Tutorial 3 Solutions: Relations

Defining Binary Relations

Concept(s)

For sets A and B, a binary relation R from A to B is a subset of $A \times B$. We write

$$a \; R \; b \text{ or } R\,(a, b) \text{ to denote } (a, b) \in R.$$

We can also represent a binary relation as a

- Graph: We draw dots for each $a \in A$ and $b \in B$. For each $(a, b) \in R$, we can draw $a \to b$.
- Matrix: We have a grid with rows labelled by elements of S and columns by elements of T. We fill in a cell with row a and column b if $(a, b) \in R$.

Exercise 1. Let $A = \{2, 3, 4, 6\}$, $B = \{1, 2, 3\}$ and $R \subseteq A \times B$ where $a \; R \; b$ means that $\gcd(a, b) = 1$.

a) Write the relation R as a set.

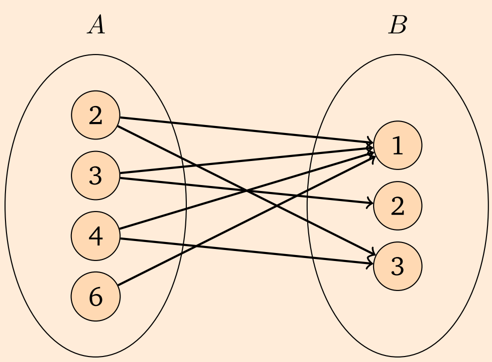
b) Draw the relation R as a graph.

c) Write the relation R as a matrix.

Answer(s)

a) $R = \{(2, 1), (2, 3), (3, 1), (3, 2), (4, 1), (4, 3), (6, 1)\}$

b) Graph representation:



c) Matrix representation:

	1	2	3
2	•	◦	•
3	•	•	◦
4	•	◦	•
6	•	◦	◦

Concept(s)

A relation on A is a binary relation R from A to A. For the graphical representation, we only draw dots for each $a \in A$ once. When $(a_1, a_2) \in R$, we can draw $a_1 \to a_2$.

Exercise 2. Let $A = \{\{\}, \{1\}, \{2\}, \{1, 2\}\}$ and $R \subseteq A \times A$ where $a\ R\ b$ means that $a \subset b$.

a) Write the relation R as a set.

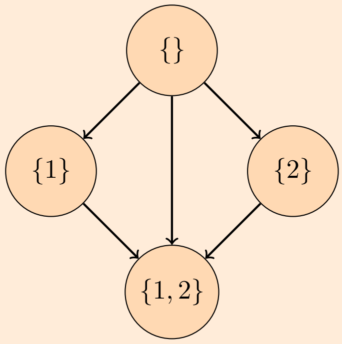
b) Draw the relation R as a graph.

c) Write the relation R as a matrix.

Answer(s)

a) $R = \{(\{\}, \{1\}), (\{\}, \{2\}), (\{\}, \{1, 2\}), (\{1\}, \{1, 2\}), (\{2\}, \{1, 2\})\}$

b) Graph representation:



c) Matrix representation:

	$\{\}$	$\{1\}$	$\{2\}$	$\{1,2\}$
$\{\}$	$\circ$	$\bullet$	$\bullet$	$\bullet$
$\{1\}$	$\circ$	$\circ$	$\circ$	$\bullet$
$\{2\}$	$\circ$	$\circ$	$\circ$	$\bullet$
$\{1,2\}$	$\circ$	$\circ$	$\circ$	$\circ$

Operations on Relations

Concept(s)

Let R be a relation from A to B and S be a relation from B to C. We define the

- Converse of R: $R^{\leftarrow} = \{(b, a) \in B \times A : a\ R\ b\}$.
- Composition of R and S: $R; S = \{(a, c) \in A \times C :$ there exists $b \in B$ such that $a\ R\ b$ and $bSc\}$.

