COMP9020

Foundations of Computer Science

Term 3, 2024

Lecture 13: Combinatorics

Combinatorics in Computer Science

Informally, **combinatorics** is the mathematics of counting.

More formally, **combinatorics** is about understanding finite systems of discrete objects.

For example:

- How many different ways are there of getting a flush in poker?

In computer science, we use combinatorics when:

- Computing cost functions in algorithmic analysis
- Identifying (in-)efficiencies in data management
- Developing effective techniques for enumerating objects
- Probability calculations

Outline

Outline

Counting Techniques

General idea: find methods, algorithms or precise formulae to count the number of elements in various sets or collections derived, in a structured way, from some basic sets.

Examples

Single base set $S = \{s_1, \ldots, s_n\}$, $|S| = n$; find the number of

- all subsets of S
- ordered selections of r different elements of S
- unordered selections of r different elements of S
- selections of r elements from S such that ...
- functions $S \longrightarrow S$ (onto, 1-1)
- partitions of S into k equivalence classes

Example

Example

A restaurant has the following menu:

Starter	Main Course	Dessert
Soup	Fish	Ice-cream
Bread	Beef	Fruit
	Pork	Cheese
	Chicken	

How many:

- 3 course meals (Starter-Main-Dessert) are possible?
- 3 course meals (Any item for each course) are possible?
- 3 course meals (Any item, no duplicates) are possible?
- Meals consisting of 3 items (order is unimportant)?

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- Any item for 3 courses?
- Any item, no duplicates, for 3 courses?
- Meals of 3 different items?

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- Any item for 3 courses?
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- Any item, no duplicates, for 3 courses? $9 \times 8 \times 7 = 504$
- Meals of 3 different items? $504/6 = 84$

Basic Counting Rules: Principles

Two simple rules:

- **Union rule** ("or"): If S and T are disjoint
  $|S \cup T| = |S| + |T|$
- **Product rule** ("followed by"): $|S \times T| = |S| \cdot |T|$

These cover many examples, though the rule application is not always obvious.

Common strategies:

- Direct application of the rule
- Relate unknown quantities to known quantities (e.g. $|S| + |T| = |S \cup T| + |S \cap T|$)
- Find a bijection to a set that can be counted

Outline

The Union Rule

Union rule — S and T *disjoint*

$$|S \cup T| = |S| + |T|$$

$S_1, S_2, \ldots, S_n$ pairwise disjoint ($S_i \cap S_j = \emptyset$ for $i \neq j$)

$$|S_1 \cup \ldots \cup S_n| = \sum |S_i|$$

Example

How many numbers in $A = [1, 2, \ldots, 999]$ are divisible by 31 or 41?

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Example

How many numbers in $A = [1, 2, \ldots, 999]$ are divisible by 31 or 41?

$\lfloor 999/31 \rfloor = 32$ numbers are divisible by 31
$\lfloor 999/41 \rfloor = 24$ numbers are divisible by 41
No number in A divisible by both 31 and 41
Hence, $32 + 24 = 56$ divisible by 31 or 41

Consequences of the Union Rule

Fact

For any sets X, Y, Z:

$$\begin{aligned}
|Y \setminus X| &= |Y| - |X \cap Y| \\
|X \cup Y| &= |X| + |Y| - |X \cap Y| \\
|X \cup Y \cup Z| &= |X| + |Y| + |Z| \\
&\quad -|X \cap Y| - |Y \cap Z| - |Z \cap X| \\
&\quad +|X \cap Y \cap Z|
\end{aligned}$$

Fact

(1) *If* $|S \cup T| = |S| + |T|$ *then* S *and* T *are disjoint*

(2) *If* $|\bigcup_{i=1}^{n} S_i| = \sum_{i=1}^{n} |S_i|$ *then* S_i *are pairwise disjoint*

(3) *If* $|T \setminus S| = |T| - |S|$ *then* $S \subseteq T$

These properties can serve to identify cases when sets are disjoint (resp. one is contained in the other).

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Proof.

We can prove these facts using the inclusion-exclusion identity for two sets. Namely, that $|S \cap T| + |S \cup T| = |S| + |T|$.

(1) Suppose $|S| + |T| = |S \cup T|$. Then inclusion-exclusion gives $|S \cap T| = |S| + |T| - |S \cup T| = 0$, so $S \cap T = \varnothing$.

(3) Suppose $|T \setminus S| = |T| - |S|$. Then inclusion-exclusion gives $|S \cap T| = |S|$, so $S \subseteq T$. □

Exercises

$\boxed{\text{RW: 5.3.1}}$ 200 people. 150 swim or jog, 85 swim and 60 do both. How many jog?

