Quiz 5 Solutions: Boolean Logic and Propositional Logic

1. Consider the following Karnaugh map for a Boolean function with variables w, x, y, and z:

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
wx	+	+		+
$w\bar{x}$	+	+		
$\bar{w}\bar{x}$			+	+
$\bar{w}x$	+	+	+	+

Which of the following expressions represents the Disjunctive Normal Form (DNF) with the minimal number of minterms for this function?

(a) $(\bar{y}\bar{w}) \vee (\bar{y}\bar{x}) \vee (\bar{z}\bar{w}) \vee (zy)$

(b) $(\bar{y}w) \vee (\bar{y}\bar{z}) \vee (\bar{x}\bar{w}) \vee (\bar{z}y)$

(c) $(\bar{w}x) \vee (\bar{z}\bar{w}) \vee (\bar{y}z) \vee (\bar{w}\bar{x})$

(d) $(\bar{y}\bar{w}) \vee (wy) \vee (zx) \vee (x\bar{w})$

Answer(s)

Let's analyze the given Karnaugh map:

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
wx	+	+		+
$w\bar{x}$	+	+		
$\bar{w}\bar{x}$			+	+
$\bar{w}x$	+	+	+	+

From this Karnaugh map, we can find the formula from the rectangles above:

- Red group (2x2): wy
- Orange group (2x2): $\bar{w}\bar{y}$
- Blue groups (1x1 each): xz (four corners)
- Cyan group (1x4): $\bar{w}x$ (bottom row)

The minimal DNF is the disjunction of these rectangles:

$$(wy) \vee (\bar{w}\bar{y}) \vee (xz) \vee (\bar{w}x)$$

We can take this disjuction in any order due to commutativity and associativity so our final expression is $(\bar{y}\bar{w}) \vee (wy) \vee (zx) \vee (x\bar{w})$. To see why the other answers are wrong, try drawing their Karnaugh Maps.

2. Which of the following are well-formed formulas (wff) according to the strictest definition?
   (a) $(\neg(p \to q) \to r)$
   (b) $((p \wedge q) \vee \neg r)$
   (c) $(p \vee q \wedge r \to)$
   (d) $(p \wedge \neg q \vee r)$
   (e) $(\neg(p \wedge q) \vee (r \wedge \leftarrow q))$
   (f) $(p \to (q \wedge r))$
   (g) $(((p \wedge q) \vee r) \to (s \wedge \neg t))$
   (h) $((\neg p \to q) \wedge r \vee s \wedge)$
   (i) $(p \to q \vee (r \wedge s))$

Answer(s)

- $(\neg(p \to q) \to r)$ is a wff.
- $((p \wedge q) \vee \neg r)$ is a wff.
- $(p \vee q \wedge r \to)$ is not a wff. The operator $\to$ requires two inputs, but only has one.
- $(p \wedge \neg q \vee r)$ is not a wff. Every binary operator requires brackets on the outside, but, the formula is missing a set of brackets. This can lead to a situation where it is unclear if this means $(p \wedge (\neg q \vee r))$ or $((p \wedge \neg q) \vee r)$.
- $(\neg(p \wedge q) \vee (r \wedge \leftarrow q))$ is not a wff. The operator $\wedge$ requires two inputs, but only has one.
- $(p \to (q \wedge r))$ is a wff.
- $(((p \wedge q) \vee r) \to (s \wedge \neg t))$ is a wff.
- $((\neg p \to q) \wedge r \vee s \wedge)$ is not a wff. The operator $\wedge$ requires two inputs, but only has one.
- $(p \to q \vee (r \wedge s))$ is not a wff. Every binary operator requires brackets on the outside, but, the formula is missing a set of brackets. This can lead to a situation where it is unclear if this means $((p \to q) \vee (r \wedge s))$ or $(p \to (q \vee (r \wedge s)))$.

3. Let $T = \{p \to q, \neg q\}$ be a set of propositions. Which of the following statements is true?
   (a) $T \models p$
   (b) $T \models \neg p$
   (c) $T \models q$
   (d) $T \models \neg q$

Answer(s)

p	q	$\neg p$	$\neg q$	$p \to q$
F	F	T	T	T
F	T	T	F	T
T	F	F	T	F
T	T	F	F	T

Since we have a theory $\{p \to q, \neg q\}$, we can only consider rows where both $p \to q, \neg q$ are true. This means that p is false and q is false. This means that (b) and (d) are true.

4. Let p, q and r be propositional variables with the following meanings:

$$p = \text{"It is sunny"}$$
$$q = \text{"we'll go to the beach"}$$
$$r = \text{"We'll need sunscreen"}$$

Consider the argument:

If it's sunny, then we'll go to the beach.
If we go to the beach, then we'll need sunscreen.
We don't need sunscreen.

It's not sunny.

Which of the following represents the correct formalization and evaluation of this argument?

(a) $p \to q, q \to r, \neg r \models \neg p$ (The argument is valid)

(b) $p \to q, q \to r, \neg r \not\models \neg p$ (The argument is not valid)

(c) $p \to q, q \to r, r \models p$ (The argument is valid)

(d) $p \to q, q \to r, r \not\models p$ (The argument is not valid)

Answer(s)

We can match premises to the logical notation in the answers:

$p \to q$	If it's sunny, then we'll go to the beach.
$q \to r$	If we go to the beach, then we'll need sunscreen.

