Tutorial 1 Solutions: Number Theory

Floor, Ceiling and Absolute Value

Concept(s)

Floor Function: $\lfloor x \rfloor$ gives the largest integer less than or equal to x.

Ceiling Function: $\lceil x \rceil$ gives the smallest integer greater than or equal to x.

Absolute Value: $|x|$ is the non-negative value of x without regard to its sign.

Exercise 1. Calculate the following:

1. $\lfloor \sqrt{10} \rfloor + \lceil \pi \rceil$
2. $\lfloor -3.01 \rfloor + \lceil 2.99 \rceil$
3. $|\lfloor -5.6 \rfloor| + \lfloor |-5.6| \rfloor$

Answer(s)

1. $\lfloor \sqrt{10} \rfloor + \lceil \pi \rceil = \lfloor 3.16\ldots \rfloor + \lceil 3.14\ldots \rceil = 3 + 4 = 7$
2. $\lfloor -3.01 \rfloor + \lceil 2.99 \rceil = -4 + 3 = -1$
3. $|\lfloor -5.6 \rfloor| + \lfloor |-5.6| \rfloor = |-6| + \lfloor 5.6 \rfloor = 6 + 5 = 11$

Exercise 2. Explain why $\lfloor x \rfloor = \lceil x \rceil$ means that x is an integer.

Answer(s)

We have $\lfloor x \rfloor \leq x$ and $x \leq \lceil x \rceil$. Since $\lfloor x \rfloor = \lceil x \rceil$, we can use substitution to find $x \leq \lfloor x \rfloor$. This means that $\lfloor x \rfloor \leq x \leq \lfloor x \rfloor$ so $x = \lfloor x \rfloor$ by Sandwich Theorem. This means x is an integer.

Exercise 3. For which numbers n is the statement $\lfloor \sqrt{n} \rfloor = \lceil \sqrt{n} \rceil$ true?

Answer(s)

For all $x \in \mathbb{R}$, $\lfloor x \rfloor = \lceil x \rceil$ if and only if x is an integer.

$$\lfloor \sqrt{n} \rfloor = \lceil \sqrt{n} \rceil \implies \sqrt{n} \in \mathbb{Z}\,.$$

This means that n has to be a square number i.e. $n = 0, 1, 4, \ldots$.

Exercise 4. Find all integer x such that the following equation is true:

$$\left\lfloor \frac{x}{2} \right\rfloor + \left\lceil \frac{x}{3} \right\rceil = 5.$$

Answer(s)

By trying integers, we find that $x = 6$ is a solution to this equation. How do we know there are no other solutions? We first find that when $x = 5$, $\lfloor \frac{x}{2} \rfloor + \lceil \frac{x}{3} \rceil = 4$ and when $x = 7$, $\lfloor \frac{x}{2} \rfloor + \lceil \frac{x}{3} \rceil = 6$.

We know that when $x \leq y$, we have $\lfloor x \rfloor \leq \lfloor y \rfloor$ and $\lceil x \rceil \leq \lceil y \rceil$. From this behaviour, we notice that when x gets smaller, the value of $\lfloor \frac{x}{2} \rfloor + \lceil \frac{x}{3} \rceil$ either stays the same or gets smaller. Specifically, when $x \leq 5$, we have $\frac{x}{2} \leq \frac{5}{2}$ and $\frac{x}{3} \leq \frac{5}{3}$ so

$$\left\lfloor \frac{x}{2} \right\rfloor + \left\lceil \frac{x}{3} \right\rceil \leq \left\lfloor \frac{5}{2} \right\rfloor + \left\lceil \frac{5}{3} \right\rceil = 4.$$

This means that there is no $x \leq 5$ such that the equation is true. Similarly, we find that when $7 \leq x$, we have

$$6 = \left\lfloor \frac{7}{2} \right\rfloor + \left\lceil \frac{7}{3} \right\rceil \leq \left\lfloor \frac{x}{2} \right\rfloor + \left\lceil \frac{x}{3} \right\rceil,$$

so there is no integer $x \geq 7$ such that the equation is true. This means that $x = 6$ is the only solution.

