COMP9020

Foundations of Computer Science
Term 3, 2024

Lecture 9: Propositional Logic

Outline

Propositional Logic, informally

Propositional Logic, formally

CNF and DNF Revisited

Beyond Propositional Logic

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Propositional Logic, informally

Propositional Logic, formally

CNF and DNF Revisited

Beyond Propositional Logic

Propositions

A **proposition** (or sentence) is a declarative statement; something that is either true or false.

Examples

- Richard Nixon was president of Ecuador.
- A square root of 16 is 4.
- Euclid's program gets stuck in an infinite loop if you input 0.
- $x^n + y^n = z^n$ has no nontrivial integer solutions for $n > 2$.
- 3 divides 24.
- K_5 is planar.

Propositions

Examples

The following are *not* declarative sentences:

- Gubble gimble goo
- For Pete's sake, take out the garbage!
- Did you watch MediaWatch last week?
- Please waive the prerequisites for this subject for me.
- x divides y.
- $x = 3$ and x divides 24.

Propositions

Examples

The following are *not* declarative sentences:

- Gubble gimble goo
- For Pete's sake, take out the garbage!
- Did you watch MediaWatch last week?
- Please waive the prerequisites for this subject for me.
- x divides y. — $R(x, y)$
- $x = 3$ and x divides 24. — $P(x)$

Logical connectives

Logical connectives join together propositions to build larger, **compound** propositions.

Examples

- Chef is a bit of a Romeo **and** Kenny is always getting killed.
- Either Bill is a liar **or** Hillary is innocent of Whitewater.
- **It is not the case that** this program always halts.
- **If** it is raining **then** I have an umbrella.

Logical connectives

Common logical connectives:

Symbol	Default	Also known as
$\wedge$	and	but, ";"
$\vee$	or	"either .. or .."
$\neg$	not	not the case
$\rightarrow$	"if .. then .."	implies whenever is sufficient for
$\leftrightarrow$	".. if and only if .."	bi-implies necessary and sufficient exactly when just in case

Compound propositions

The **truth** of a compound proposition depends on the truth of its components (**atomic propositions**):

Example

P: Chef is a bit of a Romeo **and** Kenny is always getting killed.

Chef is a bit of a Romeo	Kenny is always getting killed	P
True	True	True
False	True	False
True	False	False
False	False	False

Compound propositions

A	B	$A \wedge B$	$A \vee B$	$\neg A$	$A \rightarrow B$	$A \leftrightarrow B$
True	True	True	True	False	True	True
False	True	False	True	True	True	False
True	False	False	True	False	False	False
False	False	False	False	True	True	True

Vacuous truth

How to interpret $A \to B$ when A is false?

$A \to B$ If A (premise) then B (conclusion)

Material implication is false *only when* the premise holds and the conclusion does not.

If the premise is false, the implication is true no matter how absurd the conclusion is.

Both the following statements are true:

- If February has 30 days then March has 31 days.
- If February has 30 days then March has 42 days.

Exercises

$\boxed{\text{LLM: 3.2}}$

$$
\begin{aligned}
p &= \text{“you get an HD on your final exam”} \\
q &= \text{“you do every exercise in the book”} \\
r &= \text{“you get an HD in the course”}
\end{aligned}
$$

Translate into logical notation:

(a) You get an HD in the course although you do not do every exercise in the book.

(c) To get an HD in the course, you must get an HD on the exam.

(d) You get an HD on your exam, but you don't do every exercise in this book; nevertheless, you get an HD in this course.

Exercises

Exercises

LLM: 3.2

$$
\begin{aligned}
p &= \text{"you get an HD on your final exam"} \\
q &= \text{"you do every exercise in the book"} \\
r &= \text{"you get an HD in the course"}
\end{aligned}
$$

Translate into logical notation:

(a) You get an HD in the course although you do not do every exercise in the book. $r \wedge \neg q$

(c) To get an HD in the course, you must get an HD on the exam. $r \rightarrow q$

(d) You get an HD on your exam, but you don't do every exercise in this book; nevertheless, you get an HD in this course. $p \wedge \neg q \wedge r$

Tautologies, Contradictions and Contingencies

Definition

A proposition is:

- a **tautology** if it is always true,
- a **contradiction** if it is always false,
- a **contingency** if it is neither a tautology or a contradiction,
- **satisfiable** if it is not a contradiction.

Example

- Contingency: It is raining
- Tautology: It is raining or it is not raining
- Contradiction: It is raining and it is not raining

Applications I: Constraint Satisfaction Problems

These are problems such as timetabling, activity planning, etc.
Many can be understood as showing that a formula is satisfiable.

Example

You are planning a party, but your friends are a bit touchy about who will be there.

1. Sarah hates John's jokes. She will not come to the party if John is invited.
2. Kim loves John's jokes, and says she will not come unless John does.
3. Sarah is shy, and will only come to the party if her best friend Kim will be there.

Who can you invite without making someone unhappy?

Translation to logic: let J, S, K represent "John (Sarah, Kim) comes to the party". Then the constraints are:

1. $J \to \neg S$
2. $K \to J$
3. $S \to K$

Thus, for a successful party to be possible, we want the formula $\varphi = (J \to \neg S) \wedge (S \to K) \wedge (K \to J)$ to be satisfiable.
Truth values for J, S, K making this true are called *satisfying assignments*, or *models*.