$\boxed{\text{RW: 5.6.38}}$ (Supp) There are 100 problems, 75 of which are 'easy' and 40 'important'. What's the smallest possible number of problems that are both easy *and* important?.

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Let $S := \{\text{people who swim}\}$ and $J := \{\text{people who jog}\}$.
Then $|S \cup J| = |S| + |J| - |S \cap J|$; thus $150 = 85 + |J| - 60$ hence $|J| = 125$.
Note that the answer *does not* depend on the number of people overall (200).

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$$|E \cap I| = |E| + |I| - |E \cup I| = 75 + 40 - |E \cup I| \geq 75 + 40 - 100 = 15$$

Outline

The Product Rule

Product rule:

$$|S_1 \times \ldots \times S_k| = |S_1| \cdot |S_2| \cdots |S_k| = \prod_{i=1}^{k} |S_i|$$

Take Notice

This counts the number of sequences where the first item is from S_1, the second is from S_2, and so on.

Special case of the product rule: If all $S_i = S$ for all i and $|S| = m$ then

$$|S_1 \times S_2 \times \cdots \times S_k| = |S \times S \times \cdots \times S| = |S^k| = m^k$$

Example

Let $\Sigma = \{a, b, c, d, e, f, g\}$.

Question. How many 5-letter words can we make?

$$|\Sigma \times \Sigma \times \Sigma \times \Sigma \times \Sigma| = |\Sigma^5| = |\Sigma|^5 = 7^5 = 16,807$$

Question. How many words with no letter repeated?

Product rule: Sequences of selections

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How can we count sequences when the underlying set changes?

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How can we count sequences when the underlying set changes?

To count sequences *without replacement*:

- Define an order on the whole underlying set
- Select from $[1, n]$, where n is the size of the "remaining" set, and a selection of i represents choosing the i-th element in that set

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Example

Let $\Sigma = \{a, b, c, d, e, f, g\}$.
How many 5-letter words with no letter repeated?

$$\prod_{i=0}^{4}(|\Sigma| - i) = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2,520$$

Exercises

S, T finite. How many functions $S \longrightarrow T$ are there?

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$$|T|^{|S|}$$

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$\boxed{\text{RW: 5.3.2}}$ $S = [100 \dots 999]$, thus $|S| = 900$.

(a) How many numbers in S contain a 3 **or** 7 in their digits?

(b) How many numbers in S have a 3 **and** a 7?

Exercise

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$\boxed{\text{RW: 5.3.2}}$ $S = [100 \dots 999]$, thus $|S| = 900$.

(a) How many numbers in S contain a 3 **or** 7 in their digits?
Let $A_3 = \{\text{at least one '3'}\}$ and $A_7 = \{\text{at least one '7'}\}$. Then

$$(A_3 \cup A_7)^c = \{\ n \in [100, 999] : n \text{ digits } \in \{0, 1, 2, 4, 5, 6, 8, 9\}\ \}$$

Note that for each number in S, there are 7 choices for the first digit and 8 choices for the later digits. So

$$|(A_3 \cup A_7)^c| = |\{1, 2, 4, 5, 6, 8, 9\}| \cdot |\{0, 1, 2, 4, 5, 6, 8, 9\}|^2$$

Therefore $|A_3 \cup A_7| = |S| - |(A_3 \cup A_7)^c| = 900 - 448 = 452$.

(b) How many numbers in S have a 3 **and** a 7?

Exercise

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Therefore $|A_3 \cup A_7| = |S| - |(A_3 \cup A_7)^c| = 900 - 448 = 452$.

(b) How many numbers in S have a 3 **and** a 7?

$$\begin{aligned} |A_3 \cap A_7| &= |A_3| + |A_7| - |A_3 \cup A_7| \\ &= (900 - 8 \cdot 9 \cdot 9) + (900 - 8 \cdot 9 \cdot 9) - 452 \\ &= 2 \cdot 252 - 452 = 52 \end{aligned}$$

Combinatorial Symmetry

A **symmetry** of a mathematical object is a bijective mapping from the object to itself which preserves "structure".

A (combinatorial) symmetry defines an equivalence relation where the equivalence classes all have the same size.

We are often interested in counting a set "up to symmetry". That is, counting the number of equivalence classes.

This can also be stated as a constraint that identifies a specific item in each equivalence class (**symmetric constraint**).

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Definition

A *k-to-1 function* is a function that maps exactly k inputs to an output.

Take Notice

A k-to-1 function defines the equivalence relation of a combinatorial symmetry and vice-versa.