Divisibility and GCD/LCM

Concept(s)

For integers m and n, we say m divides n when $n = k \cdot m$ for some integer k. We write $m \mid n$.

Exercise 5. Are the following statements true?

$$5 \mid 35, \ 8 \mid 35, \ 2 \mid -14, \ -2 \mid 14.$$

Answer(s)

- $5 \mid 35$ is true
- $8 \mid 35$ is false
- $2 \mid -14$ is true
- $-2 \mid 14$ is true

Exercise 6. How many numbers between 1 and 653 are divisible by 3 or 5?

Answer(s)

Let $N(k)$ denote the number of multiples of k that exist between 1 and 653. From the lecture, we have the formula

$$N(k) = \left\lfloor \frac{653}{k} \right\rfloor - \left\lfloor \frac{1-1}{k} \right\rfloor = \left\lfloor \frac{653}{k} \right\rfloor.$$

We first count the multiples of 3, given by $N(3)$, then we count the multiples of 5, given by $N(5)$. The answer is not $N(3) + N(5)$ since there exists some overlap between the two groups. This means that some numbers are being counted twice e.g. $15, 30, 45, \ldots$. We are overcounting the multiples of 15 so our final answer is

$$N(3) + N(5) - N(15) = 304.$$

Exercise 7. For positive integers a, b, c where $ab \mid bc$, show that $a \mid c$.

Answer(s)

We have $ab \mid bc$ so there exists some integer k such that $bc = kab$. We can divide both sides by b to get $c = ka$. By definition, we can conclude that $a \mid c$.

Concept(s)

Consider two integers m and n.

The largest integer d such that $d \mid m$ and $d \mid n$ is called the gcd(m, n).

The smallest integer k such that $m \mid k$ and $n \mid k$ is called the lcm(m, n).

$$\text{gcd}(m, n) \cdot \text{lcm}(m, n) = |m|\,|n|\,.$$

We say that m and n are coprime if gcd$(m, n) = 1$.

Exercise 8. Suppose that n is a positive integer. Explain why n and $n + 1$ are coprime.

Answer(s)

Define $d = \text{gcd}(n, n + 1)$. By definition, we have $d \mid n$ and $d \mid n + 1$. This means there exist integers j and k such that

$$n = j \cdot d \ (1) \text{ and } n + 1 = k \cdot d \ (2).$$

Subtracting (1) from (2), we get that $(n + 1) - n = k \cdot d - j \cdot d$. We can factorise out d to find that

$$1 = d \cdot (k - j)$$

where $k - j$ is an integer. We see d must be 1 or -1 since $d \mid 1$. We take $d = 1$ since the gcd is the largest integer that fits this condition.

Modular Arithmetic

Concept(s)

For integers m and n, we define the following operations:

$$m \text{ div } n = \left\lfloor \frac{m}{n} \right\rfloor \text{ and } m \,\%\, n = m - n \cdot \left\lfloor \frac{m}{n} \right\rfloor.$$

This gives us

$$m = q \cdot n + r, \text{ where } q = m \text{ div } n \text{ and } r = m \,\%\, n.$$

Exercise 9. Find the last two digits of 7^{7^7}.

Answer(s)

To get the last two digits, we can get the remainder after dividing by 100. We examine the following powers of 7 to find that

$$\begin{aligned}
7^1 \ \% \ 100 &= 7, \\
7^2 \ \% \ 100 &= 49, \\
7^3 \ \% \ 100 &= 343 \ \% \ 100 = 43, \\
7^4 \ \% \ 100 &= (43 \cdot 7) \ \% \ 100 = 301 \ \% \ 100 = 1.
\end{aligned}$$

Since the remainder of 7^4 is 1, we can exploit the property that

$$7^{n+4} \ \% \ 100 = 7^n 7^4 \ \% \ 100 = 7^n(1) \ \% \ 100 = 7^n \ \% \ 100.$$