We can use logical reasoning to work out what options are available:

- If Kim comes, then John must, and Sarah must not.
- If Kim doesn't come, then Sarah cannot come. John may or may not come.

Conclusion: a party satisfying the constraints can be held. Invite nobody, or invite John only, or invite Kim and John.

Logical equivalence

Definition

Two propositions are **logically equivalent** if they are true for the same truth values of their atomic propositions.

Example

$$A: \text{ “It is raining”}$$

is logically equivalent to ‘

$$\neg(\neg A): \text{ “It is not the case that it is not raining”}$$

A	$\neg A$	$\neg(\neg A)$
True	False	True
False	True	False

Applications II: Program Logic

Example

`if` $x > 0$ `or` $(x <= 0$ `and` $y > 100)$`:`

Let $p \stackrel{\text{def}}{=} (\ x > 0)$ and $q \stackrel{\text{def}}{=} (y > 100)$

$p \ \vee \ (\neg p \wedge q)$

p	q	$\neg p$	$\neg p \wedge q$	$p \vee (\neg p \wedge q)$
F	F	T	F	F
F	T	T	T	T
T	F	F	F	T
T	T	F	F	T

This is equivalent to $p \vee q$. Hence the code can be simplified to

`if` $x > 0$ `or` $y > 100$`:`

Entailment and Validity

An **argument** consists of a set of propositions called **premises** and a declarative sentence called the **conclusion**.

Example

Premises:	Frank took the Ford or the Toyota.
	If Frank took the Ford he will be late.
	Frank is not late.
Conclusion:	Frank took the Toyota

Entailment and Validity

An argument is **valid if the conclusions are true whenever all the premises are true**. Thus: if we believe the premises, we should also believe the conclusion.
(Note: we don't care what happens when one of the premises is false.)
Other ways of saying the same thing:

- The conclusion *logically follows* from the premises.
- The conclusion is a *logical consequence* of the premises.
- The premises **entail** the conclusion.

Entailment and Validity

The argument above is valid. The following is invalid:

Example

Premises:	Frank took the Ford or the Toyota.
	If Frank took the Ford he will be late.
	Frank is late.
Conclusion:	Frank took the Ford.

Example

Example

You are on a spaceship with **crewmates** – who always tell the truth; and **imposters** – who always lie.

Premises:	Red says:	"Blue is an imposter"
	Green says:	"Red and Blue are both crewmates"
	Blue says:	"Red is a crewmate, or Green is an imposter"
	Everyone is either a crewmate, or an imposter, but not both	
Conclusion:	Green is an imposter.	

Proof: ...

Applications III: Reasoning About Requirements/Specifications

Suppose a set of English language requirements R for a software/hardware system can be formalised by a set of formulas $\{\varphi_1, \ldots \varphi_n\}$.
Suppose C is a statement formalised by a formula ψ. Then

1. The requirements cannot be implemented if $\varphi_1 \wedge \ldots \wedge \varphi_n$ is not satisfiable.
2. If $\varphi_1, \ldots \varphi_n$ entails ψ then every correct implementation of the requirements R will be such that C is always true in the resulting system.
3. If $\varphi_1, \ldots \varphi_{n-1}$ entails φ_n, then the condition φ_n of the specification is redundant and need not be stated in the specification.

Example

Example

Requirements R: A burglar alarm system for a house is to operate as follows. The alarm should not sound unless the system has been armed or there is a fire. If the system has been armed and a door is disturbed, the alarm should ring. Irrespective of whether the system has been armed, the alarm should go off when there is a fire.

Conclusion C: If the alarm is ringing and there is no fire, then the system must have been armed.

Questions

1. Will every system correctly implementing requirements R satisfy C?
2. Is the final sentence of the requirements redundant?

Example

Example

Expressing the requirements as formulas of propositional logic, with

- S = the alarm sounds = the alarm rings
- A = the system is armed
- D = a door is disturbed
- F = there is a fire

we get

Requirements:

1. $S \to (A \vee F)$
2. $(A \wedge D) \to S$
3. $F \to S$

Conclusion: $(S \wedge \neg F) \to A$

Example

Example

Our two questions then correspond to

1. Does $S \rightarrow (A \vee F)$, $(A \wedge D) \rightarrow S$, $F \rightarrow S$ entail $(S \wedge \neg F) \rightarrow A$?
2. Does $S \rightarrow (A \vee F)$, $(A \wedge D) \rightarrow S$ entail $F \rightarrow S$?

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Syntax vs Semantics

The first step in the formal definition of logic is the separation of **syntax** and **semantics**

- Syntax is how things are written: what *defines* a formula
- Semantics is what things mean: what does it mean for a formula to be "true"?

Example

"Rabbit" and "Bunny" are syntactically different, but semantically the same.

Syntax: Well-formed formulas

Let $\textsc{Prop} = \{p, q, r, \ldots\}$ be a set of propositional letters.
Consider the alphabet

$$\Sigma = \textsc{Prop} \cup \{\top, \bot, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, (,)\}.$$

The **well-formed formulas** (wffs) over $\textsc{Prop}$ is the smallest set of words over Σ such that:

- $\top$, $\bot$ and all elements of $\textsc{Prop}$ are wffs
- If φ is a wff then $\neg\varphi$ is a wff
- If φ and ψ are wffs then $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$, and $(\varphi \leftrightarrow \psi)$ are wffs.

