Unit 01: Discrete Math

Anthony Estey

CSC 225: Algorithms and Data Structures I

University of Victoria

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Unit 01 Overview

- Supplemental Reading:
  - Discrete and Combinatorial Mathematics. Fifth Ed. *Ralph P. Grimaldi*
    - Chapter 1, Sections 1.1 – 1.4
- Learning Objectives: (You should be able to…)
  - identify when to use the rule of product or rule of sum to solve a problem
  - identify when to use permutations or combinations to solve a problem
  - use the binomial theorem to expand and/or determine coefficient(s) in a polynomial expression
  - use the pigeonhole principle to solve counting problems

The Rule of Sum

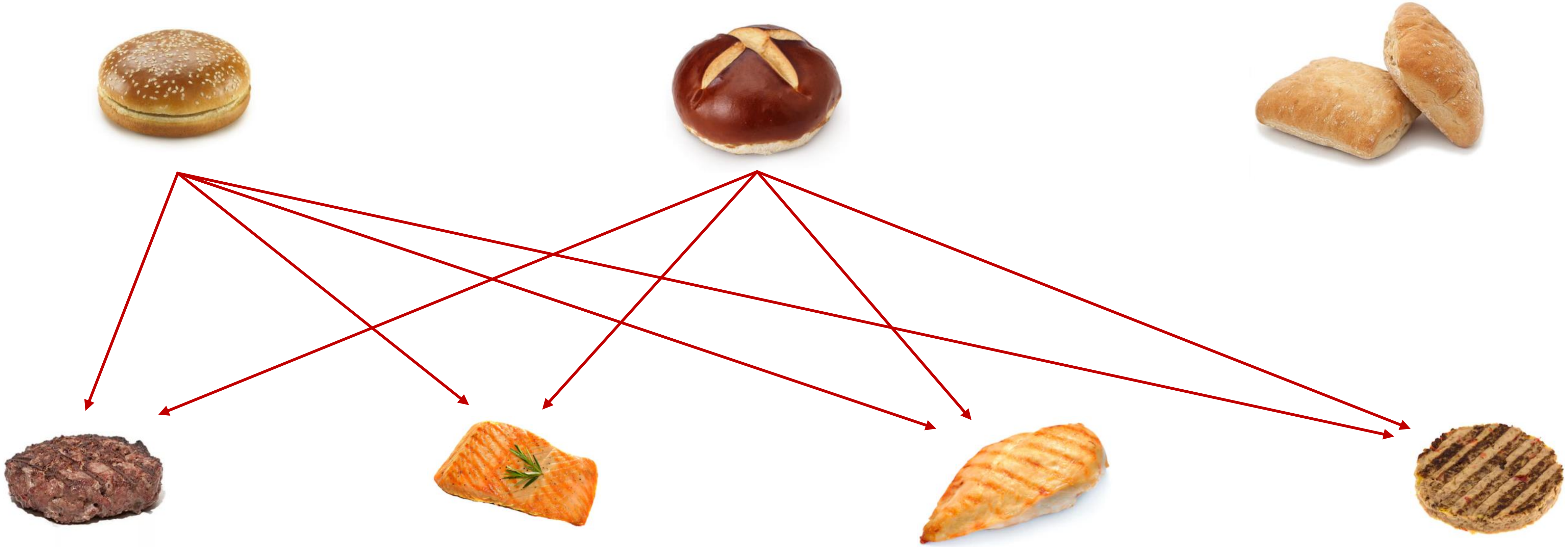
- From textbook:
  - If a first task can be performed in m ways, while a second task can be performed in n ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in any of $m + n$ ways
- From Wikipedia:
  - it is the idea that if we have A ways of doing something and B ways of doing another thing and we can not do both at the same time, then there are $A + B$ ways to choose one of the actions.
- Example:
  - If I choose to eat at a restaurant there are 3 different burgers on the menu, whereas a different restaurant has 2 burgers on the menu. Overall, there are 5 (3 + 2) different burgers I can choose from.