This means we can divide by 7^4 as much as we want and the remainder will stay the same. We find that $7^n \ \% \ 100 = 7^{(n \ \% \ 4)} \ \% \ 100$, since n and $n \ \% \ 4$ differ by a multiple of 4. Our goal becomes to find $7^7 \ \% \ 4$. Through a similar process, we have

$$7^1 \ \% \ 4 = 3 \text{ and } 7^2 \ \% \ 4 = 1.$$

We can use the same logic to conclude that $7^7 \ \% \ 4 = 7^{(7 \ \% \ 2)} \ \% \ 4 = 7^1 \ \% \ 4 = 3$. Hence, we have

$$7^{7^7} \ \% \ 100 = 7^{(7^7 \ \% \ 4)} \ \% \ 100 = 7^3 \ \% \ 100 = 43.$$

Exercise 10. Find the least positive integer n for which $5^n \ \% \ 17 = 16$. Hence, evaluate $5^{200} \ \% \ 17$.

Answer(s)

We calculate remainders for powers of 5 to see that

$$\begin{aligned}
5^1 \ \% \ 17 &= 5, \\
5^2 \ \% \ 17 &= 8, \\
5^3 \ \% \ 17 &= (5 \cdot 8) \ \% \ 17 = 6, \\
5^4 \ \% \ 17 &= (5 \cdot 6) \ \% \ 17 = 13, \\
5^5 \ \% \ 17 &= (5 \cdot 13) \ \% \ 17 = 14, \\
5^6 \ \% \ 17 &= (5 \cdot 14) \ \% \ 17 = 2, \\
5^7 \ \% \ 17 &= (5 \cdot 2) \ \% \ 17 = 10, \\
5^8 \ \% \ 17 &= (5 \cdot 10) \ \% \ 17 = 16.
\end{aligned}$$

We know that 16 has the same remainder as -1, which allows us to substitute -1 in place of 16. This makes remainders easier to work with since 16^2 will have a remainder of 1. In particular,

$$(5^8)^2 \ \% \ 17 = 16^2 \ \% \ 17 = (-1)^2 \ \% \ 17 = 1.$$

We have shown that 5^{16} has a remainder of 1. Using the same logic as the previous exercise, we can subtract multiples of 16 from the power and the remainder will remain the same. We see that

$$5^{200} \ \% \ 17 = 5^{(200 \ \% \ 16)} \ \% \ 17 = 5^8 \ \% \ 17 = 16.$$

Concept(s)

We denote $m =_{(n)} p$ to mean that $(m \ \% \ n) = (p \ \% \ n)$.

Exercise 11. Suppose that $k \mid n$ and $a =_{(n)} b$ for positive integers a, b, k, n. Show that $a =_{(k)} b$.

Answer(s)

We see that a and b have the same remainder after dividing by n. We have

$$a = q_1 n + r \text{ and } b = q_2 n + r$$

for integers r, q_1 and q_2. Since $k \mid n$, we can rewrite $n = mk$ for some integer m. This gives us

$$a = q_1 mk + r \text{ and } b = q_2 mk + r.$$

We can taken the remainder of each equation to find that $a \% k = r \% k$ and $b \% k = r \% k$ so $a \% k = b \% k$. We conclude that $a =_{(k)} b$.

Euclidean Algorithm

Concept(s)

The Euclidean algorithm provides us a way to calculate the gcd:

$$\text{gcd}(m, n) = \begin{cases} m & \text{if } n = 0 \\ n & \text{if } m = 0 \\ \text{gcd}(m \% n, n) & \text{if } m > n > 0 \\ \text{gcd}(m, n \% m) & \text{if } n > m > 0 \end{cases}$$

Exercise 12. Calculate $\text{gcd}(a, b)$ and $\text{lcm}(a, b)$ for the following pairs (a, b):

1. $(44, 17)$
2. $(56, 72)$
3. $(123, 321)$

Answer(s)

We can use the Euclidean algorithm to get the gcd and then use the formula $\text{gcd}(a, b) \cdot \text{lcm}(a, b) = |a|\,|b|$ to find the lcm.

