COMP9020

Foundations of Computer Science
Term 3, 2024

Lecture 11: Recursion

Recursion in Computer Science

Fundamental concept in Computer Science

- Defining complex objects from simpler ones
- Unbounded complexity with a finite description

Recursive Data Structures:
Finite definitions of **arbitrarily large** objects

- Natural numbers
- Words
- Linked lists
- Formulas
- Binary trees

Recursion in Computer Science

Recursive Algorithms:
Solving problems/calculations by reducing to smaller cases

- Factorial
- Euclidean gcd algorithm
- Towers of Hanoi
- Mergesort, Quicksort

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Analysis of Recursion:
Reasoning about recursive objects

- Induction, Structural Induction
- Recursive sequences (e.g. Fibonacci sequence)
- Asymptotic analysis of recursive functions

Outline

Outline

Recursion

Consists of a basis (B) and recursive process (R).

A sequence/object/algorithm is recursively defined when (typically)

- (B) some initial terms are specified, perhaps only the first one;
- (R) later terms stated as functional expressions of the earlier terms.

Take Notice

(R) also called **recurrence formula (especially when dealing with sequences)**

Example: Factorial

Example

Factorial:

$(B) \quad 0! = 1$

$(R) \quad (n+1)! = (n+1) \cdot n!$

$ \quad \texttt{fact}(n)$:

$(B) \quad\quad \texttt{if}(n = 0)$: 1

$(R) \quad\quad \texttt{else}$: $n * \texttt{fact}(n-1)$

Example: Euclid's gcd algorithm

Example

$$\gcd(m,n) = \begin{cases} m & \text{if } m = n \\ \gcd(m-n, n) & \text{if } m > n \\ \gcd(m, n-m) & \text{if } m < n \end{cases}$$

Example: Towers of Hanoi

- There are 3 towers (pegs)
- n disks of decreasing size placed on the first tower
- You need to move all disks from the first tower to the last tower
- Larger disks cannot be placed on top of smaller disks
- The third tower can be used to temporarily hold disks

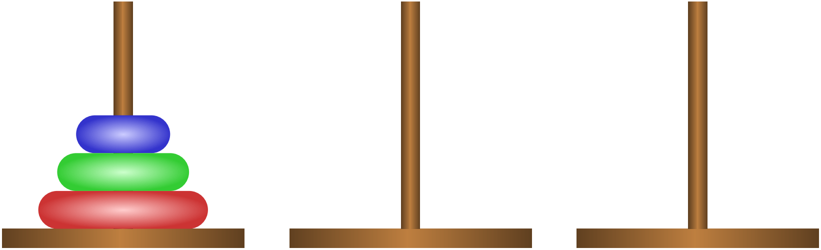
Try it out here

Example: Towers of Hanoi

Questions

- Describe a general solution for n disks
- How many moves does it take?

Example: Towers of Hanoi



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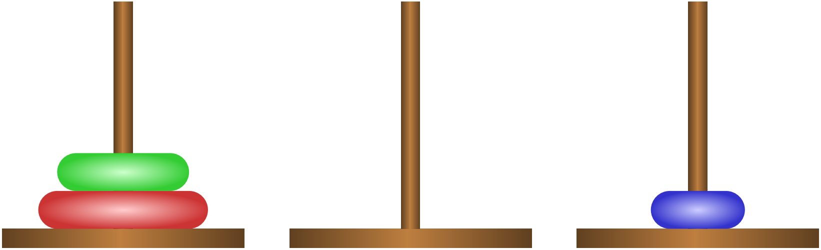

