COMP9020

Foundations of Computer Science
Term 3, 2024

Lecture 6: Equivalence Relations and Partial Orders

Outline

Outline

Equivalence relations

Equivalence relations capture a general notion of "equality". They are relations which are:

- Reflexive (R): Every object should be "equal" to itself
- Symmetric (S): If x is "equal" to y, then y should be "equal" to x
- Transitive (T): If x is "equal" to y and y is "equal" to z, then x should be "equal" to z.

Equivalence relations

Equivalence relations capture a general notion of "equality". They are relations which are:

- Reflexive (R): Every object should be "equal" to itself
- Symmetric (S): If x is "equal" to y, then y should be "equal" to x
- Transitive (T): If x is "equal" to y and y is "equal" to z, then x should be "equal" to z.

Definition

A binary relation $R \subseteq S \times S$ is *equivalence relation* if it satisfies (R), (S), (T).

Example

Partition of $\mathbb{Z}$ into classes of numbers with the same remainder on division by p; it is particularly important for p prime

$$\mathbb{Z}(p) = \mathbb{Z}_p = \{0, 1, \ldots, p-1\}$$

One can define all four arithmetic operations (with the usual properties) on $\mathbb{Z}_p$ for a prime p; division has to be restricted when p is not prime.

Take Notice

$(\mathbb{Z}_p, +, \cdot, 0, 1)$ *are fundamental algebraic structures known as* **rings**. *These structures are very important in coding theory and cryptography.*

Equivalence Classes and Partitions

Suppose $R \subseteq S \times S$ is an equivalence relation
The **equivalence class** $[s]$ (w.r.t. R) of an element $s \in S$ is

$$[s] = \{t : t \in S \text{ and } sRt\}$$

Fact

$s \; R \; t$ if and only if $[s] = [t]$.

Equivalence classes: Proof example

Proof

Suppose $[s] = [t]$. Recall $[s] = \{x \in S : (s, x) \in R\}$. We will show that $(s, t) \in R$.

Because R is reflexive, $(t, t) \in R$.

Therefore $t \in [t]$.

Because $[t] = [s]$, it follows that $t \in [s]$.

But then $(s, t) \in R$ by the definition of $[s]$.

Equivalence classes: Proof example

Proof

Now suppose $(s, t) \in R$. We will show $[s] = [t]$ by showing $[s] \subseteq [t]$ and $[t] \subseteq [s]$.

Take any $x \in [s]$.

By the definition of $[s]$, $(s, x) \in R$.

Since R is symmetric $(x, s) \in R$.

Since R is transitive and $(s, t) \in R$ we have that $(x, t) \in R$.

Since R is symmetric $(t, x) \in R$.

Therefore, $x \in [t]$.

Therefore $[s] \subseteq [t]$.

Equivalence classes: Proof example

Proof

Now suppose $(s, t) \in R$. We will show $[s] = [t]$ by showing $[s] \subseteq [t]$ and $[t] \subseteq [s]$.

Take any $x \in [t]$.

By the definition of $[t]$, $(t, x) \in R$.

Since R is transitive and $(s, t) \in R$ we have that $(s, x) \in R$.

Therefore $x \in [s]$.

Therefore $[t] \subseteq [s]$. $\square$

Partitions

Definition

A **partition** of a set S is a collection of sets $S_1, \ldots, S_k$ such that

- S_i and S_j are disjoint (for $i \neq j$)
- $S = S_1 \cup S_2 \cup \cdots \cup S_k = \bigcup_{i=1}^{k} S_i$

The collection of all equivalence classes $\{[s] : s \in S\}$ forms a partition of S.
In the opposite direction, a partition of a set defines the equivalence relation on that set. If $S = S_1 \cup \cdots \cup S_k$, then we can define $\sim \subseteq S \times S$ as:

$$s \sim t \text{ exactly when } s \text{ and } t \text{ belong to the same } S_i.$$

Exercises

Exercises

RW: 3.6.6 (supp)

(d) Show that $m \sim n$ iff $m^2 =_{(5)} n^2$ is an equivalence on $S = \{1, \ldots, 7\}$.

Find all the equivalence classes.

Exercises

$\boxed{\text{RW: 3.6.6 (supp)}}$

(d) Show that $m \sim n$ iff $m^2 =_{(5)} n^2$ is an equivalence on $S = \{1, \ldots, 7\}$.

