Quiz 1 Solutions: Number Theory

1. Suppose that a, b are positive real numbers and c is a positive integer. Compute $\left\lfloor \frac{\left\lfloor \frac{a}{b} \right\rfloor}{c} \right\rfloor$:

   a) $\left\lfloor \dfrac{a}{bc} \right\rfloor$

   b) $\left\lfloor \dfrac{a}{\lfloor bc \rfloor} \right\rfloor$

   c) $\left\lceil \dfrac{a}{bc} \right\rceil$

   d) $\left\lceil \dfrac{a}{\lceil bc \rceil} \right\rceil$

Answer(s)

We can use the fact that $\left\lfloor \frac{\lfloor r \rfloor}{n} \right\rfloor = \left\lfloor \frac{r}{n} \right\rfloor$ for $r \in \mathbb{R}$ and $n \in \mathbb{N}$.

Proof: Let $\left\lfloor \frac{\lfloor r \rfloor}{n} \right\rfloor = x \in \mathbb{Z}$. Then we get

$$\frac{\lfloor r \rfloor}{n} = x + y$$

where $y \in \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{(n-1)}{n}\right\}$. Hence,

$$\lfloor r \rfloor = n(x + y) \text{ so } r = n(x + y) + z$$

where $z \in [0, 1)$. We get

$$\frac{r}{n} = x + y + \frac{c}{n}.$$

We find that $y + \frac{c}{n} \in \left\{\frac{c}{n}, \frac{c+1}{n}, \frac{c+1}{n}, \ldots, \frac{c+(n-1)}{n}\right\} \subseteq [0, 1)$. Therefore

$$\left\lfloor \frac{r}{n} \right\rfloor = x = \left\lfloor \frac{\lfloor r \rfloor}{n} \right\rfloor.$$

This would give us the answer that $\left\lfloor \frac{\left\lfloor \frac{a}{b} \right\rfloor}{c} \right\rfloor = \left\lfloor \frac{a}{bc} \right\rfloor$.

2. Suppose that a, b are positive real numbers and c is a positive integer. Compute $\left\lceil \frac{\left\lceil \frac{a}{b} \right\rceil}{c} \right\rceil$:

   a) $\left\lfloor \frac{a}{bc} \right\rfloor$

   b) $\left\lfloor \frac{a}{\lfloor bc \rfloor} \right\rfloor$

   c) $\left\lceil \frac{a}{bc} \right\rceil$

   d) $\left\lceil \frac{a}{\lceil bc \rceil} \right\rceil$

Answer(s)

The answer is $\left\lceil \frac{\lceil \frac{a}{b} \rceil}{c} \right\rceil = \lceil \frac{a}{bc} \rceil$. This question uses a property and proof similar to the question above. See if you can write out the property and proof!

3. $13 \ \% \ 3 =$

   $155 \ \% \ 19 =$

   $(-97) \ \% \ 11 =$

Answer(s)

$13 \ \% \ 3 = 1$ as $13 = 4 \times 3 + 1$

$155 \ \% \ 19 = 3$ as $155 = 8 \times 19 + 3$

$(-97) \ \% \ 11 = 2$ as $-97 = -9 \times 11 + 2$

4. Evaluate $2^4 \ \% \ 18$ and hence evaluate $2^{300} \ \% \ 18$.

Answer(s)

Calculating the remainder for powers of 2:

$$\begin{aligned}
2^1 \ \% \ 18 &= 2, \\
2^2 \ \% \ 18 &= 4, \\
2^3 \ \% \ 18 &= 8, \\
2^4 \ \% \ 18 &= 16, \\
2^5 \ \% \ 18 &= 14, \\
2^6 \ \% \ 18 &= 10, \\
2^7 \ \% \ 18 &= 2.
\end{aligned}$$

We have $2^4 \ \% \ 18 = 16$. We also find that the remainder will repeat every time the power increases by 6. To calculate 2^{300}, we find that $300 =_6 6$ and so $2^{300} \ \% \ 18 = 2^6 \ \% \ 18 = 10$.

5. Select True/False for each of the following.

   - 52 is divisible by 13
   - $7 \mid 161$
   - $7 \mid 162$
   - $17 \mid 1001$

   Find the quotient and (non-negative) remainder when

   19 is divided by 7

   -111 is divided by 11

   1001 is divided by 13

Answer(s)

We have $52 = 4 \times 13$ so 52 is divisible by 13.

We have $161 = 23 \times 7$ so $7 \mid 161$.

We have $162 = 23 \times 7 + 1$ so $7 \nmid 162$.

We have $1001 = 58 \times 17 + 15$ so $17 \nmid 1001$.

We have $19 = 2 \times 7 + 5$ so the quotient is 2 and the remainder is 5.

We have $-111 = -11 \times 11 + 10$ so the quotient is -11 and the remainder is 10.

We have $1001 = 77 \times 13$ so the quotient is 77 and the remainder is 0.

