3. szeminárium

Monday, March 7, 2022 2:46 PM

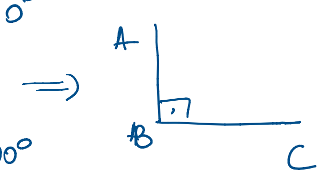
2. Adottak az $\overrightarrow{AB}(1,2,-2)$, $\overrightarrow{BC}(2,1,2)$, $\overrightarrow{CD}(-1,-2,2)$ vektorok. Igazoljuk, hogy $ABCD$ négyzet!

$$\boxed{\begin{array}{l}\vec{a}(a_1,a_2,a_3)\\ \vec{b}(b_1,b_2,b_3)\end{array}\Bigg| \Rightarrow \vec{a}\cdot\vec{b} = a_1b_1+a_2b_2+a_3b_3}$$

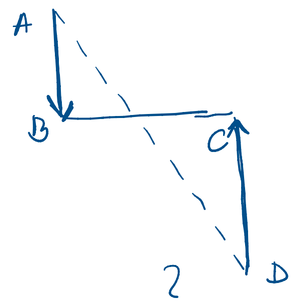
- $\overrightarrow{AB}\cdot\overrightarrow{BC} = 1\cdot 2 + 2\cdot 1 - 2\cdot 2 = 0$

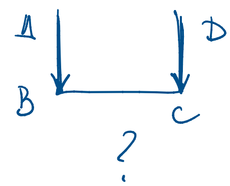
$$\Downarrow$$

$$\overrightarrow{AB}\perp\overrightarrow{BC} \Rightarrow m(\widehat{ABC}) = 90^\circ \Rightarrow$$



- $\overrightarrow{BC}\cdot\overrightarrow{CD} = -2-2+4 = 0 \Rightarrow \overrightarrow{BC}\perp\overrightarrow{CD} \Rightarrow m(\widehat{BCD}) = 90^\circ$

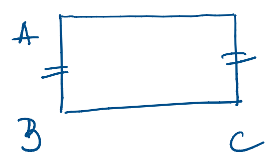




- $\left.\begin{array}{l}\overrightarrow{AB}(1,2,-2)\\ \overrightarrow{CD}(-1,-2,2)\end{array}\right| \Rightarrow \overrightarrow{AB} = -\overrightarrow{CD} = \overrightarrow{DC} \Rightarrow$

$$\Downarrow$$

$$AB = DC$$



$$\left.\begin{array}{l}AB\perp BC\\ CD\perp BC\\ AB = CD\\ \overrightarrow{AB} = \overrightarrow{DC}\end{array}\right\} \Rightarrow ABCD \text{ téglalap! } (*)$$

- $\|\overrightarrow{AB}\| \overset{?}{=} \|\overrightarrow{BC}\|$

$\overrightarrow{AB}(1,2,-2) \Rightarrow \|\overrightarrow{AB}\| = \sqrt{1^2+2^2+(-2)^2} = 3$

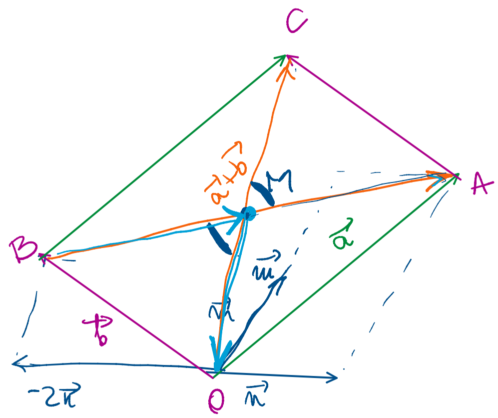
$$\boxed{\|\vec{a}\| = \sqrt{\vec{a}\cdot\vec{a}} = \sqrt{(a_1)^2+(a_2)^2+(a_3)^2}}$$

$\overrightarrow{BC}(2,1,2) \Rightarrow \|\overrightarrow{BC}\| = \sqrt{2^2+1^2+2^2} = 3$

$$\left.\right\} \overset{(*)}{\Rightarrow} ABCD_{\square} \text{ négyzet}$$

3. Határozzuk meg az $\vec{a} = 2\vec{m} + \vec{n}$ és $\vec{b} = \vec{m} - 2\vec{n}$ vektorokra epített paralelogramma átlóinak hosszát, ahol az $\vec{m}$ és $\vec{n}$ vektorok hossza 1 és a közrezárt szögük mértéke 60°.

