COMP9313: Big Data Management

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Course web site: http://www.cse.unsw.edu.au/~cs9313/

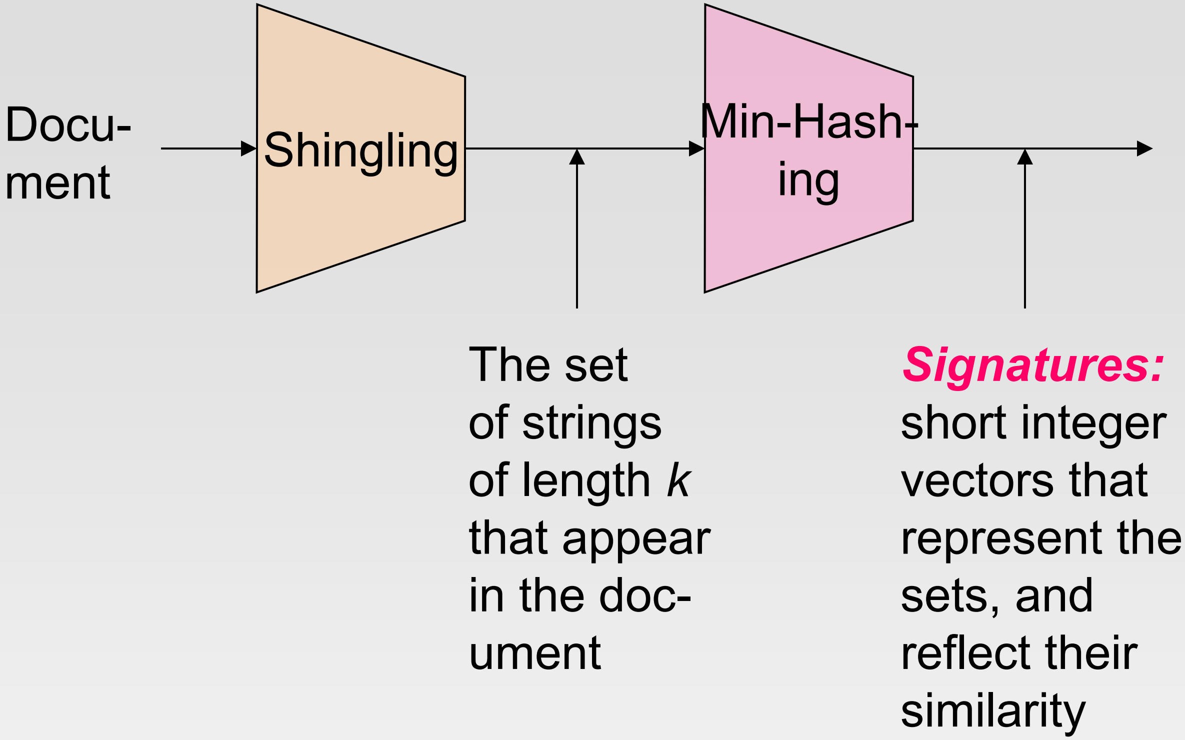
Chapter 7.2: Finding Similar Items

Motivation for Minhash/LSH

- **Suppose we need to find near-duplicate documents among $N = 1$ million documents**

- Naïvely, we would have to compute **pairwise Jaccard similarities** for **every pair of docs**
  - $\boldsymbol{N(N-1)/2}$ **≈ 5*10^{11}** comparisons
  - At 10^5 secs/day and 10^6 comparisons/sec, it would take **5 days**

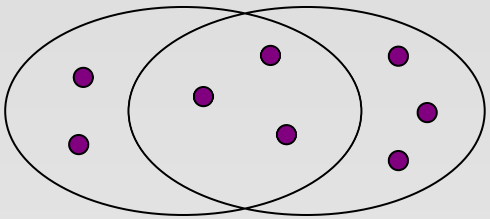
- For $\boldsymbol{N = 10}$ million, it takes more than a year…



Step 2: ***Minhashing:*** Convert **large sets** to **short signatures**, while **preserving similarity**

Encoding Sets as Bit Vectors

- Many similarity problems can be formalized as **finding subsets that have significant intersection**



- **Encode sets using 0/1 (bit, boolean) vectors**
  - One dimension per element in the universal set

- Interpret set intersection as bitwise **AND**, and set union as bitwise **OR**

- **Example:** $\mathbf{C_1}$ = 10111; $\mathbf{C_2}$ = 10011
  - Size of intersection **= 3**; size of union **= 4**,
  - **Jaccard similarity** (not distance) **= 3/4**
  - **Distance: $d(C_1,C_2)$ = 1 – (Jaccard similarity) = 1/4**

From Sets to Boolean Matrices

- **Rows** = elements (shingles)
- **Columns** = sets (documents)
  - 1 in row ***e*** and column ***s*** if and only if ***e*** is a member of ***s***
  - Column similarity is the Jaccard similarity of the corresponding sets (rows with value *1)*
  - **Typical matrix is sparse!**
- **Each document is a column:**

	Documents			
Shingles	1	1	1	0
	1	1	0	1
	0	1	0	1
	0	0	0	1
	1	0	0	1
	1	1	1	0
	1	0	1	0

From Sets to Boolean Matrices

- **Example:** S_1 = {a, d}, S_2 = {c}, S_3 = {b, d, e}, and S_4 = {a, c, d}

$Element$	S_1	S_2	S_3	S_4
a	1	0	0	1
b	0	0	1	0
c	0	1	0	1
d	1	0	1	1
e	0	0	1	0

- **sim(S_1, S_3) = ?**
  - Size of intersection = 1; size of union = 4, Jaccard similarity (not distance) = 1/4
  - **d(S_1, S_3) = 1 – (Jaccard similarity) = 3/4**

Outline: Finding Similar Columns

- **So far:**
  - Documents $\rightarrow$ Sets of shingles
  - Represent sets as boolean vectors in a matrix
- **Next goal: Find similar columns while computing small signatures**
  - **Similarity of columns == similarity of signatures**

Outline: Finding Similar Columns

- **Next Goal: Find similar columns, Small signatures**
- **Naïve approach:**
  - **1) Signatures of columns:** small summaries of columns
  - **2) Examine pairs of signatures** to find similar columns
    - **Essential:** Similarities of signatures and columns are related
  - **3) Optional:** Check that columns with similar signatures are really similar
- **Warnings:**
  - Comparing all pairs may take too much time: **Job for LSH**
    - These methods can produce false negatives, and even false positives (if the optional check is not made)

Hashing Columns (Signatures)

- **Key idea:** "hash" each column C to a small ***signature*** $h(C)$, such that:
  - **(1)** $h(C)$ is small enough that the signature fits in RAM
  - **(2)** $sim(C_1, C_2)$ is the same as the "similarity" of signatures $h(C_1)$ and $h(C_2)$

- **Goal: Find a hash function** $h(\cdot)$ **such that:**
  - If $sim(C_1,C_2)$ is high, then with high prob. $h(C_1) = h(C_2)$
  - If $sim(C_1,C_2)$ is low, then with high prob. $h(C_1) \neq h(C_2)$

- **Hash docs into buckets. Expect that "most" pairs of near duplicate docs hash into the same bucket!**

Min-Hashing

- **Goal: Find a hash function $h(\cdot)$ such that:**
  - if $sim(C_1,C_2)$ is high, then with high prob. $h(C_1) = h(C_2)$
  - if $sim(C_1,C_2)$ is low, then with high prob. $h(C_1) \neq h(C_2)$
- **Clearly, the hash function depends on the similarity metric:**
  - Not all similarity metrics have a suitable hash function
- **There is a suitable hash function for the Jaccard similarity: Min-Hashing**

Min-Hashing

- Imagine the rows of the boolean matrix permuted under **random permutation** π

- Define a **"hash" function** $h_\pi(C)$ = the index of the **first** (in the permuted order π) row in which column ***C*** has value **1**:

$$h_\pi(C) = min_\pi \pi(C)$$

- Use several (e.g., 100) independent hash functions (that is, permutations) to create a signature of a column

Min-Hashing Example

1	0	1	1	0
2	0	0	1	1
3	1	0	0	0
4	0	1	0	1
5	0	0	0	1
6	1	1	0	0
7	0	0	1	0

Input Matrix

3	1	1	2

Signature Matrix

Min-Hashing Example

7	1	0	1	1	0
6	2	0	0	1	1
5	3	1	0	0	0
4	4	0	1	0	1
3	5	0	0	0	1
2	6	1	1	0	0
1	7	0	0	1	0

Input Matrix

3	1	1	2
2	2	1	3

Signature Matrix

Min-Hashing Example

6	7	1	0	1	1	0
3	6	2	0	0	1	1
1	5	3	1	0	0	0
7	4	4	0	1	0	1
2	3	5	0	0	0	1
5	2	6	1	1	0	0
4	1	7	0	0	1	0

