Tutorial 2 Solutions: Set Theory and Formal Languages

Set Notation and Concepts

Concept(s)

A set is a collection of objects. Let A, B, C be sets and x be an object.

x is an element of A	$x \in A$	x is in A
A is a subset of B	$A \subseteq B$	If $x \in A$ then $x \in B$
A is a proper subset of B	$A \subset B$	$A \subseteq B$ and $A \neq B$
A is equal to B	$A = B$	$A \subseteq B$ and $B \subseteq A$
Empty Set	$\varnothing$, $\{\}$	Set containing nothing
Universe	$\mathcal{U}$	Set containing all possible elements

Exercise 1. Let $A = \{1, 2, 3, 4\}$, $B = \{2, 4\}$, and $C = \{1, \{1\}, 2\}$.

Determine whether the following statements are true or false:

$$2 \in A, \quad \{1\} \in C, \quad \{1\} \subset C, \quad A \subseteq B, \quad B \subseteq A, \quad \varnothing \subseteq C.$$

Answer(s)

The true statements are: $2 \in A$, $\{1\} \in C$, $\{1\} \subset C$, $B \subseteq A$ and $\varnothing \subseteq C$.

Set Operations

Concept(s)

Union of A and B	$A \cup B = \{x : x \in A$ or $x \in B\}$
Intersection of A and B	$A \cap B = \{x : x \in A$ and $x \in B\}$
Complement of A	$A^c = \{x : x \notin A$ and $x \in \mathcal{U}\}$
A but not B	$A \setminus B = A \cap (B^c)$
Symmetric Difference of A and B	$A \oplus B = (A \setminus B) \cup (B \setminus A)$

Exercise 2. Prove that for any sets A and B, if $A \subseteq B$, then $A \cap B = A$.

Answer(s)

Let A and B be sets such that $A \subseteq B$. To prove that $A \cap B = A$, we need to prove two statements: $A \cap B \subseteq A$ and $A \subseteq A \cap B$.

Proof of $A \cap B \subseteq A$: Let x be an element of $A \cap B$. Then, by definition, $x \in A$. Therefore, $A \cap B \subseteq A$.

Proof of $A \subseteq A \cap B$: Let x be an element of A. Since $A \subseteq B$, by definition, we have $x \in B$. This means that $x \in A \cap B$. Therefore $A \subseteq A \cap B$.

Exercise 3. Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ and $\mathcal{U} = \{1, 2, 3, 4, 5\}$. Calculate:

$$A \cup B, \quad (A^c) \cap B, \quad A \oplus B, \quad (A \cup B) \setminus (A \cap B), \quad (A \setminus B) \cap (B \setminus A).$$

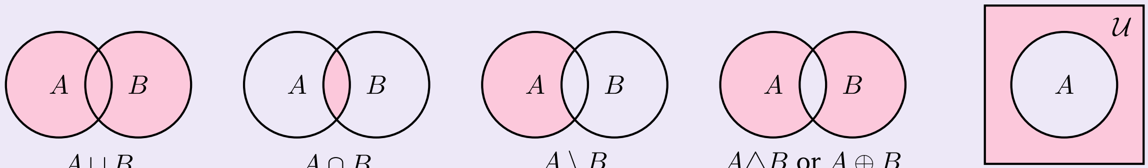
Answer(s)

$$\begin{aligned} A \cup B &= \{1, 2, 3, 4\}, \\ (A^c) \cap B &= \{4\}, \\ A \oplus B &= \{1, 4\}, \\ (A \cup B) \setminus (A \cap B) &= \{1, 4\}, \\ (A \setminus B) \cap (B \setminus A) &= \{\}. \end{aligned}$$

Venn Diagram

Concept(s)

A Venn diagram uses overlapping circles to represent sets and their relationships.



