Quiz 6 Solutions: Recursion and Induction

1. Suppose that $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is defined recursively as follows for all $m, n \in \mathbb{N}$:

$$\text{(B)}\, f(0, m) = m$$
$$\text{(R)}\, f(n+1, m) = 1 + f(n, m)$$

Which of the following statements are true?

(a) For all $m, n \in \mathbb{N}$, we have $f(n, m) = f(m, n)$.

(b) For all $a, b, c \in \mathbb{N}$, we have $f(a, f(b, c)) = f(f(a, b), c)$.

(c) For all $a, b, c \in \mathbb{N}$, we have $af(b, c) = f(ab, ac)$.

(d) For all n, we have $f(n, n) = n^2$

Answer(s)

From unwinding, we find that

$$\begin{aligned} f(n, m) &= 1 + f(n-1, m) \\ &= 2 + f(n-2, m) \\ &\vdots \\ &= k + f(n-k, m) \\ &\vdots \\ &= n + f(0, m) \\ &= n + m. \end{aligned}$$

This function is a recursive definition of addition.

(a) True, as commutativity holds in addition.

(b) True, as associativity holds in addition.

(c) True, as distribution holds in addition.

(d) False. Consider $f(1, 1) = 2 \neq 1^2$.

2. Consider the following recurrence:

$$T(n) = T(n/3) + T(n/4) + T(n/5) + 3n,$$

where $T(n)$ increases as n increases. True or false: $T(n) \in O(n \log n)$.

(a) True

(b) False

Answer(s)

True. Since $T(n)$ is increasing, we know that

$$T(n/5) \leq T(n/4) \leq T(n/3)$$

which means that

$$T(n) = T(n/3) + T(n/4) + T(n/5) + 3n \leq 3T(n/3) + 3n.$$

We can use master theorem to solve the recurrence $3T(n/3) + 3n$. We have $d = 1$ and $c = 1$ so from case 2, we get

$$T(n) \leq 3T(n/3) + 3n \in \Theta(n \log n)$$

so we conclude that $T(n) \in O(n \log n)$.

3. Let a_n and b_n be integer sequences defined recursively:

(B)	$a_1 = 1,\ a_2 = 2$	(B)	$b_1 = 1,\ b_2 = k$
(R)	$a_n = a_{n-1} + a_{n-2}$	(R)	$b_n = 2b_{n-1} - b_{n-2}$

where $k \in \mathbb{Z}$. For what value of k will $a_n = b_n$ for all $n \geq 1$?

(a) 1

(b) 2

(c) 3

(d) No such k exists

Answer(s)

We want $a_n = b_n$ for all $n \geq 1$, we need $a_2 = b_2$. This mean that $b_2 = k$ can only be 2. We compare the first couple terms of a_n and b_n to get

$$\begin{array}{ll} a_1 = 1 & b_1 = 1 \\ a_2 = 2 & b_2 = 2 \\ a_3 = 3 & b_3 = 3 \\ a_4 = 5 & b_4 = 4 \end{array}$$

Since $a_n \neq b_n$ when $n = 4$, we find that a_n and b_n are different sequences. As k cannot be 2, there is no such k value.

4. Let $\Sigma = \{a, b\}$. Consider the language $L \subseteq \Sigma^*$ defined recursively:

(B) $\lambda \in L$

(R1) If $w \in L$, then $awb \in L$ and $bwa \in L$

(R2) If $w_1, w_2 \in L$, then $w_1 w_2 \in L$

where $|w|_a$ and $|w|_b$ denotes the number of a's and number of b's in word w respectively.

Which of the following properties holds for all $w \in L$?

(a) For all $w \in L$, $|w|_a = |w|_b$

(b) For all $w \in L$, if $w = w_1 a w_2 b w_3$ then $|w_1|_b = 0$, where $w_1, w_2, w_3 \in L$.

(c) For all $w \in L$, $|w| =_{(2)} 0$

(d) For all $w \in L$, if $|w| > 0$ then $w = aw'$ or $w = bw'$ for some $w' \in \Sigma^*$

Answer(s)

(a) True, provable by structural induction.

(b) False, consider $w = abab$, where $w_1 = ab$ and $w_2 = w_3 = \lambda$. We have $|w_1|_b \neq 0$.

(c) True, provable by structural induction.

(d) True, provable by structural induction.

5. We want to prove by induction that for all integers $n \geq 4$, $2^n < (n+1)!$.

   Which combination of base case(s) and inductive step would be valid?