Examples

The following are well-formed formulas:

- $(p \wedge \neg\top)$
- $\neg(p \wedge \neg\top)$
- $\neg\neg(p \wedge \neg\top)$

The following are **not** well-formed formulas:

- $p \wedge \wedge$
- $p \wedge \neg\top$
- $(p \wedge q \wedge r)$
- $\neg(\neg p)$

Syntax: Conventions

To aid readability some conventions and binding rules can and will be used [not in proof assistant].

- Parentheses omitted if there is no ambiguity (e.g. $p \wedge q$)
- $\neg$ binds more tightly than $\wedge$ and $\vee$, which bind more tightly than $\rightarrow$ and $\leftrightarrow$ (e.g. $p \wedge q \rightarrow r$ instead of $((p \wedge q) \rightarrow r)$
- $\wedge$ and $\vee$ associate to the left: $p \vee q \vee r$ instead of $((p \vee q) \vee r)$

Syntax: Conventions

To aid readability some conventions and binding rules can and will be used [not in proof assistant].

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- $\wedge$ and $\vee$ associate to the left: $p \vee q \vee r$ instead of $((p \vee q) \vee r)$

Other conventions (rarely used/assumed in this lecture):

- $'$ or $\bar{\cdot}$ for $\neg$
- $+$ for $\vee$
- $\cdot$ or juxtaposition for $\wedge$
- $\wedge$ binds more tightly than $\vee$
- $\rightarrow$ and $\leftrightarrow$ associate to the right: $p \rightarrow q \rightarrow r$ instead of $(p \rightarrow (q \rightarrow r))$

Syntax: Parse trees

The structure of well-formed formulas (and other grammar-defined syntaxes) can be shown with a **parse tree**.

Example

$$((P \land \neg Q) \lor \neg(Q \to P))$$

$$\lor$$

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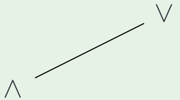

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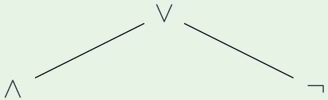
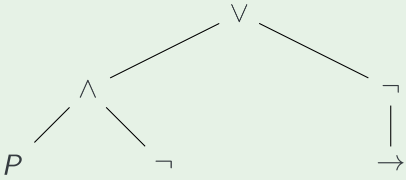


Syntax: Parse trees

The structure of well-formed formulas (and other grammar-defined syntaxes) can be shown with a **parse tree**.

Example

$$((P \wedge \neg Q) \vee \neg(Q \rightarrow P))$$



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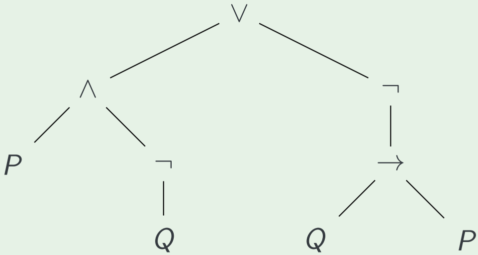

Syntax: Parse trees formally

Formally, we can define a parse tree as follows:
A parse tree is either:

- (B) A node containing $\top$;
- (B) A node containing $\bot$;
- (B) A node containing a propositional variable;
- (R) A node containing $\neg$ with a single parse tree child;
- (R) A node containing $\wedge$ with two parse tree children;
- (R) A node containing $\vee$ with two parse tree children;
- (R) A node containing $\rightarrow$ with two parse tree children; or
- (R) A node containing $\leftrightarrow$ with two parse tree children.

Semantics: Boolean Algebras

Recall the two-element Boolean Algebra $\mathbb{B} = \{\text{true}, \text{false}\} = \{T, F\} = \{1, 0\}$ together with the operations !, &&, $\|$.

Define $\rightsquigarrow$, $\leftrightsquigarrow$ as derived boolean functions:

- $x \rightsquigarrow y = (!x) \parallel y = \max\{1 - x, y\}$
- $x \leftrightsquigarrow y = (x \rightsquigarrow y) \mathbin{\&\&} (y \rightsquigarrow x) = (1 + x + y) \mathbin{\%} 2)$

Semantics: Truth valuations

A *truth assignment* is a function $v : Prop \to \mathbb{B}$.

We can extend a *truth valuation*, v, which assigns a value to *all wffs* of propositional logic as follows:

- $v(\top) = \texttt{true}$,
- $v(\bot) = \texttt{false}$,
- $v(\neg\varphi) = !v(\varphi)$,
- $v(\varphi \wedge \psi) = v(\varphi) \mathbin{\&\&} v(\psi)$
- $v(\varphi \vee \psi) = v(\varphi) \parallel v(\psi)$
- $v(\varphi \to \psi) = v(\varphi) \rightsquigarrow v(\psi)$
- $v(\varphi \leftrightarrow \psi) = v(\varphi) \leftrightsquigarrow v(\psi)$

Semantics: Truth valuations

A *truth assignment* is a function $v : Prop \to \mathbb{B}$.