$A \cup B$ $\quad$ $A \cap B$ $\quad$ $A \setminus B$ $\quad$ $A \triangle B$ or $A \oplus B$ $\quad$ A^c

Exercise 4. For any sets A, B, and C, prove or disprove:

a) $A \cap (B \cup C) = (A \cap B) \cup C$

b) $(A \setminus B) \setminus C = A \setminus (B \setminus C)$

c) $A \setminus (B \cap C) = (A \setminus B) \cap (A \setminus C)$

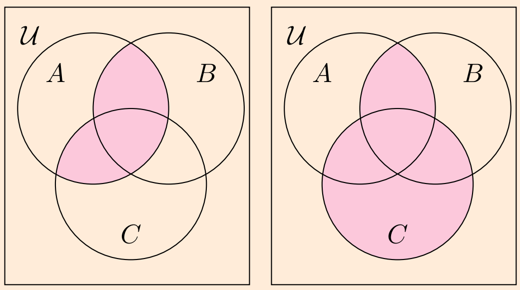
Hint: Use Venn diagrams to help you out!

Answer(s)

We can draw the Venn diagrams for the left-hand side and right-hand side to quickly check whether the equation is true or not. This will also help us come up with an example to disprove the equation.

a) The Venn diagrams are as follows:

$A \cap (B \cup C)$ $\quad$ $(A \cap B) \cup C$



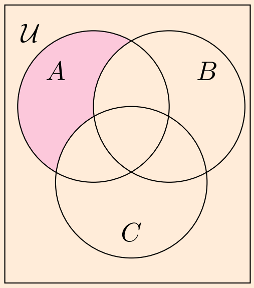
These Venn diagrams are different so the equation must not be true. We can create an example with different outcomes by having an element in C that is not in A or B. Consider $A = \varnothing$, $B = \varnothing$ and $C = \{1\}$. We have

$$A \cap (B \cup C) = \varnothing \text{ and } (A \cap B) \cup C = \{1\}$$

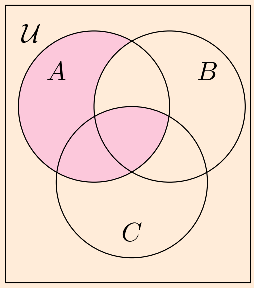
so we can conclude that $A \cap (B \cup C) \neq (A \cap B) \cup C$.

b) The Venn diagrams are as follows:

$(A \setminus B) \setminus C$



$A \setminus (B \setminus C)$



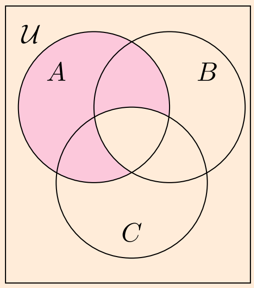
These Venn diagrams are different so the equation must not be true. We can create an example with different outcomes by having an element in A and C that is not in B . Consider $A = \{1\}$, $B = \varnothing$ and $C = \{1\}$. We have

$$(A \setminus B) \setminus C = \varnothing \text{ and } A \setminus (B \setminus C) = \{1\}$$

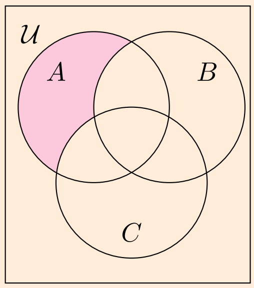
so we can conclude that $(A \setminus B) \setminus C \neq A \setminus (B \setminus C)$.

c) The Venn diagrams are as follows:

$A \setminus (B \cap C)$



$(A \setminus B) \cap (A \setminus C)$



These Venn diagrams are different so the equation must not be true. We can create an example with different outcomes by having an element in A and B that is not in C . Consider $A = \{1\}$, $B = \{1\}$ and $C = \varnothing$. We have

$$A \setminus (B \cap C) = \{1\} \text{ and } (A \setminus B) \cap (A \setminus C) = \varnothing$$

so we can conclude that $A \setminus (B \cap C) \neq (A \setminus B) \cap (A \setminus C)$.

Power Sets and Cardinality

Concept(s)

Power Set of A	$\mathrm{Pow}(A) = \{X : X \subseteq A\}$
Cardinality of A	$\|A\|$ is the number of elements in A
Cartesian Product of A and B	$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

Exercise 5. Let $A = \{1, 2\}$. Calculate:

$$\mathrm{Pow}(A), \quad |\mathrm{Pow}(A)|, \quad A \times A, \quad |A \times A|.$$

Answer(s)

$$\begin{aligned}
\mathrm{Pow}(A) &= \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}, \\
|\mathrm{Pow}(A)| &= 4, \\
A \times A &= \{(1, 1), (1, 2), (2, 1), (2, 2)\}, \\
|A \times A| &= 4.
\end{aligned}$$

Exercise 6. Let A be a finite set with n elements. Prove that $|\mathrm{Pow}(A)| = 2^n$.

