Quiz 3 Solutions: Advanced Relations, Orders

1. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and R be an equivalence relation on A. Given the following partial information about the equivalence classes:

   - $[1]_R = \{1, 4, 7\}$
   - $[2]_R = \{2, 5\}$
   - $3 \in [6]_R$

   (a) What is the maximum number of elements in R?

   Answer(s)

   When one element is added to an equivalence class, a new pair has to be added for each element that is already in the equivalence class. To maximise the number of elements, we can maximise the size of $[6]_R$, the unknown equivalence class:

   - $[1]_R = \{1, 4, 7\}$ (3 elements)
   - $[2]_R = \{2, 5\}$ (2 elements)
   - $[6]_R = \{3, 6, 8, 9\}$ (4 elements)

   The number of elements in R is: $3^2 + 2^2 + 4^2 = 9 + 4 + 16 = 29$.

   (b) Which of the following must be true? (Select all that apply)

   Answer(s)

   Correct answers:

   - $(8, 8) \in R$ (reflexivity)
   - $(3, 6) \in R$ (given that $3 \in [6]_R$)
   - $(1, 7) \in R$ (given that $1, 7 \in [1]_R$)
   - $(2, 2) \in R$ (reflexivity)

   (c) What is the maximum number of distinct equivalence classes that R can have?

   Answer(s)

   The maximum number of distinct equivalence classes is 5. We know of 3 equivalence classes already: $[1]_R$, $[2]_R$, and $[6]_R$. The remaining elements 8 and 9 could each be in their own equivalence class, giving a maximum of 5 distinct equivalence classes.

   (d) If we know that R has exactly 3 equivalence classes, which of the following statements must be true? (Select all that apply)

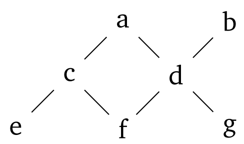
   Answer(s)

   Correct answers:

   - $9 \in [6]_R$

- $|[6]_R| = 4$
- $[8]_R = [9]_R$
- $[3]_R = [6]_R$
- $|[2]_R| = 2$

2. Consider the following Hasse diagram of a poset $(P, \leq)$:



Which of the following statements are true? (Select all that apply)

Answer(s)

- $\text{lub}(\{c, f\})$ exists and is c.
- $\text{glb}(\{c, b\})$ exists and is f.

3. Let R be a relation on set $A = \{4, -2, -1, 3, 2, -4, -3, 1\}$ defined as $R = \{(x, y) : x =_{(x)} y\}$. How many ordered pairs are in R?

Answer(s)

The relation $x =_{(x)} y$ means that x and y are equivalent modulo the absolute value of x. Since $x \,\%\, x = 0$, we can simplify this to $y \,\%\, x = 0$. We can list all the pairs:

- For $x = \pm 4$: $(4, 4), (4, -4), (-4, 4), (-4, -4)$
- For $x = \pm 3$: $(3, 3), (3, -3), (-3, 3), (-3, -3)$
- For $x = \pm 2$: $(2, 2), (2, -2), (-2, 2), (-2, -2), (2, 4), (2, -4), (-2, 4), (-2, -4)$
- For $x = \pm 1$: We find that $n \,\% \, \pm 1 = 0$ for all integers n and so there is an ordered pair for each element in A. We have $|A| = 8$ so we have 8 ordered pairs in the form $(1, n)$ and 8 in the form $(-1, n)$.

In total, there are $4 + 4 + 8 + 16 = 32$ ordered pairs.

4. Let R be a relation on $\mathbb{N}$ where xRy means that there exists $k \in \mathbb{N}$ such that $x = 2^k y$. Which of the following properties does R have? (Select all that apply)

Answer(s)

R has the following properties:

- Reflexive: For any $x \in \mathbb{N}$, $x = 2^0 x$, so xRx.
- Transitive: If xRy and yRz, then $x = 2^k y$ and $y = 2^m z$ for some $k, m \in \mathbb{N}$. Therefore, $x = 2^k(2^m z) = 2^{k+m} z$, so xRz.
- Antisymmetric: If xRy and yRx, then $x = 2^k y$ and $y = 2^m x$ for some $k, m \in \mathbb{N}$. This

implies $x = 2^{k+m}x$, which is only possible if $k+m=0$ or $x=0$. Since $x \in \mathbb{N}$, we must have $k=m=0$, implying $x=y$.

- Total: For all $x \in \mathbb{N}$, we have xRx.

Note that R is not symmetric.

5. Let R and S be binary relations on a set A. Which of the following statements are always true? (Select all that apply)

Answer(s)

The correct answers are:

(a) $(R \cup S)^{\leftarrow} = R^{\leftarrow} \cup S^{\leftarrow}$

(b) $(R \cap S)^{\leftarrow} = R^{\leftarrow} \cap S^{\leftarrow}$

(c) $(R;S)^{\leftarrow} = S^{\leftarrow};R^{\leftarrow}$

These properties of converse relations can be proved using set theory and the definitions of union, intersection, and composition of relations.

6. Let X be the set of all integers x such that $|x| \leq 11$ and $5 \mid x$. How many different equivalence relations can be defined on X?

Answer(s)

First, we determine the elements of $X = \{-10, -5, 0, 5, 10\}$.

We can look at the ways we can add to get 5. This will give us an idea of how we can form the equivalence relations. In particular, we have

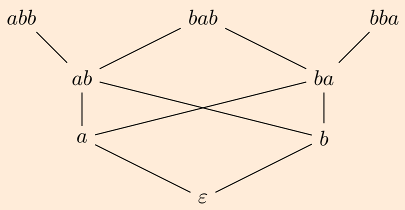
- $1+1+1+1+1$: There is 1 way to place all elements in their own equivalence class.
- $2+1+1+1$: There are 10 ways to choose 2 elements for the equivalence class of size 2. The remaining elements will go into their own equivalence class of are 10 possible equivalence relations.
- $2+2+1$: We have 5 ways to choose the element for the equivalence class of size 1. Then we have 3 ways to split the remaining 4 elements into 2 equivalence classes of size 2. This gives us 15 possible equivalence relations.
- $3+1+1$: We have 10 ways to choose 3 elements for the equivalence class of size 3. The remaining two elements will go into their own equivalence class so we have 10 possible relations.
- $3+2$: There are 10 ways to choose 2 elements for the equivalence class of size 2. Once we choose these two elements, the remaining elements will go into the equivalence class of size 3 so there are 10 possible equivalence relations.
- $4+1$: We have 5 choices for elements to place into the equivalence class of size 1. This means we have 5 possible equivalence relations.
- 5: There is 1 way to place all elements in 1 equivalence class.

Therefore, there are $1 + 10 + 15 + 10 + 10 + 5 + 1 = 52$ different equivalence relations that can be defined on X.

7. Consider the poset $(P, \leq)$ where $P = \{\varepsilon, a, b, ab, ba, bba, bab, abb\}$ and $x \leq y$ if and only if y contains the word x. Which of the following Hasse diagrams correctly represent this poset? (Select all that apply)

Answer(s)

None of the given Hasse diagrams correctly represent this poset. The correct Hasse diagram would have the following structure:



This diagram correctly represents the containment relationships between the words in the poset.