CSC 225
Practice Midterm Exam 1B

Name:______SOLUTION KEY______(please print clearly!)

UVic ID number:________________________

Signature:________________________

Exam duration: 50 minutes

Instructor: Anthony Estey

Students must check the number of pages in this examination paper before beginning to write, and report any discrepancy immediately.

- We will not answer questions during the exam. If you feel there is an error or ambiguity, write your assumption and answer the question based on that assumption.
- Answer all questions on this exam paper.
- The exam is closed book. No books or notes are permitted.
  No electronic devices of any type are permitted.
- The marks assigned to each question and to each part of a question are printed within brackets. Partial marks are available.
- There are eight (8) pages in this document, including this cover page.
- Page 8 is left blank for scratch work. If you write an answer on that page, clearly indicate this for the grader under the corresponding question.
- Clearly indicate only one answer to be graded. Questions with more than one answer will be given a **zero grade**.
- It is strongly recommended that you read the entire exam through from beginning to end before beginning to answer the questions.
- Please have your ID card available on the desk.

Part 1: Discrete Math (10 marks)

1) Ali rolls a set of six-sided dice that are indistinguishable from one another.

   Ali rolls 5 twos, 6 threes, and 7 sixes. Answer the following questions about the ways Ali can arrange the dice in a line on a table. Assume the dice are never re-rolled.

   You do not need to show work, but incorrect answers may receive part marks. You also do not need to calculate answers, and can leave them in factorial and/or 'n choose r' form. For example: 2! * C(3,2)

   a) How many arrangements of the dice are there?

There are $5+6+7=18$ total dice rolled, but Ali can't differentiate between dice rolls of the same number:

$$\frac{\mathbf{(5+6+7)!}}{\mathbf{5!\,6!\,7!}}=\frac{\mathbf{18!}}{\mathbf{5!\,6!\,7!}}$$

   b) How many arrangements are there with all of the threes together?

Group all the threes together, call it X, and $X=1$, now there are $5+1+7$ items to arrange in a line. Ali still cannot differentiate between the twos or sixes:

$$\frac{\mathbf{(5+1+7)!}}{\mathbf{5!\,1!\,7!}}=\frac{\mathbf{13!}}{\mathbf{5!\,7!}}$$

   c) How many arrangements can Ali arrange the dice in ascending order (from lowest to highest number rolled)?

There is only one way to arrange the dice in ascending order if we cannot differentiate between dice that rolled the same number: the twos come first, then threes, then sixes.

22222 333333 6666666

There is **1** arrangement.

   d) How many ways could Ali arrange the dice in ascending order if Ali could differentiate between the dice that rolled the same number?

Now Ali can differentiate between the dice that rolled the same number. There are 5! ways to organize the twos, 6! ways to organize the threes, and 7! ways to organize the sixes:

$$\mathbf{5!\,6!\,7!}$$

2) Determine the number of times the Print("first") and Print("second") statements execute relative to input size n.

```
Algorithm RuntimeAnalysis(n)
  for i ← 1 to n do
    for j ← 1 to 3i do
      Print("first")
    end
    for j ← 1 to n do
      Print("second")
    end
  end
```

Number of times "first" is printed in each iteration of outer loop:
$3, 6, 9, 12, \dots, 3n$
This is $n/2$ pairs that sum to $3n + 3$: $\frac{n(3n+3)}{2} = \boldsymbol{\frac{3n^2+3n}{2}}$

Number of times "second" is printed in each iteration of outer loop:
$n, n, n, \dots, n$
So second is printed n times for each of the n iterations of the outer loop: $\boldsymbol{n^2}$

3) For this question we will only count **assignment statements.** Count a **return** statement as **one** assignment statement.

Write the recurrence equation for the following algorithm:

Algorithm recursiveFn(n):
    if $n \leq 1$ **then**
        $base \leftarrow 3$
        return $base$
    else
        $result1 \leftarrow$ recursiveFn$(n-1)$
        $result2 \leftarrow$ recursiveFn$(n-1)$
        return $result1 + result2$
    end

Case when $n = 0$ has 2 assignments: assigning $base$ to 3, and the return.
Case when $n \geq 1$ has 3 assignments and 2 recursive calls:

$$T(n) = \begin{cases} 2, & n = 1 \\ 2T(n-1) + 3, & n \geq 2 \end{cases}$$

4) Given the following recurrence equation:

$$T(n) = \begin{cases} 8, & n = 1 \\ T(n-1) + 4, & n \geq 2 \end{cases}$$

a) What is T(4)?

$T(4) = T(3) + 4 = (T(2) + 4) + 4 = (T(1) + 4) + 4 + 4) = (8) + 4 + 4 + 4 = 20$

b) Solve the recurrence equation by repeated substitution using a top-down approach. Show all work.