It just means that $m =_{(5)} n$ or $m =_{(5)} -n$,
e.g. $1 =_{(5)} -4$.
This satisfies (R), (S), (T).

Find all the equivalence classes.

We have
$[1] = \{1, 4, 6\}$
$[2] = \{2, 3, 7\}$
$[5] = \{5\}$

Outline

Equivalence Relations

Partial Orders

Partial Order

A **partial order** $\preceq$ on S satisfies (R), (AS), (T).
We call $(S, \preceq)$ a **poset** — partially ordered set

Examples

Posets:

- $(\mathbb{Z}, \leq)$
- $(\mathrm{Pow}(X), \subseteq)$ for some set X
- $(\mathbb{N}, |)$

Not posets:

- $(\mathbb{Z}, <)$
- $(\mathbb{Z}, |)$

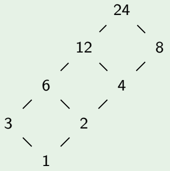
Hasse diagram

Every finite poset $(S, \preceq)$ can be represented with a **Hasse diagram**:

- Nodes are elements of S
- An edge is drawn *upward* from x to y if $x \prec y$ and there is no z such that $x \prec z \prec y$

Example

Hasse diagram for positive divisors of 24 ordered by $|$:



Ordering Concepts

Definition

Let $(S, \preceq)$ be a poset.

- **Minimal** element: x such that there is no $y \neq x$ with $y \preceq x$
- **Maximal** element: x such that there is no $y \neq x$ with $x \preceq y$
- **Minimum (least)** element: x such that $x \preceq y$ for all $y \in S$
- **Maximum (greatest)** element: x such that $y \preceq x$ for all $y \in S$

Take Notice

- *There may be multiple minimal/maximal elements.*
- *Minimum/maximum elements are the unique minimal/maximal elements if they exist.*
- *Minimal/maximal elements always exist in finite posets, but not necessarily in infinite posets.*

Examples

Examples

- $\mathrm{Pow}(\{a, b, c\})$ with the order $\subseteq$
  $\emptyset$ is minimum; $\{a, b, c\}$ is maximum
- $\mathrm{Pow}(\{a, b, c\}) \setminus \{\{a, b, c\}\}$ (proper subsets of $\{a, b, c\}$)
  Each two-element subset $\{a, b\}, \{a, c\}, \{b, c\}$ is maximal.
  - But there is no maximum

Ordering Concepts

Definition

Let $(S, \preceq)$ be a poset.

- x is an **upper bound** for A if $a \preceq x$ for all $a \in A$
- x is a **lower bound** for A if $x \preceq a$ for all $a \in A$
- The **set of upper bounds** for A is defined as $\mathrm{ub}(A) = \{x : a \preceq x \text{ for all } a \in A\}$
- The **set of lower bounds** for A is defined as $\mathrm{lb}(A) = \{x : x \preceq a \text{ for all } a \in A\}$
- The **least upper bound** of A, $\mathrm{lub}(A)$, is the minimum of $\mathrm{ub}(A)$ (if it exists)
- The **greatest lower bound** of A, $\mathrm{glb}(A)$ is the maximum of $\mathrm{lb}(A)$ (if it exists)

glb and lub

To show x is $\mathrm{glb}(A)$ you need to show:

- x is a lower bound: $x \preceq a$ for all $a \in A$.
- x is the greatest of all lower bounds: If $y \preceq a$ for all $a \in A$ then $y \preceq x$.

Example

$\mathrm{Pow}(X)$ ordered by $\subseteq$.

- $\mathrm{glb}(A, B) = A \cap B$
- $\mathrm{lub}(A, B) = A \cup B$

Definition

Let $(S, \preceq)$ be a poset.

- $(S, \preceq)$ is a **lattice** if $\mathrm{lub}(x, y)$ and $\mathrm{glb}(x, y)$ exist for every pair of elements $x, y \in S$.
- $(S, \preceq)$ is a **complete lattice** if $\mathrm{lub}(A)$ and $\mathrm{glb}(A)$ exist for every subset $A \subseteq S$.

Take Notice

A finite lattice is always a complete lattice.

