Tutorial 4 Solutions: Functions and Boolean Logic

Boolean Logic

Concept(s)

The set of Booleans $\mathbb{B} = \{0, 1\}$ with functions $! : \mathbb{B} \to \mathbb{B}$, $\&\& : \mathbb{B}^2 \to \mathbb{B}$ and $|| : \mathbb{B}^2 \to \mathbb{B}$ defined as

$$!x = 1 - x, \quad x \,\&\&\, y = \min\{x, y\}, \quad x \,||\, y = \max\{x, y\}.$$

There are laws for these operations, which are similar to the ones in set theory:

Commutativity	$x \,\|\|\, y = y \,\|\|\, x$	$x \,\&\&\, y = y \,\&\&\, x$
Associativity	$(x \,\|\|\, y) \,\|\|\, z = x \,\|\|\, (y \,\|\|\, z)$	$(x \,\&\&\, y) \,\&\&\, z = x \,\&\&\, (y \,\&\&\, z)$
Distribution	$x \,\|\|\, (y \,\&\&\, z) = (x \,\|\|\, y) \,\&\&\, (x \,\|\|\, z)$	$x \,\&\&\, (y \,\|\|\, z) = (x \,\&\&\, y) \,\|\|\, (x \,\&\&\, z)$
Identity	$x \,\|\|\, 0 = x$	$x \,\&\&\, 1 = x$
Complement	$x \,\|\|\, (!x) = 1$	$x \,\&\&\, (!x) = 0$
Idempotence	$x \,\|\|\, x = x$	$x \,\&\&\, x = x$

Exercise 1. Calculate the value of the following terms:

$$(1 \,||\, 1) \,\&\&\, (1 \,||\, 0), \quad (!0 \,\&\&\, 0) \,||\, (!1 \,\&\&\, 1), \quad !(!0 \,\&\&\, ((0 \,\&\&\, !1) \,\&\&\, 1)).$$

Answer(s)

$$(1 \,||\, 1) \,\&\&\, (1 \,||\, 0) = 1, \quad (!0 \,\&\&\, 0) \,||\, (!1 \,\&\&\, 1) = 0, \quad !(!0 \,\&\&\, ((0 \,\&\&\, !1) \,\&\&\, 1)) = 1.$$

Exercise 2. Using the laws of Boolean algebra, show that

$$[x \,\&\&\, (x \,\&\&\, !y)] \,||\, [(x \,\&\&\, y) \,||\, (y \,\&\&\, !x)] = x \,||\, y.$$

Answer(s)

$$\begin{aligned}
& [x \,\&\&\, (x \,\&\&\, !y)] \,||\, [(x \,\&\&\, y) \,||\, (y \,\&\&\, !x)] & \\
&= [x \,\&\&\, (x \,\&\&\, !y)] \,||\, [(y \,\&\&\, x) \,||\, (y \,\&\&\, !x)] & \text{(Commutativity)} \\
&= [x \,\&\&\, (x \,\&\&\, !y)] \,||\, [y \,\&\&\, (x \,||\, !x)] & \text{(Distribution)} \\
&= [x \,\&\&\, (x \,\&\&\, !y)] \,||\, [y \,\&\&\, 1] & \text{(Complement)} \\
&= [x \,\&\&\, (x \,\&\&\, !y)] \,||\, y & \text{(Identity)} \\
&= [(x \,\&\&\, x) \,\&\&\, !y] \,||\, y & \text{(Associativity)} \\
&= [x \,\&\&\, !y] \,||\, y & \text{(Idempotence)} \\
&= y \,||\, [x \,\&\&\, !y] & \text{(Commutativity)} \\
&= (y \,||\, x) \,\&\&\, (y \,||\, !y) & \text{(Distribution)} \\
&= (y \,||\, x) \,\&\&\, 1 & \text{(Complement)} \\
&= y \,||\, x & \text{(Identity)} \\
&= x \,||\, y & \text{(Commutativity)}
\end{aligned}$$

