Quiz 2 Solutions: Set Theory

1. Let A be the set $\{f, o, r, t, u, n, a, t, e, l, y\}$ and B be the set $\{b, r, i, m, s, t, o, n, e\}$. Which of the following are subsets of $A \oplus B$?

   - $\{b, a, b, y\}$
   - $\{f, a, m, i, l, y\}$
   - $\varnothing$
   - $\{\}$
   - $\{m, a, s, t, e, r\}$
   - $\{b, b, b, l, s, s, a, a, b, u, l\}$
   - $\{\varnothing\}$
   - $\{\{\}\}$
   - a, i

   Answer(s)

   We have $A \oplus B = \{i, a, b, f, l, m, s, u, y\}$. This means that

   - $\{b, a, b, y\} = \{b, a, y\}$ since we can ignore repetition. As all elements are in $A \oplus B$, this is a subset.
   - $\{f, a, m, i, l, y\}$ is a subset as all elements of this set are present in $A \oplus B$.
   - $\varnothing$ is a subset.
   - $\{\}$ is a subset.
   - $\{m, a, s, t, e, r\}$ is not a subset since r is not an element in $A \oplus B$.
   - $\{b, b, b, l, s, s, a, a, b, u, l\} = \{b, l, s, a, u\}$ since we can ignore repetition. As order does not matter, this is a subset of $A \oplus B$.
   - $\{\varnothing\}$ is not a subset since $\varnothing$ is not an element of $A \oplus B$.
   - $\{\{\}\}$ is not a subset since $\{\}$ is not an element of $A \oplus B$.
   - a, i is not a subset since there are no brackets, indicating it is not a set.

2. Let $A = \{a\}$, $B = \{b\}$ and $C = \{c\}$. Which of the following strings are not in the language $(A^*B^*C^*)^*$?

   - abc
   - $abcabc$
   - $aabbcc$
   - $abacbc$
   - λ
   - None of the other options.

Answer(s)

We know that, by definition,

$$(A^*B^*C^*)^* = (A^*B^*C^*)^0 \cup (A^*B^*C^*)^1 \cup (A^*B^*C^*)^2 \cup \dots.$$

We find that

- abc is in the language. We have $a \in A^*$, $b \in B^*$ and $c \in C^*$ so $abc \in A^*B^*C^* = (A^*B^*C^*)^1$. This means $abc \in (A^*B^*C^*)^*$.
- $abcabc$ is in the language since $abc \in A^*B^*C^*$. This means $abcabc \in (A^*B^*C^*)^2 \subseteq (A^*B^*C^*)^*$.
- $aabbcc$ is in the language. We have $aa \in A^*$, $bb \in B^*$ and $cc \in C^*$ so $aabbcc \in A^*B^*C^* = (A^*B^*C^*)^1$. This means $aabbcc \in (A^*B^*C^*)^*$.
- $abacbc$ is in the language. We have $ab \in A^*B^*C^*$, $ac \in A^*B^*C^*$ and $bc \in A^*B^*C^*$ so $abacbc \in (A^*B^*C^*)^3 \subseteq (A^*B^*C^*)^*$.
- λ is in $(A^*B^*C^*)^0$ and so it is in the language.
- all other options are in the language.

3. Let A and B be sets, and define the operation $\diamond$ between two sets as follows

$$A \diamond B = ((A \oplus B) \cup ((A \cap B) \times (A^c \cap B^c))).$$

Now, consider the following statements. Which of the following are always true?