Input Matrix

3	1	1	2
2	2	1	3
1	5	3	2

Signature Matrix

Min-Hashing Exa

Note: Another (equivalent) way is to store row indexes:

1	5	1	5
2	3	1	3
6	4	6	4

2nd element of the permutation is the first to map to a 1

Permutation π

2	4	3
3	2	4
7	1	7
6	3	2
1	6	6
5	7	1
4	5	5

Input matrix (Shingles x Documents)

1	0	1	0
1	0	0	1
0	1	0	1
0	1	0	1
0	1	0	1
1	0	1	0
1	0	1	0

Signature matrix *M*

2	1	2	1
2	1	4	1
1	2	1	2

4th element of the permutation is the first to map to a 1

The Min-Hash Property

0	0
0	0
1	**1**
0	0
0	1
1	0

- Choose a random permutation π
- <u>Claim:</u> $\Pr[h_\pi(C_1) = h_\pi(C_2)] = sim(C_1, C_2)$
- Why?
  - Let **X** be a doc (set of shingles), $y \in X$ is a shingle
  - Then: $\Pr[\pi(y) = \min(\pi(X))] = 1/|X|$
    - It is equally likely that any $y \in X$ is mapped to the ***min*** element
  - Let y be s.t. $\pi(y) = \min(\pi(C_1 \cup C_2))$
  - Then either: $\pi(y) = \min(\pi(C_1))$ if $y \in C_1$, **or**

    $\pi(y) = \min(\pi(C_2))$ if $y \in C_2$
  - So the prob. that **both** are true is the prob. $y \in C_1 \cap C_2$
  - $\Pr[\min(\pi(C_1))=\min(\pi(C_2))]=|C_1 \cap C_2|/|C_1 \cup C_2|= sim(C_1, C_2)$

One of the two cols had to have 1 at position y

Four Types of Rows

- **Given cols C_1 and C_2, rows may be classified as:**

	C_1	C_2
A	1	1
B	1	0
C	0	1
D	0	0

  - **a** = # rows of type A, etc.
- **Note:** **sim(C_1, C_2) = a/(a +b +c)**
- **Then:** **Pr**[$h(C_1) = h(C_2)$] = $Sim(C_1, C_2)$
  - Look down the cols C_1 and C_2 until we see a 1
  - If it's a type-*A* row, then $h(C_1) = h(C_2)$
    If a type-*B* or type-*C* row, then not

Similarity for Signatures

Permutation π

2	4	3
3	2	4
7	1	7
6	3	2
1	6	6
5	7	1
4	5	5

Input matrix (Shingles x Documents)

1	0	1	0
1	0	0	1
0	1	0	1
0	1	0	1
0	1	0	1
1	0	1	0
1	0	1	0

Signature matrix *M*

2	1	2	1
2	1	4	1
1	2	1	2

Similarities:

	1-3	2-4	1-2	3-4
Col/Col	0.75	0.75	0	0
Sig/Sig	0.67	1.00	0	0

Similarity for Signatures

- We know: $\Pr[h_\pi(C_1) = h_\pi(C_2)] = sim(C_1, C_2)$
- Now generalize to multiple hash functions

- **The *similarity of two signatures* is the fraction of the hash functions in which they agree**

- **Note:** Because of the Min-Hash property, the similarity of columns is the same as the expected similarity of their signatures

Min-Hash Signatures

- **Pick K=100 random permutations of the rows**
- Think of ***sig*(C)** as a column vector
- **s*ig*(C)[i] =** according to the *i*-th permutation, the index of the first row that has a 1 in column *C*

$$sig(C)[i] = \min(\pi_i(C))$$

- **Note:** The sketch (signature) of document *C* is small **~100 bytes!**

- **We achieved our goal! We "compressed" long bit vectors into short signatures**

Implementation Trick

- **Permuting rows even once is prohibitive**
- **Row hashing!**
  - Pick **K = 100** hash functions k_i
  - Ordering under k_i gives a random row permutation!
- **One-pass implementation**
  - For each column C and hash-func. k_i keep a “slot” for the min-hash value
  - Initialize all $sig(C)[i] = \infty$
  - **Scan rows looking for 1s**
    - Suppose row j has 1 in column C
    - Then for each k_i :
      - If $k_i(j) < sig(C)[i]$, then $sig(C)[i] \leftarrow k_i(j)$

How to pick a random hash function h(x)?
Universal hashing:
$h_{a,b}(x)=((a \cdot x+b) \bmod p) \bmod N$
where:
a,b … random integers
p … prime number (p > N)

Implementation Example

Row	S_1	S_2	S_3	S_4	$x+1 \mod 5$	$3x+1 \mod 5$
0	1	0	0	1	1	1
1	0	0	1	0	2	4
2	0	1	0	1	3	2
3	1	0	1	1	4	0
4	0	0	1	0	0	3

- 0. Initialize all ***sig(C)[i]* = ∞**

	S_1	S_2	S_3	S_4
h_1	∞	∞	∞	∞
h_2	∞	∞	∞	∞

- Row 0: we see that the values of $h_1(0)$ and $h_2(0)$ are both 1, thus $sig(S_1)[0] = 1$, $sig(S_1)[1] = 1$, $sig(S_4)[0] = 1$, $sig(S_4)[1] = 1$,

	S_1	S_2	S_3	S_4
h_1	1	∞	∞	1
h_2	1	∞	∞	1

- Row 1, we see $h_1(1) = 2$ and $h_2(1) = 4$, thus $sig(S_3)[0] = 2$, $sig(S_3)[1] = 4$

	S_1	S_2	S_3	S_4
h_1	1	∞	2	1
h_2	1	∞	4	1

Implementation Example

Row	S_1	S_2	S_3	S_4	$x+1 \mod 5$	$3x+1 \mod 5$
0	1	0	0	1	1	1
1	0	0	1	0	2	4
2	0	1	0	1	3	2
3	1	0	1	1	4	0
4	0	0	1	0	0	3

- Row 2: $h_1(2) = 3$ and $h_2(2) = 2$, thus
  $sig(S_2)[0] = 3$, $sig(S_2)[1] = 2$, no update for S_4

	S_1	S_2	S_3	S_4
h_1	1	3	2	1
h_2	1	2	4	1

- Row 3: $h_1(3) = 4$ and $h_2(3) = 0$, update
  $sig(S_1)[1] = 0$, $sig(S_3)[1] = 0$, $sig(S_4)[1] = 0$,

	S_1	S_2	S_3	S_4
h_1	1	3	2	1
h_2	0	2	0	0

- Row 4: $h_1(4) = 0$ and $h_2(4) = 3$, update
  $sig(S_3)[0] = 0$,

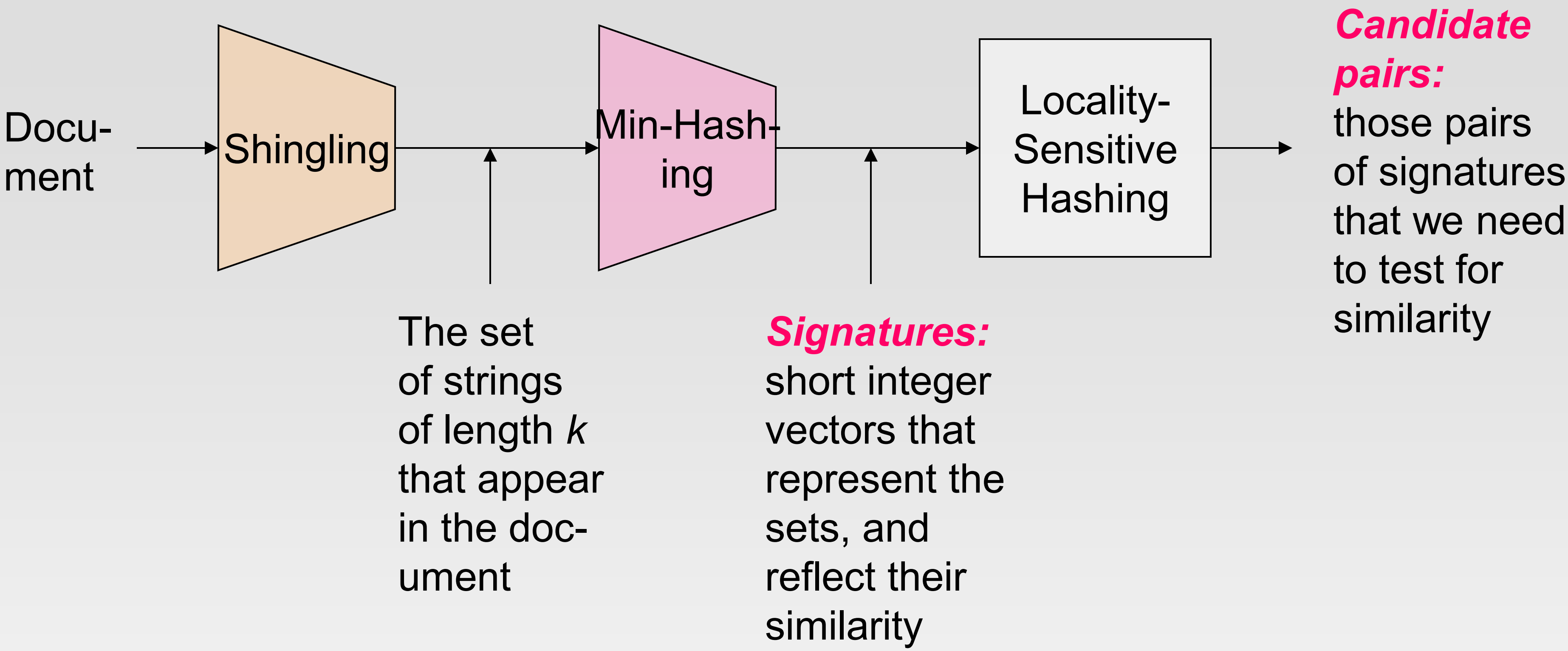
	S_1	S_2	S_3	S_4
h_1	1	3	0	1
h_2	0	2	0	0