Exercise 3. Let $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$. Consider the relation from A to B,

$$R = \{(1, x), (2, y), (3, z), (1, y)\}.$$

Compute $R; R^{\leftarrow}$. What is $R; R^{\leftarrow}$ a subset of?

Answer(s)

We first compute $R^{\leftarrow} = \{(x, 1), (y, 2), (z, 3), (y, 1)\}$. Now, we compute $R; R^{\leftarrow}$, where

$$R; R^{\leftarrow} = \{(a, c) \in A \times A : \text{there exists } b \in B \text{ such that } (a, b) \in R \text{ and } (b, c) \in R^{\leftarrow}\}.$$

We now consider each possible pair of (a, c). When $a = 1$, we can get either $b = x$ or $b = y$ such that $(a, b) \in R$. If $b = x$, then we have $c = 1$ such that $(b, c) \in R^{\leftarrow}$. This means that $(1, 1) \in R; R^{\leftarrow}$. If $b = y$, then we can have $c = 1$ or $c = 2$ such that $(b, c) \in R^{\leftarrow}$. This means that $(1, 1), (1, 2) \in R; R^{\leftarrow}$. We can continue this method to get all pairs $(a, c) \in R; R^{\leftarrow}$. Hence,

$$R; R^{\leftarrow} = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}.$$

The resulting relation $R; R^{\leftarrow}$ is a subset of $A \times A$.

Properties of Relations $R \subseteq A \times A$

Concept(s)

(R)	Reflexive	For all $a \in A$, we have $(a, a) \in R$
(AR)	Antireflexive	For all $a \in A$, we have $(a, a) \notin R$
(S)	Symmetric	For all $a, b \in A$, if $(a, b) \in R$ then $(b, a) \in R$
(AS)	Antisymmetric	For all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$ then $a = b$
(T)	Transitive	For all $a, b, c \in A$, if $(a, b), (b, c) \in R$ then $(a, c) \in R$

Exercise 4. Which of the properties (R), (AR), (S), (AS), (T) does R satisfy? Explain why.

a) $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a > b\}$

b) $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a \leq b\}$

c) $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : |a - b| \leq 2\}$

Answer(s)

a) $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a > b\}$

- (R) No. We see that $1 \not> 1$, so $(1, 1) \notin R$.
- (AR) Yes. For all $a \in \mathbb{Z}$, $a \not> a$, so $(a, a) \notin R$.
- (S) No. If $(a, b) \in R$, then $a > b$, but this implies $b < a$, so $(b, a) \notin R$.
- (AS) Yes. Our condition is never fulfilled since $a > b$ and $b < a$ can never happen together. If our condition can never be fulfilled, our statement is true by vacuous truth.
- (T) Yes. If $(a, b) \in R$ and $(b, c) \in R$, then $a > b$ and $b > c$, which implies $a > c$, so $(a, c) \in R$.

b) $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a \leq b\}$

- (R) Yes. For all $a \in \mathbb{Z}$, $a \leq a$, so $(a, a) \in R$.
- (AR) No. We see that $1 \leq 1$, so $(1, 1) \in R$.
- (S) No. If $(a, b) \in R$ and $a < b$, then $b \not\leq a$, so $(b, a) \notin R$.
- (AS) Yes. If $(a, b) \in R$ and $(b, a) \in R$, then $a \leq b$ and $b \leq a$, which implies $a = b$.
- (T) Yes. If $(a, b) \in R$ and $(b, c) \in R$, then $a \leq b$ and $b \leq c$, which implies $a \leq c$, so $(a, c) \in R$.

c) $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : |a - b| \leq 2\}$

- (R) Yes. For all $a \in \mathbb{Z}$, $|a - a| = 0 \leq 2$, so $(a, a) \in R$.

(AR) No. We see that $|1-1| = 0 \leq 2$, so $(1,1) \in R$.

(S) Yes. If $(a,b) \in R$, then $|a-b| \leq 2$, which implies $|b-a| \leq 2$, so $(b,a) \in R$.