Examples

Examples

- $\{1, 2, 3, 4, 6, 8, 12, 24\}$ partially ordered by divisibility is a lattice
  - e.g. $\text{lub}(\{4, 6\}) = 12$; $\text{glb}(\{4, 6\}) = 2$
- $\{1, 2, 3\}$ partially ordered by divisibility is not a lattice
  - $\{2, 3\}$ has no lub
- $\{2, 3, 6\}$ partially ordered by divisibility
  - $\{2, 3\}$ has no glb
- $\{1, 2, 3, 12, 18, 36\}$ partially ordered by divisibility
  - $\{2, 3\}$ has no lub ($12, 18$ are minimal upper bounds)

Take Notice

An infinite lattice need not have a lub (or no glb) for an arbitrary infinite subset of its elements, in particular no such bound may exist for **all** *its elements.*

Examples

- $(\mathbb{Z}, \leq)$: neither $\mathrm{lub}(\mathbb{Z})$ nor $\mathrm{glb}(\mathbb{Z})$ exist
- $(\mathcal{F}(\mathbb{N}), \subseteq)$ [all finite subsets of $\mathbb{N}$]: lub exists for pairs of elements but not generally for (infinite) sets of elements. glb exists for any set of elements: intersection of a set of finite sets is finite.
- $(\mathcal{I}(\mathbb{N}), \subseteq)$ [all infinite subsets of $\mathbb{N}$]: glb does not exist for some pairs of elements (e.g. odds and evens). lub exists for any set of elements: union of a set of infinite sets is always infinite.

Exercises

Exercises

$\boxed{\text{RW: 11.1.5}}$ Consider poset $(\mathbb{R}, \leq)$

(a) Is this a lattice?

(b) Give an example of a non-empty subset of $\mathbb{R}$ that has no upper bound.

(c) Find $\mathrm{lub}(\{\, x \in \mathbb{R} : x < 73 \,\})$

(d) Find $\mathrm{lub}(\{\, x \in \mathbb{R} : x \leq 73 \,\})$

(e) Find $\mathrm{lub}(\{\, x : x^2 < 73 \,\})$

(f) Find $\mathrm{glb}(\{\, x : x^2 < 73 \,\})$

Exercises

$\boxed{\text{RW: 11.1.5}}$ Consider poset $(\mathbb{R}, \leq)$

(a)	Is this a lattice?	Yes
(b)	Give an example of a non-empty subset of $\mathbb{R}$ that has no upper bound.	$\{ r \in \mathbb{R} : r > 0 \} = (0, \infty)$
(c)	Find $\text{lub}(\{ x \in \mathbb{R} : x < 73 \})$	73
(d)	Find $\text{lub}(\{ x \in \mathbb{R} : x \leq 73 \})$	73
(e)	Find $\text{lub}(\{ x : x^2 < 73 \})$	$\sqrt{73}$
(f)	Find $\text{glb}(\{ x : x^2 < 73 \})$	$-\sqrt{73}$

Total orders

Definition

A **total order** is a partial order that also satisfies:

(L) *Linearity* (any two elements are comparable):

$$\text{For all } x, y \text{ either: } x \leq y \text{ or } y \leq x \text{ (or both if } x = y)$$

Take Notice

On a finite set all total orders are "isomorphic"
On an infinite set there is quite a variety of possibilities.

Examples

Examples

- $\mathbb{Z}$ with $\leq$:
  (no minimum/maximum element)
- $\mathbb{Z}$ with
  $\{(x, y) : (xy \leq 0 \text{ and } x \leq y) \text{ or } (xy > 0 \text{ and } |x| \leq |y|)\}$:
  (no maximum element, minimum element is -1)
- $\mathbb{Z}$ with $\{(x, y) : (xy \leq 0 \text{ and } x \geq y) \text{ or } (xy > 0 \text{ and } x \leq y)\}$:
  (minimum element 1, maximum element -1)

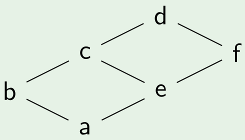
Ordering of a Poset — Topological Sort

Definition

For a poset $(S, \preceq)$ any total order $\leq$ that is consistent with $\preceq$ (if $a \preceq b$ then $a \leq b$) is called a **topological sort**.

Example

Consider



The following all are topological sorts:

$a \leq b \leq e \leq c \leq f \leq d$

$a \leq e \leq b \leq f \leq c \leq d$

$a \leq e \leq f \leq b \leq c \leq d$

Well-Ordered Sets

Definition

A **well-ordered set** is a poset where every subset has a least element.

Take Notice

The greatest element is not required.

