Big-Oh, Big-Omega, Big-Theta, Little-Oh, and Little-Omega

1) Show that $n\log n \in O(n^2)$

$n\log n \le n^2$ for all $n \ge 1$

$\therefore f(n)\in O(n^2)$ with $c, n_0 = 1$

$\lim_{n\to\infty}\frac{n\log n}{n^2}=\lim_{n\to\infty}\frac{\log n}{n}=\frac{\infty}{\infty}$. Use L'Hospital's rule:

In the case $\lim_{n\to\infty}\frac{f(n)}{g(n)}=\frac{\infty}{\infty}$, we have: $\lim_{n\to\infty}\frac{f(n)}{g(n)}=\lim_{n\to\infty}\frac{f'(n)}{g'(n)}$

So, with the derivatives: $\lim_{n\to\infty}\frac{\left(\frac{1}{n\ln 2}\right)}{(1)}=\lim_{n\to\infty}\frac{1}{n\ln 2}=0$

So, $n\log n\in o(n^2)$
This means $n\log n\notin \Omega(n^2)$, and so $n\log n \notin \Theta(n^2)$

2) Show that $10^{-10}n^2 \in \Theta(n^2)$

Let $c = 10^{-10}$ and $n_0 = 0$.
$T(n)\le cn^2$ for all $n\ge n_0$. $\therefore 10^{-10}n^2\in O(n^2)$
$T(n)\ge cn^2$ for all $n\ge n_0$. $\therefore 10^{-10}n^2\in \Omega(n^2)$

Using limits:
$\lim_{n\to\infty}\frac{10^{-10}n^2}{n^2}=\lim_{n\to\infty}\frac{10^{-10}}{1}=10^{-10}$

$\therefore 10^{-10}n^2\in\Theta(n^2)$

Since $10^{-10}n^2\in O(n^2)$ and $10^{-10}n^2\in\Omega(n^2)$, $\underline{10^{-10}n^2\in\Theta(n^2)}$

3) Show that $n\log n \in \Omega(n)$

$n\log n\ge n$ for all $n\ge 2$

$\therefore n\log n\in\Omega(n)$ with $c=1, n_0=2$

$\lim_{n\to\infty}\frac{n\log n}{n}=\lim_{n\to\infty}\frac{\log n}{1}=\infty$

$\therefore n\log n\in\omega(n^2)$, so
$n\log n\notin O(n^2)$, and so $n\log n\notin\Theta(n^2)$

4) If $d(n)$ is $\Omega(f(n))$ and $e(n)$ is $\Omega(f(n))$ then is it always true that $d(n)+e(n)$ is $\Omega(f(n))$? If so, prove it. If not, provide a counterexample.

$d(n)\in\Omega(f(n))$ means there exists positive constants c and n_0 such that:
$d(n)\ge cf(n)$ for all $n\ge n_0$

Similarly, $e(n)\in\Omega(f(n))$ means there exists positive constants c' and n_0' such that:
$e(n)\ge c'f(n)$ for all $n\ge n_0'$

This means that $d(n)+e(n)\ge cf(n)+c'f(n)$ for all $n\ge \max\{n_0, n_0'\}$
$=(c+c')(f(n))$ so our new constant is $c+c'$

Therefore, we have satisfied the definition for Big-Omega.

5) Show that $n^3+4\in o(n^4)$

$$\lim_{n\to\infty}\frac{T(n)}{f(n)}=\lim_{n\to\infty}\frac{n^3+4}{n^4}=\lim_{n\to\infty}\frac{n^3}{n^4}+\lim_{n\to\infty}\frac{4}{n^4}=0+0=0$$

$\therefore n^3+4\in o(n^4)$

6) Show that $n^3+4\in\omega(n^2)$

$$\lim_{n\to\infty}\frac{T(n)}{f(n)}=\lim_{n\to\infty}\frac{n^3+4}{n^2}=\lim_{n\to\infty}\frac{n^3}{n^2}+\lim_{n\to\infty}\frac{4}{n^2}=\lim_{n\to\infty}n+\lim_{n\to\infty}\frac{4}{n^2}=\infty+0=\infty$$

$\therefore n^3+4\in\omega(n^2)$

7) Find a tight bound on $T(n)=\log n!$

1) Big-Oh: $T(n)=\log(n*(n-1)*(n-2)*\ldots*2*1)$
$=\log n+\log n-1+\log n-2+\cdots+\log 2+\log 1$
$\le \log n+\log n+\log n+\cdots+\log n+\log n$
$=n\log n$
$\therefore T(n)\in O(n\log n)$ with $c, n=1$

2) Big-Omega: $T(n)=\log n+\log n-1+\log n-2+\cdots+\log 2+\log 1$
$\ge \log\frac{n}{2}+\log\frac{n}{2}+\log\frac{n}{2}+\cdots+0+0$
$=n/2(\log n/2)$
$=n/2(\log n-\log 2)$
$=n/2(\log n-1)$
$\ge n/4\log n$ for all $n\ge 4$
$\therefore T(n)\in O(n\log n)$ with $c=1/4$ and $n_0=4$

Since $T(n)\in O(n\log n)$ and $T(n)\in O(n\log n)$, $T(n)\in\Theta(n\log n)$