(AS) No. For example, $(0,2) \in R$ and $(2,0) \in R$, but $0 \neq 2$.

(T) No. For example, $(0,2) \in R$ and $(2,4) \in R$, but $(0,4) \notin R$ since $|0-4| > 2$.

Exercise 5. Let R be a relation on a set A. Prove or disprove the following:

a) If R is symmetric and transitive, then R is reflexive.

b) If R is antireflexive and transitive, then R is antisymmetric.

Answer(s)

a) Consider $R = \{\}$. A relation with no transitions is symmetric and transitive but not reflexive. This means that this statement is not always true.

b) Let R be antireflexive and transitive. Suppose that R is not antisymmetric. This means there exists $(a,b) \in R$ such that $(b,a) \in R$ and $a \neq b$. By transitivity, we have $(a,a) \in R$, from $(a,b) \in R$ and $(b,a) \in R$. This is a contradiction since R is antireflexive. Therefore, our assumption must be false and R is antisymmetric.

Equivalence Relations

Concept(s)

An equivalence relation R is a relation on A that is (R), (S) and (T).

The equivalence class of $a \in A$ is $[a] = \{b \in A : a \; R \; b\}$.

Exercise 6. Let $\Sigma = \{a,b\}$. We define the relation $\sim$ on Σ^*, where $w_1 \sim w_2$ means that w_1 and w_2 have the same number of letters. Explain why $\sim$ is an equivalence relation.

Answer(s)

To prove that $\sim$ is an equivalence relation, we need to show that it is (R), (S) and (T).

(R) For any $w \in \Sigma^*$, $w \sim w$ because w has the same number of letters as itself.

(S) For any $w_1, w_2 \in \Sigma^*$, if $w_1 \sim w_2$, then w_1 and w_2 have the same number of letters. This implies that w_2 and w_1 also have the same number of letters, so $w_2 \sim w_1$.

(T) For any $w_1, w_2, w_3 \in \Sigma^*$, if $w_1 \sim w_2$ and $w_2 \sim w_3$, then w_1 and w_2 have the same number of letters, and so do w_2 and w_3. This implies that w_1 and w_3 must also have the same number of letters, so $w_1 \sim w_3$.

Since $\sim$ satisfies all three properties, it is an equivalence relation on Σ^*.

Exercise 7. Consider the relation F on $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ where $(a,b)F(c,d)$ means that $ad = bc$.

a) Prove that F is an equivalence relation.

b) Describe the equivalence class $[(1,2)]$.

Answer(s)

a) To prove that F is an equivalence relation, we need to show that it is (R), (S) and (T).

(R) For any $(a, b) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$, $(a, b)F(a, b)$ because $ab = ba$.

(S) If $(a, b)F(c, d)$, then $ad = bc$. We can rearrange our equation to get $cb = da$ so $(c, d)F(a, b)$.

(T) If $(a, b)F(c, d)$ and $(c, d)F(e, f)$, then $ad = bc$ and $cf = de$. Multiplying these equations, we get $(ad)(cf) = (bc)(de)$. Dividing both sides by cd (which is possible since $c, d \in \mathbb{Z}_{>0}$), we get $af = be$. This means $(a, b)F(e, f)$.

Since F satisfies all three properties, it is an equivalence relation on $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$.

b) The equivalence class $[(1, 2)]$ consists of all pairs $(a, b) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ such that $(1, 2)F(a, b)$. This means if $1b = 2a$ then $(a, b) \in [(1, 2)]$. Every element is a pair where their second number is double their first. Therefore,

$$[(1, 2)] = \{(1, 2), (2, 4), (3, 6), (4, 8), \ldots\} = \{(n, 2n) : n \in \mathbb{Z}_{>0}\}.$$

Partial Orders

Concept(s)

A partial order $\preceq$ is a relation on A that is (R), (AS) and (T). We call $(A, \preceq)$ a poset.

A Hasse diagram is a graph if $a \preceq b$ and $a \neq b$, then there is an edge drawn upward from a to b.