Functions and Their Properties

Concept(s)

A binary relation $f \subseteq X \times Y$ is a function if for all $x \in X$, there is exactly one $y \in Y$ such that $(x, y) \in f$. We write $f(x) = y$ when $(x, y) \in f$.

f is Injective	For all $a, b \in X$, if $f(a) = f(b)$, then $a = b$
f is Surjective	For all $y \in Y$, there exists $x \in X$ such that $f(x) = y$
f is Bijective	Injective and Surjective

We define $\mathrm{Dom}(f) = X$, $\mathrm{Codom}(f) = Y$ and $\mathrm{Im}(f) = \{f(x) : x \in \mathrm{Dom}(f)\}$.

Exercise 3. Which of the following binary relations are functions? If it is a function, find the domain, codomain and image.

a) $f \subseteq \mathbb{Z} \times \mathbb{N}$, where $(x, y) \in f$ if and only if $y = |x|$.

b) $f \subseteq \mathbb{N} \times \mathbb{Z}$, where $(x, y) \in f$ if and only if $|y| = x$.

c) $f \subseteq \mathbb{N} \times \mathbb{Z}$, where $(x, y) \in f$ if and only if $y = 2x + 1$.

d) $f \subseteq \mathbb{B} \times \mathbb{B}$, where $f = \{(1, 1)\}$.

e) $f \subseteq \mathbb{B} \times \mathbb{B}$, where $(x, y) \in f$ if and only if $y = x \mid\mid !x$.

Answer(s)

a) Every x value has one associated y value, $|x|$. This means that f is a function with $\mathrm{Dom}(f) = \mathbb{Z}$ and $\mathrm{Codom}(f) = \mathbb{N}$. We have $\mathrm{Im}(f) = \mathbb{N}$ as $(n, n) \in f$ for all $n \in \mathbb{N}$.

b) An x value can have multiple associated y values e.g. $(1, 1), (1, -1) \in f$ so f is not a function.

c) Every x value has one associated y value, $2x+1$. This means that f is a function with $\mathrm{Dom}(f) = \mathbb{N}$ and $\mathrm{Codom}(f) = \mathbb{Z}$. We have $f(0) = 1$, $f(1) = 3$ and increasing x by 1 increases $f(x)$ by 2. This means $\mathrm{Im}(f) = \{x \in \mathbb{N} : x \text{ is an odd integer}\}$.

d) There are no values for an input of 0 in the relation so f is not a function.

e) Every x value has one associated y value, $x \mid\mid !x$. This means that f is a function with $\mathrm{Dom}(f) = \mathbb{B}$ and $\mathrm{Codom}(f) = \mathbb{B}$. We have $f(1) = 1$ and $f(0) = 1$ so $\mathrm{Im}(f) = \{1\}$.

Exercise 4. Determine which of the following functions are injective, surjective or bijective.

a) $f : \mathbb{R} \to \mathbb{Z}$, $f(x) = \lfloor x + 1 \rfloor$.

b) Let $X = \{x\}$ and $f : X^* \to \mathbb{N}$, $f(w) = \mathrm{length}(w)$.

c) $f : \mathbb{N} \to \mathbb{N}$, $f(x) = x^2$.

d) Let $\Sigma = \{a, b\}$ and $\mathrm{A} : \Sigma^* \to \mathbb{N}$, where $\mathrm{A}(w)$ is the number of a's in the word w.

e) $f : \mathbb{B} \to \mathbb{B}$, $f(x) = !x$.