We can extend a truth valuation, v, to all wffs of propositional logic as follows:

- $v(\top) = 1$,
- $v(\bot) = 0$,
- $v(\neg\varphi) = 1 - v(\varphi)$,
- $v(\varphi \wedge \psi) = \min\{v(\varphi), v(\psi)\}$
- $v(\varphi \vee \psi) = \max\{v(\varphi), v(\psi)\}$
- $v(\varphi \to \psi) = \max\{1 - v(\varphi), v(\psi)\}$
- $v(\varphi \leftrightarrow \psi) = (1 + v(\varphi) + v(\psi))\ \%\ 2$

Exercises

Evaluate the following formulas with the truth assignment $v(p) = v(q) = \text{false}$

- $p \to q$
- $(p \to q) \to (p \to q)$
- $\neg\neg p$
- $\top \wedge \neg\bot \to p$

Semantics: Exercises

Exercises

Evaluate the following formulas with the truth assignment
$v(p) = v(q) = \text{false}$

• $p \to q$	true
• $(p \to q) \to (p \to q)$	true
• $\neg\neg p$	false
• $\top \wedge \neg\bot \to p$	false

Semantics: Truth tables

- Row for every **truth assignment** — assignment of T/F to elements of *Prop*
- Columns for subformulas

Example

p	q	$\neg p$	$\neg p \wedge q$	$p \vee (\neg p \wedge q)$
F	F	T	F	F
F	T	T	T	T
T	F	F	F	T
T	T	F	F	T

Satisfiability, Validity and Equivalence

A formula φ is

- **satisfiable** if $v(\varphi) = \texttt{true}$ for some truth assignment v (v **satisfies** φ)
- a **tautology** if $v(\varphi) = \texttt{true}$ for all truth assignments v
- **unsatisfiable** or a **contradiction** if $v(\varphi) = \texttt{false}$ for all truth assignments v

Example: Party invitations

Translation to logic: let J, S, K represent "John (Sarah, Kim) comes to the party". Then the constraints are:

1. $J \rightarrow \neg S$
2. $S \rightarrow K$
3. $K \rightarrow J$

Thus, for a successful party to be possible, we want the formula $\phi = (J \rightarrow \neg S) \wedge (S \rightarrow K) \wedge (K \rightarrow J)$ to be satisfiable.
Truth values for J, S, K making this true are called *satisfying assignments*, or *models*.

We figure out where the conjuncts are false, below. (so blank = T)

J	K	S	$J \to \neg S$	$S \to K$	$K \to J$	ϕ
F	F	F				
F	F	T		F		F
F	T	F			F	F
F	T	T			F	F
T	F	F				
T	F	T	F	F		F
T	T	F				
T	T	T	F			F

Conclusion: a party satisfying the constraints can be held. Invite nobody, or invite John only, or invite Kim and John.

Exercise

Exercises

RW: 2.7.14 (supp)

Which of the following formulas are *always* true?

(a) $(p \wedge (p \rightarrow q)) \rightarrow q$

(b) $((p \vee q) \wedge \neg p) \rightarrow \neg q$

(e) $((p \rightarrow q) \vee (q \rightarrow r)) \rightarrow (p \rightarrow r)$

(f) $(p \wedge q) \rightarrow q$

Exercise

Exercises

RW: 2.7.14 (supp)

Which of the following formulas are *always* true?

(a) $(p \wedge (p \rightarrow q)) \rightarrow q$ — always true

(b) $((p \vee q) \wedge \neg p) \rightarrow \neg q$ — not always true

(e) $((p \rightarrow q) \vee (q \rightarrow r)) \rightarrow (p \rightarrow r)$ — not always true

(f) $(p \wedge q) \rightarrow q$ — always true

Logical equivalence

Definition

Two formulas, φ and ψ, are **logically equivalent**, $\varphi \equiv \psi$, if $v(\varphi) = v(\psi)$ for all truth assignments v.

Fact

$\equiv$ *is an equivalence relation.*

Logical equivalence

Example

For all propositions P, Q, R:

$$\begin{array}{lrcl}
\text{Commutativity:} & P \vee Q & \equiv & Q \vee P \\
 & P \wedge Q & \equiv & Q \wedge P \\
\text{Associativity:} & (P \vee Q) \vee R & \equiv & P \vee (Q \vee R) \\
 & (P \wedge Q) \wedge R & \equiv & P \wedge (Q \wedge R) \\
\text{Distributivity:} & P \vee (Q \wedge R) & \equiv & (P \vee Q) \wedge (P \vee R) \\
 & P \wedge (Q \vee R) & \equiv & (P \wedge Q) \vee (P \wedge R) \\
\text{Identity:} & P \vee \bot & \equiv & P \\
 & P \wedge \top & \equiv & P \\
\text{Complement:} & P \vee \neg P & \equiv & \top \\
 & P \wedge \neg P & \equiv & \bot
\end{array}$$

Logical equivalence

Example

Other properties:

- Implication: $p \to q \equiv \neg p \vee q$
- Double negation: $\neg\neg p \equiv p$
- Contrapositive: $(p \to q) \equiv (\neg q \to \neg p)$
- De Morgan's: $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Logical equivalence

Fact

$\varphi \equiv \psi$ *if, and only if,* $(\varphi \leftrightarrow \psi)$ *is a tautology.*

Strategies for showing logical equivalence:

- Compare all rows of truth table.
- Show $(\varphi \leftrightarrow \psi)$ is a tautology.
- Use transitivity of $\equiv$.