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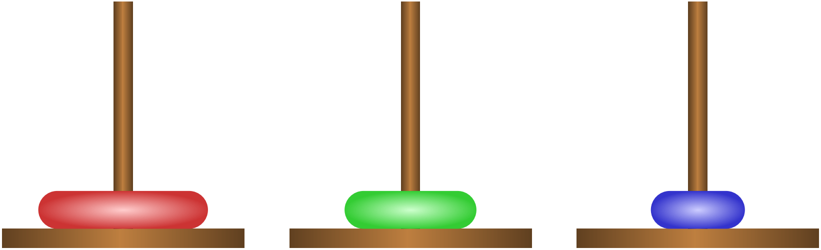

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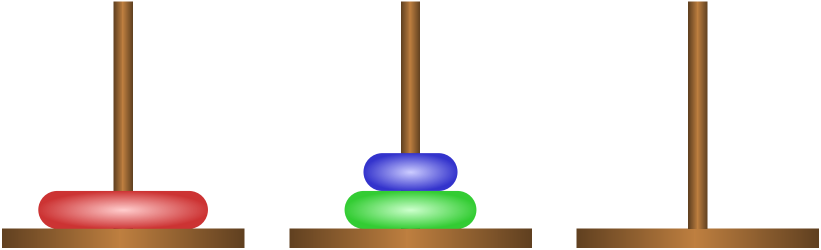

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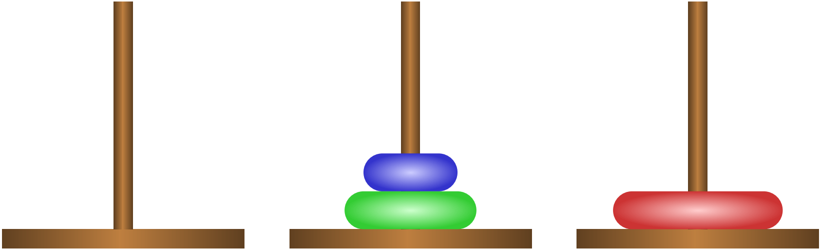

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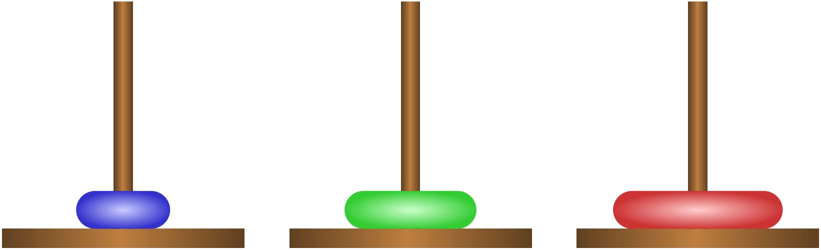

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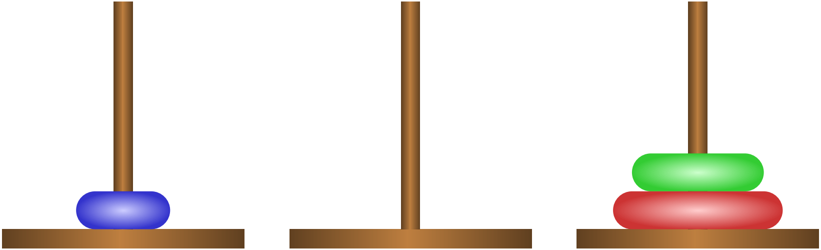

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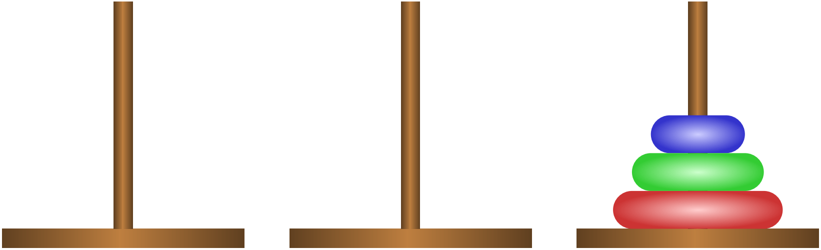