Answer(s)

a) Since $f(1) = f(1.5) = 2$, f is not injective and therefore not bijective. For all $n \in \mathbb{Z}$, we have $f(n - 1) = n$ so f is surjective.

b) We have $X^* = \{\lambda, x, xx, xxx, \ldots\}$. If $f(w_1) = f(w_2)$ where $w_1, w_2 \in X^*$, this means w_1 and w_2 have the same length. There is one word associated with each length so $w_1 = w_2$. This means f is injective. For all $n \in \mathbb{N}$, we have $f(x^n) = n$ so f is surjective and bijective.

c) If we have $f(x) = f(y)$, then $x^2 = y^2$ so $x = y$, as x and y are non-negative. This means f is injective. Since there is no $n \in \mathbb{N}$ such that $n^2 = 2$, f is not surjective and not bijective.

d) Since $f(a) = f(ab)$, we find f is not injective and therefore not bijective. For all $n \in \mathbb{N}$, we have $f(a^n) = n$ so f is surjective.

e) Considering all input, we have $f = \{(0,1),(1,0)\}$. We find that f is bijective.

Inverse Functions

Concept(s)

For $f : X \to Y$ and $g : Y \to Z$, the composition of f and g is $g \circ f$, where $(g \circ f)(x) = g(f(x))$.

The identity function on X, $\mathrm{Id} : X \to X$, is defined as $\mathrm{Id}_X(x) = x$.

Exercise 5. Define $f : \mathbb{Z} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{Z}$ where $f(x) = x^2 + 3$ and $g(x) = (-1)^x x$.

a) Compute $f \circ g$ and $g \circ f$.

b) What is the domain, codomain and image of $f \circ g$?

c) Is $g \circ f$ injective, surjective or bijective?

Answer(s)

a) We have
$$(f \circ g)(x) = f(g(x)) = f((-1)^x x) = ((-1)^x x)^2 + 3 = x^2 + 3$$
and
$$(g \circ f)(x) = g(f(x)) = g(x^2 + 3) = (-1)^{x^2+3}(x^2 + 3).$$

b) The domain and codomain of $f \circ g$ is $\mathbb{N}$. We find that $\{n^2 : n \in \mathbb{N}\} = \{0, 1, 4, 9, 16, \ldots\}$ so $\mathrm{Im}(f) = \{3, 4, 7, 12, \ldots\}$.

c) Since $(g \circ f)(1) = (g \circ f)(-1)$, $g \circ f$ is not injective. We also do not have $x \in \mathbb{Z}$ such that $(g \circ f)(x) = 0$, since $|(g \circ f)(x)| = \left|(-1)^{x^2+3}(x^2 + 3)\right| = x^2 + 3 \geq 3$.

Exercise 6. Prove if $f : A \to B$ and $g : B \to C$ are both injective, then $g \circ f$ is injective.

Answer(s)

Suppose that we have $x, y \in A$, where $(g \circ f)(x) = (g \circ f)(y)$. We have $g(f(x)) = g(f(y))$. Since g is injective, we find that $f(x) = f(y)$. Since f is injective, we find that $x = y$. Therefore, if $(g \circ f)(x) = (g \circ f)(y)$, then $x = y$, meaning $g \circ f$ is injective.

Concept(s)

For $f : X \to Y$, if $f^{\leftarrow}$ is a function, we call it inverse function of f.

The function f has an inverse if and only if f is bijective.

For $f : X \to Y$ and $g : Y \to X$, we have $g = f^{-1}$ the inverse of f whenever

$$g \circ f = \mathrm{Id}_X \text{ and } f \circ g = \mathrm{Id}_Y.$$

Exercise 7. Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 3x + 5$ and $g(x) = \frac{x-5}{3}$. Show that g is the inverse of f.

Answer(s)

We can show that g is an inverse of f by computing $g \circ f$ and $f \circ g$. First, we have

$$(g \circ f)(x) = g(f(x)) = g(3x + 5) = \frac{(3x + 5) - 5}{3} = x \text{ so } g \circ f = \mathrm{Id}_{\mathbb{R}}.$$

We also have

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{x-5}{3}\right) = 3\frac{x-5}{3} + 5 = x \text{ so } f \circ g = \mathrm{Id}_{\mathbb{R}}.$$

Therefore, g is the inverse of f.

Exercise 8. Let $f, g : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n$ and

$$g(n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (n-1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

Show that $g \circ f = \mathrm{Id}_{\mathbb{N}}$. Is g an inverse of f?