Examples

- $\mathbb{N} = \{0, 1, \ldots\}$
- Disjoint union of copies of $\mathbb{N}$:

$$\mathbb{N}_1 \mathbin{\dot\cup} \mathbb{N}_2 \mathbin{\dot\cup} \mathbb{N}_3 \mathbin{\dot\cup} \ldots,$$

  where each $\mathbb{N}_i \simeq \mathbb{N}$ and $\mathbb{N}_1 < \mathbb{N}_2 < \mathbb{N}_3 \cdots$

Take Notice

Well-ordered sets are an important mathematical tool to prove termination of programs.

Orders for Cartesian products and languages

There are several practical ways of combining orders:

- **Product order**: Given posets $(S, \preceq_S)$ and $(T, \preceq_T)$, define:

$$(s, t) \preceq (s', t') \quad \text{if } s \preceq_S s' \text{ and } t \preceq_T t'$$

Notes

- No implicit weighting.
- No bias toward any component.
- In general, it is only a partial order, even if combining total orders.

Orders for Cartesian products and languages

There are several practical ways of combining orders:

- **Product order**: Given posets $(S, \preceq_S)$ and $(T, \preceq_T)$, define:

$$(s, t) \preceq (s', t') \quad \text{if } s \preceq_S s' \text{ and } t \preceq_T t'$$

- **Lexicographic order** Given posets $(S, \preceq_S)$ and $(T, \preceq_T)$, define:

$$(s, t) \leq_{\mathsf{lex}} (s', t') \quad \text{if } s \preceq_S s' \text{ or } (s = s' \text{ and } t \preceq_T t')$$

Notes

- No implicit weighting.
- Gives total order when combining total orders.
- Can be sensibly extended to words.

Orders for Cartesian products and languages

There are several practical ways of combining orders:

- **Product order**: Given posets $(S, \preceq_S)$ and $(T, \preceq_T)$, define:

$$(s,t) \preceq (s',t') \quad \text{if } s \preceq_S s' \text{ and } t \preceq_T t'$$

- **Lexicographic order** Given posets $(S, \preceq_S)$ and $(T, \preceq_T)$, define:

$$(s,t) \leq_{\mathsf{lex}} (s',t') \quad \text{if } s \preceq_S s' \text{ or } (s = s' \text{ and } t \preceq_T t')$$

  Extension to words: $\lambda \leq_{\mathsf{lex}} w$ for all words

Notes

- No implicit weighting.
- Gives total order when combining total orders.
- Can be sensibly extended to words.
- Not ideal for enumeration.

Orders for Cartesian products and languages

There are several practical ways of combining orders:

- **Product order**: Given posets $(S, \preceq_S)$ and $(T, \preceq_T)$, define:

$$(s, t) \preceq (s', t') \quad \text{if } s \preceq_S s' \text{ and } t \preceq_T t'$$

- **Lexicographic order** Given posets $(S, \preceq_S)$ and $(T, \preceq_T)$, define:

$$(s, t) \leq_{\mathsf{lex}} (s', t') \quad \text{if } s \preceq_S s' \text{ or } (s = s' \text{ and } t \preceq_T t')$$

  Extension to words: $\lambda \leq_{\mathsf{lex}} w$ for all words

- **Lenlex order**: Lexicographic ordering, but order by length first.

Notes

- Only applicable for languages (subsets of Σ^*).
- Gives total order when Σ is totally ordered.
- Gives an enumeration of Σ^*.

Example

Example

$\boxed{\text{RW: 11.2.5}}$ Let $\mathbb{B} = \{0, 1\}$ with the usual order $0 < 1$. List the elements $101, 010, 11, 000, 10, 0010, 1000$ of $\mathbb{B}^*$ in the

(a) Lexicographic order

(b) Lenlex order

$\boxed{\text{RW: 11.2.8}}$ When are the lexicographic order and *lenlex* on Σ^* the same?

Example

Example

$\boxed{\text{RW: 11.2.5}}$ Let $\mathbb{B} = \{0, 1\}$ with the usual order $0 < 1$. List the elements $101, 010, 11, 000, 10, 0010, 1000$ of $\mathbb{B}^*$ in the

(a) Lexicographic order

$000, 0010, 010, 10, 1000, 101, 11$

(b) Lenlex order

$10, 11, 000, 010, 101, 0010, 1000$

$\boxed{\text{RW: 11.2.8}}$ When are the lexicographic order and *lenlex* on Σ^* the same?

Only when $|\Sigma| = 1$.