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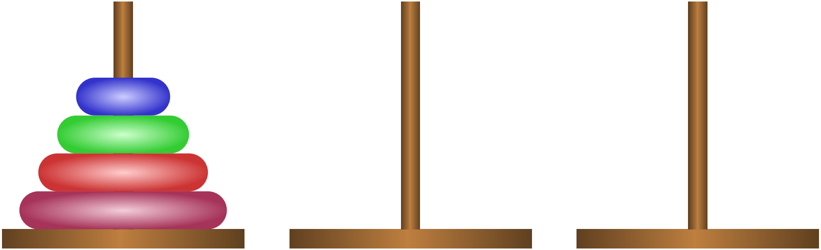

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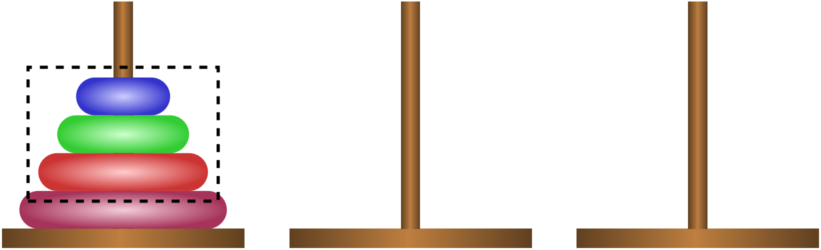

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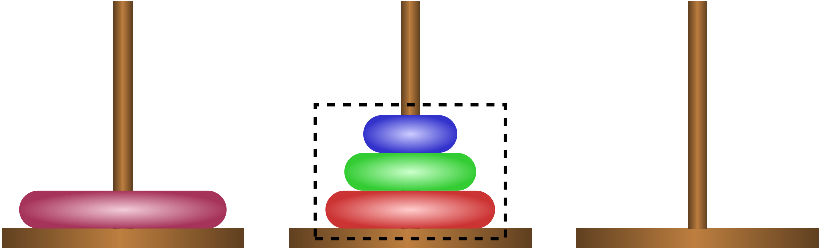

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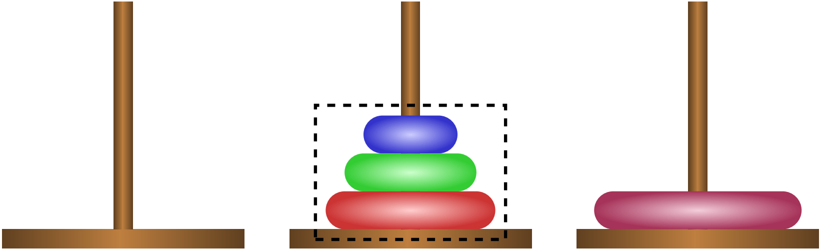

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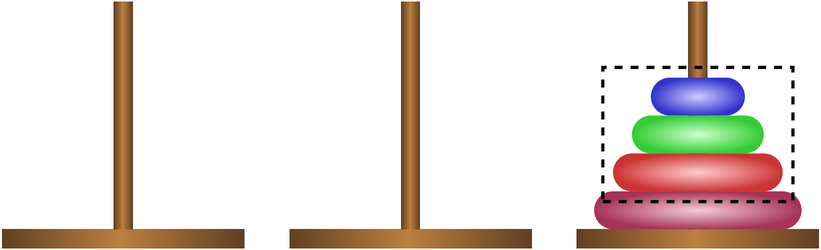

Example: Towers of Hanoi

Questions

- Describe a general solution for n disks
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Example: Towers of Hanoi

Questions

- Describe a general solution for n disks
- How many moves does it take? $M(n) \leq 2M(n-1) + 1$

Outline

Recursion

Recursive Data Structures

Recursive Programming

Solving Recurrences

Example: Natural numbers

Example

A natural number is either 0 (B) or one more than a natural number (R).