Product rule: Symmetries and duplications

Question

- *How can we count sequences when we have symmetric constraints?*
- *How can we count sequences when we have duplicates?*

Example

Let $\Sigma = \{a, b, c, d, e\}$.

- How many 5-letter words with no letter repeated and a before b before c?
- How many 5-letter words can be made from a, a, a, d, e?

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- How many 5-letter words with no letter repeated and a before b before c?
- How many 5-letter words can be made from a, a, a, d, e?

Take Notice

The answer will be the same.

Product rule: Symmetries and duplications

- S_1 = {sequences accounting for symmetry},
- S_2 = {symmetries},
- S = {sequences without symmetry}

$$S = S_1 \times S_2,$$

so

$$|S_1| = |S|/|S_2|$$

Alternatively, $\frac{1}{|S_2|}$ of the $|S|$ sequences meet the symmetric constraint.

Product rule: Symmetries and duplications

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Product rule: Symmetries and duplications

Example

Question. Let $\Sigma = \{a, b, c, d, e\}$. How many 5-letter words with no letter repeated and a before b before c?

Answer. Let $\Sigma' = \{a, b, c\}$. Then

$$\begin{aligned}|S| &= |\{\text{5 letter words using letters from } \Sigma \text{ with no repeats}\}| \\ &= \prod_{i=0}^{4}(|\Sigma| - i) = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120\end{aligned}$$

and

$$\begin{aligned}|S_2| &= |\{\text{orderings of elements in } \Sigma'\}| \\ &= \prod_{i=0}^{2}(|\Sigma'| - i) = 3 \cdot 2 \cdot 1 = 6\end{aligned}$$

So

$$|S_1| = |\{\text{words in } S \text{ containing } a, b, c \text{ in order}\}| = \frac{120}{6} = 20$$

Outline

Combinatorial Objects: How Many?

permutations
Ordering of all objects from a set S; equivalently: Selecting all objects while *recognising* the order of selection.

The number of permutations of n elements is

$$n! = n \cdot (n-1) \cdots 1, \quad 0! = 1! = 1$$

r-**permutations (sequences without repetition)**
Selecting any r objects from a set S of size n without repetition while *recognising* the order of selection.

Their number is

$$(n)_r = {}^nP_r = n \cdot (n-1) \cdots (n-r+1) = \frac{n!}{(n-r)!}$$

Permutations with duplicates

Example

How many anagrams of ASSESS?

Permutations with duplicates

Example

How many anagrams of ASSESS?

Label S's: $AS_1S_2ES_3S_4$: $6!$

In each anagram we can label the S's in $4!$ ways.

Suppose there are m anagrams. So $m \cdot 4! = 6!$, i.e. $m = \frac{6!}{4!}$

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Example

Number of anagrams of MISSISSIPPI? $\frac{11!}{4!4!2!}$

r-**selections** (or: r-**combinations**)

Collecting any r distinct objects without repetition; equivalently: selecting r objects from a set S of size n and *not* recognising the order of selection.

Their number is

$$\binom{n}{r} = \frac{(n)_r}{r!} = \frac{n!}{(n-r)!r!} = \frac{n \cdot (n-1)\cdots(n-r+1)}{1 \cdot 2 \cdots r}$$

Take Notice

These numbers are usually called binomial coefficients due to

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + b^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

Also defined for any $\alpha \in \mathbb{R}$ *as* $\displaystyle\binom{\alpha}{r} = \frac{\alpha(\alpha-1)\cdots(\alpha-r+1)}{r!}$

Simple Counting Problems

Example

$\boxed{\text{RW: 5.1.2}}$ Give an example of a counting problem whose answer is

(a) $(26)_{10}$

(b) $\binom{26}{10}$

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(b) $\binom{26}{10}$

Draw 10 cards from a half deck (eg. black cards only)
(a) the cards are recorded in the order of appearance
(b) only the complete draw is recorded

Examples

- Number of diagonals in a convex polygon
- Number of poker hands
- Decisions in games, lotteries etc.

Exercises

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$\boxed{\text{RW: 5.1.6}}$ From a group of 12 men and 16 women, how many committees can be chosen consisting of

(a) 7 members?

(b) 3 men and 4 women?

(c) 7 women or 7 men?

$\boxed{\text{RW: 5.1.7}}$ As above, but any 4 people (male or female) out of 9 and two, Alice and Bob, unwilling to serve on the same committee.