Answer(s)

Consider that

$$(g \circ f)(n) = g(f(n)) = g(2n) = 2n/2 = n,$$

where $2n$ is even so we always take the even branch of g. Therefore, $g \circ f = \mathrm{Id}_{\mathbb{R}}$.

We find that f is not surjective as there is no $n \in \mathbb{N}$ such that $f(n) = 1$. This means f is not bijective and so f cannot have an inverse.

Conjunctive and Disjunctive Normal Form

Concept(s)

Literal	A function $\mathbb{B} \to \mathbb{B}$ (All l_i are literals in the following definitions)
Minterm	A function $\mathbb{B}^n \to \mathbb{B}$ of the form $(\ldots(l_1(x_1) \;\&\&\; l_2(x_2)) \;\&\&\; \ldots \;\&\&\; l_n(x_n))$
Maxterm	A function $\mathbb{B}^n \to \mathbb{B}$ of the form $(\ldots(l_1(x_1) \;\|\|\; l_2(x_2)) \;\|\|\; \ldots \;\|\|\; l_n(x_n))$
CNF Boolean Function	A function $(\ldots(m_1 \;\&\&\; m_2) \;\&\&\; \ldots \;\&\&\; m_n)$ where m_i are maxterms
DNF Boolean Function	A function $(\ldots(m_1 \;\|\|\; m_2) \;\|\|\; \ldots \;\|\|\; m_n)$ where m_i are minterms

Exercise 9. Determine if the following terms are CNF, DNF or neither:

a) $((!x_2 \;\&\&\; (x_3 \;\&\&\; !x_1)) \;\|\|\; x_3) \;\|\|\; (x_1 \;\&\&\; !x_2)$

b) $((!x_1 \mid\mid x_2) \;\&\&\; (x_3 \mid\mid x_4)) \mid\mid (!x_3 \;\&\&\; x_4)$

c) $!x_1 \;\&\&\; (x_3 \mid\mid (x_2 \;\&\&\; !x_3))$

d) $((x_2 \mid\mid !x_3) \;\&\&\; (x_1 \mid\mid (x_5 \mid\mid x_6))) \;\&\&\; ((!x_2 \mid\mid !x_5) \mid\mid x_6)$

Answer(s)

a) DNF: The minterms are $!x_2 \;\&\&\; (x_3 \;\&\&\; !x_1)$, x_3 and $x_1 \;\&\&\; !x_2$.

b) Neither: The operator applied last determines if the term is a CNF or DNF. Since the operator applied last is $\mid\mid$ as shown below, the term is either DNF or neither.

$$((!x_1 \mid\mid x_2) \;\&\&\; (x_3 \mid\mid x_4)) \mid\mid (!x_3 \;\&\&\; x_4)$$

The term to the left of $\mid\mid$ has the operator applied last as $\&\&$ but is not a minterm so this formula is neither.

c) Neither: The operator that is applied last is $\&\&$ so the term is not a DNF, as shown below:

$$!x_1 \;\&\&\; (x_3 \mid\mid (x_2 \;\&\&\; !x_3))$$

The term to the right of $\&\&$ must be a maxterm as the operator applied last is $\mid\mid$. Since $x_3 \mid\mid (x_2 \;\&\&\; !x_3)$ is not a maxterm, it is not a CNF.

d) CNF: The maxterms are $x_2 \mid\mid !x_3$, $x_1 \mid\mid (x_5 \mid\mid x_6)$ and $(!x_2 \mid\mid !x_5) \mid\mid x_6$.

Concept(s)

For a term f of the form $\mathbb{B}^n \to \mathbb{B}$, we can consider all $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{B}^n$, and define

$$m_{\mathbf{b}} = (\ldots(l_1(x_1) \;\&\&\; l_2(x_2)) \;\&\&\; \ldots \;\&\&\; l_n(x_n)) \text{ where } l_i(x_i) = \begin{cases} x_i & \text{if } b_i = 1, \\ !x_i & \text{if } b_i = 0. \end{cases}$$

The formula in Disjunctive Normal Form is the disjunction (or) over all min terms where $f(\mathbf{b}) = 1$.