Answer(s)

Let $a_1, a_2, a_3, \ldots, a_n$ be the elements of A. To form a subset, we first choose whether to include a_1 or not. We then choose whether to include a_2 or not. We can do this for the remaining elements $a_3, \ldots, a_n$. For each of the n elements, we have two possibilities. This gives us

$$2 \times 2 \times \cdots \times 2 = 2^n$$

possible ways to create a subset. The power set is the set containing all subsets so the power set of A must have 2^n elements.

Exercise 7. For sets A and B, prove or disprove $\mathrm{Pow}(A \cup B) = \mathrm{Pow}(A) \cup \mathrm{Pow}(B)$

Answer(s)

Consider $A = \{1\}$, $B = \{2\}$, where

$$\mathrm{Pow}(A \cup B) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\} \text{ and } \mathrm{Pow}(A) \cup \mathrm{Pow}(B) = \{\varnothing, \{1\}, \{2\}\}.$$

This disproves our claim.

Set Theory Laws

Concept(s)

Commutativity	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Distribution	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity	$A \cup \varnothing = A$	$A \cap \mathcal{U} = A$
Complement	$A \cup (A^c) = \mathcal{U}$	$A \cap (A^c) = \varnothing$
Idempotence	$A \cup A = A$	$A \cap A = A$
Double Complement	$(A^c)^c = A$	
Annihilation	$A \cup \mathcal{U} = \mathcal{U}$	$A \cap \varnothing = \varnothing$
de Morgan's	$(A \cap B)^c = (A^c) \cup (B^c)$	$(A \cup B)^c = (A^c) \cap (B^c)$

The principle of duality states that any law still holds true when you swap $\cap$ with $\cup$ and $\varnothing$ with $\mathcal{U}$.

Exercise 8. Prove that for any sets A and B, $A \setminus (A \cap B) = A \setminus B$.

Answer(s)

$$\begin{aligned}
A \setminus (A \cap B) &= A \cap (A \cap B)^c && \text{(By definition)}\\
&= A \cap (A^c \cup B^c) && \text{(de Morgan's Law)}\\
&= (A \cap A^c) \cup (A \cap B^c) && \text{(Distributive Law)}\\
&= \varnothing \cup (A \cap B^c) && \text{(Complement Law)}\\
&= A \cap B^c && \text{(Identity Law)}\\
&= A \setminus B && \text{(By definition)}
\end{aligned}$$

Exercise 9. Prove or disprove that for any sets A, B, and C, $((A \setminus B) \cap (B \setminus C)) \cap (C \setminus A) = \varnothing$.

Answer(s)

The key is to realise that $A \setminus B$ contains B^c and $B \setminus A$ contains B.

$$\begin{aligned}
((A \setminus B) \cap (B \setminus C)) \cap (C \setminus A) &= ((A \cap B^c) \cap (B \setminus C)) \cap (C \setminus A) && \text{(By definition)}\\
&= ((A \cap B^c) \cap (B \cap C^c)) \cap (C \setminus A) && \text{(By definition)}\\
&= (((A \cap B^c) \cap B) \cap C^c) \cap (C \setminus A) && \text{(Associativity)}\\
&= ((A \cap (B^c \cap B)) \cap C^c) \cap (C \setminus A) && \text{(Associativity)}\\
&= ((A \cap (B \cap B^c)) \cap C^c) \cap (C \setminus A) && \text{(Commutativity)}\\
&= ((A \cap \varnothing) \cap C^c) \cap (C \setminus A) && \text{(Complement Law)}\\
&= (\varnothing \cap C^c) \cap (C \setminus A) && \text{(Annihilation Law)}\\
&= \varnothing \cap (C \setminus A) && \text{(Annihilation Law)}\\
&= \varnothing && \text{(Annihilation Law)}
\end{aligned}$$

Exercise 10. Simplify the expression $[A \cap (A \cap B^c)] \cup [(A \cap B) \cup (B \cap A^c)]$.