In particular, we have r = "We'll need sunscreen". This means the final premise translates to $\neg r$. The premises are $p \to q, q \to r, \neg r$. Note that only options a and b have these premises so the conclusion has to be $\neg p$. Let's draw a truth table

p	q	r	$\neg r$	$p \to q$	$q \to r$	$\neg p$
F	F	F	T	T	T	T
F	F	T	F	T	T	T
F	T	F	T	T	F	T
F	T	T	F	T	T	T
T	F	F	T	F	T	F
T	F	T	F	F	T	F
T	T	F	T	T	F	F
T	T	T	F	T	T	F

From the truth table, all premises are true, $\neg p$ is true. This means that $p \to q, q \to r, \neg r \models \neg p$.

5. Which of the following expressions are equivalent to the Boolean expression $(x \wedge y) \vee (x \wedge \neg y) \vee (\neg x \wedge y)$?

(a) $x \vee y$

(b) $x \wedge y$

(c) $x \to y$

(d) $x \leftrightarrow y$

(e) None of these options

Answer(s)

x	y	$(x \wedge y) \vee (x \wedge \neg y) \vee (\neg x \wedge y)$	$x \vee y$	$x \wedge y$	$x \to y$	$x \leftrightarrow y$
F	F	F	F	F	T	T
F	T	T	T	F	T	F
T	F	T	T	F	F	F
T	T	T	T	T	T	T

The only one that matches the expression is $x \vee y$.

6. Let $\Sigma = \{0, 1\}$ and let L_1, L_2 be languages under Σ. Which of the following propositional formulas correctly expresses the statement

“If a word is in L_1 and L_2, then it’s length is less than 2”

Let p, q and r be propositional variables with the following meanings:

$$
\begin{aligned}
p &= \text{“The word is in } L_1\text{”} \\
q &= \text{“The word is in } L_2\text{”} \\
r &= \text{“The word has length at least 2”}
\end{aligned}
$$

Note that conventions from lecture 9 can apply.

(a) $(p \vee q) \to r$

(b) $(p \vee q) \to \neg r$

(c) $(p \wedge q) \to r$

(d) $(p \wedge q) \to \neg r$

Answer(s)

- “If a word is in both L_1 and L_2”: $p \wedge q$
- “it either has length less than 2”: The variable r means $|w| \geq 2$, but, the statement wants $|w| < 2$ so this translates to $\neg r$

The entire proposition

“If a word is in both L_1 and L_2, then it’s length is less than 2”

can be translated into $(p \wedge q) \to \neg r$.

7. Let $f : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ and $g : \mathbb{B} \to \mathbb{B}$ be Boolean functions defined as follows:

x	y	$f(x,y)$
0	0	1
0	1	0
1	0	0
1	1	1

x	$g(x)$
0	1
1	0

Let $h(x, y) = g(f(x, y))$. Which of the following truth tables are correct?

(a)

x	y	$h(x,y)$
0	0	0
0	1	1
1	0	1
1	1	0

(b)

x	$h(g(x), g(!g(x)))$
0	0
1	0

(c)

x	y	$f(h(x,y), h(x,y))$
0	0	1
0	1	1
1	0	1
1	1	1

(d)

x	y	$f(x,y)$
0	0	1
0	1	0
1	0	0
1	1	1

Answer(s)

We note that $g(x) = !x$. This means $h(x,y) = !f(x,y)$. Consider the following table of output.

x	y	$f(x,y)$	$h(x,y)$
0	0	1	0
0	1	0	1
1	0	0	1
1	1	1	0

Hence, we find that (a) is correct. We now simplify

$$h(g(x), g(!g(x))) = !f(!x, !!!x) = !f(!x, !x)$$

by the double complement law. Note that $f(x,y) = 1$ when $x = y$. This means

$$!f(!x, !x) = !1 = 0.$$

Therefore, the truth table in (b) is true. Similarly, we have $f(h(x,y), h(x,y)) = 1$ so the table in (c) is also true. The table in (d) is the same as the definition so it is true.

All the options are true.

8. Which of the following options are Boolean algebras?

(a)

$$\begin{aligned} T &= \{0, 1\} \\ x \vee y &= (x + y) \ \% \ 2 \\ x \wedge y &= (xy) \ \% \ 2 \\ x' &= 1 - x \\ \mathbb{0} &= 0 \\ \mathbb{1} &= 1 \end{aligned}$$

(b)

$$\begin{aligned} T &= \big\{\varnothing, \{a\}, \{b\}, \{a, b\}\big\} \\ x \vee y &= x \cap y \\ x \wedge y &= x \cup y \\ x' &= \{a, b\} \setminus x \\ \mathbb{0} &= \{a, b\} \\ \mathbb{1} &= \varnothing \end{aligned}$$

(c)

$$\begin{aligned} T &= \{0, 1, 2, 3\} \\ x \vee y &= \max(x, y) \\ x \wedge y &= \min(x, y) \\ x' &= 3 - x \\ \mathbb{0} &= 0 \\ \mathbb{1} &= 3 \end{aligned}$$

(d)

$$\begin{aligned} T &= \{1, 3, 5, 15\} \\ x \vee y &= \mathrm{lcm}(x, y) \\ x \wedge y &= \gcd(x, y) \\ x' &= 15/x \\ \mathbb{0} &= 1 \\ \mathbb{1} &= 15 \end{aligned}$$

Answer(s)

Both (b) and (d) are Boolean algebras.

(a) Consider that $1 \vee (0 \wedge 1) = 1$ but $(1 \vee 0) \wedge (1 \vee 1) = 0$ so the distributive law does not hold.

(b) You can verify that all the properties of a Boolean algebra hold :)

(c) From the complement law, we need $x \wedge x' = \mathbb{0}$. This is not true as $1 \wedge 1' = \min(1, 2) = 1 \neq 0$.

(d) You can verify that all the properties of a Boolean algebra hold :)