Implementation Example

Row	S_1	S_2	S_3	S_4	$x+1 \mod 5$	$3x+1 \mod 5$
0	1	0	0	1	1	1
1	0	0	1	0	2	4
2	0	1	0	1	3	2
3	1	0	1	1	4	0
4	0	0	1	0	0	3

	S_1	S_2	S_3	S_4
h_1	1	3	0	1
h_2	0	2	0	0

- We can estimate the Jaccard similarities of the underlying sets from this signature matrix.
  - Signature matrix: SIM(S_1, S_4) = 1.0
  - Jaccard Similarity: SIM(S_1, S_4) = 2/3



Step 3: ***Locality-Sensitive Hashing:***
Focus on pairs of signatures likely to be from similar documents

LSH: First Cut

2	1	4	1
1	2	1	2
2	1	2	1

- **Goal:** Find documents with Jaccard similarity at least ***s*** (for some similarity threshold, e.g., ***s***=0.8)

- **LSH – General idea:** Use a function ***f(x,y)*** that tells whether ***x*** and ***y*** is a ***candidate pair:*** a pair of elements whose similarity must be evaluated

- **For Min-Hash matrices:**
  - Hash columns of signature matrix ***M*** to many buckets
  - Each pair of documents that hashes into the same bucket is a **candidate pair**

Candidates from Min-Hash

2	1	4	1
1	2	1	2
2	1	2	1

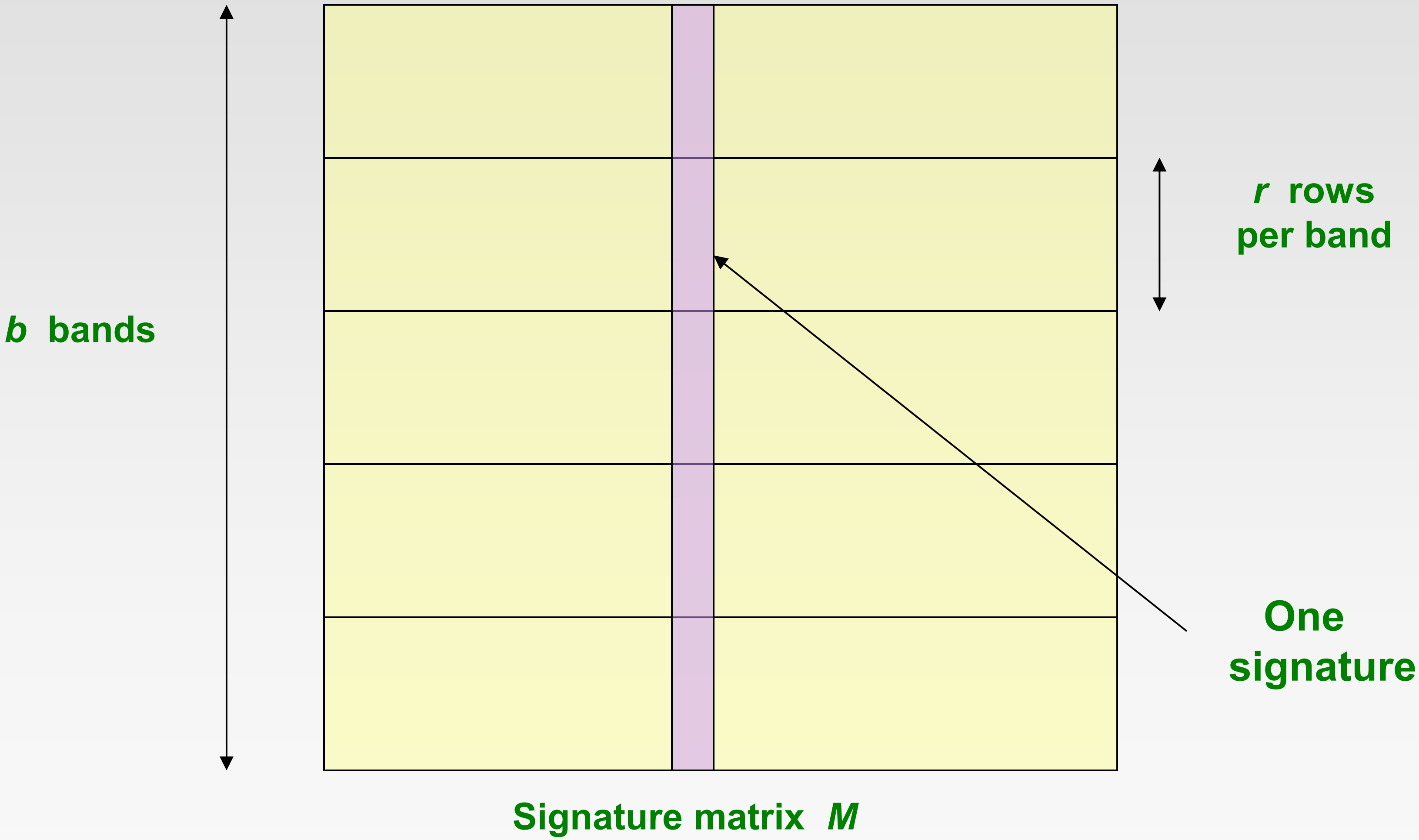
- **Pick a similarity threshold *s* (0 < s < 1)**
- Columns ***x*** and ***y*** of ***M*** are a **candidate pair** if their signatures agree on at least fraction ***s*** of their rows:
  $M(i, x) = M(i, y)$ for at least frac. ***s*** values of ***i***
  - We expect documents ***x*** and ***y*** to have the same (Jaccard) similarity as their signatures

LSH for Min-Hash

2	1	4	1
1	2	1	2
2	1	2	1

- **Big idea: Hash columns of signature matrix *M* several times**
- Arrange that (only) **similar columns** are likely to **hash to the same bucket**, with high probability
- **Candidate pairs are those that hash to the same bucket**