- $A \diamond B = (A \cup B) \setminus (A \cap B)$
- $A \diamond B = (A \times B) \cup (B \times A)$
- $A \diamond B = B \diamond A$
- $A \diamond A = A \times A^c$
- None of these options

Answer(s)

- Consider $A = \{1\}, B = \{1\}$ and $\mathcal{U} = \{1, 2\}$. We get $(A \cup B) \setminus (A \cap B) = \{\}$ but $A \diamond B = \{(1, 2)\}$, since $A \oplus B = \{\}$, $A \cap B = \{1\}$ and $A^c \cap B^c = \{2\}$. Therefore, $A \diamond B = (A \times B) \cup (B \times A)$ is not true.
- Consider $A = \{1\}$ and $B = \{\}$. We get $A \times B = \{\} = B \times A$. But, we find that $A \diamond B = \{1\}$. This means $A \diamond B = (A \times B) \cup (B \times A)$ is not true.
- We find that $A \diamond B = B \diamond A$ as $A \oplus B = B \oplus A$, $A \cap B = B \cap A$ and $A^c \cap B^c = B^c \cap A^c$.
- We find that
$$A \diamond A = ((A \oplus A) \cup ((A \cap A) \times (A^c \cap A^c))) = A \times A^c,$$
using the fact that $A \oplus A = \{\}$, $A \cap A = A$ and $A^c \times A^c = A^c$.

4. Let $X = \{0, 1, 2, 3, 4\}$. Define $Z = \{(x, y) : x, y \in X$ and $x^2 - y$ is a perfect square. What is $|Z|$?

Answer(s)

We simply have to go through all the possibilities:

$$Z = \{(0,0),(1,0),(2,0),(3,0),(4,0),(1,1),(2,3),(2,4)\}.$$

We have $|Z| = 8$.

5. Let $\Sigma = \{g,c,d\}$ and $\psi = \{d,u,c,k\}$. How many words are in the set $\Sigma^{=2} \cup \Psi^{\leq 3}$.

Answer(s)

To calculate $\Sigma^{=2}$, we want the number of 2-letter words we can make. We have 3 choices ($\{g,c,d\}$) for the first letter and 3 choices ($\{g,c,d\}$) for the second letter. We have $3 \times 3 = 9$ words in $\Sigma^{=2}$.

Now, we will calculate the size of $\Psi^{\leq 3}$. This is the set of words with length 0, 1, 2 and 3. There is only 1 word of length 0, the empty word λ. There are 4 letters so 4 words of length 1. For words of length 2, we have $4 \times 4 = 16$ words. For words of length 3, we have $4 \times 4 \times 4 = 64$ words. This gives us $\left|\Psi^{\leq 3}\right| = 85$.

We will now calculate the overlap. We find that $\Sigma \cap \Psi = \{c,d\}$ so the overlap between the two sets will be two letter words made up of c and d. This gives us $2 \times 2 = 4$ words. Hence,

$$|\Sigma^{=2} \cup \Psi^{\leq 3}| = |\Sigma^{=2}| + |\Psi^{\leq 3}| - |\Sigma^{=2} \cap \Psi^{\leq 3}| = 9 + 85 - 4 = 90.$$

6. Let $A = \{\varnothing, \{\varnothing\}\}$ and $B = \mathrm{Pow}(\mathrm{Pow}(A))$. Which of the following statements are true?

- $|\mathrm{Pow}(A)| = 2^{|A|}$
- $\{\emptyset, \{\{\emptyset\}\}\} \in B$
- $\mathrm{Pow}(A) \subset B$
- $\{\{A\}\} \in B$
- None of these options.

Answer(s)

We can calculate $\mathrm{Pow}(A) = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$. We then find that $|\mathrm{Pow}(A)| = 2^{|A|}$.

Now, we know that $\varnothing \in B$ as $\varnothing$ is a subset of $\mathrm{Pow}(A)$. We also know that $\{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}$ are all subsets of $\mathrm{Pow}(A)$ and therefore elements of B. This means that $Pow(A) \subset B$ and $\{\varnothing, \{\{\varnothing\}\} \in B$.

We have $A \in \mathrm{Pow}(A)$ so $\{A\} \in B$, but $\{\{A\}\} \notin B$.

7. Define the operation $\diamondsuit$ on sets as follows: For any set X, $\diamondsuit X = \{Y : Y \subseteq X \text{ and } |Y| \text{ is prime}\}$. Let $C = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$. Which of the following is true?

