Tutorial 3: Relations

Defining Binary Relations

Concept(s)

For sets A and B, a binary relation R from A to B is a subset of $A \times B$. We write

$$a\ R\ b \text{ or } R(a,b) \text{ to denote } (a,b) \in R.$$

We can also represent a binary relation as a

- Graph: We draw dots for each $a \in A$ and $b \in B$. For each $(a,b) \in R$, we can draw $a \to b$.
- Matrix: We have a grid with rows labelled by elements of S and columns by elements of T. We fill in a cell with row a and column b if $(a,b) \in R$.

Exercise 1. Let $A = \{1,2,3,4\}$, $B = \{1,3,5\}$ and $R \subseteq A \times B$ where aRb means that $\gcd(a,b) = 1$.

a) Write the relation R as a set.

b) Draw the relation R as a graph.

c) Write the relation R as a matrix.

Concept(s)

A relation on A is a binary relation R from A to A. For the graphical representation, we only draw dots for each $a \in A$ once. When $(a_1, a_2) \in R$, we can draw $a_1 \to a_2$.

Exercise 2. Let $A = \{\{\}, \{1\}, \{2\}, \{1,2\}\}$ and $R \subseteq A \times A$ where aRb means that $a \subset b$.

a) Write the relation R as a set.

b) Draw the relation R as a graph.

c) Write the relation R as a matrix.

Operations on Relations

Concept(s)

Let R be a relation from A to B and S be a relation from B to C. We define the

- Converse of R: $R^{\leftarrow} = \{(b,a) \in B \times A : aRb\}$.
- Composition of R and S: $R;S = \{(a,c) \in A \times C : \text{there exists } b \in B \text{ such that } aRb \text{ and } bSc\}$.

Exercise 3. Let $A = \{1,2,3,4\}$ and $B = \{x,y,z\}$. Consider the relation from A to B,

$$R = \{(1,x),(2,y),(3,z),(1,y)\}.$$

Compute $R;R^{\leftarrow}$. What is $R;R^{\leftarrow}$ a subset of?

Properties of Relations $R \subseteq A \times A$

Concept(s)

(R)	Reflexive	For all $a \in A$, we have $(a, a) \in R$
(AR)	Antireflexive	For all $a \in A$, we have $(a, a) \notin R$
(S)	Symmetric	For all $a, b \in A$, if $(a, b) \in R$ then $(b, a) \in R$
(AS)	Antisymmetric	For all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$ then $a = b$
(T)	Transitive	For all $a, b, c \in A$, if $(x, y), (y, z) \in R$ then $(x, z) \in R$

Exercise 4. Which of the properties (R), (AR), (S), (AS), (T) does R satisfy? Explain why.

a) $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a > b\}$

b) $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a \leq b\}$

c) $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : |a - b| \leq 2\}$

Exercise 5. Let R be a relation on a set A. Prove or disprove the following:

a) If R is symmetric and transitive, then R is reflexive.

b) If R is antireflexive and transitive, then R is antisymmetric.

Equivalence Relations

Concept(s)

An equivalence relation R is a relation on A that is (R), (S) and (T).

The equivalence class of $a \in A$ is $[a] = \{b \in A : aRb\}$.

Exercise 6. Let $\Sigma = \{a, b\}$. We define the relation $\sim$ on Σ^*, where $w_1 \sim w_2$ means that w_1 and w_2 have the same number of letters. Explain why $\sim$ is an equivalence relation.

Exercise 7. Consider the relation F on $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ where $(a, b)F(c, d)$ means that $ad = bc$.

a) Prove that F is an equivalence relation.

b) Describe the equivalence class $[(1, 2)]$.

Partial Orders

Concept(s)

A partial order $\preceq$ is a relation on A that is (R), (AS) and (T). We call $(A, \preceq)$ a poset.

A Hasse diagram is a graph if $a \preceq b$ and $a \neq b$, then there is an edge drawn upward from a to b.

Minimal	$a \in A$ such that there is no $a \neq b$ where $b \preceq a$
Minimum	$a \in A$ such that $a \preceq b$ for all $b \in A$
Maximal	$a \in A$ such that there is no $a \neq b$ where $a \preceq b$
Maximum	$a \in A$ such that $b \preceq a$ for all $b \in A$

Exercise 8. For the poset $(\{2, 4, 6, 9, 12, 36, 72\}, |)$:

a) Draw a Hasse diagram.

b) Find the maximal/minimal elements.

c) Is there a minimum element? Is there a maximum element?

Exercise 9. Let $(A, \preceq)$ be a poset. Prove that if A has a maximum then it is unique.

Concept(s)

Let $(A, \preceq)$ be a poset.