   (a) **[B]** Show $P(4)$ holds
   [I] Show for all integers $k \geq 4$, if $P(k)$ holds then $P(k+1)$ holds

   (b) **[B]** Show $P(4)$ and $P(5)$ hold
   [I] Show for all integers $k \geq 4$, if $P(k)$ holds then $P(k+2)$ holds

   (c) **[B]** Show $P(4)$ holds
   [I] Show for all integers $k \geq 4$, if $P(i)$ holds for all integers $4 \leq i \leq k$ then $P(k+1)$ holds

   (d) **[B]** Show $P(4)$, $P(5)$ and $P(6)$ hold
   [I] Show for all integers $k \geq 6$, if $P(k)$ holds then $P(k+1)$ holds

Answer(s)

(a) This proof is induction for $n \geq 4$ and valid.

(b) This proof method is induction with larger steps for $n \geq 4$ and valid.

(c) This proof is strong induction for $n \geq 4$ and valid.

(d) This proof uses induction to prove the property for all $n \geq 6$. We also prove the $n = 4$ and $n = 5$ cases to show the property is true for all $n \geq 6$. This proof method is valid.

6. Consider the recurrence relation:

$$\begin{aligned} &(\mathrm{B}) \quad T(1) = 1 \\ &(\mathrm{R}) \quad T(n) = 2T(\sqrt{n}) + n \log n \end{aligned}$$

   Which statement is correct about solving this recurrence?

   (a) Case 1 of Master Theorem applies.

   (b) Case 2 of Master Theorem applies.

   (c) Case 3 of Master Theorem applies.

   (d) The Master Theorem cannot be applied to this recurrence.

Answer(s)

Master Theorem cannot be applied to this recurrence as it only applies when we divide n by some constant in the recursive rule. The recursive rule instead takes the square root of n.

7. Let $\mathrm{Prop} = \{p, q, r, s\}$ be a set of letters that represent propositions. Consider

$$\Sigma = \mathrm{Prop} \cup \{\top, \bot, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, (,), [,], \diamond\}.$$

The better-formed formulas (bffs) over Prop is the smallest subset of Σ^* where:

- (B1) $\top, \bot$ are bffs
- (B2) All elements of Prop are bffs
- (B3) $[p]$ is a bff for all $p \in \mathrm{Prop}$
- (R1) If φ is a bffs, then $\neg\varphi$ is a bff
- (R2) If φ and ψ are bffs, then $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$ and $(\varphi \leftrightarrow \psi)$ are bffs
- (R3) If φ is a bff, then $[\varphi]$ is a bff
- (R4) If $\varphi_1, \varphi_2, \varphi_3$ are bffs, then $(\diamond\varphi_1\varphi_2\varphi_3)$ is a bff

Which of the following are better-formed formulas according to this definition?

(a) $(\neg[p] \wedge (\diamond qrs))$

(b) $(\diamond[p]q[\neg r])$

(c) $((\diamond pq) \rightarrow s)$

(d) $(\neg(\diamond pqr) \vee [\diamond pqr])$

Answer(s)

- $(\neg[p] \wedge (\diamond qrs))$: This is a bff.
- $(\diamond[p]q[\neg r])$: This is a bff.
- $((\diamond pq) \rightarrow s)$: This is not a bff. The $\diamond$ operator requires 3 inputs but we have 2 in $(\diamond pq)$.
- $(\neg(\diamond pqr)\vee[\diamond pqr])$: This is not a bff. The term $[\diamond pqr]$ is not a bff as the rule requires brackets $()$ around the $\diamond$ operator.

8. Let $\Sigma = \{a, b\}$. We define function $f : \Sigma^* \rightarrow \mathbb{N}$ as

$$\begin{array}{ll} \text{(B)} & f(\lambda) = 0 \\ \text{(R)} & f(aw) = 2 + f(w) \\ \text{(R)} & f(bw) = 1 + f(w) \end{array}$$

Where $|w|_a, |w|_b$ denotes the number of a's and b's in word w respectively.

Which of the following properties are true?

(a) $f(w) = 2|w|_a + |w|_b$

(b) $f(w_1w_2) = f(w_1) + f(w_2)$

(c) $f(w_1w_2) = f(w_1) + f(w_2) + |w_1|_a$

(d) $f(w_1w_2) = f(w_1) + f(w_2) + |w_2|_a$

Answer(s)

(a) We find that $f(w) = 2|w|_a + |w|_b$ is true. This can be proven by structural induction.

(b) This can be proven with the closed form of $f(w)$.

(c) Consider the counterexample $f(a) \neq f(a) + f(\lambda) + |a|_a$.

(d) Consider the counterexample $f(a) \neq f(\lambda) + f(a) + |a|_a$.