Exercises

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$\boxed{\text{RW: 5.1.6}}$ From a group of 12 men and 16 women, how many committees can be chosen consisting of

(a) 7 members? $\binom{12+16}{7}$

(b) 3 men and 4 women? $\binom{12}{3}\binom{16}{4}$

(c) 7 women or 7 men? $\binom{12}{7} + \binom{16}{7}$

$\boxed{\text{RW: 5.1.7}}$ As above, but any 4 people (male or female) out of 9 and two, Alice and Bob, unwilling to serve on the same committee.

$\{$all committees$\}$ $-$ $\{$committees with both A and $B\}$
$= \binom{9}{4} - \binom{7}{2} = 126 - 21 = 105$

equivalently, $\{$A in, B out$\}$ $+$ $\{$A out, B in$\}$ $+$ $\{$none in$\}$

$$= \binom{7}{3} + \binom{7}{3} + \binom{7}{4} = 35 + 35 + 35 = 105$$

Counting Poker Hands

Exercises

$\boxed{\text{RW: 5.1.15}}$ A poker hand consists of 5 cards drawn without replacement from a standard deck of 52 cards

$\{A, 2\text{-}10, J, Q, K\} \times \{\text{club } \clubsuit, \text{spade } \spadesuit, \text{heart } \heartsuit, \text{diamond } \diamondsuit\}$

(a) Number of "4 of a kind" hands (e.g. 4 Jacks)

(b) Number of non-straight flushes, i.e. all cards of same suit but *not* consecutive (e.g. 8,9,10,J,K)

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(a) Number of "4 of a kind" hands (e.g. 4 Jacks)
$|\text{rank of the 4-of-a-kind}| \cdot |\text{any other card}| = 13 \cdot (52 - 4)$

(b) Number of non-straight flushes, i.e. all cards of same suit but *not* consecutive (e.g. 8,9,10,J,K)
$|\text{all flush}| - |\text{straight flush}|$
$= |\text{suit}| \cdot |\text{5-hand in a given suit}| -$
$\quad |\text{suit}| \cdot |\text{rank of a straight flush in a given suit}|$
$= 4 \cdot \binom{13}{5} - 4 \cdot 10$

Selecting items summary

Selecting k items from a set of n items:

With replacement	Order matters	Examples	Formula
Yes	Yes	Words of length k (sequences of length k)	n^k
No	Yes	k-permutations	$(n)_k$
No	No	Subsets of size k	$\binom{n}{k}$
Yes	No		

In a multiset, I am allowed to choose the same number more than once.

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No	No	Subsets of size k	$\binom{n}{k}$
Yes	No	Multisets of size k	$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}$

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"Balls in boxes"

Have n "distinguishable" boxes.
Have k balls which are either:

1. Indistinguishable
2. Distinguishable

How many ways to place balls in boxes with

A. At most one
B. Any number of

balls per box?

Take Notice

Suppose K is a set with $|K| = k$ and N is a set with $|N| = n$:

- *2A counts the number of injective functions from K to N*
- *2B counts the number of functions from K to N*

"Balls in boxes"

Case	Balls	Balls per box	Number
1A	Indist.	At most 1	
1B	Indist.	Any number	
2A	Dist.	At most 1	
2B	Dist.	Any number	

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“Balls in boxes”

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2A	Dist.	At most 1	$(n)_k$
2B	Dist.	Any number	n^k

Outline

Alternative techniques

What if the current techniques are unwieldy?
Other techniques for obtaining an exact count:

- Find a different approach for counting
- Make use of symmetries
- Make use of recursion
- Write a program (running time?)

Example

How many sequences of 15 coin flips have an even number of heads?

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- Using "balls in boxes": $\binom{15}{0} + \binom{15}{2} + \ldots + \binom{15}{14}$

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- Using "balls in boxes": $\binom{15}{0} + \binom{15}{2} + \ldots + \binom{15}{14}$
- Use symmetry: $\frac{1}{2} \times 2^{15}$
- Use recursion: $\mathrm{Even}(n) = \mathrm{Odd}(n-1) + \mathrm{Even}(n-1)$; $\mathrm{Odd}(n) = \mathrm{Even}(n-1) + \mathrm{Odd}(n-1)$
  where,
  - $\mathrm{Even}(n)$ is the number of sequences with an even number of heads after n flips, which comes from $\mathrm{Even}(n-1)$ (if the last flip was tail) and $\mathrm{Odd}(n-1)$ (if the last flip was head).
  - $\mathrm{Odd}(n)$ as the number of sequences with an odd number of heads after n flips, which comes from $\mathrm{Even}(n-1)$ (if the last flip was head) and $\mathrm{Odd}(n-1)$ (if the last flip was tail).

Example

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How many sequences of n coin flips contain HH?

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Example

How many sequences of n coin flips contain HH?