Logical equivalence: Examples

Examples

$\boxed{\text{RW: 2.2.18}}$ Prove or disprove:

(a) $p \to (q \to r) \;\equiv\; (p \to q) \to (p \to r)$

(c) $(p \to q) \to r \;\equiv\; p \to (q \to r)$

Logical equivalence: Examples

Examples

(a) $(p \to q) \to (p \to r)$

$\equiv$

Logical equivalence: Examples

Examples

(a) $(p \to q) \to (p \to r)$

$\equiv \neg(p \to q) \vee (p \to r)$ [Implication]

$\equiv$

Logical equivalence: Examples

Examples

(a) $(p \to q) \to (p \to r)$

$$
\begin{aligned}
&\equiv \neg(p \to q) \vee (p \to r) && \text{[Implication]} \\
&\equiv \neg(\neg p \vee q) \vee (\neg p \vee r) && \text{[Implication]} \\
&\equiv
\end{aligned}
$$

Logical equivalence: Examples

Examples

(a) $(p \to q) \to (p \to r)$

$\equiv \neg(p \to q) \vee (p \to r)$ [Implication]

$\equiv \neg(\neg p \vee q) \vee (\neg p \vee r)$ [Implication]

$\equiv (\neg\neg p \wedge \neg q) \vee (\neg p \vee r)$ [De Morgan's]

$\equiv$

Logical equivalence: Examples

Examples

(a) $(p \to q) \to (p \to r)$

$$
\begin{aligned}
&\equiv \neg(p \to q) \lor (p \to r) && \text{[Implication]}\\
&\equiv \neg(\neg p \lor q) \lor (\neg p \lor r) && \text{[Implication]}\\
&\equiv (\neg\neg p \land \neg q) \lor (\neg p \lor r) && \text{[De Morgan's]}\\
&\equiv (p \lor (\neg p \lor r)) \land (\neg q \lor (\neg p \lor r)) && \text{[Distributivity]}\\
&\equiv
\end{aligned}
$$

Logical equivalence: Examples

Examples

(a) $(p \to q) \to (p \to r)$

$$
\begin{aligned}
&\equiv \neg(p \to q) \lor (p \to r) && \text{[Implication]}\\
&\equiv \neg(\neg p \lor q) \lor (\neg p \lor r) && \text{[Implication]}\\
&\equiv (\neg\neg p \land \neg q) \lor (\neg p \lor r) && \text{[De Morgan's]}\\
&\equiv (p \lor (\neg p \lor r)) \land (\neg q \lor (\neg p \lor r)) && \text{[Distributivity]}\\
&\equiv ((p \lor \neg p) \lor r) \land ((\neg q \lor \neg p) \lor r) && \text{[Associativity]}\\
&\equiv
\end{aligned}
$$

Logical equivalence: Examples

Examples

(a) $(p \to q) \to (p \to r)$

$$
\begin{aligned}
&\equiv \neg(p \to q) \vee (p \to r) && \text{[Implication]}\\
&\equiv \neg(\neg p \vee q) \vee (\neg p \vee r) && \text{[Implication]}\\
&\equiv (\neg\neg p \wedge \neg q) \vee (\neg p \vee r) && \text{[De Morgan's]}\\
&\equiv (p \vee (\neg p \vee r)) \wedge (\neg q \vee (\neg p \vee r)) && \text{[Distributivity]}\\
&\equiv ((p \vee \neg p) \vee r) \wedge ((\neg q \vee \neg p) \vee r) && \text{[Associativity]}\\
&\equiv \top \wedge ((\neg q \vee \neg p) \vee r)) && \text{[Complement]}\\
&\equiv
\end{aligned}
$$

Logical equivalence: Examples

Examples

(a) $(p \to q) \to (p \to r)$

$$\begin{aligned}
&\equiv \neg(p \to q) \lor (p \to r) && \text{[Implication]}\\
&\equiv \neg(\neg p \lor q) \lor (\neg p \lor r) && \text{[Implication]}\\
&\equiv (\neg\neg p \land \neg q) \lor (\neg p \lor r) && \text{[De Morgan's]}\\
&\equiv (p \lor (\neg p \lor r)) \land (\neg q \lor (\neg p \lor r)) && \text{[Distributivity]}\\
&\equiv ((p \lor \neg p) \lor r) \land ((\neg q \lor \neg p) \lor r) && \text{[Associativity]}\\
&\equiv \top \land ((\neg q \lor \neg p) \lor r)) && \text{[Complement]}\\
&\equiv (\neg q \lor \neg p) \lor r && \text{[Identity]}\\
&\equiv
\end{aligned}$$

Logical equivalence: Examples

Examples

(a) $(p \to q) \to (p \to r)$

$$
\begin{aligned}
&\equiv \neg(p \to q) \vee (p \to r) && \text{[Implication]}\\
&\equiv \neg(\neg p \vee q) \vee (\neg p \vee r) && \text{[Implication]}\\
&\equiv (\neg\neg p \wedge \neg q) \vee (\neg p \vee r) && \text{[De Morgan's]}\\
&\equiv (p \vee (\neg p \vee r)) \wedge (\neg q \vee (\neg p \vee r)) && \text{[Distributivity]}\\
&\equiv ((p \vee \neg p) \vee r) \wedge ((\neg q \vee \neg p) \vee r) && \text{[Associativity]}\\
&\equiv \top \wedge ((\neg q \vee \neg p) \vee r)) && \text{[Complement]}\\
&\equiv (\neg q \vee \neg p) \vee r && \text{[Identity]}\\
&\equiv (\neg p \vee \neg q) \vee r && \text{[Commutativity]}\\
&\equiv
\end{aligned}
$$