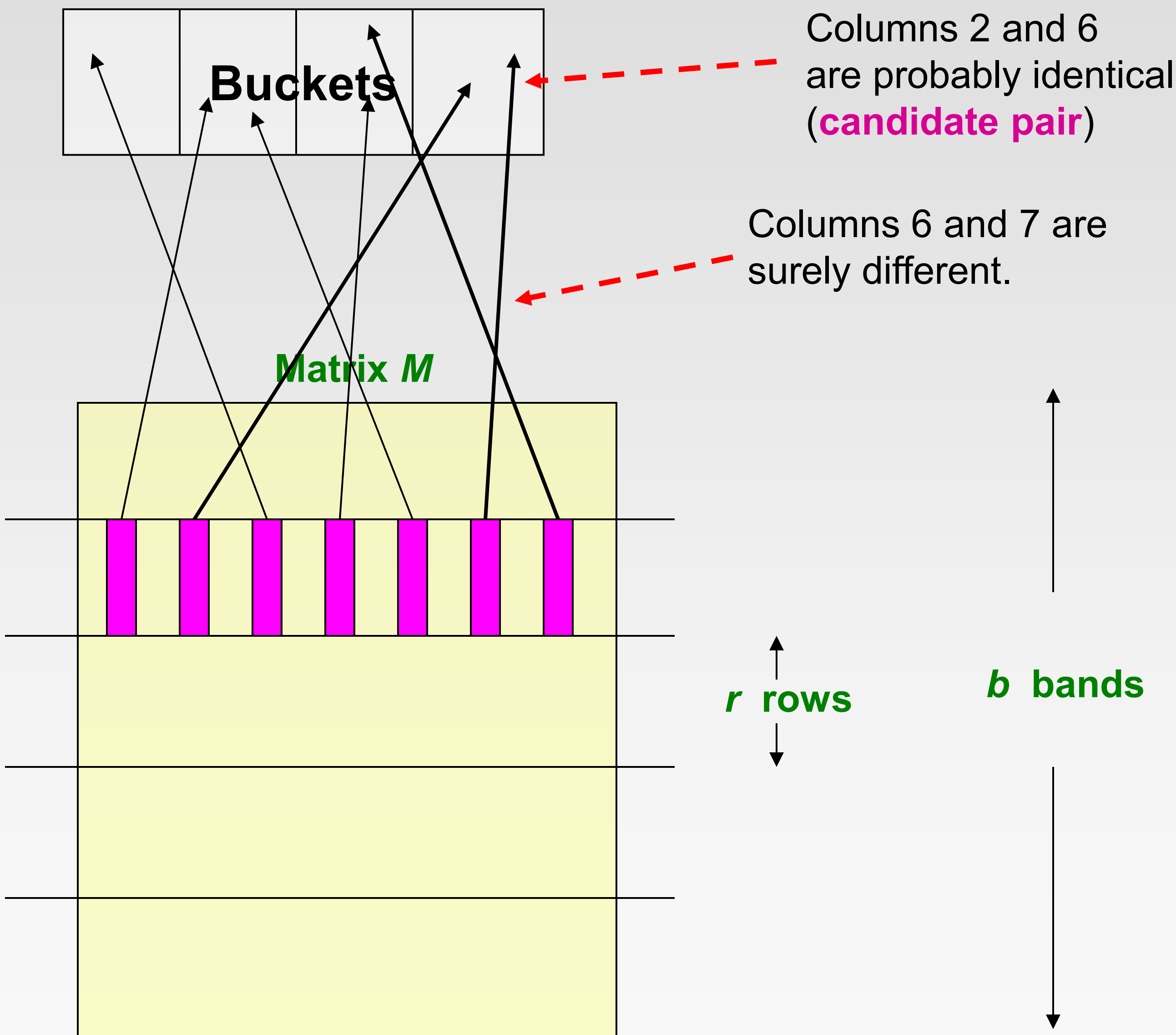
Partition *M* into *b* Bands



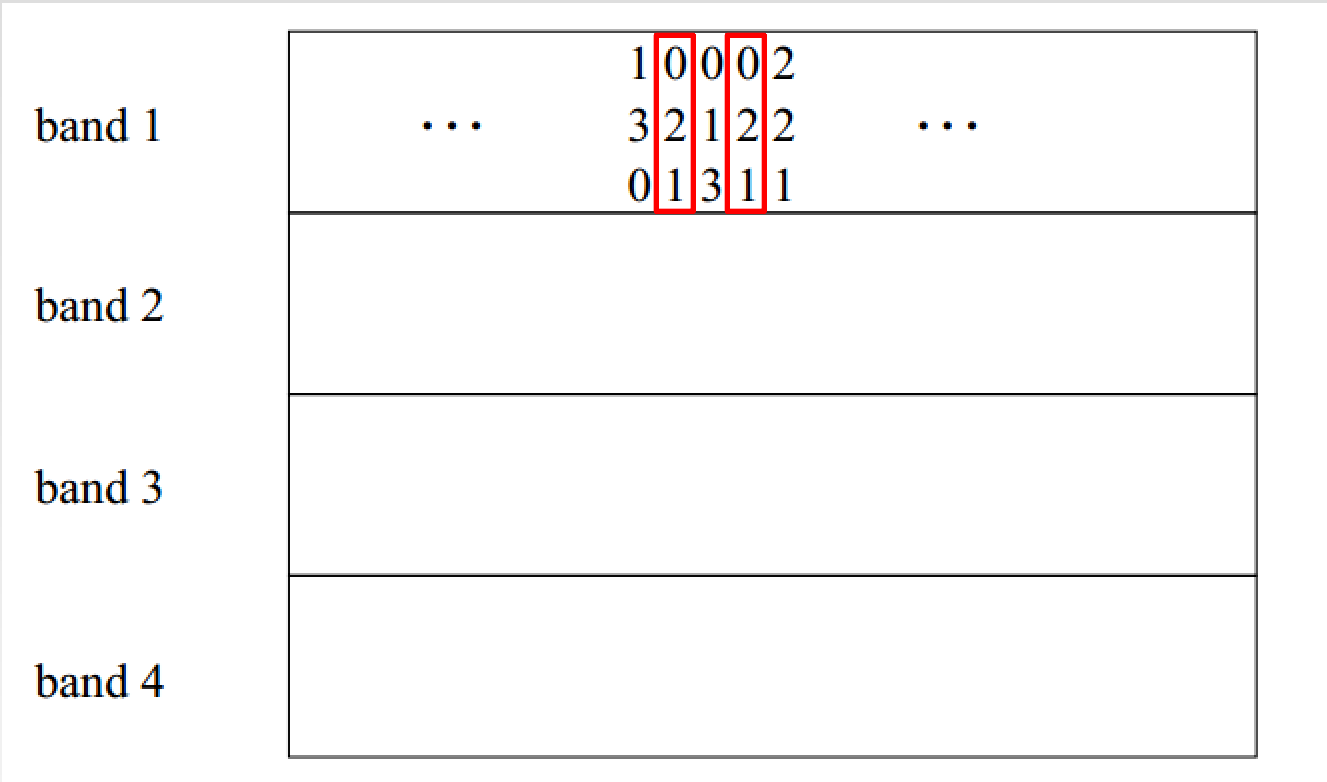
Partition M into Bands

- Divide matrix ***M*** into ***b*** bands of ***r*** rows
- For each band, hash its portion of each column to a hash table with ***k*** buckets
  - Make ***k*** as large as possible
- ***Candidate*** column pairs are those that hash to the same bucket for ≥ **1** band
- Tune ***b*** and ***r*** to catch most similar pairs, but few non-similar pairs

Hashing Bands



Hashing Bands



- Regardless of what those columns look like in the other three bands, this pair of columns will be a candidate pair
- Two columns that do not agree in band 1 have three other chances to become a candidate pair; they might be identical in any one of these other bands.

Simplifying Assumption

- There are **enough buckets** that columns are unlikely to hash to the same bucket unless they are **identical** in a particular band

- Hereafter, we assume that "**same bucket**" means "**identical in that band**"

- Assumption needed only to simplify analysis, not for correctness of algorithm

Example of Bands

Assume the following case:

- Suppose 100,000 columns of ***M*** (100k docs)
- Signatures of 100 integers (rows)
- Therefore, signatures take 40Mb
- Choose ***b*** = 20 bands of ***r*** = 5 integers/band

- **Goal:** Find pairs of documents that are at least ***s*** = *0.8* similar

C_1, C_2 are 80% Similar

- **Find pairs of** $\geq$ ***s***=0.8 similarity, set **b**=20, **r**=5
- **Assume:** sim(C_1, C_2) = 0.8
  - Since sim(C_1, C_2) $\geq$ **s**, we want C_1, C_2 to be a **candidate pair**: We want them to hash to at **least 1 common bucket** (at least one band is identical)
- **Probability C_1, C_2 identical in one particular band:** $(0.8)^5 = 0.328$
- Probability C_1, C_2 are ***not*** similar in all of the 20 bands: $(1-0.328)^{20} = 0.00035$
  - i.e., about 1/3000th of the 80%-similar column pairs are **false negatives** (we miss them)
  - **We would find 99.965% pairs of truly similar documents**