The Rule of Product

- From textbook:
  - If a first task can be performed in m ways, while a second task can be performed in n ways, and the two tasks can be performed simultaneously, then performing both tasks can be accomplished in any one of $m \cdot n$ ways
- From Wikipedia:
  - it is the idea that if there are *A* ways of doing one thing and *B* ways of doing another thing, then there are *A·B* ways of performing both actions.
- Example:
  - If the burger I choose to order has *3* different options for the type of bun, and *4* different options for the "meat", then there are *12 (3x4)* different ways I could customize my burger

Rule of Product: Visualization

$4 + 4 + 4 = 4 * 3 = 12$



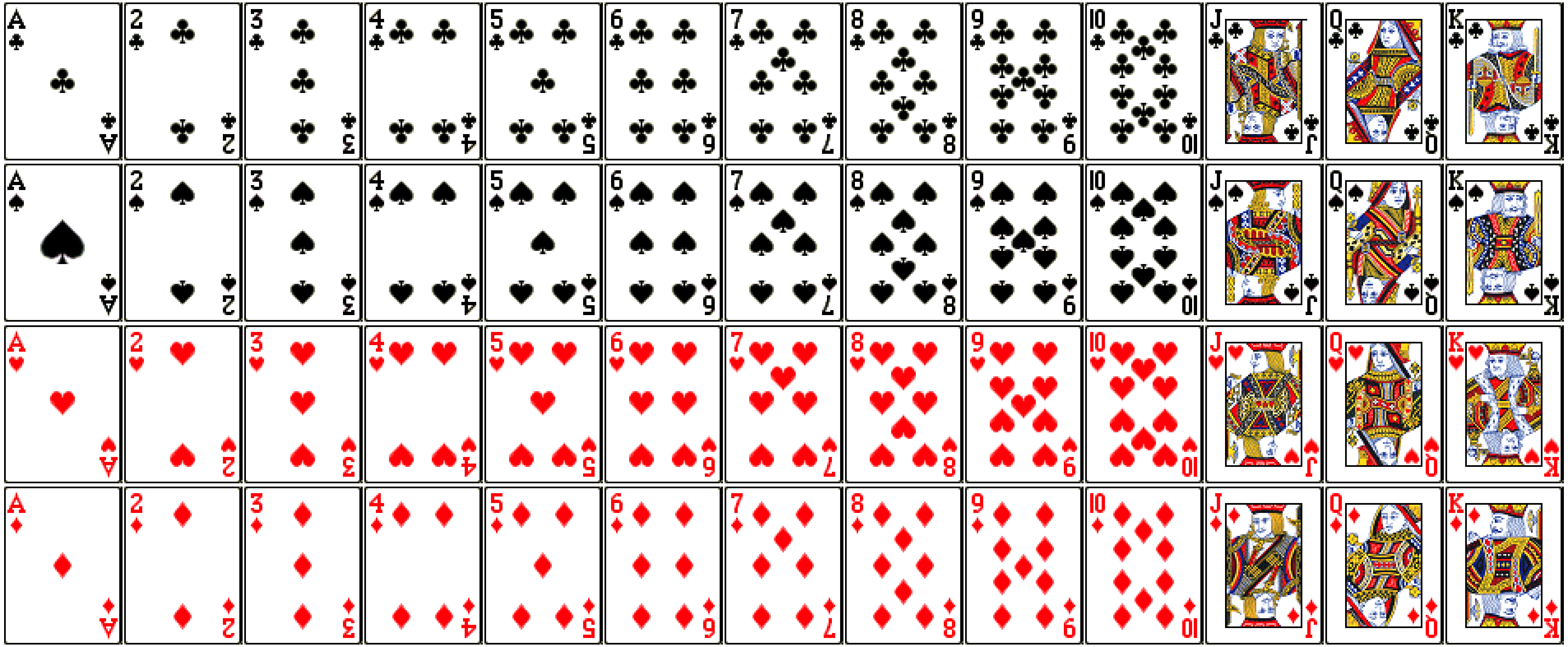
bun images taken from https://www.themanual.com/food-and-drink/best-burger-buns/

Exercises

- A computer science course is made up of two groups of students: 30 students majoring in computer science, and 20 students who are majoring in a different program.
    a) How many options are there to pick a student representative for the class?
    b) How many ways could I choose a pair of representatives, where there must be 1 student from both groups?

- Assume a third group of 10 students is added, for students who are majoring in the Health Informatics program
    a) How many options are there to pick a student representative for the class?
    b) How many ways could I choose a group of three representatives, where there must be 1 student from each group?

Exercises



- How many ways can I choose 1 card?
- How many ways can I choose a black card followed by a red card?