6. For integers a, b and c, if $a \mid b$ and $b \mid c$, which of the following are true?
   - $a \leq b$.
   - $\frac{a}{b} \leq \frac{b}{c}$.
   - $a \mid c$.
   - $a \mid i \cdot b + j \cdot c$ for all integers i and j.

Answer(s)

We find that $a \leq b$ is not true as $1 \mid -1$ so we could have $a = 1$ and $b = -1$.

We find that $\frac{a}{b} \leq \frac{b}{c}$ is not true when $a = 1$, $b = 1$ and $c = -1$. We have $a \mid b$ and $b \mid c$, but

$$\frac{a}{b} = 1 \geq -1 = \frac{b}{c}.$$

The statement $a \mid c$ is true.

Proof: Since $a \mid b$, there is an integer k such that $b = ka$. Similarly, since $b \mid c$, there is an integer j such that $c = jb$. We can substitute our first equation into our second to get $c = jka$. Since jk is an integer, $a \mid c$.

The statement $a \mid i \cdot b + j \cdot c$ for all integers i and j is true.

Proof: Let i and j be integers. Since $a \mid b$, there is an integer k such that $b = ka$. Similarly, since $b \mid c$, we have $a \mid c$, using the previous option. This means that $c = la$ for some integer l. Therefore,

$$i \cdot b + j \cdot c = ika + jla = (ik + jl)a.$$

Since $ik + jl$ is an integer, we have $a \mid i \cdot b + j \cdot c$.

7. Let $c = \gcd(a, b)$, then $\gcd(\frac{a}{c}, \frac{b}{c}) =$

Answer(s)

Let $d = \gcd(\frac{a}{c}, \frac{b}{c})$. By definition, $d \mid \frac{a}{c}$ and $d \mid \frac{b}{c}$. Hence, $dc \mid a$ and $dc \mid b$. We find that dc is a common divisor of a and b. As c is the greatest common divisor, we must have $dc \leq c$ and so we get $d \leq 1$. As the $\gcd$ must be a positive integer by definition, the only choice is $d = 1$.

8. Give the value of each of the following.
   - $\lfloor 0.53 \rfloor + \lceil -0.92 \rceil + \lfloor -1.27 \rfloor$
   - $\lfloor |-1.7| \rfloor$
   - $|\lfloor -1.7 \rfloor|$

Answer(s)

We have $\lfloor 0.53 \rfloor + \lceil -0.92 \rceil + \lfloor -1.27 \rfloor = 0 + 0 + -2 = -2$.

We have $\lfloor |-1.7| \rfloor = \lfloor 1.7 \rfloor = 1$.

We have $|\lfloor -1.7 \rfloor| = |-2| = 2$.

9. Which of the following is true for all real numbers x?
   - $\lfloor \lceil x \rceil \rfloor \leq \lceil \lfloor x \rfloor \rceil$
   - $\lceil x \rceil = \lceil \lceil x \rceil \rceil$
   - $\lceil \lfloor x \rfloor \rceil \leq \lfloor \lceil x \rceil \rfloor$
   - $\lfloor x \rfloor < \lceil x \rceil$

Answer(s)

This question mainly uses the fact that when n is an integer, $\lfloor n \rfloor = n = \lceil n \rceil$. This means that since $\lceil x \rceil$ is an integer, $\lfloor \lceil x \rceil \rfloor = \lceil x \rceil$.

- We find that $\lfloor \lceil x \rceil \rfloor \leq \lceil \lfloor x \rfloor \rceil$ is the same as saying $\lceil x \rceil \leq \lfloor x \rfloor$. This is not true when $x = 0.5$.
- Since $\lceil x \rceil$ is an integer, $\lceil \lceil x \rceil \rceil = \lceil x \rceil$.
- We also have $\lceil \lfloor x \rfloor \rceil \leq \lfloor \lceil x \rceil \rfloor$ which simplifies to $\lfloor x \rfloor \leq \lceil x \rceil$, which is always true.
- The statement $\lfloor x \rfloor < \lceil x \rceil$ is not true for integers.

10. Find the greatest common divisor (gcd) and least common multiple (lcm) for the following pairs.

    1372 and 5733

    1001 and 121

Answer(s)

We can use the Euclidean algorithm to calculate $\gcd$. We have

$$\begin{aligned}\gcd(5733, 1372) &= \gcd(245, 1372)\\ &= \gcd(245, 147)\\ &= \gcd(98, 147)\\ &= \gcd(98, 49)\\ &= \gcd(0, 49) = 49.\end{aligned}$$

We use the formula $\mathrm{lcm}(a, b) = \frac{|ab|}{\gcd(a,b)}$ to find that $\mathrm{lcm}(5733, 1372) = 160524$.

We can use the same method for the second part to get

$$\gcd(1001, 121) = 11 \text{ and } \mathrm{lcm}(1001, 121) = 11011.$$