Quiz 4 Solutions: Functions, Boolean Logic

1. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined as $f(n) = n^2 - 7n + 1 \text{ div } 11$ and $g : \mathbb{Z} \to \mathbb{Z}$ be defined as $g(n) = n^2 - n^3$. Calculate $(f \circ g)(3)$.

Answer(s)

First, calculate $g(3)$:

$$g(3) = 3^2 - 3^3 = 9 - 27 = -18.$$

Now calculate $f(-18)$:

$$f(-18) = \left\lfloor \frac{(-18)^2 - 7(-18) + 1}{11} \right\rfloor = \left\lfloor \frac{324 + 126 + 1}{11} \right\rfloor = \left\lfloor \frac{451}{11} \right\rfloor = 41.$$

2. Given the following matrices:

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$$

Find the result of $(\mathbf{AB})^T - \mathbf{C}$, which will be a 2×2 matrix of the form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Answer(s)

Step 1: Calculate AB

$$AB = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 + 1 \cdot 4 & 2 \cdot 2 + 1 \cdot 3 \\ 3 \cdot 1 + 0 \cdot 4 & 3 \cdot 2 + 0 \cdot 3 \end{pmatrix} = \begin{pmatrix} 6 & 7 \\ 3 & 6 \end{pmatrix}$$

Step 2: Calculate $(AB)^T$

$$(AB)^T = \begin{pmatrix} 6 & 7 \\ 3 & 6 \end{pmatrix}^T = \begin{pmatrix} 6 & 3 \\ 7 & 6 \end{pmatrix}$$

Step 3: Calculate $(AB)^T - C$

$$(AB)^T - C = \begin{pmatrix} 6 & 3 \\ 7 & 6 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 6-1 & 3-2 \\ 7-3 & 6-1 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 4 & 5 \end{pmatrix}$$

Where $a = 5$, $b = 1$, $c = 4$, and $d = 5$.

3. Which of the following is true about the function $g(n) = 2^n + n^3$?

   (a) $g(n) \in O(2^n)$

   (b) $g(n) \in O(n^3)$

   (c) $g(n) \in \Theta(2^n)$

(d) $g(n) \in \Theta(n^3)$

(e) $g(n) \in \Omega(2^n)$

(f) $g(n) \in \Omega(n^3)$

(g) $g(n) \in O(2^n + n^3)$

(h) $g(n) \in \Theta(2^n + n^3)$

Answer(s)

- True: As n grows, 2^n dominates n^3, so $g(n) \leq c \cdot 2^n$ for some constant c and large enough n.
- False: 2^n grows faster than n^3, so $g(n)$ cannot be bounded by n^3.
- True: We have $g(n) \in O(2^n)$ and $g(n) \in \Omega(2^n)$
- False: $g(n)$ grows faster than n^3 due to the 2^n term.
- True: $g(n) \geq 2^n$ for all $n \geq 0$.
- True: $g(n) \geq n^3$ for all $n \geq 0$.
- True: $g(n) = 2^n + n^3$, so it's exactly in $O(2^n + n^3)$.
- True: $g(n) = 2^n + n^3$, so it's exactly in $\Theta(2^n + n^3)$.

4. Which of the following functions grows the fastest as n approaches infinity?

(a) $f(n) = n^2 \log n$

(b) $g(n) = 2^{\sqrt{n}}$

(c) $h(n) = n!$

(d) $k(n) = n^{\log n}$

Answer(s)

- Compare $n^2 \log n$ and $2^{\sqrt{n}}$: For all $n \geq 16$, we have $n^2 \log n \leq 2^{\sqrt{n}}$. Therefore, $n^2 \log n \in O(2^{\sqrt{n}})$.
- Compare $2^{\sqrt{n}}$ and $n!$: For all $n \geq 5$, we have $2^{\sqrt{n}} \leq n!$. Therefore, $2^{\sqrt{n}} \in O(n!)$.
- Compare $n!$ and $n^{\log n}$: For all $n \geq 3$, we have $n^{\log n} \leq n!$. Therefore, $n^{\log n} \in O(n!)$.