Minimal	$a \in A$ such that there is no $a \neq b$ where $b \preceq a$
Minimum	$a \in A$ such that $a \preceq b$ for all $b \in A$
Maximal	$a \in A$ such that there is no $a \neq b$ where $a \preceq b$
Maximum	$a \in A$ such that $b \preceq a$ for all $b \in A$

Exercise 8. For the poset $(\{2, 4, 6, 9, 12, 36, 72\}, |)$:

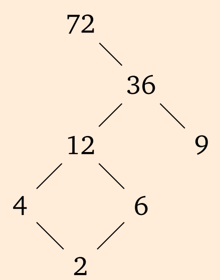
a) Draw a Hasse diagram.

b) Find the maximal/minimal elements.

c) Is there a minimum element? Is there a maximum element?

Answer(s)

a) Hasse diagram:



b) A maximal element is one that points to no other element. Therefore, 72 is our only maximal element. A minimal element is an element that no other element points to. Our two minimal elements are 2 and 9.

c) If every element points to one element, that is our maximum. In this case, we have a maximum at 72. A minimum is an element that points to every other element. There is no such element so there is no minimum.

Exercise 9. Let $(A, \preceq)$ be a poset. Prove that if A has a maximum then the maximum is unique.

Answer(s)

To do so, we can prove that if we have two maximums, then they must be the same. Suppose that a, b are maximum elements. By definition, for all $x \in A$, we have $x \preceq a$ and $x \preceq b$. This means that $a \preceq b$ and $b \preceq a$. Since $\preceq$ is a partial order, we can use the antisymmetry property to conclude that $a = b$. This means that a maximum element is unique.

Concept(s)

Let $(A, \preceq)$ be a poset.

- a is an upper bound for B if $b \preceq a$ for all $b \in B$
- The least upper bound of B, $\text{lub}(B)$, is the minimum of $\{a : a \text{ is an upper bound of } B\}$
- a is a lower bound for B if $a \preceq b$ for all $b \in B$
- The greatest lower bound of B, $\text{glb}(B)$, is the maximum of $\{a : a \text{ is a lower bound of } B\}$

A lattice is a poset where each pair of elements has a lub and glb.

Exercise 10. For the poset $(\{2, 4, 6, 9, 12, 36, 72\}, |)$:

a) Find $\text{lub}(\{6, 9\})$ and $\text{glb}(\{6, 9\})$

b) Is this poset a lattice?

Answer(s)

a) Our Hasse diagram will help us find the lub between 6 and 9. We need the smallest number that both 6 and 9 point to. From our diagram, this is 36 so $\text{lub}(\{6, 9\}) = 36$. The greatest lower bound is the biggest number that points to both 6 and 9, but from our diagram, there are no numbers that point to both so $\text{glb}(\{6, 9\})$ does not exist.

b) This poset is not a lattice. For a poset to be a lattice, every pair of elements must have both a least upper bound and a greatest lower bound within the set. We've shown that the pair $\{6, 9\}$ does not have a greatest lower bound in the set, so this poset is not a lattice.

Exercise 11. For the poset $(Pow(\{a, b, c\}), \subseteq)$:

a) Prove that $\subseteq$ is a partial order.

b) Draw a Hasse diagram.

c) Find the maximal/minimal elements.

d) Is there a minimum element? Is there a maximum element?

e) Find $\text{lub}(\{\{a\}, \{b, c\}\})$ and $\text{glb}(\{\{a\}, \{b, c\}\})$, if they exist.

f) Is this poset a lattice?

Answer(s)

a) To prove that $\subseteq$ is a partial order, we need to show it is (R), (AS), (T).