Formal definition of $\mathbb{N}$:

- (B) $0 \in \mathbb{N}$
- (R) If $n \in \mathbb{N}$ then $(n + 1) \in \mathbb{N}$

Example: Odd/Even numbers

Example

The set of even numbers can be defined as:

- (B) 0 is an even number
- (R) If n is an even number then $n + 2$ is an even number

Example: Odd/Even numbers

Example

The set of odd numbers can be defined as:

- (B) 1 is an odd number
- (R) If n is an odd number then $n + 2$ is an odd number

Example: Fibonacci numbers

Example

The Fibonacci sequence starts $0, 1, 1, 2, 3, \ldots$ where, after $0, 1$, each term is the sum of the previous two terms.

Formally, the sequence of Fibonacci numbers: $F_0, F_1, F_2, \ldots$ where the n-th Fibonacci number F_n is defined as:

- (B) $F_0 = 0$,
- (B) $F_1 = 1$,
- (R) $F_n = F_{n-1} + F_{n-2}$

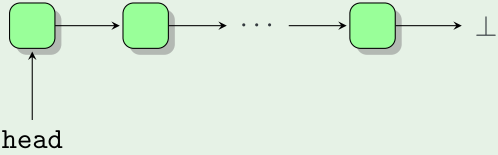
Take Notice

Could also define the Fibonacci sequence as a function $\textsc{fib} : \mathbb{N} \to \mathbb{F}$.

Example: Linked lists

Example

A linked list is zero or more linked list nodes:



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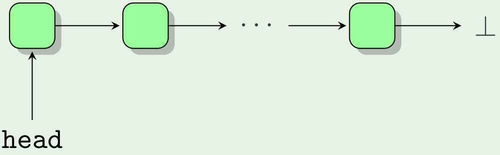

In C:

```
struct node{
  int data;
  struct node *next;
}
```

Example: Linked lists

Example

We can view the linked list **structure** abstractly. A linked list is either:

- (B) an empty list, or
- (R) an ordered pair $(\texttt{Data}, \mathsf{List})$.

Example: Words over Σ

Example

A word over an alphabet Σ is either λ (B) or a symbol from Σ followed by a word (R).

Formal definition of Σ^*:

- (B) $\lambda \in \Sigma^*$
- (R) If $w \in \Sigma^*$ then $aw \in \Sigma^*$ for all $a \in \Sigma$

Take Notice

This matches the recursive definition of a **Linked List** *data type.*

Example: Expressions in the Proof Assistant

Example

- (B) $A, B, \ldots, Z, a, b, \ldots z$ are expressions
- (B) $\emptyset$ and $\mathcal{U}$ are expressions
- (R) If E is an expression then so is (E) and E^c
- (R) If E_1 and E_2 are expressions then:
    - $(E_1 \cup E_2)$,
    - $(E_1 \cap E_2)$,
    - $(E_1 \setminus E_2)$, and
    - $(E_1 \oplus E_2)$ are expressions.

Example: Propositional formulas

Example

A well-formed formula (wff) over a set of propositional variables, $\textsc{Prop}$ is defined as:

- (B) $\top$ is a wff
- (B) $\bot$ is a wff
- (B) *p* is a wff for all $p \in \textsc{Prop}$
- (R) If φ is a wff then $\neg\varphi$ is a wff
- (R) If φ and ψ are wffs then:
  - $(\varphi \wedge \psi)$,
  - $(\varphi \vee \psi)$,
  - $(\varphi \rightarrow \psi)$, and
  - $(\varphi \leftrightarrow \psi)$ are wffs.

Exercises

$\boxed{\text{RW: 4.4.4}}$ (a) Give a recursive definition for the sequence

$$(2,\ 4,\ 16,\ 256,\ \ldots)$$

(b) Give a recursive definition for the sequence

$$(2,\ 4,\ 16,\ 65536,\ \ldots)$$

Exercises

$\boxed{\text{RW: 4.4.4}}$ (a) Give a recursive definition for the sequence

$$(2,\ 4,\ 16,\ 256,\ \ldots)$$

To generate $a_n = 2^{2^n}$ use $a_n = (a_{n-1})^2$.
(The related "Fermat numbers" $F_n = 2^{2^n} + 1$ are used in cryptography.)

