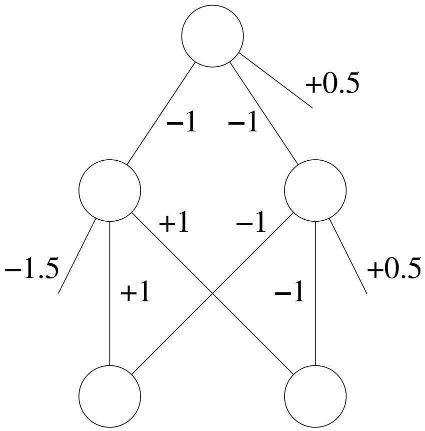
Possible solution:

x_1 XOR x_2 can be written as: (x_1 AND x_2) NOR (x_1 NOR x_2)

Recall that AND, OR and NOR can be implemented by perceptrons.

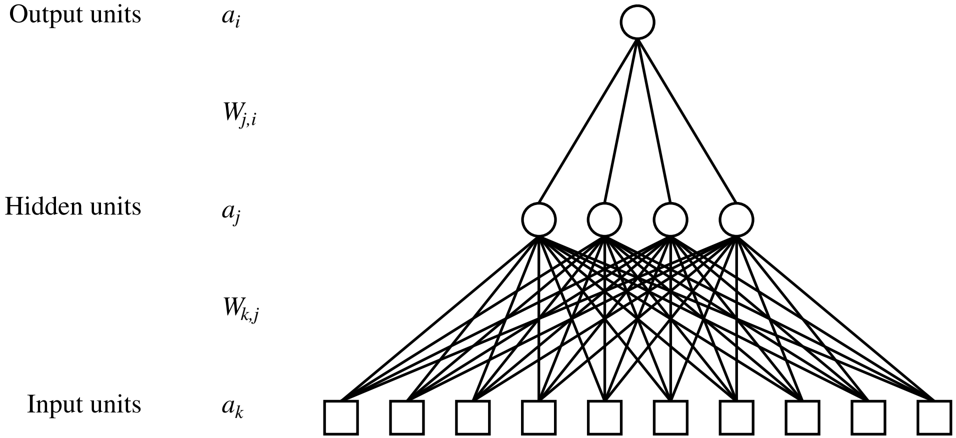
Multi-Layer Neural Networks





Problem: How can we train it to learn a new function? (credit assignment)

Two-Layer Neural Network



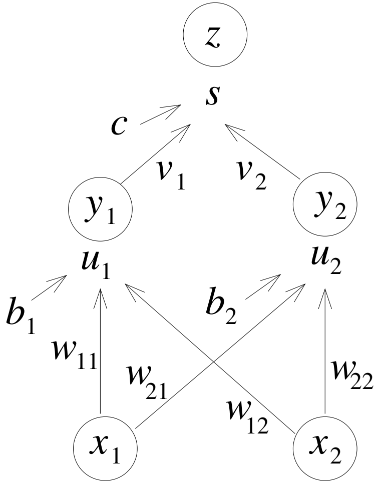
Normally, the numbers of input and output units are fixed,
but we can choose the number of hidden units.

The XOR Problem

x_1	x_2	target
0	0	0
0	1	1
1	0	1
1	1	0

XOR data cannot be learned with a perceptron, but it can be achieved using a 2-layer network with two hidden units.

Neural Network Equations



$$
\begin{aligned}
u_1 &= b_1 + w_{11}x_1 + w_{12}x_2 \\
y_1 &= g(u_1) \\
s &= c + v_1 y_1 + v_2 y_2 \\
z &= g(s)
\end{aligned}
$$

We sometimes use w as a shorthand for any of the trainable weights $\{c, v_1, v_2, b_1, b_2, w_{11}, w_{21}, w_{12}, w_{22}\}$.