Logical equivalence: Examples

Examples

(a) $(p \to q) \to (p \to r)$

$$
\begin{aligned}
&\equiv \neg(p \to q) \vee (p \to r) && \text{[Implication]}\\
&\equiv \neg(\neg p \vee q) \vee (\neg p \vee r) && \text{[Implication]}\\
&\equiv (\neg\neg p \wedge \neg q) \vee (\neg p \vee r) && \text{[De Morgan's]}\\
&\equiv (p \vee (\neg p \vee r)) \wedge (\neg q \vee (\neg p \vee r)) && \text{[Distributivity]}\\
&\equiv ((p \vee \neg p) \vee r) \wedge ((\neg q \vee \neg p) \vee r) && \text{[Associativity]}\\
&\equiv \top \wedge ((\neg q \vee \neg p) \vee r)) && \text{[Complement]}\\
&\equiv (\neg q \vee \neg p) \vee r && \text{[Identity]}\\
&\equiv (\neg p \vee \neg q) \vee r && \text{[Commutativity]}\\
&\equiv \neg p \vee (\neg q \vee r) && \text{[Associativity]}\\
&\equiv
\end{aligned}
$$

Logical equivalence: Examples

Examples

(a) $(p \rightarrow q) \rightarrow (p \rightarrow r)$

$$
\begin{aligned}
&\equiv \neg(p \rightarrow q) \vee (p \rightarrow r) && \text{[Implication]}\\
&\equiv \neg(\neg p \vee q) \vee (\neg p \vee r) && \text{[Implication]}\\
&\equiv (\neg\neg p \wedge \neg q) \vee (\neg p \vee r) && \text{[De Morgan's]}\\
&\equiv (p \vee (\neg p \vee r)) \wedge (\neg q \vee (\neg p \vee r)) && \text{[Distributivity]}\\
&\equiv ((p \vee \neg p) \vee r) \wedge ((\neg q \vee \neg p) \vee r) && \text{[Associativity]}\\
&\equiv \top \wedge ((\neg q \vee \neg p) \vee r)) && \text{[Complement]}\\
&\equiv (\neg q \vee \neg p) \vee r && \text{[Identity]}\\
&\equiv (\neg p \vee \neg q) \vee r && \text{[Commutativity]}\\
&\equiv \neg p \vee (\neg q \vee r) && \text{[Associativity]}\\
&\equiv p \rightarrow (q \rightarrow r) && \text{[Implication]}
\end{aligned}
$$

Logical equivalence: Examples

Examples

(a) $(p \to q) \to (p \to r)$

$\equiv \neg(p \to q) \vee (p \to r)$	[Implication]
$\equiv \neg(\neg p \vee q) \vee (\neg p \vee r)$	[Implication]
$\equiv (\neg\neg p \wedge \neg q) \vee (\neg p \vee r)$	[De Morgan's]
$\equiv (p \vee (\neg p \vee r)) \wedge (\neg q \vee (\neg p \vee r))$	[Distributivity]
$\equiv ((p \vee \neg p) \vee r) \wedge ((\neg q \vee \neg p) \vee r)$	[Associativity]
$\equiv \top \wedge ((\neg q \vee \neg p) \vee r))$	[Complement]
$\equiv (\neg q \vee \neg p) \vee r$	[Identity]
$\equiv (\neg p \vee \neg q) \vee r$	[Commutativity]
$\equiv \neg p \vee (\neg q \vee r)$	[Associativity]
$\equiv p \to (q \to r)$	[Implication]

(c) $(p \to q) \to r \not\equiv p \to (q \to r)$

Counterexample:

p	q	r	$(p \to q) \to r$	$p \to (q \to r)$
F	T	F	F	T

Theories and entailment

A set of formulas is a **theory**

A truth assignment v *satisfies* a theory T if $v(\varphi) = \texttt{true}$ for all $\varphi \in T$

A theory T **entails** a formula φ, $T \models \varphi$, if $v(\varphi) = \texttt{true}$ for all truth assignments v which satisfy T

Take Notice

Other notation (when $T = \{\varphi_1, \varphi_2, \ldots, \varphi_n\}$*)*

- $\varphi_1, \varphi_2, \ldots, \varphi_n \models \varphi$
- $\varphi_1, \varphi_2, \ldots, \varphi_n, \quad \therefore \varphi$
- $\varphi_1, \varphi_2, \ldots, \varphi_n \Longrightarrow \varphi$

Entailment and Implication

Theorem

The following are equivalent:

- $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$
- $\emptyset \models ((\varphi_1 \wedge \varphi_2) \wedge \ldots \varphi_n) \to \psi$
- $((\varphi_1 \wedge \varphi_2) \wedge \ldots \varphi_n) \to \psi$ *is a tautology*
- $\emptyset \models \varphi_1 \to (\varphi_2 \to (\ldots \to \varphi_n) \to \psi)) \ldots)$
- $\varphi_1 \models \varphi_2 \to (\ldots \to \varphi_n) \to \psi)) \ldots)$

These last two equivalences can be proven using the following fact.

Fact

Let T be a theory, and A and B be propositions. Then $T \models A \to B$ is equivalent to $T \cup \{A\} \models B$.