Permutations

- From textbook:
  - An application of the Rule of Product when counting linear arrangements of distinct objects.
- From Wikipedia:
  - a permutation of a set is an arrangement of its members into a sequence or linear order

Permutations

What are some display options?

- Example:
  - Assume a restaurant has 5 daily specials (call them A, B, C, D, and E) and they want to layout information about the specials along their advertisement banner. How many different arrangements are possible?

DAILY SPECIALS

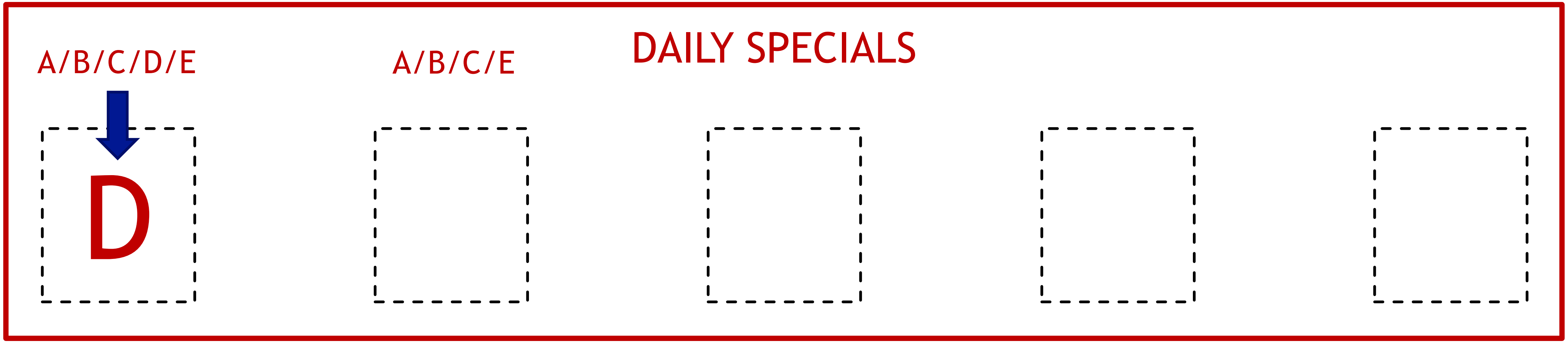
B	A	E	C	D

Permutations

How many options do we have for the first box?

Assume we choose D (but this choice is arbitrary), how many choices do we have for the second box?

- Example:
  - Assume a restaurant has 5 daily specials (call them A, B, C, D, and E) and they want to layout information about the specials along their advertisement banner. How many different arrangements are possible?



5 4

Permutations

- Example:
  - Assume a restaurant has 5 daily specials (call them A, B, C, D, and E) and they want to layout information about the specials along their advertisement banner. How many different arrangements are possible?

- Answer:
  - Each of the 5 specials could go in the first spot. For the next spot, we would select one of the remaining 4 specials. Continuing this way, there will be 3 options, then 2 options, and finally 1 option for the final spot.
  - This yields a total of 120 different arrangements (5 x 4 x 3 x 2 x 1)

Factorial

- For an integer $n \geq 0$, *n factorial* (denoted *n!*) is defined as:

  0! = 1

  n! = $(n)x(n-1)x(n-2)\ldots(3)x(2)x(1)$, for $n \geq 1$

- Examples:

  3! = $(3)x(2)x(1)$ = 6

  8! = $(8)x(7)x(6)x(5)x(4)x(3)x(2)x(1)$ = 40,320

- We could write our solution to the last problem as 5!

Factorial (cont'd)

- Observe:

$$\frac{8!}{4!} = 8x7x6x5$$

$$= \frac{8x7x6x5x4x3x2x1}{4x3x2x1}$$

$$= 8x7x6x5x\frac{\cancel{4x3x2x1}}{\cancel{4x3x2x1}}$$

$$= \frac{8!}{4!}$$

Fundamental Principles of Counting

- In general, the number of permutations of size r from n distinct objects, where $0 \leq r \leq n$, is given by:

$$P(n,r) = \frac{n!}{(n-r)!}$$

5 different daily specials

We wanted to place 5 items on the banner

Restaurant example:

Assume a restaurant has 5 daily specials (call them A, B, C, D, and E) and they want to layout information about the specials along their advertisement banner. How many different arrangements are possible?