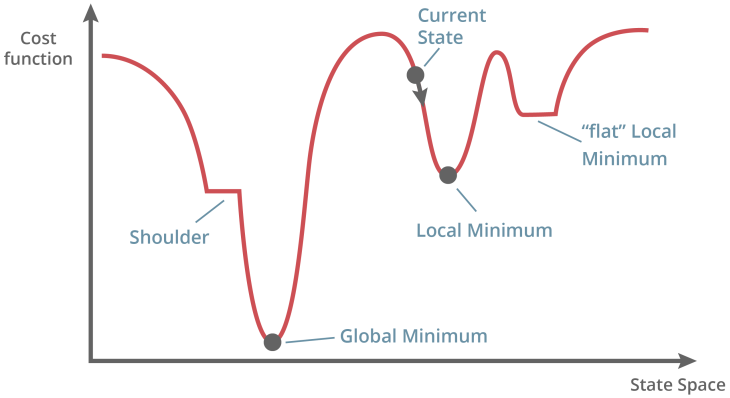
NN Training as Cost Minimization

We define an **error** function or **loss** function E to be (half) the sum over all input patterns of the square of the difference between actual output and **target** output

$$E = \frac{1}{2}\sum_i (z_i - t_i)^2$$

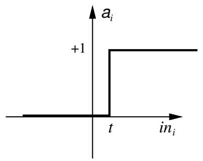
If we think of E as height, it defines an error **landscape** on the weight space. The aim is to find a set of weights for which E is very low.

Local Search in Weight Space

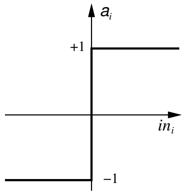


Problem: because of the step function, the landscape will not be smooth, but will instead consist almost entirely of flat local regions and "shoulders", with occasional discontinuous jumps.

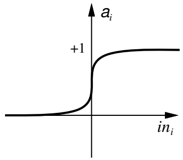
Continuous Activation Functions



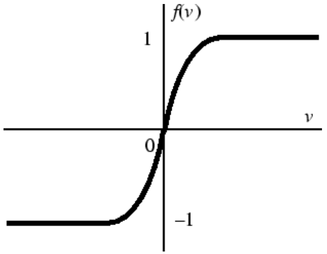
(a) Step function



(b) Sign function



(c) Sigmoid function



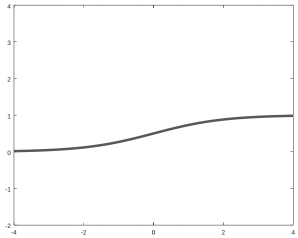
Key Idea: Replace the (discontinuous) step function with a differentiable function, such as the sigmoid:

$$g(s) = \frac{1}{1+e^{-s}}$$

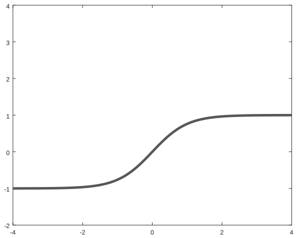
or hyperbolic tangent

$$g(s) = \tanh(s) = \frac{e^s - e^{-s}}{e^s + e^{-s}} = 2\Big(\frac{1}{1+e^{-2s}}\Big) - 1$$

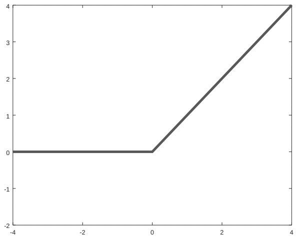
Activation Functions



Sigmoid



Hyperbolic Tangent



Rectified Linear Unit (ReLU)

Gradient Descent

Recall that the **loss** function E is (half) the sum over all input patterns of the square of the difference between actual output and target output

$$E = \frac{1}{2} \sum_i (z_i - t_i)^2$$

The aim is to find a set of weights for which E is very low.

If the functions involved are smooth, we can use multi-variable calculus to adjust the weights in such a way as to take us in the steepest downhill direction.

$$w \leftarrow w - \eta \, \frac{\partial E}{\partial w}$$

Parameter η is called the **learning rate**.

Chain Rule

If, say

$$y = y(u)$$

$$u = u(x)$$

Then

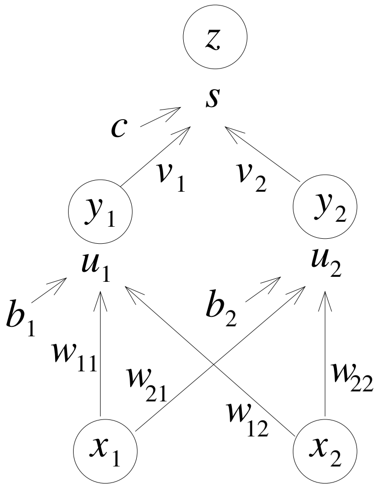
$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial x}$$

This principle can be used to compute the partial derivatives in an efficient and localized manner. Note that the transfer function must be differentiable (usually sigmoid, or tanh).