This means that $n!$ grows the fastest.

5. Which of the following functions is in $O(n^2)$?

(a) $f(n) = 100n \log n$

(b) $g(n) = n^2 / \log n$

(c) $h(n) = n^2 + n \log n$

(d) $k(n) = n^3$

(e) None of the Above

Answer(s)

- For all $n \geq 1$, we have $100n \log n \leq 100n^2$ so $f(n) \in O(n^2)$.
- For all $n \geq 10$, we have $\log n \geq 1$ so we find that $n^2 / \log n \leq n^2$. Therefore $g(n) \in O(n^2)$.
- For all $n \geq 1$, we have $n \log n \leq n^2$ so $n^2 + n \log n \leq 2n^2$. Therefore $h(n) \in O(n^2)$.
- Suppose that $n^3 \in O(n^2)$. Then, there exists $n_0 \in \mathbb{N}$ and real number $c > 0$ such that $n^3 \leq cn^2$ when $n \geq n_0$. This means that $n \leq c$, but, this is not true for all $n \geq n_0$. This is a contradiction so $k(n) \notin O(n^2)$.

6. Let $\Sigma = \{a, b\}$ be an alphabet and $\Sigma^{\leq 2} = \{\lambda, a, b, aa, ab, ba, bb\}$.

Define a function $f : \Sigma^{\leq 2} \to \mathbb{N}$ as follows:

$$f(w) = \begin{cases} 0 & \text{if } w = \varepsilon \\ 2^{|w|} & \text{if } w \text{ ends with 'a'} \\ 2^{|w|} - 1 & \text{if } w \text{ ends with 'b'} \end{cases}$$

where $|w|$ denotes the length of the word w. Which of the following statements are true?

(a) f is an injective function

(b) f is invertible and f^{-1} exists

(c) f is a bijection from $\Sigma^{\leq 2}$ to $\{0, 1, 2, 3, 4, 7, 8\}$

(d) None of the other options

Answer(s)

Let's analyze the function f

x	λ	a	b	aa	ab	ba	bb
$f(x)$	0	2	1	4	3	4	3

- f is not injective as $f(aa) = f(ba) = 4$ and $f(ab) = f(bb) = 3$
- f is not invertible as it's not injective
- f is not surjective as the image of f is $\{0, 1, 2, 3, 4\}$, not $\{0, 1, 2, 3, 4, 7, 8\}$

Therefore, none of the other options are true.

7. Which of the following options are correct?

(a) $((p \mid\mid q)\ \&\&\ (!r \mid\mid s))\ \&\&\ ((!p \mid\mid t)\ \&\&\ (q \mid\mid !s))$ is in CNF

(b) $((x\ \&\&\ y)\ \&\&\ !z) \mid\mid ((!x\ \&\&\ y)\ \&\&\ w) \mid\mid ((x\ \&\&\ !y)\ \&\&\ z)$ is in DNF

(c) $((a \mid\mid b) \mid\mid c)\ \&\&\ (((!a \mid\mid d) \mid\mid e)\ \&\&\ ((b \mid\mid !c) \mid\mid !e))$ is in CNF

(d) $(((g\ \&\&\ h)\ \&\&\ i) \mid\mid ((!g\ \&\&\ h)\ \&\&\ !i)) \mid\mid ((g\ \&\&\ !h)\ \&\&\ i)$ is in DNF

(e) None of these options

Answer(s)

- True: It's a conjunction of four clauses, each clause being a disjunction of literals.
- True: It's a disjunction of three terms, each term being a conjunction of literals.
- True: It's a conjunction of three clauses, each clause being a disjunction of literals.
- True: It's a disjunction of three terms, each term being a conjunction of literals.

All options are correct.