$$P(n,r) = P(5,5) = \frac{5!}{(5-5)!} = \frac{5!}{(0)!} = \frac{5!}{1} = 5!$$

Fundamental Principles of Counting

- In general, the number of permutations of size r from n distinct objects, where $0 \leq r \leq n$, is given by:

$$P(n,r) = \frac{n!}{(n-r)!}$$

20 different daily specials

We need to place 4 items on the banner

Restaurant example (extended):

Assume a restaurant has 20 daily specials and they want to layout information about the 4 of the specials along their advertisement banner. How many different arrangements are possible?

$$P(n,r) = \boxed{P(20,4)} = \frac{20!}{(20-4)!} = \frac{20!}{(16)!} = 20x19x18x17 = 116{,}280$$

Permutations

- Exercise #2:
- Assume I have 45 pictures and a picture frame that holds 4. How many different ways can I arrange pictures within the picture frame?

- Approach:
  - What is n? What is r?
  - There are 45 total pictures (n)
  - And our picture frame holds 4 (r)
  - Answer: $P(n,r) = P(45,4) = 3{,}575{,}880$

Another example

- The number of permutations of the word COMPUTER is 8!
- If only 5 letters are used, we have 8 distinct letters and need a permutation of size 5 → $P(n, r) = P(8{,}5) = \frac{8!}{(8-5)!} = \frac{8!}{3!} = 6720$
- What is the number of permutations of the word BALL?
  - We might expect it to be 4! (24), but it is not
  - It is just 12
  - This is because we do not have 4 distinct letters to arrange

Example taken from textbook: Discrete and Combinatorial Mathematics. Fifth Ed. *Ralph P. Grimaldi*. Pearson Education, 2004

Another example

- Permutations of the word BALL
  - It is 12, not 4! (24).
  - We do not have 4 distinct letters to arrange

Table 1.1

A	B	L	L	A	B	L_1	L_2	A	B	L_2	L_1
A	L	B	L	A	L_1	B	L_2	A	L_2	B	L_1
A	L	L	B	A	L_1	L_2	B	A	L_2	L_1	B
B	A	L	L	B	A	L_1	L_2	B	A	L_2	L_1
B	L	A	L	B	L_1	A	L_2	B	L_2	A	L_1
B	L	L	A	B	L_1	L_2	A	B	L_2	L_1	A
L	A	B	L	L_1	A	B	L_2	L_2	A	B	L_1
L	A	L	B	L_1	A	L_2	B	L_2	A	L_1	B
L	B	A	L	L_1	B	A	L_2	L_2	B	A	L_1
L	B	L	A	L_1	B	L_2	A	L_2	B	L_1	A
L	L	A	B	L_1	L_2	A	B	L_2	L_1	A	B
L	L	B	A	L_1	L_2	B	A	L_2	L_1	B	A

(a) (b)

- Reasoning
  - If we could distinguish the two L's as L_1 and L_2 then we would have 4!
  - But the column on the left shows that for each of the two ways we can organize L_1 and L_2, there is only one sequence when the L's are not disginguishable

- Answer: The number of permutations of BALL is $\frac{4!}{2} = 12$

Example taken from textbook: Discrete and Combinatorial Mathematics. Fifth Ed. *Ralph P. Grimaldi*. Pearson Education, 2004

Expanding on this idea

- Let's now consider the arrangements of letters in DATABASES
  - There are 3! = 6 arrangements of the A's if they are distinguishable for the same single arrangement if the A's are not distinguishable:
  - Example: There are 6 different arrangements of DATABASES if A's are distinguishable: $DA_1TA_2BA_3SES$ $DA_1TA_3BA_2SES$ $DA_2TA_1BA_3SES$ $DA_2TA_3BA_1SES$ $DA_3TA_1BA_2SES$ $DA_3TA_2BA_1SES$
  - For each of the cases where A's are distinguishable, there are also 2 different arrangements if S's are distinguishable: $DA_1TA_2BA_3S_1ES_2$ $DA_1TA_2BA_3S_2ES_1$
- Answer: The number of permutations of DATABASES is $\frac{9!}{(2!3!)} = 30{,}240$

Example taken from textbook: Discrete and Combinatorial Mathematics. Fifth Ed. *Ralph P. Grimaldi*. Pearson Education, 2004