Note: if $z(s) = \dfrac{1}{1+e^{-s}}, \qquad z'(s) = z(1-z).$

if $z(s) = \tanh(s), \qquad z'(s) = 1 - z^2.$

Forward Pass



$$\begin{aligned}
u_1 &= b_1 + w_{11}x_1 + w_{12}x_2 \\
y_1 &= g(u_1) \\
s &= c + v_1 y_1 + v_2 y_2 \\
z &= g(s) \\
E &= \frac{1}{2}\sum(z-t)^2
\end{aligned}$$

Backpropagation

Partial Derivatives

$$\begin{aligned}
\frac{\partial E}{\partial z} &= z - t \\
\frac{dz}{ds} &= g'(s) = z(1-z) \\
\frac{\partial s}{\partial y_1} &= v_1 \\
\frac{dy_1}{du_1} &= y_1(1-y_1)
\end{aligned}$$

Useful notation

$$\delta_{\text{out}} = \frac{\partial E}{\partial s} \quad \delta_1 = \frac{\partial E}{\partial u_1} \quad \delta_2 = \frac{\partial E}{\partial u_2}$$

Then

$$\begin{aligned}
\delta_{\text{out}} &= (z-t)\, z\, (1-z) \\
\frac{\partial E}{\partial v_1} &= \delta_{\text{out}}\, y_1 \\
\delta_1 &= \delta_{\text{out}}\, v_1\, y_1\, (1-y_1) \\
\frac{\partial E}{\partial w_{11}} &= \delta_1\, x_1
\end{aligned}$$

Partial derivatives can be calculated efficiently by packpropagating deltas through the network.

Two-Layer NN's – Applications

- ➛ Medical Dignosis
- ➛ Autonomous Driving
- ➛ Game Playing
- ➛ Credit Card Fraud Detection
- ➛ Handwriting Recognition
- ➛ Financial Prediction

Example: Pima Indians Diabetes Dataset

	Attribute	mean	stdv
1.	Number of times pregnant	3.8	3.4
2.	Plasma glucose concentration	120.9	32.0
3.	Diastolic blood pressure (mm Hg)	69.1	19.4
4.	Triceps skin fold thickness (mm)	20.5	16.0
5.	2-Hour serum insulin (mu U/ml)	79.8	115.2
6.	Body mass index (weight in kg/(height in m)^2)	32.0	7.9
7.	Diabetes pedigree function	0.5	0.3
8.	Age (years)	33.2	11.8

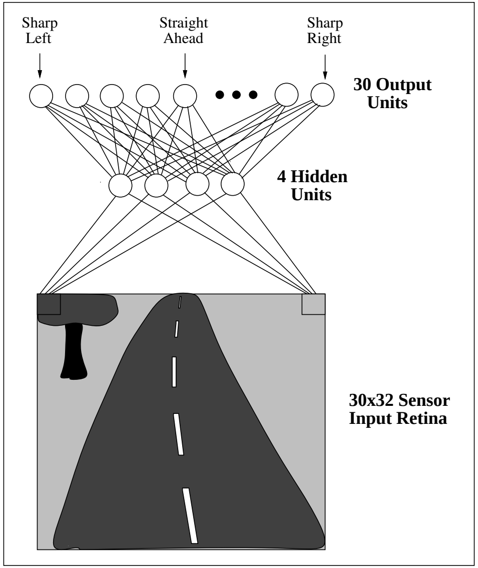
Based on these inputs, try to predict whether the patient will develop Diabetes (1) or Not (0).