Showing entailment

Strategies for showing $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$:

- Draw a truth table with columns for $\varphi_1, \ldots, \varphi_n$ and φ. Check φ is `true` in rows where **all** the φ_i are true.
- Show $((\varphi_1 \wedge \varphi_2) \wedge \ldots \varphi_n) \rightarrow \psi$ is a tautology.
- Show $\varphi_1 \rightarrow (\varphi_2 \rightarrow (\ldots \rightarrow \varphi_n) \rightarrow \psi)) \ldots)$ is a tautology.
- Show $\varphi_1 \models \varphi_2 \rightarrow (\ldots \rightarrow \varphi_n) \rightarrow \psi)) \ldots)$

- Syntactic techniques: Natural deduction, Resolution, etc (not covered here)

Entailment example

Example

Premises:	Frank took the Ford or the Toyota.
	If Frank took the Ford he will be late.
	Frank is not late.
Conclusion:	Frank took the Toyota

Entailment example

Example

We mark only true locations (blank = F)

Frd	$Tyta$	$Late$	$Frd \vee Tyta$	$Frd \rightarrow Late$	$\neg Late$	$Tyta$
F	F	F		T	T	
F	F	T		T		
F	T	F	T	T	T	T
F	T	T	T	T		T
T	F	F	T		T	
T	F	T	T	T		
T	T	F	T		T	T
T	T	T	T	T		T

This shows $Frd \vee Tyta, Frd \rightarrow Late, \neg Late \models Tyta$

Entailment example

Example

The following row shows $Frd \vee Tyta,\ Frd \to Late,\ Late \not\models Frd$

Frd	$Tyta$	$Late$	$Frd \vee Tyta$	$Frd \to Late$	$Late$	Frd
F	T	T	T	T	T	F

Example: Crewmates and Imposters

Example

Premises:	Everyone is either a crewmate, or an imposter, but not both
	Red: "Blue is an imposter"
	Green: "Red and Blue are both crewmates"
	Blue: "Red is a crewmate, or Green is an imposter"
Conclusion:	Green is an imposter

Example: Crewmates and Imposters

Example

Translation to logic: Let R, G, B represent "Red (Green, Blue) is a crewmate".

Then the constraints are:

Premises:	Everyone is either a crewmate, or an imposter, but not both
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Example: Crewmates and Imposters

Example

Translation to logic: Let R, G, B represent "Red (Green, Blue) is a crewmate".

Then the constraints are:

Premises:	Everyone is either a crewmate, or an imposter, but not both
	$\varphi_1 = R \leftrightarrow \neg B$
	Green: "Red and Blue are both crewmates"
	Blue: "Red is a crewmate, or Green is an imposter"
Conclusion:	Green is an imposter

Example: Crewmates and Imposters

Example

Translation to logic: Let R, G, B represent "Red (Green, Blue) is a crewmate".

Then the constraints are:

Premises:	Everyone is either a crewmate, or an imposter, but not both
	$\varphi_1 = R \leftrightarrow \neg B$
	$\varphi_2 = G \leftrightarrow (R \wedge B)$
	Blue: "Red is a crewmate, or Green is an imposter"
Conclusion:	Green is an imposter

Example: Crewmates and Imposters

Example

Translation to logic: Let R, G, B represent "Red (Green, Blue) is a crewmate".

Then the constraints are:

Premises: Everyone is either a crewmate, or an imposter, but not both

$\varphi_1 = R \leftrightarrow \neg B$

$\varphi_2 = G \leftrightarrow (R \wedge B)$

$\varphi_3 = B \leftrightarrow (R \vee \neg G)$

Conclusion: Green is an imposter

Example: Crewmates and Imposters

Example

Translation to logic: Let R, G, B represent "Red (Green, Blue) is a crewmate".

Then the constraints are:

Premises:	Everyone is either a crewmate, or an imposter, but not both
	$\varphi_1 = R \leftrightarrow \neg B$
	$\varphi_2 = G \leftrightarrow (R \land B)$
	$\varphi_3 = B \leftrightarrow (R \lor \neg G)$
Conclusion:	$\psi = \neg G$

Example: Crewmates and Imposters

G	R	B	φ_1	$R \wedge B$	φ_2	$R \vee \neg G$	φ_3	ψ
F	F	F						T
F	F	T						T
F	T	F						T
F	T	T						T
T	F	F	F					F
T	F	T	T	F	F			F
T	T	F	T	F	F			F
T	T	T	F					F

Example

Recall the alarm specification:

- Requirement 1: $R_1 = S \to (A \vee F)$
- Requirement 2: $R_2 = (A \wedge D) \to S$
- Requirement 3: $R_3 = F \to S$
- Conclusion: $C = (S \wedge \neg F) \to A$

Questions:

1. Does $R_1, R_2, R_3 \models C$?
2. Does $R_1, R_2 \models R_3$?

Example

Example

1. Does $R_1, R_2, R_3 \models C$?
2. Does $R_1, R_2 \models R_3$?

-: not relevant

A	D	F	S	R_1	R_2	R_3	C
F	-	-	T	F	-	-	-
-	-	F	T	F	-	-	-
T	T	-	F	-	F	-	-
-	-	T	F	-	-	F	-

Example

Example

1. Does $R_1, R_2, R_3 \models C$?
2. Does $R_1, R_2 \models R_3$?

-: not relevant

A	D	F	S	R_1	R_2	R_3	C
F	-	-	T	F	-	-	-
-	-	F	T	F	-	-	-
T	T	-	F	-	F	-	-
-	-	T	F	-	-	F	-
-	-	-	F	-	-	-	T

Example

Example

1. Does $R_1, R_2, R_3 \models C$? Yes
2. Does $R_1, R_2 \models R_3$?

-: not relevant

A	D	F	S	R_1	R_2	R_3	C
F	-	-	T	F	-	-	-
-	-	F	T	F	-	-	-
T	T	-	F	-	F	-	-
-	-	T	F	-	-	F	-
-	-	-	F	-	-	-	T
T	T	T	T	T	T	T	T
T	F	T	T	T	T	T	T

