CSC 225
Practice Midterm Exam 1A

Name:_________SOLUTION KEY_________(please print clearly!)

UVic ID number:______________________________

Signature:______________________________

Exam duration: 50 minutes

Instructor: Anthony Estey

Students must check the number of pages in this examination paper before beginning to write, and report any discrepancy immediately.

- We will not answer questions during the exam. If you feel there is an error or ambiguity, write your assumption and answer the question based on that assumption.
- Answer all questions on this exam paper.
- The exam is closed book. No books or notes are permitted.
  No electronic devices of any type are permitted.
- The marks assigned to each question and to each part of a question are printed within brackets. Partial marks are available.
- There are eight (8) pages in this document, including this cover page.
- Page 8 is left blank for scratch work. If you write an answer on that page, clearly indicate this for the grader under the corresponding question.
- Clearly indicate only one answer to be graded. Questions with more than one answer will be given a **zero grade**.
- It is strongly recommended that you read the entire exam through from beginning to end before beginning to answer the questions.
- Please have your ID card available on the desk.

Part 1: Discrete Math (10 marks)

1) Ali buys a bag of sour candies. When Ali pours all of the candies out, there are 3 red candies, 2 blue candies, and 4 orange candies. Ali decides to arrange the candies in a line on the table (and is unable to differentiate between candies of the same colour).

   Answer the following questions about the ways Ali can arrange the candies in a line on the table. You do not need to show work, but incorrect answers may receive part marks.

   a) How many arrangements of the candies are there?

Remember: Ali cannot differentiate between the groups of candies of the same colour, therefore we must make sure we do not over count:

$$\frac{(\mathbf{3}+\mathbf{2}+\mathbf{4})!}{\mathbf{3}!\,\mathbf{2}!\,\mathbf{4}!}=\frac{\mathbf{9}!}{\mathbf{3}!\,\mathbf{2}!\,\mathbf{4}!}$$

   b) How many arrangements are there with all of the red candies together?

Group all the red candies together. Still cannot differentiate between blue and orange candies:

$$\frac{(\mathbf{1}+\mathbf{2}+\mathbf{4})!}{\mathbf{1}!\,\mathbf{2}!\,\mathbf{4}!}=\frac{\mathbf{7}!}{\mathbf{2}!\,\mathbf{4}!}$$

   c) How many arrangements are there with no red candies beside one another?

First we organize the blue and orange candies: $\frac{(2+4)!}{2!4!}$

Next, the red candies can be placed on the outside or between any of the candies (but not together). There are $(2+4+1)$ total spots to place the red candies, and there are 3 red candies to place: so there are $\binom{7}{3}$ ways to place the red candies.

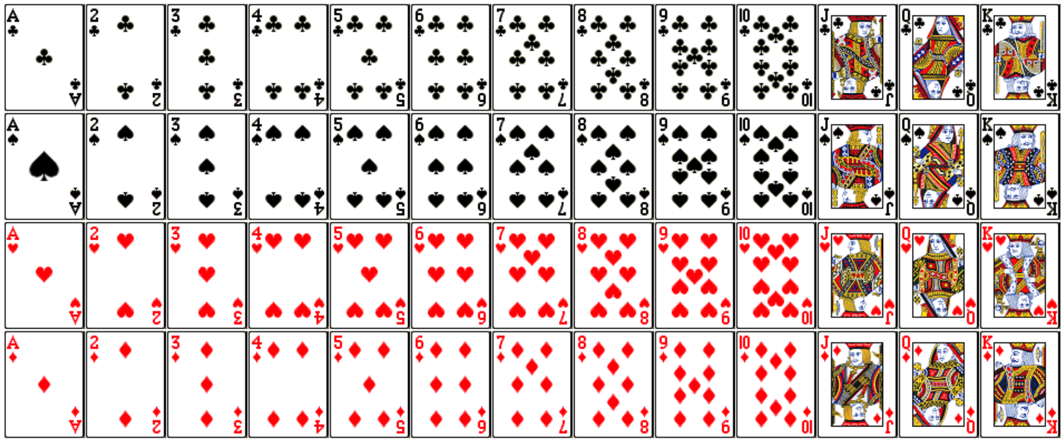
$$\frac{\mathbf{6}!}{\mathbf{2}!\,\mathbf{4}!}*\binom{7}{3}$$

   d) Ali now finds a way to differentiate between candies of the same colour. How many ways can Ali arrange the candies now?