C_1, C_2 are 30% Similar

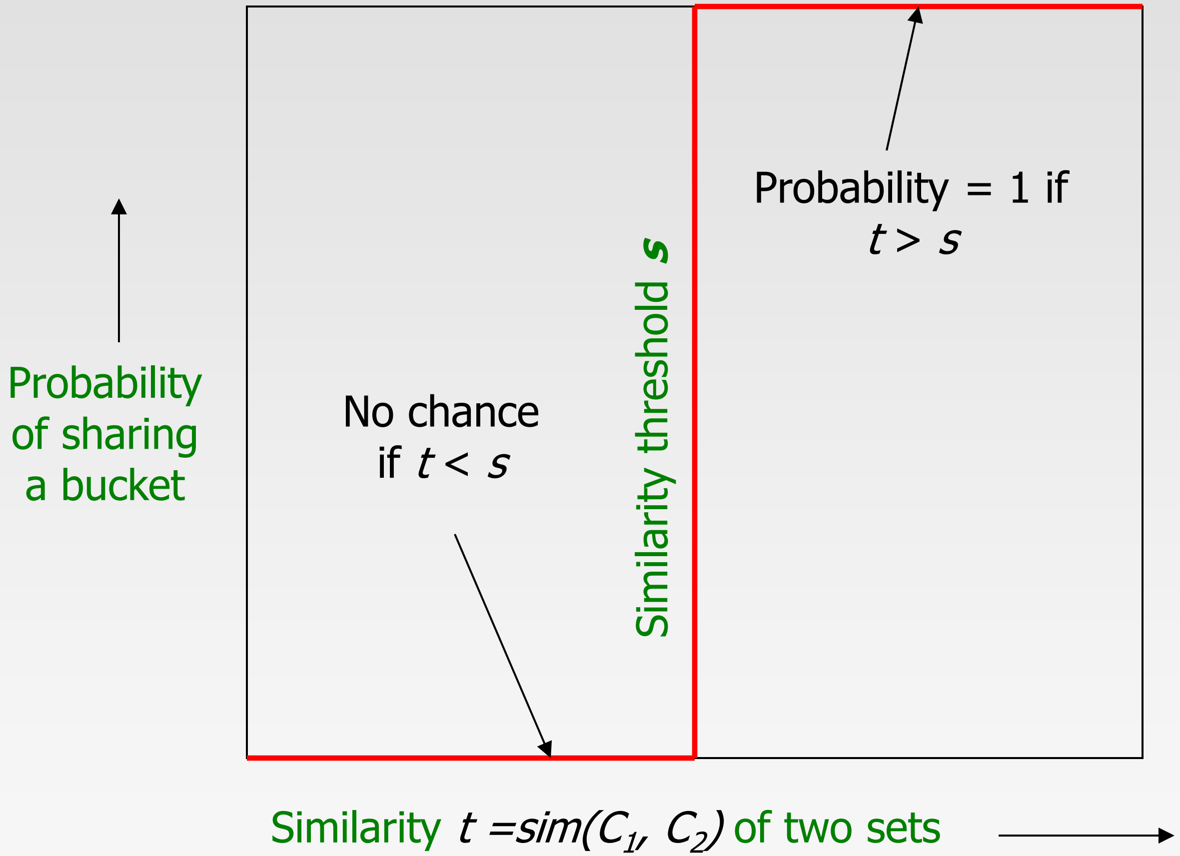
- **Find pairs of** ≥ ***s***=0.8 similarity, set **b**=20, **r**=5
- **Assume:** sim(C_1, C_2) = 0.3
  - Since sim(C_1, C_2) < **s** we want C_1, C_2 to hash to **NO common buckets** (all bands should be different)
- **Probability C_1, C_2 identical in one particular band:** $(0.3)^5 = 0.00243$
- Probability C_1, C_2 identical in at least 1 of 20 bands: $1 - (1 - 0.00243)^{20} = 0.0474$
  - In other words, approximately 4.74% pairs of docs with similarity 0.3% end up becoming **candidate pairs**
    - They are **false positives** since we will have to examine them (they are candidate pairs) but then it will turn out their similarity is below threshold **s**

LSH Involves a Tradeoff

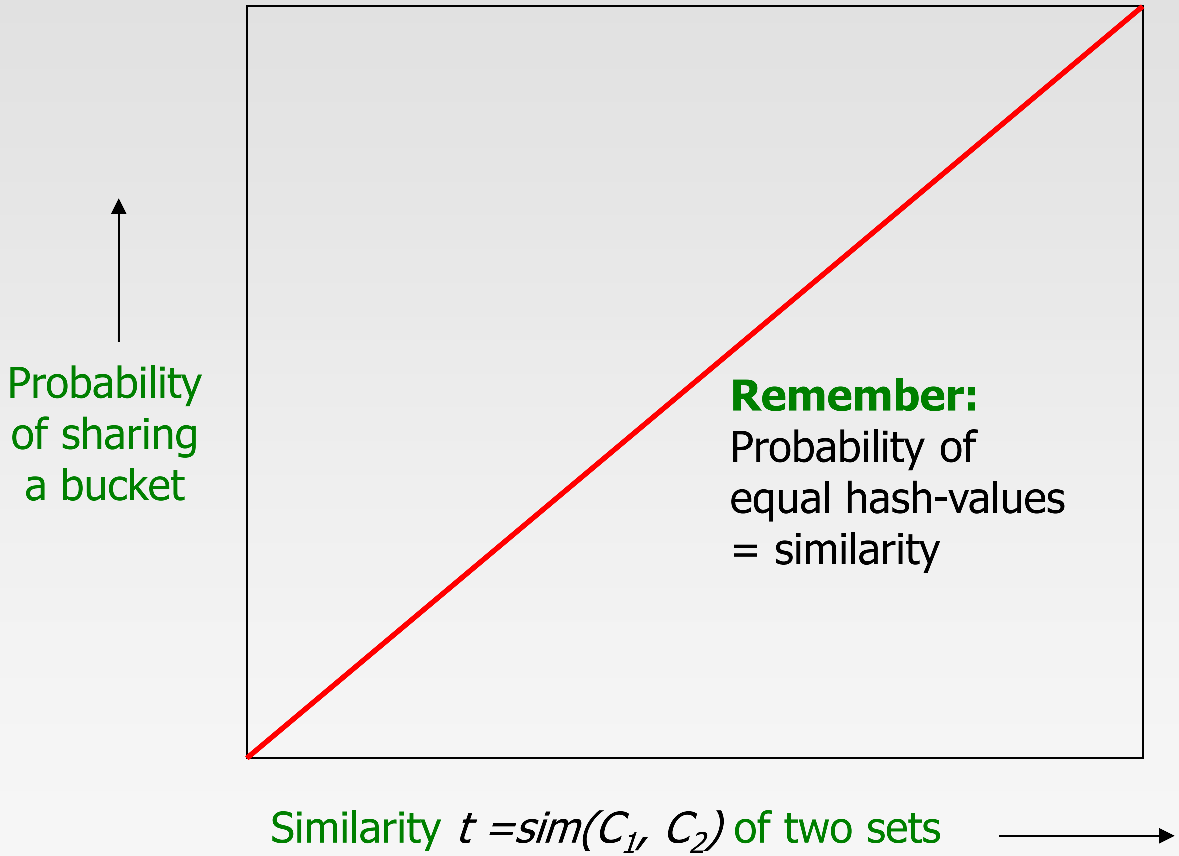
- **Pick:**
  - The number of Min-Hashes (rows of ***M***)
  - The number of bands ***b***, and
  - The number of rows ***r*** per band to balance false positives/negatives

- **Example:** If we had only 15 bands of 5 rows, the number of false positives would go down, but the number of false negatives would go up

Analysis of LSH – What We Want



What 1 Band of 1 Row Gives You



b bands, *r* rows/band

- The probability that the minhash signatures for the documents agree in any one particular row of the signature matrix is ***t*** ($sim(C_1, C_2)$)

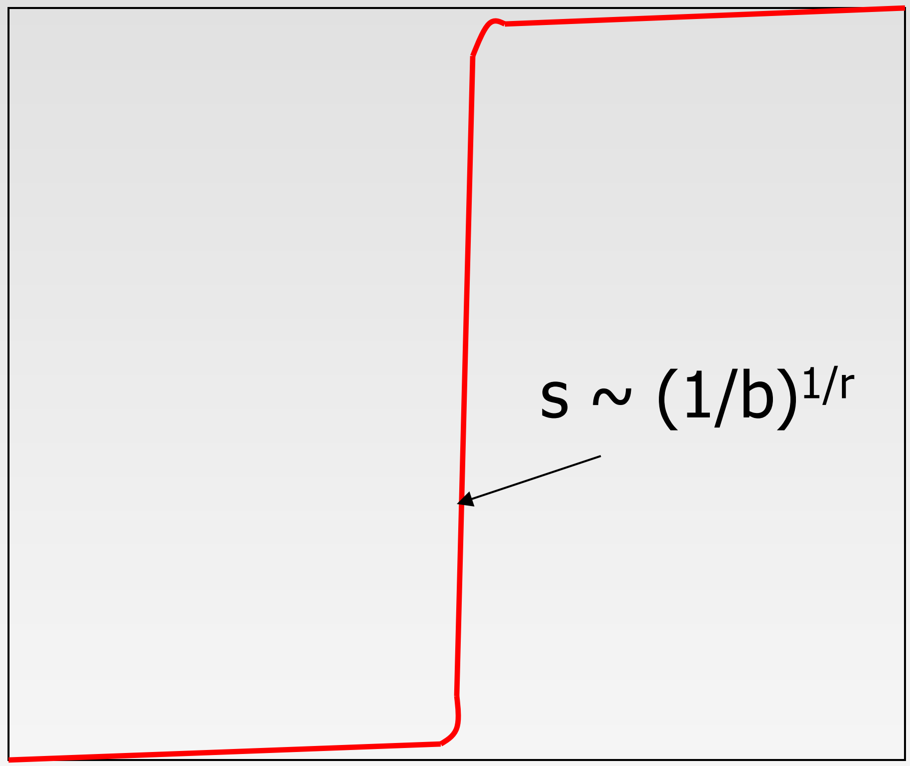
- Pick any band (***r*** rows)
  - Prob. that all rows in band equal = t^r
  - Prob. that some row in band unequal = $1 - t^r$