(b) Give a recursive definition for the sequence

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Exercises

$\boxed{\text{RW: 4.4.4}}$ (a) Give a recursive definition for the sequence

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(b) Give a recursive definition for the sequence

$$(2,\ 4,\ 16,\ 65536,\ \ldots)$$

To generate a "stack" of n 2's use $b_n = 2^{b_{n-1}}$.

Outline

Programming over recursive datatypes

Recursive datatypes make recursive programming/functions easy.

Example

The factorial function:

$$\begin{array}{ll} & \texttt{fact}(n): \\ (B) & \quad \texttt{if}(n = 0)\text{: } 1 \\ (R) & \quad \texttt{else}\text{: } n * \texttt{fact}(n - 1) \end{array}$$

Programming over recursive datatypes

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Example

Summing the first n natural numbers:

$$
\begin{array}{ll}
 & \mathtt{sum}(n)\text{:} \\
(B) & \quad \mathtt{if}(n = 0)\text{: } 0 \\
(R) & \quad \mathtt{else}\text{: } n + \mathtt{sum}(n-1)
\end{array}
$$

Programming over recursive datatypes

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Example

Summing elements of a linked list:

```
        sum(L):
(B)       if(L.isEmpty()):
            return 0
(R)       else:
            return L.data + sum(L.next)
```

Programming over recursive datatypes

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Example

Sorting elements of a linked list (insertion sort):

```
        sort(L):
(B)       if(L.isEmpty()):
            return L
          else:
(R)         L2 = sort(L.next)
            insert L.data into L2
            return L2
```

Programming over recursive datatypes

Recursive datatypes make recursive programming/functions easy.

Example

Concatenation of words (defining wv):

$$\begin{array}{ll} & \text{For all } w, v \in \Sigma^* \text{ and } a \in \Sigma : \\ (B) & \lambda v = v \\ (R) & (aw)v = a(wv) \end{array}$$

Programming over recursive datatypes

Recursive datatypes make recursive programming/functions easy.

Example

Length of words:

$$\begin{array}{ll} (B) & \mathrm{length}(\lambda) = 0 \\ (R) & \mathrm{length}(aw) = 1 + \mathrm{length}(w) \end{array}$$

Programming over recursive datatypes

Recursive datatypes make recursive programming/functions easy.

Example

"Evaluation" of a propositional formula

Exercise

Exercise

Let Σ be a finite set.
Define $\mathsf{append} : \Sigma^* \times \Sigma \to \Sigma^*$ by

$$\mathsf{append}(w, a) = wa$$

Give a (direct) definition of append [i.e. only concatenates symbols on the left].

Exercise

Let Σ be a finite set.
Define append : $\Sigma^* \times \Sigma \to \Sigma^*$ by

$$\text{append}(w, a) = wa$$

Give a (direct) definition of append [i.e. only concatenates symbols on the left].

$$\begin{array}{ll} & \text{For all } w \in \Sigma^* \text{ and } a, x \in \Sigma : \\ (B) & \text{append}(\lambda, x) = x \\ (R) & \text{append}(aw, x) = a\,\text{append}(w, x) \end{array}$$

Pitfall: Correctness of Recursive Definition

A recurrence formula is correct if the computation of any later term can be reduced to the initial values given in (B).

Example (Incorrect definition)

- Function $g(n)$ is defined recursively by

$$g(n) = g(g(n-1)-1)+1, \qquad g(0) = 2.$$

The definition of $g(n)$ is incomplete — the recursion may not terminate:
Attempt to compute $g(1)$ gives

$$g(1) = g(g(0)-1)+1 = g(1)+1 = \ldots = g(1)+1+1+1\ldots$$

When implemented, it leads to an overflow; most static analyses cannot detect this kind of ill-defined recursion.