1. We know that 17 is a prime number. The divisors of 17 are 1 and 17. Since 17 does not divide 44, we have $\text{gcd}(44, 17) = 1$.
2. For the pair $(56, 72)$, we have

$$\begin{aligned} \text{gcd}(56, 72) &= \text{gcd}(56, 16) \\ &= \text{gcd}(8, 16) \\ &= \text{gcd}(8, 0) = 8. \end{aligned}$$

   This means that $\text{lcm}(56, 72) = \frac{|56||72|}{8} = 504$.

3. For the pair $(123, 321)$, we have

$$\begin{aligned}\gcd(123,321) &= \gcd(123,75)\\ &= \gcd(48,75)\\ &= \gcd(48,27)\\ &= \gcd(21,27)\\ &= \gcd(21,6)\\ &= \gcd(3,6)\\ &= \gcd(3,0) = 3.\end{aligned}$$

This means that $\text{lcm}(123, 321) = \frac{|123||321|}{3} = 13161$.

Exercise 13. Find $\gcd(615, 220)$. Are there integers x and y such that $4 = 615x + 220y$? Explain why.

Answer(s)

Suppose that $4 = 615x + 220y$. We calculate $\gcd(615, 220) = 5$. Since 5 is a factor of both 615 and 220, we get

$$5 \mid 615x + 220y \implies 5 \mid 4,$$

by substitution. This cannot happen since $5 > 4$ so we cannot have $4 = 615x + 220y$.

Extra Practice Problems

Note(s)

These practice questions are designed to help deepen your understanding. No answers will be provided, as the goal is to encourage independent problem-solving and reinforce key concepts.

1. Find the following values:

   $$\lfloor 17.73 \rfloor, \quad \lceil 73.17 \rceil, \quad \lceil \lfloor 1 \rfloor \rceil, \quad \left| -\frac{1}{2} \right|, \quad \left\lfloor \sqrt{59} \right\rfloor, \quad \left| \left\lceil -\frac{222}{10} \right\rceil \right|, \quad \left\lceil \left| -\frac{222}{10} \right| \right\rceil.$$

2. a) Provide an example of numbers x and y such that $\lceil x \rceil + \lceil y \rceil = \lceil x + y \rceil$.
   b) Provide an example of numbers x and y such that $\lceil x \rceil + \lceil y \rceil > \lceil x + y \rceil$.
   c) Give an argument that $\lceil x \rceil + \lceil y \rceil \geq \lceil x + y \rceil$ for all numbers x and y.

3. If t is an integer, then $\lceil x + t \rceil = \lceil x \rceil + t$ for every number x.
   a) State a similar fact about the floor function $\lfloor \cdot \rfloor$.
   b) Explain why this fact is true.

4. Is there an example of numbers x and y such that $|x| + |y| < |x + y|$?

5. Which of the following are true?

   $$7 \mid 161, \quad 7 \mid 162, \quad 8 \mid 4, \quad 17 \mid 68, \quad 3 \mid 10^{400}, \quad 11 \mid 1001.$$

6. Find the following values:

   $$100 \text{ div } 13, \quad 100 \,\%\, 13, \quad 67 \text{ div } -22, \quad 67 \,\%\, -22, \quad (-238) \text{ div } 11, \quad (-238) \,\%\, 11.$$

7. Suppose that $a \mid c$ and $b \mid d$. Explain why $ab \mid cd$.

8. Find the least positive integer n for which $3^n \,\%\, 7 = 1$. Evaluate $3^{100} \,\%\, 7$.

9. Find the following gcd values and use them to find the corresponding lcm:
   a) $\gcd(12, 18)$ and $\text{lcm}(12, 18)$
   b) $\gcd(83, 36)$ and $\text{lcm}(83, 36)$
   c) $\gcd(533, 182)$ and $\text{lcm}(533, 182)$
   d) $\gcd(112, 629)$ and $\text{lcm}(112, 629)$

10. Which pairs of numbers from the previous question are coprime?

11. The amount of integers between integers m and n, where $n > m$ is $n - m + 1$. How many integers are there between two real numbers x, y where $x > y$?

12. Suppose that $a =_{(n)} b$ for positive integers a, b, n. Show that $\gcd(a, n) = \gcd(b, n)$.