Exercise 10. Convert $(x \mid\mid y) \;\&\&\; (!x \mid\mid !y)$ into Disjunctive Normal Form (DNF).

Answer(s)

We have two inputs for this term, x and y. There are four possibilities for x and y:

x	y	$(x \mid\mid y) \;\&\&\; (!x \mid\mid !y)$
0	0	0
0	1	1
1	0	1
1	1	0

Since the term is only true when $(x, y) = (0, 1)$ and $(x, y) = (1, 0)$, our DNF becomes

$$(!x \;\&\&\; y) \mid\mid (x \;\&\&\; !y).$$

Exercise 11. Convert $(x \mid\mid y) \;\&\&\; (x \;\&\&\; (!y \mid\mid z))$ into Disjunctive Normal Form (DNF).

Answer(s)

Let $f(x,y,z) = (x \mid\mid y)$ && $(x$ && $(!y \mid\mid z))$. We have three inputs for this term, x, y and z. There are eight possibilities for x, y and z:

x	y	z	$f(x,y,z)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

The formula f is true for $(x,y,z) \in \{(1,0,0),(1,0,1),(1,1,1)\}$. Our DNF is

$$(((x \;\&\&\; !y) \;\&\&\; !z) \mid\mid ((x \;\&\&\; !y) \;\&\&\; z) \mid\mid ((x \;\&\&\; y) \;\&\&\; z).$$

Big-O Notation

Concept(s)

Let $f, g : \mathbb{N} \to \mathbb{R}_{\geq 0}$.

$f(n) \in O(g(n))$ means there exists $n_0 \in \mathbb{N}$ and $c \in \mathbb{R}_{>0}$ where for all $n \geq n_0$, $f(n) \leq c \cdot g(n)$
$f(n) \in \Omega(g(n))$ means there exists $n_0 \in \mathbb{N}$ and $c \in \mathbb{R}_{>0}$ where for all $n \geq n_0$, $f(n) \geq c \cdot g(n)$
$f(n) \in \Theta(g(n))$ means that $f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$

Exercise 12. Consider $f, g : \mathbb{N} \to \mathbb{R}$ and determine if $f(n) \in O(g(n))$, $f(n) \in \Omega(g(n))$ or $f(n) \in \Theta(g(n))$.

a) $f(n) = 4n + 2$, $g(n) = n^2 - 4$

b) $f(n) = 2n^2 - n$, $g(n) = n^2 - 4$

c) $f(n) = 2^{3n}$, $g(n) = 2^{2n}$

Answer(s)

a) We have $f(n) \in O(g(n))$. Consider that $4n + 2 \leq n^2 - 4$ for all $n \geq 6$.

Suppose that $f(n) \in \Omega(g(n))$. Then there exists $n_0 \in \mathbb{N}$ and real number $c > 0$, such that $4n + 2 \geq cn^2 - 4c$ for all $n \geq n_0$. Then, $4 + \frac{2}{n} \geq cn - \frac{4c}{n}$. For all $n \geq 1$, we have $\frac{2}{n} \leq 2$ and $\frac{4c}{n} \leq 4c$. This means that $6 \geq 4 + \frac{2}{n} \geq cn - \frac{4c}{n} g \geq cn - 4c$.

Since for all $n \geq 1$, we have $\frac{2}{n} \leq 2$ and $\frac{4c}{n} \leq 4c$, we have $6 \leq cn - c4$ when $n \geq 1$. But, this is not true when $n > \frac{6}{c}$ which is a contradiction. Therefore, $f(n) \in \Omega(g(n))$ is not possible and $f(n) \notin \Theta(g(n))$.

b) We have $f(n) \in \Theta(g(n))$. Consider that when $n \geq 3$, we have $n^2 - 4 \leq 2n^2 - n \leq 3(n^2 - 4)$.

c) We have $f(n) \in \Omega(g(n))$. Consider that $2^{2n} \leq 2^{3n}$ for all $n \geq 0$.