Exercises

RW: 11.6.6 True or false?

(a) If a set Σ is totally ordered, then the corresponding lexicographic partial order on Σ^* also must be totally ordered.

(b) If a set Σ is totally ordered, then the corresponding lenlex order on Σ^* also must be totally ordered.

(c) Every finite poset has a Hasse diagram.

(d) Every finite poset has a topological sorting.

(e) Every finite poset has a minimum element.

(f) Every finite totally ordered set has a maximum element.

(g) An infinite poset cannot have a maximum element.

Exercises

RW: 11.6.6 True or false?

(a) If a set Σ is totally ordered, then the corresponding lexicographic partial order on Σ^* also must be totally ordered. True

(b) If a set Σ is totally ordered, then the corresponding lenlex order on Σ^* also must be totally ordered. True

(c) Every finite poset has a Hasse diagram.

(d) Every finite poset has a topological sorting.

(e) Every finite poset has a minimum element.

(f) Every finite totally ordered set has a maximum element.

(g) An infinite poset cannot have a maximum element.

Exercises

Exercises

RW: 11.6.6 True or false?

(a) If a set Σ is totally ordered, then the corresponding lexicographic partial order on Σ^* also must be totally ordered. True

(b) If a set Σ is totally ordered, then the corresponding lenlex order on Σ^* also must be totally ordered. True

(c) Every finite poset has a Hasse diagram.

(d) Every finite poset has a topological sorting.

(e) Every finite poset has a minimum element.

(f) Every finite totally ordered set has a maximum element.

(g) An infinite poset cannot have a maximum element.

Exercises

Exercises

RW: 11.6.6 True or false?

(a) If a set Σ is totally ordered, then the corresponding lexicographic partial order on Σ^* also must be totally ordered. True

(b) If a set Σ is totally ordered, then the corresponding lenlex order on Σ^* also must be totally ordered. True

(c) Every finite poset has a Hasse diagram. True

(d) Every finite poset has a topological sorting.

(e) Every finite poset has a minimum element.

(f) Every finite totally ordered set has a maximum element.

(g) An infinite poset cannot have a maximum element.

Exercises

Exercises

RW: 11.6.6 True or false?

(a) If a set Σ is totally ordered, then the corresponding lexicographic partial order on Σ^* also must be totally ordered. True

(b) If a set Σ is totally ordered, then the corresponding lenlex order on Σ^* also must be totally ordered. True

(c) Every finite poset has a Hasse diagram. True

(d) Every finite poset has a topological sorting. True

(e) Every finite poset has a minimum element.

(f) Every finite totally ordered set has a maximum element.

(g) An infinite poset cannot have a maximum element.

Exercises

Exercises

RW: 11.6.6 True or false?

(a)	If a set Σ is totally ordered, then the corresponding lexicographic partial order on Σ^* also must be totally ordered.	True
(b)	If a set Σ is totally ordered, then the corresponding lenlex order on Σ^* also must be totally ordered.	True
(c)	Every finite poset has a Hasse diagram.	True
(d)	Every finite poset has a topological sorting.	True
(e)	Every finite poset has a minimum element.	False
(f)	Every finite totally ordered set has a maximum element.	
(g)	An infinite poset cannot have a maximum element.	

Exercises

Exercises

$\boxed{\text{RW: 11.6.6}}$ True or false?

(a)	If a set Σ is totally ordered, then the corresponding lexicographic partial order on Σ^* also must be totally ordered.	True
(b)	If a set Σ is totally ordered, then the corresponding lenlex order on Σ^* also must be totally ordered.	True
(c)	Every finite poset has a Hasse diagram.	True
(d)	Every finite poset has a topological sorting.	True
(e)	Every finite poset has a minimum element.	False
(f)	Every finite totally ordered set has a maximum element.	True
(g)	An infinite poset cannot have a maximum element.	

Exercises

RW: 11.6.6 True or false?

(a)	If a set Σ is totally ordered, then the corresponding lexicographic partial order on Σ^* also must be totally ordered.	True
(b)	If a set Σ is totally ordered, then the corresponding lenlex order on Σ^* also must be totally ordered.	True
(c)	Every finite poset has a Hasse diagram.	True
(d)	Every finite poset has a topological sorting.	True
(e)	Every finite poset has a minimum element.	False
(f)	Every finite totally ordered set has a maximum element.	True
(g)	An infinite poset cannot have a maximum element.	False