- Prob. that no band identical = $(1 - t^r)^b$

- Prob. that at least 1 band identical = $1 - (1 - t^r)^b$

What b Bands of r Rows Gives You

Probability of sharing a bucket



Similarity $t=sim(C_1, C_2)$ of two sets

At least one band identical

No bands identical

$$\mathbf{1 - (1 - t^r)^b}$$

Some row of a band unequal

All rows of a band are equal

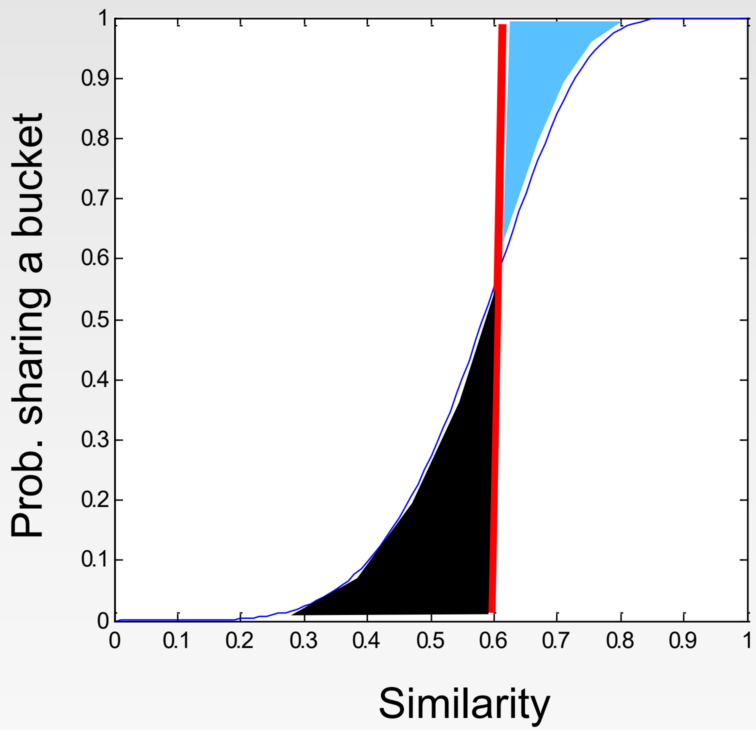
Example: $b = 20$, $r = 5$

- Similarity threshold s
- Prob. that at least 1 band is identical:

s	$1-(1-s^r)^b$
.2	.006
.3	.047
.4	.186
.5	.470
.6	.802
.7	.975
.8	.9996

Picking *r* and *b*: The S-curve

- **Picking *r* and *b* to get the best S-curve**
  - 50 hash-functions (r=5, b=10)



Blue area: False Negative rate
Black area: False Positive rate

LSH Summary

- Tune ***M, b, r*** to get almost all pairs with similar signatures, but eliminate most pairs that do not have similar signatures

- Check in main memory that **candidate pairs** really do have **similar signatures**

- **Optional:** In another pass through data, check that the remaining candidate pairs really represent similar documents

Summary: 3 Steps

- **Shingling:** Convert documents to sets
  - We used hashing to assign each shingle an ID
- **Min-Hashing:** Convert large sets to short signatures, while preserving similarity
  - We used **similarity preserving hashing** to generate signatures with property $\mathbf{Pr}[h_\pi(C_1) = h_\pi(C_2)] = sim(C_1, C_2)$
  - We used hashing to get around generating random permutations
- **Locality-Sensitive Hashing:** Focus on pairs of signatures likely to be from similar documents
  - We used hashing to find **candidate pairs** of similarity $\geq$ **s**

Distance Measures

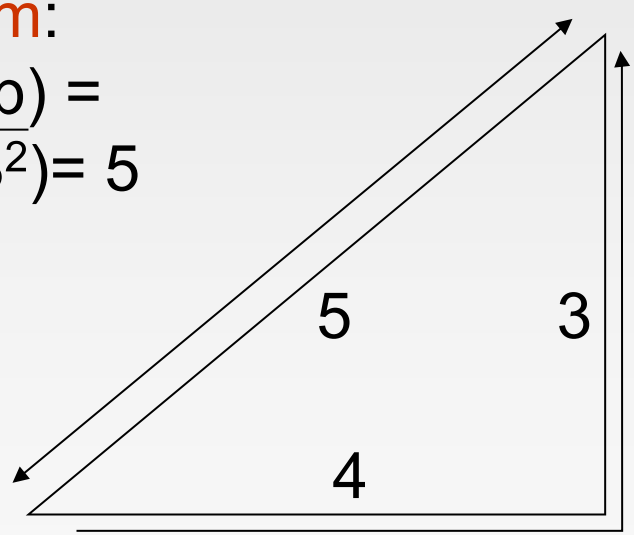
- Generalized LSH is based on some kind of "distance" between points.
  - Similar points are "close."
- Example: Jaccard similarity is not a distance; 1 minus Jaccard similarity is.
- *d* is a *distance measure* if it is a function from pairs of points to real numbers such that:
  1. $d(x,y) \geq 0$.
  2. $d(x,y) = 0$ iff $x = y$.
  3. $d(x,y) = d(y,x)$.
  4. $d(x,y) \leq d(x,z) + d(z,y)$ (*triangle inequality*).

Some Euclidean Distances

- *L_2 norm*: d(x,y) = square root of the sum of the squares of the differences between *x* and *y* in each dimension.
  - The most common notion of "distance."
- *L_1 norm*: sum of the differences in each dimension.
  - *Manhattan distance* = distance if you had to travel along coordinates only.

b = (9,8)

L_2-norm:
dist(a,b) =
$\sqrt{(4^2+3^2)} = 5$



a = (5,5)

L_1-norm:
dist(a,b) =4+3 = 7

Some Non-Euclidean Distances

- *Jaccard distance* for sets = 1 minus Jaccard similarity.
- *Cosine distance* for vectors = angle between the vectors.
- *Edit distance* for strings = number of inserts and deletes to change one string into another.

Cosine Distance

- Think of a point as a vector from the origin [0,0,…,0] to its location.
- Two points' vectors make an angle, whose cosine is the normalized dot-product of the vectors: $p_1.p_2/|p_2||p_1|$.
  - Example: p_1 = [1,0,2,-2,0]; p_2 = [0,0,3,0,0].
  - $p_1.p_2 = 6$; $|p_1| = |p_2| = \sqrt{9} = 3$.
  - $\cos(\theta) = 6/9$; θ is about 48 degrees.

Edit Distance

- The *edit distance* of two strings is the number of inserts and deletes of characters needed to turn one into the other.
- An equivalent definition: d(x,y) = |x| + |y| - 2|LCS(x,y)|.
  - LCS = *longest common subsequence* = any longest string obtained both by deleting from *x* and deleting from *y*.
- Example:
  - *x* = *abcde* ; *y* = *bcduve*.
  - Turn *x* into *y* by deleting *a*, then inserting *u* and *v* after *d*.
    - Edit distance = 3.
  - Or, computing edit distance through the LCS, note that LCS(x,y) = *bcde*.
  - Then:|x| + |y| - 2|LCS(x,y)| = 5 + 6 –2*4 = 3 = edit distance.