Fundamental Principles of Counting

- In general, the number of linear arrangements of n objects is $\frac{n!}{n_1!n_2! \dots nr!}$ where there are n_1 indistinguishable objects of a first type, n_2 indistinguishable objects of a second type, …, and n_r indistinguishable objecs of an rth type, and $n_1 + n_2 + \dots + n_r = n$

- When everything is distinguishable, the divisor is simply 1:
  - Arrangements of ABCD: $\frac{4!}{1!1!1!1!} = 4! = 24$
- Otherwise, we put the number of indistinguishable objects in divisor:
  - Arrangements of BALL: $\frac{4!}{1!1!2!} = \frac{4!}{2!} = 12$
  - Arrangements of BANANA: $\frac{6!}{3!2!} = 60$

Counting where order does not matter

Counting when order does **not** matter

- Example:
  - Assume a restaurant has 5 daily specials (call them A, B, C, D, and E) and they want to layout information about the specials along their advertisement banner. How many different arrangements are possible?

Counting when order does **not** matter

- Example:
  - Assume a restaurant has 5 daily specials (call them A, B, C, D, and E) and **3** need to be chosen for the advertisement banner.

$$P(5,3) = \frac{5!}{(5-3)!} = \frac{5!}{(2)!} = 5x4x3 = 60$$

  - But what if order does not matter?

$$\frac{P(5,3)}{3!} = \frac{5!}{3!(5-3)!} = \frac{5!}{3!(2)!} = \frac{60}{3!} = 10$$

A	B	E	A	E	B
B	E	A	B	A	E
E	A	B	E	B	A

Combinations

- In general, the number of selections (combinations) of size r with no reference to order, taken from n distinct objects is given by

$$\binom{n}{r} = C(n, r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}, \qquad 0 \le r \le n$$

- $\binom{n}{r}$, which we we call "n choose r" is what we use whenever we want to choose r items from n distinct objects, and order is not important

Another example

- Lotto 6/49
  - All entrants choose 6 out of 49 possible numbers, and if your numbers match the 6 picked, you win jackpot!

- How many ways can we pick 6 out of 49?
  - In this case, order doesn't matter!
  - It doesn't matter if we choose 2 as our first number, or our sixth

- Answer: $\binom{49}{6} = \frac{49!}{6!(49-6)!} = \frac{49!}{6!(43)!} = 13{,}983{,}816$

Sigma Notation

- A concise way of writing the sum of a list of terms

- For example:
  - $3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2$ can be written much more concisely as:
  - $\sum_{i=3}^{9} i^2$

Sum all values from 3^2 up to 9^2

- Another example:
  - $\sum_{i=7}^{12} 2i = 2(7) + 2(8) + 2(9) + 2(10) + 2(11) + 2(12)$

Sum all values from $2(7)$ up to $2(12)$

Binomial Theorem

- The binomial theorem (also called binomial expansion) provides a way of expanding polynomial expressions $(x + y)^n$ into a sum
- Pascal's triangle:

row 0	$0 + 1$
row 1	$\boxed{1}\ \boxed{1}$
row 2	$1\quad 2\quad 1 \longleftrightarrow (x+y)^2 = x^2 + 2xy + y^2$
row 3	$1\quad 3 + 3\quad 1$
row 4	$1\quad 4\quad \boxed{6}\quad 4\quad 1 \longleftrightarrow (x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
…	$1\quad 5 + 10\quad 10\quad 5\quad 1$
	$1\quad 6\quad \boxed{15}\quad 20\quad 15\quad 6\quad 1$
	$1\quad 7\quad 21\quad 35\quad 35\quad 21\quad 7\quad 1$

Binomial Theorem

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$= \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$

$$= \binom{4}{0}x^4y^0 + \binom{4}{1}x^3y^1 + \binom{4}{2}x^2y^2 + \binom{4}{3}x^1y^3 + \binom{4}{4}x^0y^4$$

$$= \sum_{k=0}^{4}\binom{4}{k}x^{4-k}y^k$$

The Binomial Theorem

- If x and y are variables and n a positive integer, then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

- $\binom{n}{k}$ is often referred to as a *binomial coefficient*

Example

- What is the coefficient of $a^{\boxed{5}}b^2$ in the expansion of $(\boxed{2a}\boxed{-3b})^{\boxed{7}}$?
- Answer:
  - What is n in this case? 7
  - What is k? 5
  - replace $2a$ with x and $-3b$ with y

$$\binom{n}{k}x^k y^{n-k} = \binom{7}{5}(2)^5(-3)^{7-5} = \binom{7}{5}(2)^5(-3)^{(2)} = (21)(32)(9) = 6048$$