Answer(s)

$$
\begin{aligned}
& [A \cap (A \cap B^c)] \cup [(A \cap B) \cup (B \cap A^c)] & \\
&= [A \cap (A \cap B^c)] \cup [(B \cap A) \cup (B \cap A^c)] & \text{(Commutative Law)}\\
&= [A \cap (A \cap B^c)] \cup [B \cap (A \cup A^c)] & \text{(Distributive Law)}\\
&= [A \cap (A \cap B^c)] \cup [B \cap \mathcal{U}] & \text{(Complement Law)}\\
&= [A \cap (A \cap B^c)] \cup B & \text{(Identity Law)}\\
&= [(A \cap A) \cap B^c] \cup B & \text{(Associativity)}\\
&= [A \cap B^c] \cup B & \text{(Indempotence)}\\
&= B \cup [A \cap B^c] & \text{(Commutative Law)}\\
&= (B \cup A) \cap (B \cup B^c) & \text{(Distributive Law)}\\
&= (B \cup A) \cap \mathcal{U} & \text{(Complement Law)}\\
&= B \cup A & \text{(Identity Law)}
\end{aligned}
$$

Formal Languages

Concept(s)

Terminology for Formal Languages:

Alphabet (Σ)	A finite, non-empty set of symbols
Word	A finite sequence of symbols from Σ
Empty Word (λ)	The word with no symbols
Σ^*	The set of all finite words
Language	A subset of Σ^*

Operations on Formal Languages: Let A, B be languages.

Concatenation	$AB = \{ab : a \in A \text{ and } b \in B\}$
Concatenation Notation	$A^0 = \{\lambda\}$, $A^{i+1} = AA^i$
Kleene Star	$A^* = A^0 \cup A^1 \cup A^2 \cup \ldots$

Since languages are defined as sets, the normal set operations also apply.

Exercise 11. Let $A = \{ab, ba\}$. Calculate A^0, A^1 and A^2.

Answer(s)

$$A^0 = \{\lambda\}, A^1 = \{ab, ba\}, A^2 = \{abab, abba, baab, baba\}$$

Exercise 12. Let A, B, C be languages with alphabet $\{a, b\}$ where $AC = BC$. Prove or disprove $A = B$?

Answer(s)

Consider the language $X = \{x\}$. Let $C = X^* = \{\lambda, x, xx, xxx, \dots\}$. Now consider when $A = \{\lambda\}$ and $B = \{\lambda, x\}$. We see that

$$AC = \{ab : a \in A \text{ and } b \in C\}.$$

Since λ is the only word in A, we find that

$$AC = \{\lambda\lambda, \lambda x, \lambda xx, \dots\} = \{\lambda, x, xx, \dots\} = C.$$

Now, consider that

$$BC = \{ab : a \in B \text{ and } b \in C\}.$$

The words in BC will be λw or xw for a word $w \in C$. This means that

$$BC = \{\lambda\lambda, \lambda x, \lambda xx, \dots\} \cup \{x\lambda, xx, xxx, \dots\} = C \cup \{x, xx, xxx, \dots\} = C,$$

as every word in $\{x, xx, xxx, \dots\}$ is also in C. Therefore, $AC = BC$ and $A \neq B$, which disproves our claim.

Exercise 13. Let L_1 and L_2 be languages over Σ. Prove that $(L_1 \cap L_2)^* \subseteq L_1^* \cap L_2^*$.

Answer(s)

Let w be a word in $(L_1 \cap L_2)^*$. Then

$$w \in (L_1 \cap L_2)^0 \cup (L_1 \cap L_2)^1 \cup (L_1 \cap L_2)^2 \cup \dots .$$

This means that there exists $n \in \mathbb{N}$ such that $w \in (L_1 \cap L_2)^n$. We can rewrite w as

$$w = w_1 w_2 \dots w_n,$$

where $w_i \in L_1 \cap L_2$ for all $1 \leq i \leq n$. This means that $w_i \in L_1$ and so $w \in L_1^n$. Since $L_1^n \subseteq L_1^*$, by definition, we have $w \in L_1^*$. We can use this same argument to show that L_2^*. Therefore $w \in L_1^* \cap L_2^*$ which proves our claim that $(L_1 \cap L_2)^* \subseteq L_1^* \cap L_2^*$.