8. Consider the system of linear equations:

$$\begin{aligned} 2x + y - z &= 3 \\ x - y + 2z &= 1 \\ 3x + 2y + z &= 4 \end{aligned}$$

Which of the following is equivalent to matrix representation of this system?

(a) $\begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} z \\ x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & -1 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ z \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} y \\ x \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & 1 & -1 \\ -1 & 2 & 1 \\ -1 & 3 & 2 \end{pmatrix} \begin{pmatrix} z \\ x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$

Answer(s)

- Not equivalent. Expanding the matrix equation gives:

$$\begin{aligned} 2z + x - y &= 1 \\ z + 2x - y &= 3 \\ z + 3x + 2y &= 4 \end{aligned}$$

- Equivalent. Expanding the matrix equation gives:

$$\begin{aligned} 2x - z + y &= 3 \\ 3x + z + 2y &= 4 \\ x + 2z - y &= 1 \end{aligned}$$

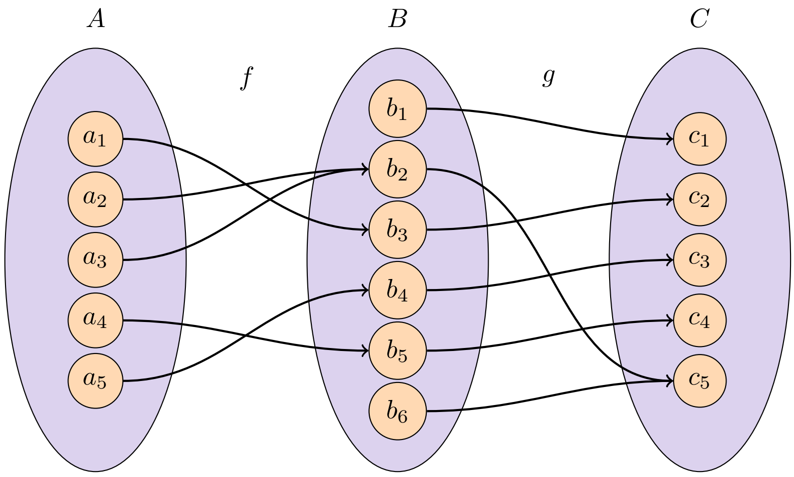
- Equivalent. Expanding the matrix equation gives:

$$\begin{aligned} y + 2x - z &= 3 \\ -x + y + 2z &= 1 \\ 2x + 3y + z &= 4 \end{aligned}$$

- Not equivalent. Expanding the matrix equation gives:

$$\begin{aligned} 2z + x - y &= 1 \\ -z + 2x + y &= 3 \\ -z + 3x + 2y &= 4 \end{aligned}$$

9. Consider the following diagram representing functions $f : A \to B$, $g : B \to C$, and $h : A \to C$, where $h = g \circ f$:



(a) Which of the following statements are true about functions f, g, and h?

i. f is injective

ii. g is surjective

iii. The converse of f is a function from B to A

iv. There exists an element in C that is not in the image of h

v. $h(a_2) = h(a_3)$

(b) Let $B' = \{b_2, b_3, b_4, b_5\}$. We define $k : A \to B'$ where $k(x) = f(x)$. Which of the following statements would be true?

i. The converse of k is a function from B' to A

ii. k is injective but not surjective

iii. The image of k would have fewer elements than its domain

Answer(s)

(a)
- False: The function f is not injective as $f(a_2) = f(a_3)$.
- True: Every element in C has at least one element from B mapping to it.

- False: The function f is not bijective so f has no inverse.
- True: There is no $a \in A$ such that $f(a) = c_1$.
- True: Both a_2 and a_3 map to b_2 under f, which then maps to c_5 under g.

(b)
- False: The function k is not bijective so k has no inverse.
- False: The function k is not injective.
- True: The image has 4 elements (b_2, b_3, b_4, b_5) while the domain has 5 elements.