Example

Example

1. Does $R_1, R_2, R_3 \models C$? Yes
2. Does $R_1, R_2 \models R_3$? No

-: not relevant

A	D	F	S	R_1	R_2	R_3	C
F	-	-	T	F	-	-	-
-	-	F	T	F	-	-	-
T	T	-	F	-	F	-	-
-	-	T	F	-	-	F	-
-	-	-	F	-	-	-	T
T	T	T	T	T	T	T	T
T	F	T	T	T	T	T	T
F	F	T	F	T	T	F	

Outline

Propositional Logic, informally

Propositional Logic, formally

CNF and DNF Revisited

Beyond Propositional Logic

CNF and DNF revisited

Definition

- A **literal** is an expression p or $\neg p$, where p is a propositional atom.
- A propositional formula is in CNF (conjunctive normal form) if it has the form
$$\bigwedge_i C_i$$
where each **clause** C_i is a disjunction of literals e.g. $p \vee q \vee \neg r$.
- A propositional formula is in DNF (disjunctive normal form) if it has the form
$$\bigvee_i C_i$$
where each **clause** C_i is a conjunction of literals e.g. $p \wedge q \wedge \neg r$.

CNF and DNF revisited

Take Notice

CNF and DNF are syntactic forms.

Theorem

For every Boolean expression φ*, there exists an equivalent expression in conjunctive normal form and an equivalent expression in disjunctive normal form.*

Outline

Propositional Logic, informally

Propositional Logic, formally

CNF and DNF Revisited

Beyond Propositional Logic

Limitations to Propositional Logic

Propositional logic is unable to capture several useful phenomena:

- Spatial/temporal dependence (e.g. P holds **after** Q holds)
- Belief and knowledge (e.g. I know that you know that X holds)
- Relationships between propositions (e.g. "The sky is blue" and "my eyes are blue")
- Quantification (e.g. "All men are mortal")

Beyond Propositional Logic

Modal logic: Introduce **modalities** to capture statement qualifying.

Example

Temporal logic:

- $\mathcal{F}\,\varphi$: φ will be true at some point in the future
- $\mathcal{G}\,\varphi$: φ will be true at all points in the future
- $\varphi\,\mathcal{U}\,\psi$: φ will be true until ψ holds

Beyond Propositional Logic

First order logic/Predicate logic: Add relations (predicates) and quantifiers to capture relationships between propositions.

Example

- P: All men are mortal:
- Q: Socrates is a man:
- R: Socrates is mortal:

In propositional logic, there is no connection between P, Q and R: it is not the case that $P, Q \models R$.

Beyond Propositional Logic

First order logic/Predicate logic: Add relations (predicates) and quantifiers to capture relationships between propositions.

Example

- P: All men are mortal: $\forall x\,\texttt{Man}(x) \to \texttt{Mortal}(x)$
- Q: Socrates is a man:
- R: Socrates is mortal:

In propositional logic, there is no connection between P, Q and R: it is not the case that $P, Q \models R$.

Beyond Propositional Logic

First order logic/Predicate logic: Add relations (predicates) and quantifiers to capture relationships between propositions.

Example

- P: All men are mortal: $\forall x\, \mathtt{Man}(x) \rightarrow \mathtt{Mortal}(x)$
- Q: Socrates is a man: $\mathtt{Man(Socrates)}$
- R: Socrates is mortal:

In propositional logic, there is no connection between P, Q and R: it is not the case that $P, Q \models R$.

Beyond Propositional Logic

First order logic/Predicate logic: Add relations (predicates) and quantifiers to capture relationships between propositions.

Example

- P: All men are mortal: $\forall x\, \texttt{Man}(x) \rightarrow \texttt{Mortal}(x)$
- Q: Socrates is a man: `Man(Socrates)`
- R: Socrates is mortal: `Mortal(Socrates)`

In propositional logic, there is no connection between P, Q and R: it is not the case that $P, Q \models R$.

In first-order logic you can show $P, Q \models R$.

Beyond Propositional Logic

First order logic/Predicate logic: Add relations (predicates) and quantifiers to capture relationships between propositions.

Example

- P: All men are mortal: $\forall x\, \mathtt{Man}(x) \rightarrow \mathtt{Mortal}(x)$
- Q: Socrates is a man: `Man(Socrates)`
- R: Socrates is mortal: `Mortal(Socrates)`

In propositional logic, there is no connection between P, Q and R: it is not the case that $P, Q \models R$.

In first-order logic you can show $P, Q \models R$.

Second order logic: Add quantification of relations.

Limitations

More expressive logics require more complex semantics.

- Logical equivalence harder to show
- Entailment harder to show
- Connections between different concepts not so straightforward

Example

In Temporal Logic, a valuation is a function $v : \textsc{Prop} \times \mathbb{N} \to \mathbb{B}$ – i.e. truth tables that change over time.