Hash Functions Decide Equality

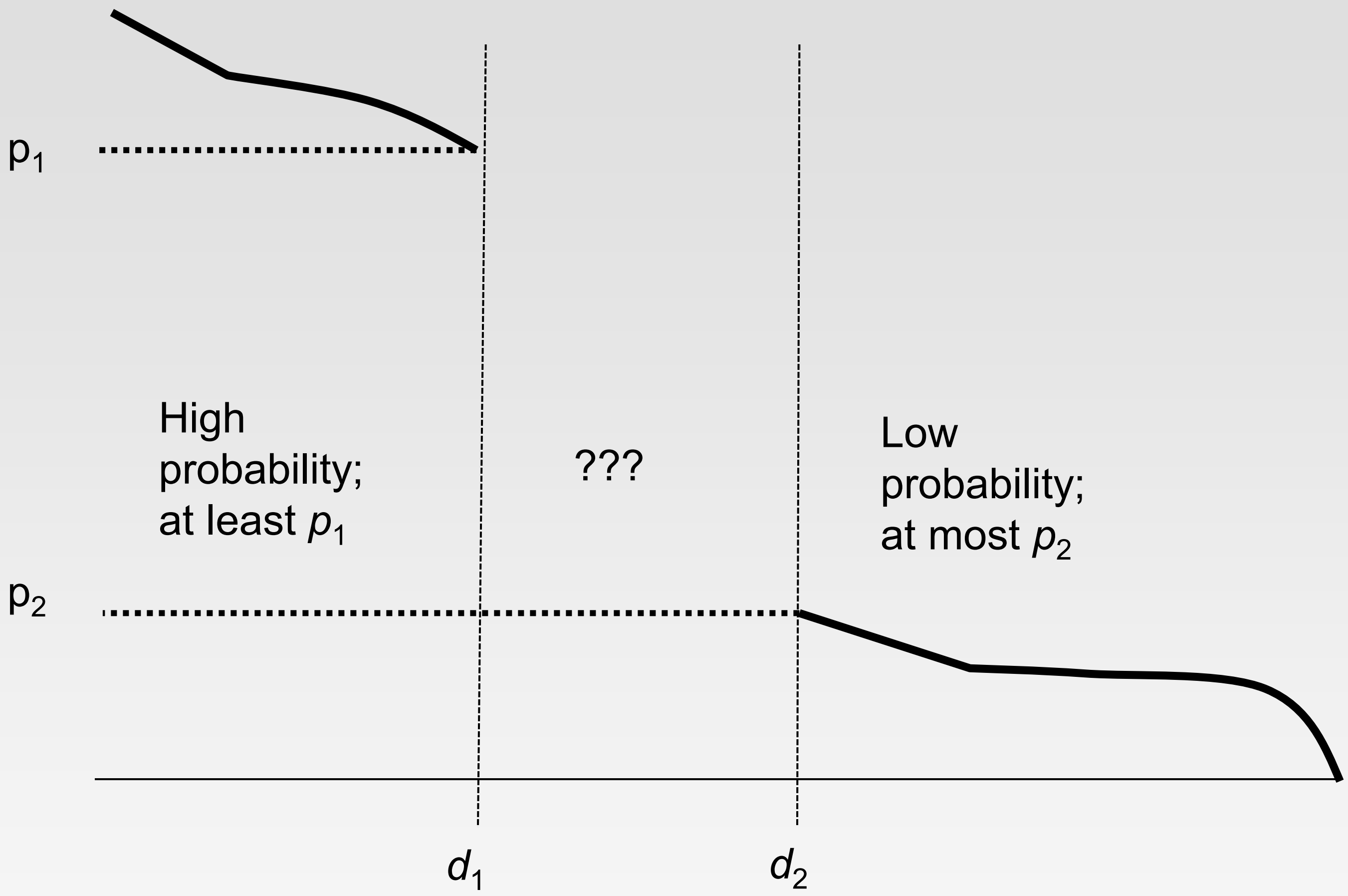
- There is a subtlety about what a "hash function" is, in the context of LSH families.
- A hash function h really takes two elements x and y, and returns a decision whether x and y are candidates for comparison.
- Example: the family of minhash functions computes minhash values and says "yes" iff they are the same.
- Shorthand: "h(x) = h(y)" means h says "yes" for pair of elements x and y.

LSH Families Defined

- Suppose we have a space *S* of points with a distance measure *d*.
- A family **H** of hash functions is said to be *(d_1,d_2,p_1,p_2)-sensitive* if for any *x* and *y* in *S*:

1. If $d(x,y) \le d_1$, then the probability over all *h* in **H**, that $h(x) = h(y)$ is at least p_1.
2. If $d(x,y) \ge d_2$, then the probability over all *h* in **H**, that $h(x) = h(y)$ is at most p_2.

LSH Families: Illustration



Example: LSH Family – (2)

- **Claim**: **H** is a ($\boxed{1/3}$, $\boxed{3/4}$, $\boxed{2/3}$, $\boxed{1/4}$)-sensitive family for *S* and *d*.

If distance $\geq$ 3/4
(so similarity $\leq$ 1/4)

Then probability
that minhash values
agree is $\leq$ 1/4

If distance $\leq$ 1/3
(so similarity $\geq$ 2/3)

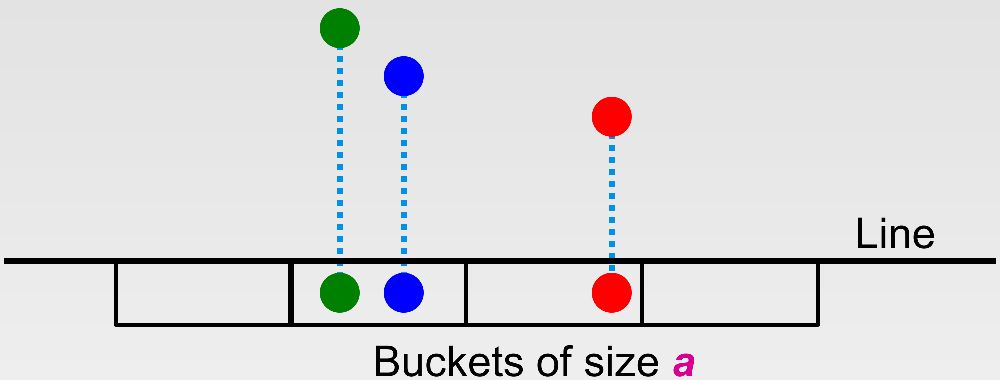
Then probability
that minhash values
agree is $\geq$ 2/3

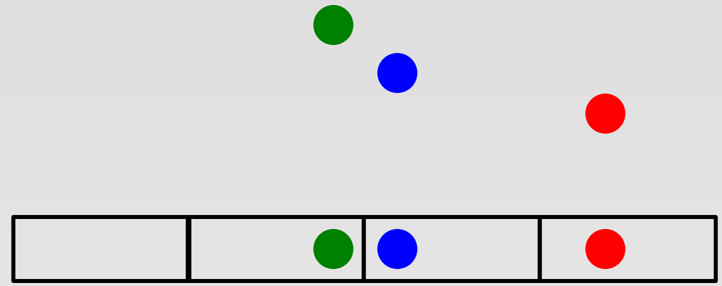
For Jaccard similarity, minhashing gives us a
$(d_1,d_2,(1-d_1),(1-d_2))$-sensitive family for any $d_1 < d_2$.

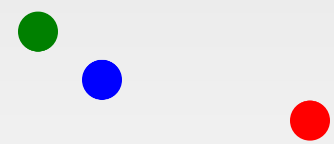
LSH for Euclidean Distance

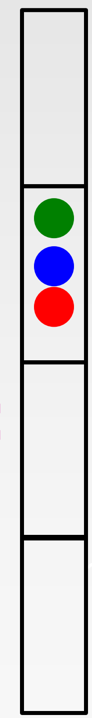
- **Idea: Hash functions correspond to lines**
- Partition the line into buckets of size ***a***
- **Hash each point to the bucket containing its projection onto the line**
  - An element of the "Signature" is a bucket id for that given projection line
- **Nearby points are always close**; distant points are rarely in same bucket