Suppose that $f(n) \in O(g(n))$. Then there exists $n_0 \in \mathbb{N}$ and real number $c > 0$, such that $2^{3n} \leq c2^{2n}$ for all $n \geq n_0$. This means that $2^n \leq c$ for all $n \geq n_0$ which is not true as $2^n > c$ for all $n \geq c$. Therefore, $f(n) \in O(g(n))$ is not possible and $f(n) \notin \Theta(g(n))$.

Exercise 13. Prove that if $f(n) \in O(g(n))$, then $g(n) \in \Omega(f(n))$, where $f, g : \mathbb{N} \to \mathbb{R}_{\geq 0}$.

Answer(s)

Suppose that $f(n) \in O(g(n))$. Then, there exists $n_0 \in \mathbb{N}$ and real number $c > 0$, such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$. This means that $\frac{1}{c} f(n) \leq g(n)$ for all $n \geq n_0$, where $\frac{1}{c} > 0$. By definition, we have $g(n) \in \Omega(f(n))$.

Extra Practice Problems

1. Determine which of the following functions are injective, surjective or bijective.
   a) $f : \mathbb{N} \to \mathbb{Z}$, $f(x) = (-1)^x x$.
   b) Let $\Sigma = \{a, b\}$ and $\text{len} : \Sigma^* \to \mathbb{N}$, where $\text{len}(w)$ is the number of symbols in the word w.
   c) Let $\Sigma = \{a, b\}$ and $f : \mathbb{N} \to Pow(\Sigma^*)$, where $f(n) = \{w \in \Sigma^* : \text{len}(w) \leq n\}$.
   d) $f : \mathbb{Z}\, to\, \mathbb{N}$, $f(x) = |x|$.
2. Let $f : X \to Y$ be bijective and $A, B \subseteq Y$. Show that $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B)$.
3. We define $f, g, h : \mathbb{Z} \to \mathbb{Z}$ as $f(x) = x^3 - 4x$, $g(x) = x \bmod 5$, and $h(x) = x^2$. Find
$$f \circ f, \quad h \circ g, \quad g \circ g, \quad f \circ g \circ h, \quad g \circ f \circ h.$$
4. Show that if f and g are both surjective, then $g \circ f$ is also surjective.
5. Show that $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = -x$ is bijective and find f^{-1}.
6. Simplify the following Boolean expressions:
   a) $(x \;\&\&\; y) \;||\; (x \;\&\&\; !y) \;||\; (!x \;\&\&\; y)$
   b) $(x \;||\; y) \;\&\&\; (x \;||\; !y) \;\&\&\; (!x \;||\; y)$
   c) $!(x \;\&\&\; y) \;||\; (x \;\&\&\; !y)$
7. Convert the following Boolean expressions to Disjunctive Normal Form (DNF):
   a) $x \;\&\&\; (y \;||\; x)$
   b) $(x \;||\; y) \;\&\&\; (!x \;||\; z)$
8. Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = n^2 + 2n + 1$. Find a function $g : \mathbb{N} \to \mathbb{N}$ such that $g \circ f = \text{Id}_{\mathbb{N}}$. Is g the inverse of f? Explain why or why not.
9. Let $\Sigma = \{0, 1\}$ and define $f : \Sigma^* \to \mathbb{Z}$ as $f(w) =$ the number of 1's in w minus the number of 0's in w. Is f injective? Is it surjective? Justify your answers.
10. For each of the following pairs of functions $f(n)$ and $g(n)$, determine whether $f(n) \in O(g(n))$, $f(n) \in \Omega(g(n))$, or $f(n) \in \Theta(g(n))$. Justify your answers.
    a) $f(n) = n \log n$, $g(n) = n^{1.5}$
    b) $f(n) = 2^n$, $g(n) = n^2$
    c) $f(n) = n!$, $g(n) = 2^n$