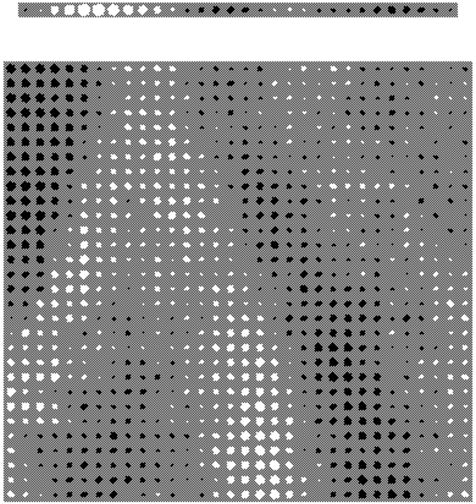
Training Tips

- re-scale inputs and outputs to be in the range 0 to 1 or -1 to 1
  - otherwise, backprop may put undue emphasis on larger values
- replace missing values with mean value for that attribute
- weight initialization
  - for shallow networks, initialize weights to small random values
  - for deep networks, more sophisticated strategies
- on-line, batch, mini-batch, experience replay
- three different ways to prevent overfitting:
  - limit the number of hidden nodes or connections
  - limit the training time, using a validation set
  - weight decay
- adjust learning rate (and other parameters) to suit the particular task

ALVINN (Pomerleau 1991, 1993)

ALVINN

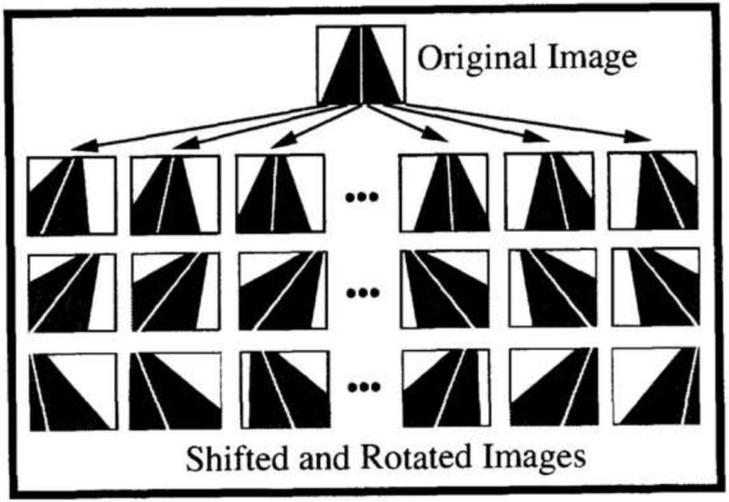

Sharp Left
Straight Ahead
Sharp Right
30 Output Units
4 Hidden Units
30x32 Sensor Input Retina



ALVINN

- Autonomous Land Vehicle In a Neural Network
- Later version included a sonar range finder
  - 8×32 range finder input retina
  - 29 hidden units
  - 45 output units
- Supervised Learning, from human actions (Behavioral Cloning)
  - Replay Memory – experiences are stored in a database and randomly shuffled for training
  - Data Augmentation – additional “transformed” training items are created, in order to cover emergency situations
- drove autonomously from coast to coast

Data Augmentation



Momentum

If the landscape is shaped like a “rain gutter”, weights will tend to oscillate without much improvement. We can add a momentum factor

$$\begin{aligned} \delta w &\leftarrow \alpha\,\delta w - \eta\,\frac{\partial E}{\partial w} \\ w &\leftarrow w + \delta w \end{aligned}$$

Hopefully, this will dampen sideways oscillations but amplify downhill motion by $\frac{1}{1-\alpha}$. Momentum can also help to escape from local minima, or move quickly across flat regions in the loss landscape.

When momentum is used, we generally reduce the learning rate at the same time, in order to compensate for the implicit factor of $\frac{1}{1-\alpha}$.

Adaptive Moment Estimation (Adam)

Maintain a running average of the gradients (m_t) and squared gradients (v_t) for each weight in the network.

$$\begin{aligned} m_t &= \beta_1 m_{t-1} + (1-\beta_1) g_t \\ v_t &= \beta_2 v_{t-1} + (1-\beta_2) g_t^2 \end{aligned}$$

To speed up training in the early stages, compensating for the fact that m_t, g_t are initialized to zero, we rescale as follows:

$$\hat{m}_t = \frac{m_t}{1-\beta_1^t}, \quad \hat{v}_t = \frac{v_t}{1-\beta_2^t}$$

Finally, each parameter is adjusted according to:

$$w_t = w_{t-1} - \frac{\eta}{\sqrt{\hat{v}_t} + \varepsilon} \hat{m}_t$$