Pitfall: Correctness of Recursive Definition

Example (continued)

However, the definition could be repaired. For example, we can add the specification specify $g(1) = 2$.

Then $g(2) = g(2-1) + 1 = 3$,
$g(3) = g(g(2) - 1) + 1 = g(3-1) + 1 = 4$,
$\ldots$

In fact, by induction ... $g(n) = n + 1$

Pitfall: Correctness of Recursive Definition

Check your base cases!

Example

Function $f(n)$ is defined by

$$f(n) = f(\lceil n/2 \rceil), \quad f(0) = 1$$

When evaluated for $n = 1$ it leads to

$$f(1) = f(1) = f(1) = \dots$$

This one can also be repaired. For example, one could specify that $f(1) = 1$.
This would lead to a constant function $f(n) = 1$ for all $n \geq 0$.

Mutual Recursion

Sometimes recursive definitions use more than one function, with each calling each other.

Example (Fibonacci, again)

Recall:

- (B) $f(0) = 0; \quad f(1) = 1,$
- (R) $f(n) = f(n-1) + f(n-2)$

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Recall:

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Alternative, mutually recursive definition:

- (B) $f(1) = 1; \quad g(1) = 0$
- (R) $f(n) = f(n-1) + g(n-1)$
- (R) $g(n) = f(n-1)$

Mutual Recursion

Sometimes recursive definitions use more than one function, with each calling each other.

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Recall:

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Alternative, mutually recursive definition:

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- (R) $f(n) = f(n-1) + g(n-1)$
- (R) $g(n) = f(n-1)$

$$\begin{pmatrix} f(n) \\ g(n) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f(n-1) \\ g(n-1) \end{pmatrix}$$

Outline

Solving recurrences

Question

How can we (asymptotically) compare recursively defined functions?

Solving recurrences

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How can we (asymptotically) compare recursively defined functions?

Some practical approaches:

- Unwinding the recurrence
- Approximating with big-O
- The Master Theorem

Take Notice

Each approach gives an informal "solution": ideally one should prove a solution is correct (using e.g. induction).

Examples

Example (Unwinding)

$$f(0) = 1 \qquad f(n) = 2f(n-1)$$

Examples

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$$f(0) = 1 \qquad f(n) = 2f(n-1)$$

Unwinding:

$$\begin{aligned}
f(n) &= 2f(n-1) \\
&= 2(2f(n-2)) = 4f(n-2) \\
&= 4(2f(n-3)) = 8f(n-3) \\
&\vdots \quad \vdots \\
&= 2^i f(n-i) \\
&\vdots \quad \vdots \\
&= 2^n f(0) = 2^n
\end{aligned}$$

Examples

Example (Unwinding)

$$f(1) = 0 \qquad f(n) = 1 + f\left(\left\lfloor \frac{n}{2} \right\rfloor\right)$$

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Unwinding:

$$\begin{array}{rcl} f(n) & = & 1 + f(n/2) \\ & = & 1 + (1 + f(n/4)) = 2 + f(n/4) \\ & = & 2 + (1 + f(n/8)) \\ & \vdots & \vdots \\ & = & i + f(n/2^i) \\ & \vdots & \vdots \\ & = & \log(n) + f(0) = \log(n) \end{array}$$

Examples

Example (Approximating with big-O)

$$f(0) = 1 \quad f(1) = 1 \quad f(n) = f(n-1) + f(n-2)$$

Examples

Example (Approximating with big-O)

$$f(0) = 1 \quad f(1) = 1 \quad f(n) = f(n-1) + f(n-2)$$

Assuming $f(n)$ is increasing:

$$f(n-2) \leq f(n-1)$$

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so:

$$f(n) \leq 2f(n-1)$$

Examples

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so (by unwinding):

$$f(n) \leq 2^n$$

Examples

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so:

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so (by unwinding):

$$f(n) \leq 2^n$$

so:

$$f(n) \in O(2^n)$$

Master Theorem

The following result covers many recurrences that arise in practice (e.g. divide-and-conquer algorithms)

Theorem

Suppose

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$

where $f(n) \in \Theta(n^c (\log n)^k)$.