$T(n) = T(n-1) + 4$
$T(n) = (T(n-2) + 4) + 4$
$T(n) = (T(n-3) + 4) + 4 + 4$
$T(n) = (T(n-4) + 4) + 4 + 4 + 4$
$T(n) = (T(n-5) + 4) + 4 + 4 + 4 + 4$
$\quad = T(n-5) + 5(4)$

General form: $T(n) = T(n-i) + i(4)$

The known case is when $n = 1$, so we need to use substitution to get the $T(n-i)$ so that it is $T(1)$. To do this, we need $n - i = 1$. Solving for i, we have $i = n - 1$

When we substitute $n - 1$ for all i in the equation above, we have:
$T(n) = T\big(n - (n-1)\big) + (n-1)4$
$\quad = T(1) + 4n - 4$
$\quad = 8 + 4n - 4$
$\quad = 4n + 4$

5) Provide constants c and n_0 to show that $5n^3 - 17n^2 + 4n - 8$ is Big-Oh(n^3).

Show all work.

$5\mathrm{n}^3 - 17\mathrm{n}^2 + 4\mathrm{n} - 8$
$\leq 5n^3 + 4n$
$\leq 5n^3 + 4n^3$ for all $n \geq 1$.
$= 9n^3$
Therefore, $5\mathrm{n}^3 - 17\mathrm{n}^2 + 4\mathrm{n} - 8$ is $O(n^3)$ with $c = 9$ and $n_0 = 1$

6) Provide constants c and n0 to show that $(13n^3 - 4)/3n$ is Big-Omega(n^2).

Show all work.

$(13n^3 - 4)/3n$
$\geq (13n^3 - 4n^3)/3n$ for all $n \geq 1$
$= (9n^3)/3n$
$= 3n^2$
Therefore, $(13n^3 - 4)/3n$ is $\Omega(n^2)$ with $c = 3$ and $n_0 = 1$

7) Assume you had to prove the following by induction:

For all $n \geq 1$, $1(1!) + 2(2!) + \ldots + n(n!) = (n+1)! - 1$.

Write the Base Case and Inductive Hypothesis below to begin the proof by induction.

Base Case (when $n = 1$):

LHS: $1(1!) = 1(1) = 1$
RHS: $(1 + 1)! - 1 = (2)! - 1 = 2 - 1 = 1$ = LHS

Inductive Hypothesis:

Assume $1(1!) + 2(2!) + \ldots + k(k!) = (k+1)! - 1$ holds for some $k \geq 1$.

8) This term we have explored a number of problems where we had to determine the number of integer solutions for a sum of terms expression. For example:

$x_1 + x_2 + x_3 + x_4 + x_5 = 18$,
where $x_i \geq 0$, for $1 \leq i \leq 5$.

For this question, you will need to need to finish writing a sum of terms expression based on the following criteria:

Ali wants to give 9 candies to 4 friends: Sam, Lee, Juan, and Tia. Ali promised Sam and Lee 2 candies each. Juan has 3 candies of his own that he said Ali could distribute in addition to the original 9.

There are multiple valid ways we could write this as a sum of integers expression to satisfy the conditions stated above (about how Ali will distribute the candies). Fill in the blanks to finish the sum of terms expression.

$x_1 + x_2 + x_3 + x_4 =$ **8**
where $x_i \geq 0$, for $1 \leq i \leq 4$.

Writing the expression in this form where all $x_i \geq 0$ must me we have already given 2 candies to both Sam and Lee (otherwise x_1 and x_2 would need to be ≥ 2. It also means we have received 3 additional candies from Juan to distribute.

So, from our original 9 we gave 4 away (2 each to Sam and Lee), and received 3 extra from Juan, leaving us with $9 - 4 + 3 = 8$ candies left to distribute among 4 people.

Alternatively, this could be written as:
$x_1 + x_2 + x_3 + x_4 = 9$
where $x_1 \geq$ **2** $x_2 \geq$ **2** $x_3 \geq$ **-3** $x_4 \geq$ **0**

Writing the expression in this form we need to think about conditions stated above.

From the original 9 candies, Sam and Lee are promised at least 2. This can be represented as $x_1, x_2 \geq 2$.

On the other hand, Juan is donating 3 candies, so at the end of the day Juan could end up with less candies than he originally came with. This is represented with $x_3 \geq -3$.

Nothing has changed with Tia, she will still get at least 0 candies.

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END OF EXAM

Question	**Value**	**Mark**
Question 1	8	
Question 2	6	
Question 3	4	
Question 4	8	
Question 5	3	
Question 6	3	
Question 7	2	
Question 8	6	
Total	**40**	