Combinations with Repetition

- Seven students stop at a restaurant that has the following options: cheeseburger, hot dog, taco, or fish sandwich. How many different purchases are possible?
  - At this point, we only care about how many of each item is purchased, not the order the items were purchased

x's denote the different meals to make, |'s break up the items

- Number of ways to arrange x's and |'s:

$$\frac{10!}{7!\,3!} = \binom{10}{7}$$

Table 1.6

<table>
<tr><td>1.</td><td>c, c, h, h, t, t, f</td><td>1.</td><td>x x | x x | x x | x</td></tr>
<tr><td>2.</td><td>c, c, c, c, h, t, f</td><td>2.</td><td>x x x x | x | x | x</td></tr>
<tr><td>3.</td><td>c, c, c, c, c, c, f</td><td>3.</td><td>x x x x x x | | | x</td></tr>
<tr><td>4.</td><td>h, t, t, f, f, f, f</td><td>4.</td><td>| x | x x | x x x x</td></tr>
<tr><td>5.</td><td>t, t, t, t, t, f, f</td><td>5.</td><td>| | x x x x x | x x</td></tr>
<tr><td>6.</td><td>t, t, t, t, t, t, t</td><td>6.</td><td>| | x x x x x x x |</td></tr>
<tr><td>7.</td><td>f, f, f, f, f, f, f</td><td>7.</td><td>| | | x x x x x x x</td></tr>
<tr><td></td><td>(a)</td><td></td><td>(b)</td></tr>
</table>

Example taken from textbook: Discrete and Combinatorial Mathematics. Fifth Ed. *Ralph P. Grimaldi*. Pearson Education, 2004

Combinations with Repetition

- In general, the number of ways of selecting r items, with repetition, from n distinct objects is

$$\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!}$$

- The number of combinations of n objects taken r at a time, with repetition, is $C(n+r-1, r)$

Combinations with Repetition

- Determine all integer solutions to the equation: $x_1 + x_2 + x_3 + x_4 = 7$, where $x_i \geq 0$ for all $1 \leq i \leq 4$

- We could think of this problem as distributing 7 identical pieces of gold to 4 distinct people.
  - If we were to draw this out with x's and |'s we would see there are
    - 7 x's (representing the gold) and
    - 3 |'s (to split up the amount of gold given to the 4 different people)

- Solution: $\binom{n+r-1}{r} = \binom{4+7-1}{7} = \binom{10}{7} = 120$

Pigeonhole Principle

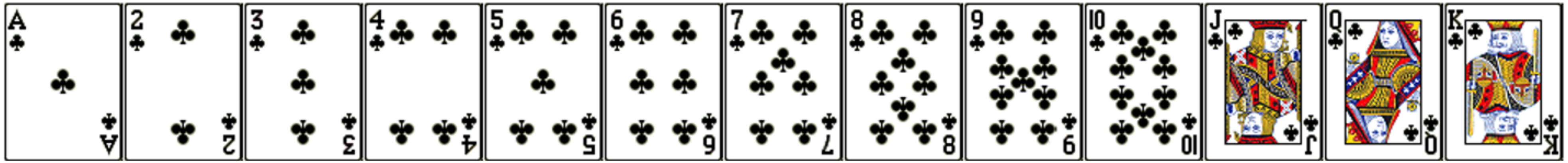
- If m pigeons occupy n pigeonholes and $m > n$, then at least one pigeonhole has two or more pigeons roosting in it.

- Why?
  - The first n can each find an unoccupied pigeonhole, but after all n pigeons have occupied a pigeon hole, pigeon $n + 1$ will need to join another pigeon

- The pigeonhole principle can be applied to solve many problems, the challenge is often determining how to associate pigeons and pigeonholes within the problem domain.

Examples

1. If I draw 14 cards from a standard deck of 52, will there be a pair?



2. Given a text file with 500,000 words of size 4 or less, is it possible that the words are distinct?

1-letter words:	2-letter words:	3-letter words:	4-letter words:
A to Z	AA to ZZ	AAA to ZZZ	AAAA to ZZZZ
26	(26)(26)	(26)(26)(26)	(26)(26)(26)(26)

$$26^1 + 26^2 + 26^3 + 26^4 = 475,254$$