Let $n = 3+2+4$. Now there are n candies that can all be uniquely identified. n ways to choose the first in line, $n-1$ to choose the second, $n-2$ to choose the third, etc.
So, there are $n*(n-1)*(n-2)*\ldots*2*1 = n!$ arrangents

$$\boldsymbol{n}! = (\mathbf{3}+\mathbf{2}+\mathbf{4})! = \mathbf{9}!$$

2) A standard deck of cards has 13 ranks (A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K) in each of the 4 suits (clubs, spades, hearts, and diamonds), as shown below.



Ali draws one card at a time from a standard deck of 52 cards. How many cards will Ali need to draw before Ali is guaranteed to have a card from 3 different suits?

Our pigeonholes are the different suits. Each pigeon hole has room for 13 cards (a spot for each rank A, 2, 3, …, J, Q, K).

By the pigeonhole principle, when the 27th card is draw, there will be pigeons in at least three pigeonholes, meaning there must be a card from at least 3 different suits (the first 26 cards could fill just two pigeonholes if all cards were drawn from just 2 suits).

Answer: 27

Part 2: Runtime Analysis (15 marks)

3) Order the following functions by order of growth from slowest to fastest.

$1000n,\qquad \log n^4,\qquad 3n^2,\qquad 2^n,\qquad 2^{\log n},\qquad n!$

$\log n^4$, $2^{\log n}$, $1000n$, $3n^2$, 2^n, $n!$

4) Which of the following for-loops will result in a $\Theta(\log n)$ runtime? (circle all that apply)

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for (int i = 1; i <= n; i++)

for (int i = 1; i <= n/2; i++)

for (int i = 1; i <= n; i+=i)

for (int i = 1; i <= n; i+=2)

for (int i = 1; i <= n; i*=2)

for (int i = 1; i <= n; i*=i)

for (int i = n; i >= 1; i--)

for (int i = n/2; i >= 1; i--)

for (int i = n; i >= 1; i-=2)

for (int i = n; i >= 1; i-=i)

for (int i = n; i >= 1; i/=2)