Exercise 14. We define a^n and b^n as n consecutive occurrences of the symbol a or b, respectively. Let L_1 and L_2 be languages over $\Sigma = \{a, b\}$ defined as

$$L_1 = \{a^n b^{n+2} : n \geq 0 \text{ and } n \text{ is a perfect square}\} \quad L_2 = \{b^{3n+1} a^{n+1} : n \geq 0\}$$

We define the language $L_3 = L_1 L_2$.

a) Give three examples of strings in L_1 and explain why they are in L_1.

b) Give three examples of strings in L_2 and explain why they are in L_2.

c) Determine which of the following strings are in L_3 and prove your answers:

- $aaaabbbbbbba$
- $aabbbbaaa$
- $aaaaaabbbbbbbbbbbbbbaaa$
- $abbbbbbbaa$

d) Describe the general form of strings in L_3.

Answer(s)

a) bb, $abbb$, $aaaabbbbbb$

b) ba, $bbbbaa$, $bbbbbbbaaa$

c)
- We find that $aaaabbbbbbba$ is in L_3. It is comprised of $aaaabbbbbb$ which is in L_1 and ba which is in L_2.
- For words in L_3, we can split it into 3 sections: some (possibly zero) amount of a's, then some amount of b's and some amount of a's. This is because words in L_3 are made of a word in L_1 (which has the form $a^n b^{n+2}$) + a word in L_2 (which has the form $b^{3n+1} a^{n+1}$). The first section of a's is always attributed to a word in L_2. Since there is no word with $n = 2$ in L_1, we cannot have any word in L_3 start with aa.
- By a similar logic to the previous string, we cannot have a string in L_3 that has 6 a's in its first section.
- We find that $abbbbbbbaa$ is in L_3. It is comprised of $abbb$ which is in L_1 and $bbbbaa$ which is in L_2.

d) $L_3 = \{a^n b^{n+2+3m+1} a^{m+1} : n \geq 0, n \text{ is a perfect square and } m \geq 0\}$

Extra Practice Problems

Note(s)

These practice questions are designed to help deepen your understanding. No answers will be provided, as the goal is to encourage independent problem-solving and reinforce key concepts.

1. Which of the following statements are true? Let $S = \{a, \{a\}, \{\{a\}\}\}$

$$\varnothing \in S, \quad a \in S, \quad \{a\} \in S, \quad \varnothing \subset S, \quad a \subset S, \quad \{a\} \subset S, \quad \varnothing \subseteq S, \quad a \subseteq S, \quad \{a\} \subseteq S.$$

2. Let $A = \{7, 8, 9\}$, $B = \{6, 9, -1\}$ and $\mathcal{U} = \{-1, 2, 3, 6, 7, 8, 9\}$. Calculate the following:

$$(A \cup B)^c, \quad (A \setminus B) \cap A, \quad \text{Pow}(A), \quad A \times B, \quad (A^c) \cap (B^c), \quad A \oplus (B^c).$$

3. Show that when $R \subseteq S$ and $R \subseteq T$, we have $R \subseteq S \cap T$.
4. Prove or disprove $A \cap \varnothing = A$ for all sets A.
5. Suppose that $S \cup T = S \cap T$. Show that $S = T$.
6. Give an example that disproves $A \setminus (B \cup C) = (A \setminus B) \cup (A \setminus C)$, where A, B, C are sets.
7. Suppose that $S \oplus T = T$. Is $S = \varnothing$? Explain your answer.
8. Let $S = \{0, 1\}$. Find $\text{Pow}(\text{Pow}(S))$.
9. For sets A, B, show that if $A \subseteq B$ then $\text{Pow}(A) \subseteq \text{Pow}(B)$.
10. Simplify $(A \setminus B^c) \cup (B \cap (A \cap B)^c)$
11. Simplify $(A \cup B) \cup (C \cap A) \cup (A \cap B)^c$. Use the principle of duality to find the dual law.
12. Let $A = \{a\}$, $B = \{b\}$. Prove or disprove that $(A^*B)^* = (A^*B^*)^*$.
13. Let L_1 and L_2 be languages over Σ. Prove that $L_1^* \cup L_2^* \subseteq (L_1 \cup L_2)^*$. Give an example where $L_1^* \cup L_2^* \neq (L_1 \cup L_2)^*$.
14. Let $A = \{\lambda, b, ab\}$ and $B = \{\lambda, a\}$. Calculate A^2, B^*, BA^*.