Projection of Points



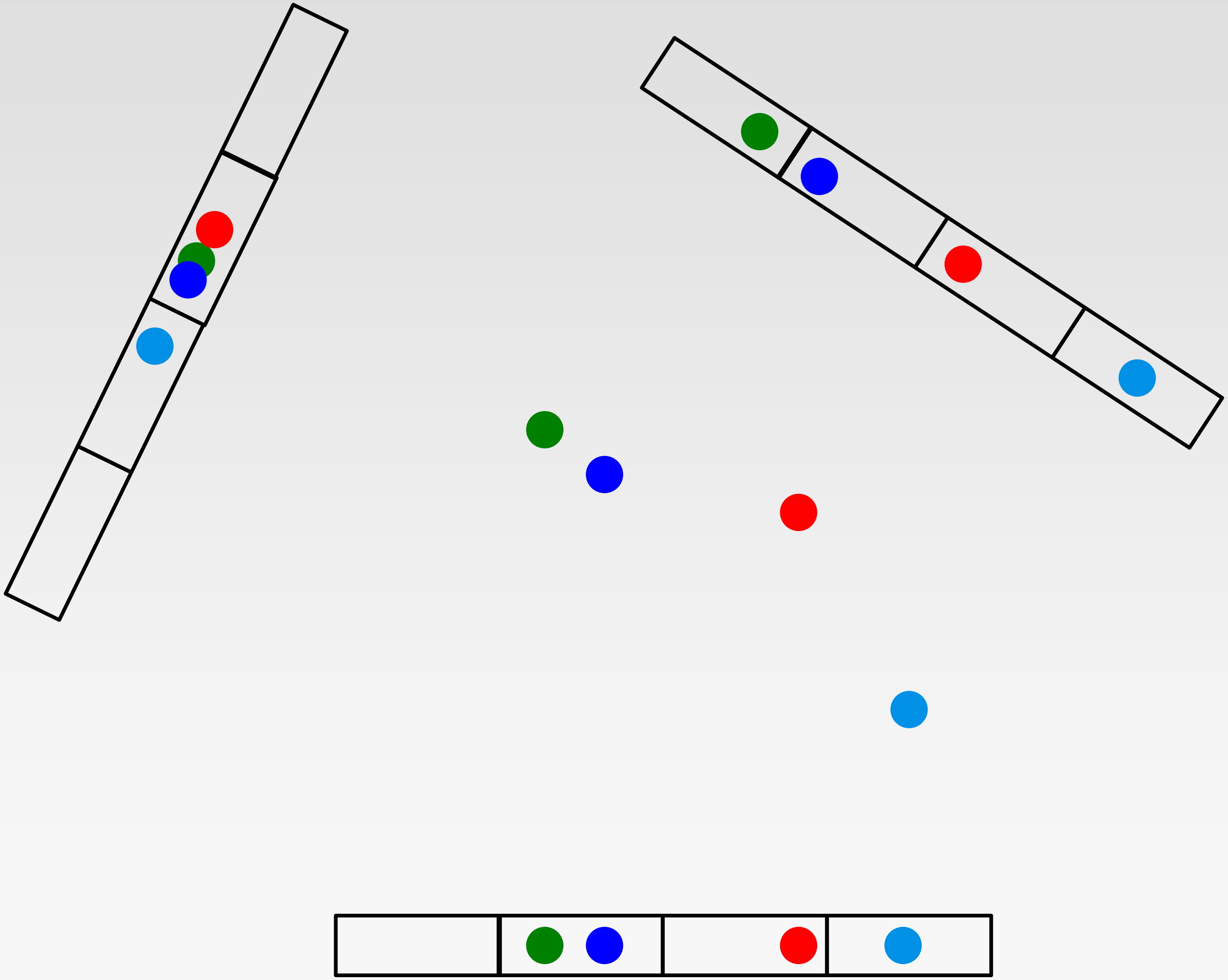




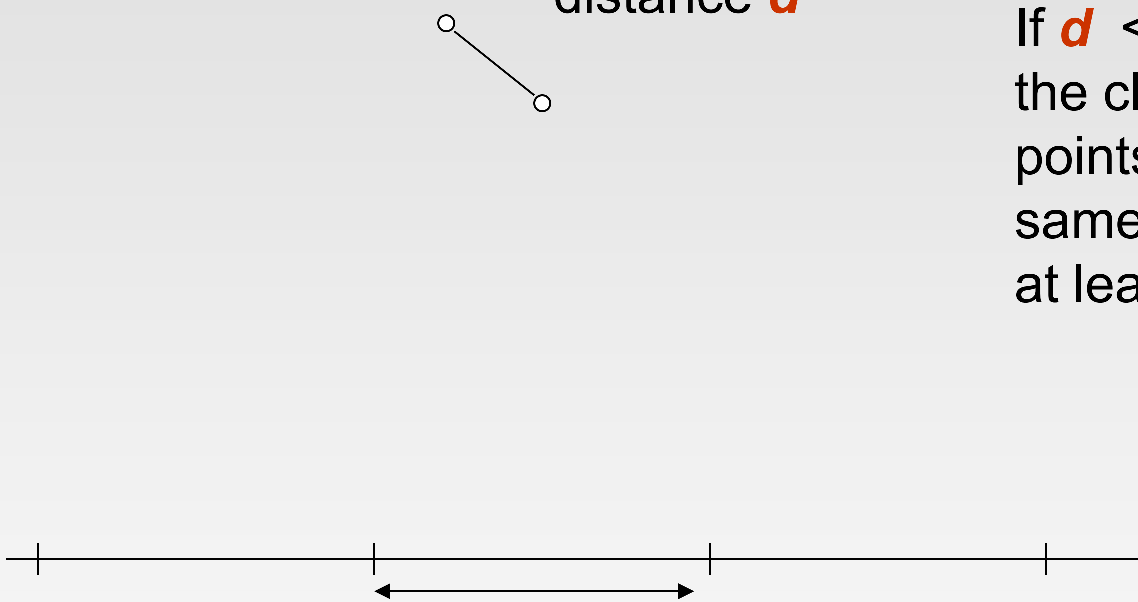


- **"Lucky" case:**
  - Points that are close hash in the same bucket
  - Distant points end up in different buckets
- **Two "unlucky" cases:**
  - **Top:** unlucky quantization
  - **Bottom:** unlucky projection

Multiple Projections



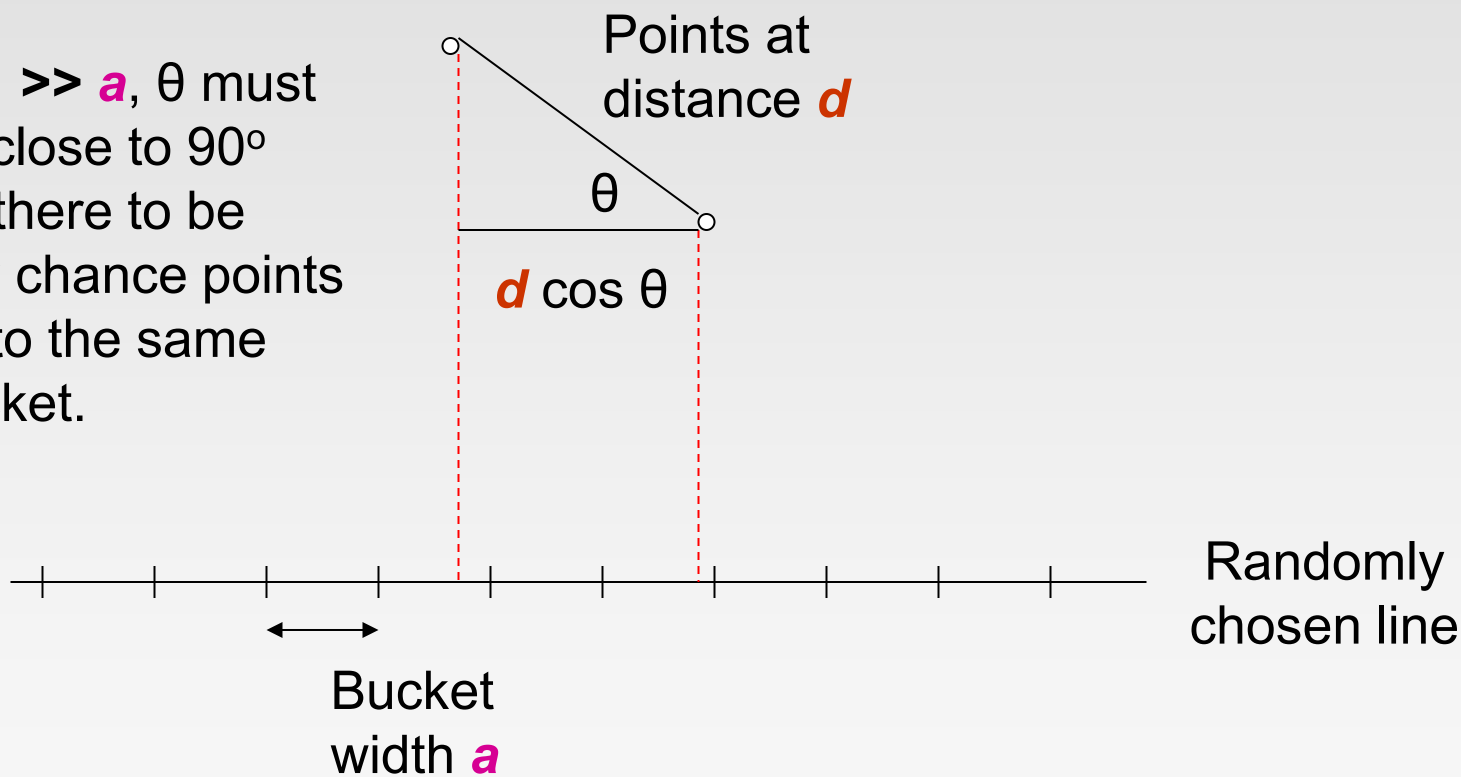
Projection of Points



If $d \ll a$, then the chance the points are in the same bucket is at least $\mathbf{1} - d/a$.

Projection of Points

If ***d*** **>>** ***a***, θ must be close to 90° for there to be any chance points go to the same bucket.



An LS-Family for Euclidean Distance

- If points are distance $d \leq a/2$, prob, they are in same bucket $\geq 1 - d/a = ½$

- If points are distance $d \geq 2a$ apart, then they can be in the same bucket only if $d \cos\theta \leq a$
  - $\cos\theta \leq ½$
  - $60 \leq \theta \leq 90$, i.e., at most 1/3 probability

- Yields a ***(a/2, 2a, 1/2, 1/3)***-*sensitive* family of hash functions for any ***a***

References

- Chapter 3 of Mining of Massive Datasets.

End of Chapter 7.2