for (int i = n; i >= 1; i/=i)
```

5) Read through the pseudocode below:

$sum \leftarrow 0$
for $i \leftarrow 1$ **to** n^3 **do**
    for $j \leftarrow 1$ **to** $4i$ **do**
        if $j < i$ **then**
            $sum \leftarrow sum + j$
        end
    end
end

a) Count the number of times the **condition** in the **if-statement** is executed. Express your answer similar to how we did during lecture this term (e.g. $T(n) = ...$)

The inner loop repeats $4 + 8 + 12 + 16 + \cdots + 4(n^3 - 1) + 4n^3$ times.
This can be written as $4(1 + 2 + 3 + 4 + \cdots + (n^3 - 1) + n^3)$ times.
This is 4 times the $\frac{n^3}{2}$ pairs that sum to $n^3 + 1$:
Which is $4\sum_{i=1}^{n^3} i = 4 * \frac{n^3(n^3+1)}{2} = 4 * \frac{n^6+n^3}{2} = 2n^6 + 2n^3$

$$\boldsymbol{T(n) = 2n^6 + 2n^3}$$

b) What is the time complexity for the number of times the condition in the if-statement is executed? Write your solution in big-Theta notation and provide constants c and n_0.

$T(n) = 2n^6 + 2n^3 \leq 2n^6 + 2n^3$ for all $n \geq 0$
$\leq 4n^6$
$T(n) \in O(n^6)$ with $c = 4$ and $n_0 = 0$

$T(n) = 2n^6 + 2n^3 \geq 2n^6$ for all $n \geq 0$
$T(n) \in \Omega(n^6)$ with $c = 2$ and $n_0 = 0$

Since $T(n) \in O(n^6)$ and $T(n) \in \Omega(n^6)$ then $T(n) \in \Theta(n^6)$

Part 3: Proofs (9 marks)

6) In some sporting events, players enter the sporting area one-by-one before the game starts. Each player enters one at a time, and when each player enters, the player high fives all of the players who have entered before them.

   The first player will not high-five anyone; the second player will high-five the first player (who had previously entered); the third player will high-five both players who previously entered (the first and second players), etc.

   Prove by induction that for a team with n players, $\frac{n(n-1)}{2}$ high-fives occur.

   a) **Base case** (a team with only 1 player):

LHS: A team with 1 player will have 0 people to high-five.
RHS $= \frac{n(n-1)}{2} = \frac{1(1-1)}{2} = \frac{1(0)}{2} = 0 =$ LHS

   b) **Inductive Hypothesis**:

Assume for a team of size k that $\frac{k(k-1)}{2}$ high-fives occur.

   c) Write the **Inductive Step** and then complete the proof:

Need to show that for a team of size $k+1$ that $\frac{(k+1)((k+1)-1)}{2} = \frac{k(k+1)}{2}$ high-fives occur.

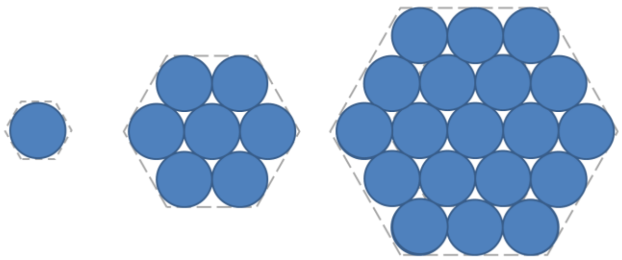
From the I.H. we know that for a team of size k that $\frac{k(k-1)}{2}$ high-fives occur.
When the $k+1$th player enters that player will high-five all k players who previously entered.

Thus, we have
$$\begin{aligned}\frac{k(k-1)}{2} + k &= \frac{k(k-1)}{2} + \frac{2k}{2} \\ &= \frac{k^2-k+2k}{2} \\ &= \frac{k^2+k}{2} \\ &= \frac{k(k+1)}{2}\end{aligned}$$

This shows that for a team of $k+1$ players that $\frac{k(k+1)}{2}$ high-fives occur, completing the proof.

Part 4: Recurrence Relations (11 marks)

$n = 1$ $n = 2$ $n = 3$



1 circle 7 circles 19 circles

Notice the pattern:

# circles	difference
1	
7	6
19	12
37	18
61	24
97	30
127	36

7) A recurrence relation can be written to express the number of circles in the illustration above. For example, we have $C(1) = 1, C(2) = 7, C(3) = 19$.

a) How many circles will there be when $n = 7$?

$C(7) = 127$

b) Fill in the blanks to complete the recurrence equation for the number of circles:

$$C(n) = \begin{cases} \boxed{1} & \text{when } n = 1 \\ \boxed{1}\,C(\boxed{n-1}) + \boxed{6(n-1)} & \text{when } n \geq 2 \end{cases}$$

c) Solve the recurrence equation by repeated substitution.

$C(n) = C(n-1) + 6(n-1)$
$C(n-1) = C(n-2) + 6(n-2)$
$C(n-2) = C(n-3) + 6(n-3)$
…

$C(n) = C(n-1) + 6(n-1)$ substituting for $(n-1)$ above we get:
$C(n) = C(n-2) + 6(n-2) + 6(n-1)$
$C(n) = C(n-3) + 6(n-3) + 6(n-2) + 6(n-1)$
…
$C(n) = C(n-i) + 6(n-i) + 6(n-i-1) + \cdots + 6(n-2) + 6(n-1)$

Let $i = n-1$
$C(n) = C(n-(n-1)) + 6(n-(n-1)) + 6(n-(n-1)-1) + \cdots + 6(n-1)$
$C(n) = C(1) + 6(1)) + 6(2) + \cdots + 6(n-2) + 6(n-1)$
$C(n) = C(1) + 6(1 + 2 + 3 + \cdots + n - 2 + n - 1)$
$C(n) = 1 + 6\left(\frac{n(n-1)}{2}\right) = 1 + 3\big(n(n-1)\big) = 1 + 3n^2 - 3n$
$\boldsymbol{C(n) = 3n^2 - 3n + 1}$

… Left blank for scratch work…

END OF EXAM

Question	**Value**	**Mark**
Part 1	10	
Part 2	15	
Part 3	9	
Part 4	11	
Total	**45**	