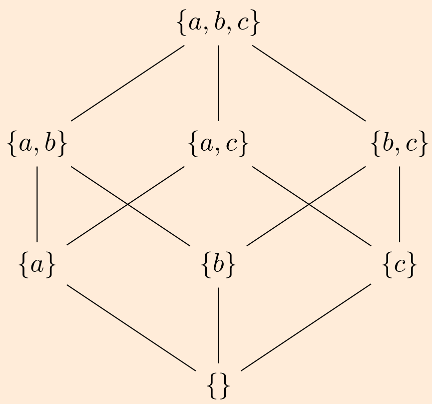
(R) For any set X, $X \subseteq X$, so $\subseteq$ is reflexive.

(AS) If $X \subseteq Y$ and $Y \subseteq X$, then $X = Y$, so $\subseteq$ is antisymmetric.

(T) Suppose that $X \subseteq Y$ and $Y \subseteq Z$. Let $x \in X$. Since $X \subseteq Y$, we find that $x \in Y$. Since $Y \subseteq Z$, we also have $x \in Z$. By definition, we can conclude that $X \subseteq Z$. Therefore, if $X \subseteq Y$ and $Y \subseteq Z$, then $X \subseteq Z$, so $\subseteq$ is transitive.

Therefore, $\subseteq$ is a partial order.

b) Hasse diagram:



c) Maximal element: $\{a, b, c\}$
Minimal element: $\{\}$

d) Minimum element: $\{\}$
Maximum element: $\{a, b, c\}$

e) $\mathrm{lub}\left(\{\{a\}, \{b, c\}\}\right) = \{a, b, c\}$
$\mathrm{glb}\left(\{\{a\}, \{b, c\}\}\right) = \{\}$

f) Yes, this poset is a lattice. For any $X, Y \in Pow(\{a, b, c\})$, $\mathrm{lub}(\{X, Y\}) = X \cup Y$ and $\mathrm{glb}(\{X, Y\}) = X \cap Y$. Since the intersection and union are from elements of $\{a, b, c\}$, they must exist in $Pow(\{a, b, c\})$. We must prove this is true.

To prove something is a lub, you must first prove it is an upper bound and then show it is the minimum upper bound. We can use a similar method for glb.

Proof that $\mathrm{lub}(\{X, Y\}) = X \cup Y$.
Since $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$, we find that $X \cup Y$ is an upper bound of X and Y. Suppose that Z is an upper bound of X and Y too. Suppose that $z \in X \cup Y$. We have two cases: either $z \in X$ or $z \in Y$. When $z \in X$, we have $z \in Z$ as $X \subseteq Z$ by definition of being an upper bound. Similarly, when $z \in Y$, we have $z \in Z$. This means that $z \in Z$ in all cases. We conclude that $X \cup Y \subseteq Z$ for any upper bound Z. This shows that $X \cup Y$ is the minimum upper bound so $\mathrm{lub}(\{X, Y\}) = X \cup Y$.

Proof that $\mathrm{glb}(\{X, Y\}) = X \cap Y$.
Since $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$, we find that $X \cap Y$ is a lower bound of X and Y. Suppose

that Z is a lower bound of X and Y too. Suppose that $z \in Z$. By definition of being a lower bound, $Z \subseteq X$ and $Z \subseteq Y$ so we have $z \in X$ and $z \in Y$. This means that $z \in X \cap Y$ and so $Z \subseteq X \cap Y$ for any lower bound Z. This shows that $X \cap Y$ is the maximum lower bound so $\text{glb}(\{X, Y\}) = X \cap Y$.

Concept(s)

A total order is a partial order that has Linearity: For all a, b, either $a \leq b$ or $b \leq a$.

A topological sort of $(A, \preceq)$ is a total order $\leq$ where if $a \preceq b$, then $a \leq b$.

Exercise 12. For the poset $(Pow(\{a, b, c\}), \subseteq)$, find a topological sort.

Answer(s)

A topological sort for the poset $(Pow(\{a, b, c\}), \subseteq)$ is

$$\{\} \leq \{a\} \leq \{b\} \leq \{c\} \leq \{a, b\} \leq \{a, c\} \leq \{b, c\} \leq \{a, b, c\}.$$

The intuition is that if $X \subseteq Y$, then we have $|X| \leq |Y|$. Therefore, we can order our sets by size to get our topological sort. As you can see, if $X \subseteq Y$, X will appear before or at the same position as Y in our total order.