Let $d = \log_b(a)$. *Then:*

- **Case 1:** *If* $c < d$ *then* $T(n) = \Theta(n^d)$
- **Case 2:** *If* $c = d$ *then* $T(n) = \Theta(n^c (\log n)^{k+1})$
- **Case 3:** *If* $c > d$ *then* $T(n) = \Theta(f(n))$

Master Theorem: Examples

Example (Master Theorem)

$$T(n) = T\left(\frac{n}{2}\right) + n^2, \quad T(1) = 1$$

Master Theorem: Examples

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$$T(n) = T\left(\frac{n}{2}\right) + n^2, \quad T(1) = 1$$

Here $a = 1$, $b = 2$, $c = 2$, $k = 0$ and $d = 0$. So we have Case 3 and the solution is

$$T(n) = \Theta(n^c) = \Theta(n^2)$$

Master Theorem: Examples

Example (Master Theorem)

Mergesort has

$$T(n) = 2T\left(\frac{n}{2}\right) + (n-1)$$

for the number of comparisons.

Master Theorem: Examples

Example (Master Theorem)

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$$T(n) = 2T\left(\frac{n}{2}\right) + (n-1)$$

for the number of comparisons.
Here $a = b = 2$, $c = 1$, $k = 0$ and $d = 1$. So we have Case 2, and the solution is

$$T(n) = \Theta(n^c \log(n)) = O(n \log(n))$$

Master Theorem: Examples

Example (Master Theorem)

Unwinding example:

$$T(1) = 0 \qquad T(n) = 1 + T(\lfloor \frac{n}{2} \rfloor)$$

Master Theorem: Examples

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Unwinding example:

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Here $a = 1$, $b = 2$, $c = 0$, $k = 0$, and $d = 0$. So we have Case 2, and the solution is

$$T(n) = \Theta(\log(n))$$

The Master Theorem: Pitfalls

Take Notice

- a, b, c, k *have to be constants (not dependent on* n*).*
- *Only one recursive term.*
- *Recursive term is of the form* $T(n/b)$*, not* $T(n-b)$*.*
- *Solution is only an asymptotic bound.*

Examples

The Master theorem does not apply to any of these:

$$\begin{aligned} T(n) &= 2^n T(n/2) + n^2 \\ T(n) &= T(n/5) + T(7n/10) + n \\ T(n) &= 2T(n-1) \end{aligned}$$

The Master Theorem: Linear differences

Take Notice

The Master Theorem applies to recurrences where $T(n)$ is defined in terms of $T(n/b)$; not in terms of $T(n-1)$.

However, the following is a consequence of the Master Theorem:

Theorem

Suppose

$$T(n) = a \cdot T(n-1) + bn^k$$

Then

$$T(n) = \begin{cases} O(n^{k+1}) & \text{if } a = 1 \\ O(a^n) & \text{if } a > 1 \end{cases}$$

Exercise

Exercise

Solve $T(n) = 3^n T\left(\frac{n}{2}\right)$ with $T(1) = 1$

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Solve $T(n) = 3^n T\left(\frac{n}{2}\right)$ with $T(1) = 1$

Let $n \geq 2$ be a power of 2 then

$$T(n) = 3^n \cdot 3^{\frac{n}{2}} \cdot 3^{\frac{n}{4}} \cdot 3^{\frac{n}{8}} \cdot \ldots = 3^{n(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots)} = O(3^{2n})$$