Tutorial 2: Set Theory and Formal Languages

Set Notation and Concepts

Concept(s)

A set is a collection of objects. Let A, B, C be sets and x be an object.

x is an element of A	$x \in A$	x is in A
A is a subset of B	$A \subseteq B$	If $x \in A$ then $x \in B$
A is a proper subset of B	$A \subset B$	$A \subseteq B$ and $A \neq B$
A is equal to B	$A = B$	$A \subseteq B$ and $B \subseteq A$
Empty Set	$\varnothing$, $\{\}$	Set containing nothing
Universe	$\mathcal{U}$	Set containing all possible elements

Exercise 1. Let $A = \{1, 2, 3, 4\}$, $B = \{2, 4\}$, and $C = \{1, \{1\}, 2\}$.

Determine whether the following statements are true or false:

$$2 \in A, \quad \{1\} \in C, \quad \{1\} \subset C, \quad A \subseteq B, \quad B \subseteq A, \quad \varnothing \subseteq C.$$

Set Operations

Concept(s)

Union of A and B	$A \cup B = \{x : x \in A \text{ or } x \in B\}$
Intersection of A and B	$A \cap B = \{x : x \in A \text{ and } x \in B\}$
Complement of A	$A^c = \{x : x \notin A \text{ and } x \in \mathcal{U}\}$
A but not B	$A \setminus B = A \cap (B^c)$
Symmetric Difference of A and B	$A \oplus B = (A \setminus B) \cup (B \setminus A)$

Exercise 2. Prove that for any sets A and B, if $A \subseteq B$, then $A \cap B = A$.

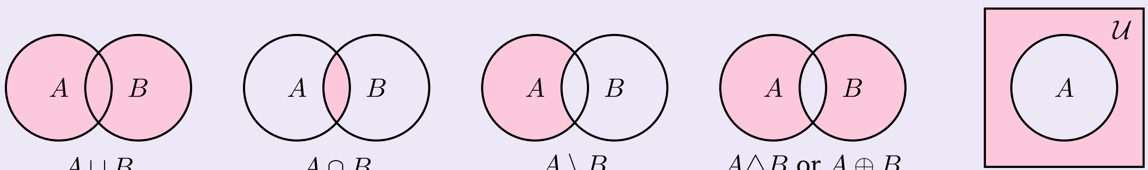
Exercise 3. Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ and $\mathcal{U} = \{1, 2, 3, 4, 5\}$. Calculate:

$$A \cup B, \quad (A^c) \cap B, \quad A \oplus B, \quad (A \cup B) \setminus (A \cap B), \quad (A \setminus B) \cap (B \setminus A).$$

Venn Diagram

Concept(s)

A Venn diagram uses overlapping circles to represent sets and their relationships.



$A \cup B$ — $A \cap B$ — $A \setminus B$ — $A \triangle B$ or $A \oplus B$ — A^c

Exercise 4. For any sets A, B, and C, prove or disprove:

a) $A \cap (B \cup C) = (A \cap B) \cup C$.

b) $(A \setminus B) \setminus C = A \setminus (B \setminus C)$.

c) $A \setminus (B \cap C) = (A \setminus B) \cap (A \setminus C)$.

Hint: Use Venn Diagrams to help you out!

Power Sets and Cardinality

Concept(s)

Power Set of A	$\mathrm{Pow}(A) = \{X : X \subseteq A\}$
Cardinality of A	$\lvert A \rvert$ is the number of elements in A
Cartesian Product of A and B	$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

Exercise 5. Let $A = \{1, 2\}$. Calculate:

$$\mathrm{Pow}(A), \quad \lvert \mathrm{Pow}(A) \rvert, \quad A \times A, \quad \lvert A \times A \rvert .$$

Exercise 6. Let A be a finite set with n elements. Prove that $\lvert \mathrm{Pow}(A) \rvert = 2^n$.

Exercise 7. For sets A and B, prove or disprove $\mathrm{Pow}(A \cup B) = \mathrm{Pow}(A) \cup \mathrm{Pow}(B)$

Set Theory Laws

Concept(s)

Commutativity	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Distribution	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity	$A \cup \varnothing = A$	$A \cap \mathcal{U} = A$
Complement	$A \cup (A^c) = \mathcal{U}$	$A \cap (A^c) = \varnothing$
Idempotence	$A \cup A = A$	$A \cap A = A$
Double Complement	$(A^c)^c = A$	
Annihilation	$A \cup \mathcal{U} = \mathcal{U}$	$A \cap \varnothing = \varnothing$
de Morgan's	$(A \cap B)^c = (A^c) \cup (B^c)$	$(A \cup B)^c = (A^c) \cap (B^c)$

The principle of duality states that any law still holds true when you swap $\cap$ with $\cup$ and $\varnothing$ with $\mathcal{U}$.