- a is an upper bound for B if $b \preceq a$ for all $b \in A$
- The least upper bound of B, $\mathrm{lub}(B)$, is the minimum of $\{a : a$ is an upper bound of $B\}$
- a is a lower bound for B if $a \preceq b$ for all $b \in A$
- The greatest lower bound of B, $\mathrm{glb}(B)$, is the maximum of $\{a : a$ is a lower bound of $B\}$

A lattice is a poset where each pair of elements has a lub and glb.

Exercise 10. For the poset $(\{2, 4, 6, 9, 12, 36, 72\}, |)$:

a) Find $\mathrm{lub}(\{6, 9\})$ and $\mathrm{glb}(\{6, 9\})$

b) Is this poset a lattice?

Exercise 11. For the poset $(Pow(\{a, b, c\}), \subseteq)$:

a) Prove that $\subseteq$ is a partial order.

b) Draw a Hasse diagram.

c) Find the maximal/minimal elements.

d) Is there a minimum element? Is there a maximum element?

e) Find $\mathrm{lub}(\{\{a\}, \{b, c\}\})$ and $\mathrm{glb}(\{\{a\}, \{b, c\}\})$, if they exist.

f) Is this poset a lattice?

Concept(s)

A total order is a partial order that has Linearity: For all a, b, either $a \leq b$ or $b \leq a$.

A topological sort of $(A, \preceq)$ is a total order $\leq$ where if $a \preceq b$, then $a \leq b$.

Exercise 12. For the poset $(Pow(\{a, b, c\}), \subseteq)$, find a topological sort.

Extra Practice Problems

Note(s)

These practice questions are designed to help deepen your understanding. No answers will be provided, as the goal is to encourage independent problem-solving and reinforce key concepts.

1. Let $A = \{1, 2, 3, 4\}$. Define the relation R on A by aRb if and only if $a + b$ is odd. Write R as a set of ordered pairs and draw its directed graph.

2. Draw graphs for relations on $\{1, 2, 3\}$ that have the following properties:
   a) Symmetric and antisymmetric and reflexive.
   b) Reflexive, but neither transitive nor symmetric.
   c) Transitive and reflexive, but not symmetric.

3. Which of the properties (R), (AR), (S), (AS), (T) does R satisfy? Prove your answers.
   a) Let $\Sigma = \{1, 2, 3\}$. Define R over Σ^* where $n_1 R n_2$ means that n_1 and n_2 have a common digit.
   b) Consider the relation $\sim$ on $\mathbb{Z}$ where $a \sim b$ means that $3 \mid a$ and $3 \mid b$ OR $3 \nmid a$ and $3 \nmid b$.
   c) The relation on $\mathbb{Z}$, $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a =_{(2)} b\}$.

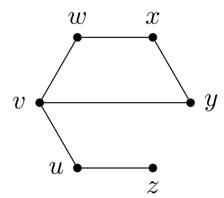
4. How many reflexive relations on A are there if $|A| = n$?

5. Let $A = \{a, b, c, d\}$ and R be the relation $\{(a, b), (b, c), (c, d), (d, a)\}$. Compute $R; R$ and $R^{\leftarrow}; R$.

6. If you take the converse of the relation $<$, what other familiar relation does it become? What if you take the converse of $=$?

7. A relation $\sim$ on $\mathbb{Z}$ is defined by $a \sim b$ when $a - b$ is either divisible by 3 or 5. Show that $\sim$ is not an equivalence relation.

8. Let $V = \{u, v, w, x, y, z\}$. The following diagram shows direct flights between the 6 cities:

   Define the relation $\sim$ on V where $a \sim b$ means that it is possible to fly from a to b with an even number of flights (including 0 flights).
   a) Prove that $\sim$ is an equivalence relation.
   b) Partition the set V into equivalence classes.

9. Let $\Sigma = \{a, b, c, d\}$. Define the relation A on Σ^* where $w_1 A w_2$ means that w_1 is an anagram of w_2.
   a) Prove that A is an equivalence relation.
   b) List out all elements of $[bad]$.

10. List all partitions of $\{a, b, c\}$.

11. With respect to the partial order $|$ on the set $\mathbb{Z}_{>0}$, find
    a) $\text{lub}(\{4, 6\})$
    b) $\text{lub}(\{3, 4, 5, 6\})$
    c) $\text{glb}(\{12, 16, 18, 24\})$
    d) $\text{glb}(\{737, 2345\})$
12. Consider the set $A = \{B \subseteq \{1, 2, 3, 4\} : |B| \neq 2\}$ with the partial order $\subseteq$.
13. Let $A = \{1, 2, 3, 4\}$ be a poset defined with $\leq$. Consider the lexicographic order $\leq_{\text{lex}}$ (check the lectures) on $A \times A$.
    - Prove that $\leq_{\text{lex}}$ is a partial order.
    - Draw a Hasse diagram.
    - Find the maximal/minimal elements.
    - Is there a maximum element? Is there a minimum element?
    - Is this poset a lattice?