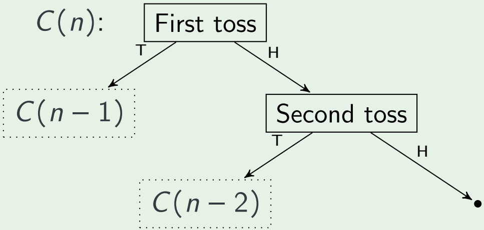
$$
\begin{aligned}
C(0) &= 0 \\
C(1) &= 0 \\
C(n) &= C(n-1) + C(n-2) + 2^{n-2}
\end{aligned}
$$

Example

Example

How many sequences of n coin flips contain HH?

We can summarise all possible outcomes in a **recursive tree**



Example

Example (cont'd)

[B]

- $C(0) = 0$: With zero flips, there is no sequence containing "HH".
- $C(1) = 0$: With one flip, there are two possible sequences ("H" and "T"), but none of them contains "HH".

Example

Example (cont'd)

[I]

- If the last flip is "T": Any sequence of $n-1$ flips that already contains "HH" can have "T" appended without changing the fact that "HH" appears. This contributes $C(n-1)$ sequences.
- If the last two flips are "HT": Any sequence of $n-2$ flips that contains "HH" can have "HT" appended, preserving the fact that "HH" appears. This contributes $C(n-2)$ sequences.
- If the last two flips are "HH": The substring "HH" itself forms the required pattern, and any sequence of $n-2$ flips (even if it does not contain "HH") will satisfy the condition once we append "HH" at the end. There are 2^{n-2} possible sequences of $n-2$ flips, as each flip can be either "H" or "T".

Example

Example

How many sequences of n coin flips do not contain HH?

$$
\begin{aligned}
N(0) &= 1 \\
N(1) &= 2 \\
N(2) &= 3 \\
N(n) &= N(n-1) + N(n-2)
\end{aligned}
$$

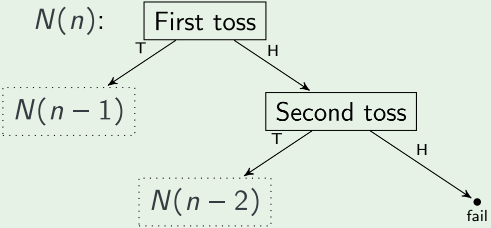
Example

Example

How many sequences of n coin flips do not contain HH?

$$\begin{aligned} N(0) &= 1 \\ N(1) &= 2 \\ N(2) &= 3 \\ N(n) &= N(n-1) + N(n-2) \end{aligned}$$

We can summarise all possible outcomes in a **recursive tree**



Example

Example (cont'd)

[B]

- $N(0) = 1$: With zero flips, there is one sequence (the empty sequence), which trivially does not contain "HH".
- $N(1) = 2$: With one flip, there are two possible sequences ("H" and "T"), neither of which contains "HH".
- $N(2) = 3$: With two flips, there are three possible sequences that do not contain "HH": "HT", "TH", and "TT".

Example (cont'd)

[I]

This recurrence relation works by considering the last flip in a sequence of n flips:

- If the last flip is "T": The remaining $n-1$ flips form a sequence of length $n-1$ that does not contain "HH". So, we can append "T" to any valid sequence of length $n-1$ without introducing "HH". This contributes $N(n-1)$ valid sequences.
- If the last two flips are "TH": The remaining $n-2$ flips form a sequence of length $n-2$ that does not contain "HH". We can append "TH" to any valid sequence of length $n-2$ without introducing "HH". This contributes $N(n-2)$ valid sequences.

Outline

Using Programs to Count

Two dice, a red die and a black die, are rolled.
(Note: one *die*, two or more *dice*)

Write a program to list all the pairs $\{(R, B) : R > B\}$

Similarly, for three dice, list all triples $R > B > G$

Generally, for n dice, all of which are m-sided ($n \leq m$), list all *decreasing* n-tuples

Take Notice

In order to just find the number of such n-tuples, it is not necessary to list them all. One can write a recurrence relation for these numbers and compute (or try to solve) it.

Approximate Counting

Take Notice

A Count may be a precise value or an **estimate***.*

The latter should be asymptotically correct or at least give a good asymptotic bound, whether upper or lower. If S is the base set, $|S| = n$ its size, and we denote by $c(S)$ some collection of objects from S we are interested in, then we seek constants a, b such that

$$a \leq \lim_{n \to \infty} \frac{est(|c(S)|)}{|c(S)|} \leq b$$

In other words $est(|c(S)|) \in \Theta(|c(S)|)$.