Exercise 8. Prove that for any sets A and B, $A \setminus (A \cap B) = A \setminus B$.

Exercise 9. Prove or disprove that for any sets A, B, and C, $(A \setminus B) \cap (B \setminus C) \cap (C \setminus A) = \varnothing$.

Exercise 10. Simplify the expression $[A \cap (A \cap B^c)] \cup [(A \cap B) \cup (B \cap A^c)]$.

Formal Languages

Concept(s)

Terminology for Formal Languages:

Alphabet (Σ)	A finite, non-empty set of symbols
Word	A finite sequence of symbols from Σ
Empty Word (λ)	The word with no symbols
Σ^*	The set of all finite words
Language	A subset of Σ^*

Operations on Formal Languages: Let A, B be languages.

Concatenation	$AB = \{ab : a \in A \text{ and } b \in B\}$
Concatenation Notation	$A^0 = \{\lambda\}$, $A^{i+1} = AA^i$
Kleene Star	$A^* = A^0 \cup A^1 \cup A^2 \cup \ldots$

Since languages are defined as sets, the normal set operations also apply.

Exercise 11. Let $A = \{ab, ba\}$. Calculate A^0, A^1 and A^2.

Exercise 12. Let A, B, C be languages with alphabet $\{a, b\}$ where $AC = BC$. Prove or disprove $A = B$?

Exercise 13. Let L_1 and L_2 be languages over Σ. Prove that $(L_1 \cap L_2)^* \subseteq L_1^* \cap L_2^*$.

Exercise 14. We define a^n and b^n as n consecutive occurrences of the symbol a or b, respectively. Let L_1 and L_2 be languages over $\Sigma = \{a, b\}$ defined as

$$L_1 = \{a^n b^{n+2} : n \geq 0 \text{ and } n \text{ is a perfect square}\} \quad L_2 = \{b^{3n+1} a^{n+1} : n \geq 0\}$$

We define the language $L_3 = L_1 L_2$.

a) Give three examples of strings in L_1 and explain why they are in L_1.

b) Give three examples of strings in L_2 and explain why they are in L_2.

c) Determine which of the following strings are in L_3 and prove your answers:

- $aaaabbbbbbba$
- $aabbbbaaa$
- $aaaaaabbbbbbbbbbbbbbaaa$
- $abbbbbbbaa$

d) Describe the general form of strings in L_3.

Extra Practice Problems

Note(s)

These practice questions are designed to help deepen your understanding. No answers will be provided, as the goal is to encourage independent problem-solving and reinforce key concepts.

1. Which of the following statements are true? Let $S = \{a, \{a\}, \{\{a\}\}\}$

$$\varnothing \in S, \quad a \in S, \quad \{a\} \in S, \quad \varnothing \subset S, \quad a \subset S, \quad \{a\} \subset S, \quad \varnothing \subseteq S, \quad a \subseteq S, \quad \{a\} \subseteq S.$$

2. Let $A = \{7, 8, 9\}$, $B = \{6, 9, -1\}$ and $\mathcal{U} = \{-1, 2, 3, 6, 7, 8, 9\}$. Calculate the following:

$$(A \cup B)^c, \quad (A \setminus B) \cap A, \quad \mathrm{Pow}(A), \quad A \times B, \quad (A^c) \cap (B^c), \quad A \oplus (B^c).$$

3. Show that when $R \subseteq S$ and $R \subseteq T$, we have $R \subseteq S \cap T$.
4. Prove or disprove $A \cap \varnothing = A$ for all sets A.
5. Suppose that $S \cup T = S \cap T$. Show that $S = T$.
6. Give an example that disproves $A \setminus (B \cup C) = (A \setminus B) \cup (A \setminus C)$, where A, B, C are sets.
7. Suppose that $S \oplus T = T$. Is $S = \varnothing$? Explain your answer.
8. Let $S = \{0, 1\}$. Find $\mathrm{Pow}(\mathrm{Pow}(S))$.
9. For sets A, B, show that if $A \subseteq B$ then $\mathrm{Pow}(A) \subseteq \mathrm{Pow}(B)$.
10. Simplify $(A \setminus B^c) \cup (B \cap (A \cap B)^c)$
11. Simplify $(A \cup B) \cup (C \cap A) \cup (A \cap B)^c$. Use the principle of duality to find the dual law.
12. Let $A = \{a\}$, $B = \{b\}$. Prove or disprove that $(A^*B)^* = (A^*B^*)^*$.
13. Let L_1 and L_2 be languages over Σ. Prove that $L_1^* \cup L_2^* \subseteq (L_1 \cup L_2)^*$. Give an example where $L_1^* \cup L_2^* \neq (L_1 \cup L_2)^*$.
14. Let $A = \{\lambda, b, ab\}$ and $B = \{\lambda, a\}$. Calculate A^2, B^*, BA^*.