Note that this is not the only possible topological sort. For example, we could swap the positions of $\{b\}$ and $\{c\}$, or $\{a, b\}$ and $\{a, c\}$, and still have a valid topological sort.

Extra Practice Problems

Note(s)

These practice questions are designed to help deepen your understanding. No answers will be provided, as the goal is to encourage independent problem-solving and reinforce key concepts.

1. Let $A = \{1, 2, 3, 4\}$. Define the relation R on A by $a\ R\ b$ if and only if $a + b$ is even. Write R as a set of ordered pairs and draw its directed graph.

2. Draw graphs for relations on $\{1, 2, 3\}$ that have the following properties:
   a) Symmetric and antisymmetric and reflexive.
   b) Reflexive, but neither transitive nor symmetric.
   c) Transitive and reflexive, but not symmetric.

3. Which of the properties (R), (AR), (S), (AS), (T) does R satisfy? Prove your answers.
   a) Let $\Sigma = \{1, 2, 3\}$. Define R over Σ^* where $n_1 R n_2$ means that n_1 and n_2 have a common digit.
   b) Consider the relation $\sim$ on $\mathbb{Z}$ where $a \sim b$ means that $3 \mid a$ and $3 \mid b$ OR $3 \nmid a$ and $3 \nmid b$.
   c) The relation on $\mathbb{Z}$, $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a =_{(2)} b\}$.

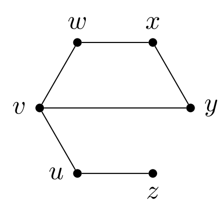
4. How many reflexive relations on A are there if $|A| = n$?

5. Let $A = \{a, b, c, d\}$ and R be the relation $\{(a, b), (b, c), (c, d), (d, a)\}$. Compute $R; R$ and $R^{\leftarrow}; R$.

6. If you take the converse of the relation $<$, what other familiar relation does it become? What if you take the converse of $=$?

7. A relation $\sim$ on $\mathbb{Z}$ is defined by $a \sim b$ when $a - b$ is either divisible by 3 or 5. Show that $\sim$ is not an equivalence relation.

8. Let $V = \{u, v, w, x, y, z\}$. The following diagram shows direct flights between the 6 cities:



   Define the relation $\sim$ on V where $a \sim b$ means that it is possible to fly from a to b with an even number of flights (including 0 flights).
   a) Prove that $\sim$ is an equivalence relation.
   b) Partition the set V into equivalence classes.

9. Let $\Sigma = \{a, b, c, d\}$. Define the relation A on Σ^* where $w_1 A w_2$ means that w_1 is an anagram of w_2.
   a) Prove that A is an equivalence relation.
   b) List out all elements of $[bad]$.

10. List all partitions of $\{a, b, c\}$.

11. With respect to the partial order $|$ on the set $\mathbb{Z}_{>0}$, find
    a) $\mathrm{lub}(\{4, 6\})$
    b) $\mathrm{lub}(\{3, 4, 5, 6\})$
    c) $\mathrm{glb}(\{12, 16, 18, 24\})$
    d) $\mathrm{glb}(\{737, 2345\})$
12. Consider the set $A = \{B \subseteq \{1, 2, 3, 4\} : |B| \neq 2\}$ with the partial order $\subseteq$.
13. Let $A = \{1, 2, 3, 4\}$ be a poset defined with $\leq$. Consider the lexicographic order $\leq_{\mathrm{lex}}$ (check the lectures) on $A \times A$.
    - Prove that $\leq_{\mathrm{lex}}$ is a partial order.
    - Draw a Hasse diagram.
    - Find the maximal/minimal elements.
    - Is there a maximum element? Is there a minimum element?
    - Is this poset a lattice?