- $\emptyset \in (\diamondsuit(\diamondsuit C))$
- $|\mathrm{Pow}(C) \setminus (\diamondsuit C)| = 2^{|C|} - |\diamondsuit C|$

- $|\Diamond C| = |\text{Pow}(C)|$
- None of the above

Answer(s)

- The diamond operator only contains sets with a prime size. Since $|\varnothing| = 0$ and 0 is not prime, $\varnothing \notin (\Diamond(\Diamond C))$.
- Since $\text{Pow}(C) = \{X : X \subseteq C\}$, we find that $\Diamond C$ is a subset of $\text{Pow}(C)$ as all elements in $\Diamond C$ are subsets of C. This means that $|\text{Pow}(C) \setminus (\Diamond C)| = 2^{|C|} - |\Diamond C|$.
- We find that $\Diamond C \subseteq \text{Pow}(C)$, but $\varnothing \notin \Diamond C$ while $\varnothing \in \text{Pow}(C)$. This means that $|\Diamond C| < |\text{Pow}(C)|$.

8. Let $\Sigma = \{a, b\}$, and define $L_1 = \{a^n b^n : n \geq 0\}$ and $L_2 = \{a^{n^2} b^n : n \geq 0\}$. Which of the following is true about $L_1 \cap L_2$?

   (a) $L_1 \cap L_2 = \{\varepsilon, ab\}$

   (b) $L_1 \cap L_2$ is finite and contains more than two elements

   (c) $L_1 \cap L_2$ is infinite

   (d) $L_1 \cap L_2 = \varnothing$

Answer(s)

To calculate $L_1 \cap L_2$, let w be a word both in L_1 and L_2. We find that $w = a^n b^n = a^{m^2} b^m$ for some non-negative integers n and m. Since $a^n b^n$ and $a^{m^2} b^m$ must have the same amount of b's, we have $n = m$. The words must also have the same number of a's so $n = m^2 = n^2$. The only numbers where $n = n^2$ are 0 and 1 so the only two words in $L_1 \cap L_2$ are λ and ab.

9. Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$. Which of the following statements is true?

   - $|\text{Pow}(A)| = |\text{Pow}(B)|$
   - $|A \times B| = |B \times A|$
   - $|\text{Pow}(A \cup B)| = |\text{Pow}(A)| + |\text{Pow}(B)|$
   - $|A \oplus B| = |A| + |B|$
   - None of the other options

Answer(s)

We know that $|\text{Pow}(X)| = 2^{|X|}$ for all sets X. This means that since $|A| \neq |B|$, we have $|\text{Pow}(A)| \neq |\text{Pow}(B)|$. Since $|A \cup B| = 7$, we can use our formula to find that $|\text{Pow}(A \cup B)| \neq |\text{Pow}(A)| + |\text{Pow}(B)|$.

We have $|A \times B| = |A| \times |B|$ so we find that $|A \times B| = |B \times A|$.

We can calculate $A \oplus B = \{1, 2, 3, 4, a, b, c\}$ to get $|A \oplus B| = 7 = |A| + |B|$.

10. Let $\Sigma = \{a, b, c\}$. For $x \in \Sigma$, we define $|w|_x$ to be the number of times x appears in the word w. Consider the language $L = \{w \in \Sigma^* : |w|_a, |w|_b, |w|_c \text{ are primes}\}$. Which of the following is in L?

- $w = aabcbababbbcb$
- $w = bcbabbaaabaaa$
- $w = cccbbbccbbcc$
- $w = abcabcabcabcabc$
- $w = accbbcacaaab$

Answer(s)

- $w = aabcbababbbcb$ has 5 a's, 5 b's and 2 c's which are all prime. This means $w \in L$.
- $w = bcbabbaaabaaa$ is not in L as there is only one c and 1 is not prime.
- $w = cccbbbccbbcc$ is not in L as there are 0 a's and 0 is not prime.
- $w = abcabcabcabcabc$ has 5 a's, 5 b's and 5 c's which are all prime. This means $w \in L$.
- $w = accbbcacaaab$ is not in L as there are 4 a's and 4 is not prime.