(és közrezárt szögüket)



$\vec{OC} = \vec{a} + \vec{b} = 3\vec{m} - \vec{n}$

$\|\vec{OC}\|^2 = \vec{OC}\cdot\vec{OC} = (3\vec{m} - \vec{n})\cdot(3\vec{m} - \vec{n}) =$

$= 9\vec{m}\cdot\vec{m} - 3\cdot\vec{m}\cdot\vec{n} - 3\vec{n}\cdot\vec{m} + \vec{n}\cdot\vec{n}$

$= 9\cdot 1\cdot 1\cdot 1 - 2\cdot 3\cdot 1\cdot 1\cdot\cos 60^\circ + 1\cdot 1\cdot 1$

$= 10 - \cancel{2}\cdot 3\cdot\frac{1}{\cancel{2}} = 7 \Rightarrow OC = \sqrt{7}$

$\vec{BA} = \vec{BO} + \vec{OA} = \vec{a} - \vec{b} = (2\vec{m} + \vec{n}) - (\vec{m} - 2\vec{n}) = \vec{m} + 3\vec{n}$

$\Rightarrow \|\vec{BA}\|^2 = \|\vec{m} + 3\vec{n}\|^2 = (\vec{m} + 3\vec{n})\cdot(\vec{m} + 3\vec{n}) =$

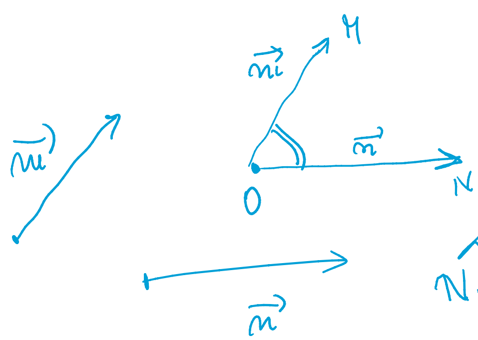
$= \vec{m}\cdot\vec{m} + 3\vec{m}\cdot\vec{n} + 3\vec{n}\cdot\vec{m} + 9\vec{n}\cdot\vec{n} =$

$= 1 + 3\cdot 1\cdot 1\cdot\frac{1}{2} + 3\cdot 1\cdot 1\cdot\frac{1}{2} + 9\cdot 1 =$

$= 13 \Rightarrow BA = \sqrt{13}.$

? $m(\widehat{BMO}) = ? = m(\widehat{AMC})$

$\widehat{BMO} \neq (\vec{BM}, \vec{MO})$

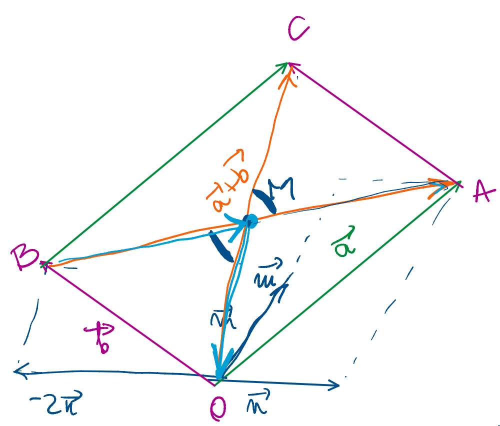


$\widehat{NOM} \overset{\text{ért}}{=} (\vec{m}, \vec{n})$

$m(\widehat{BMO}) = m(\widehat{\vec{MB}, \vec{MO}}) \Rightarrow \vec{MB}\cdot\vec{MO} = \|\vec{MB}\|\cdot\|\vec{MO}\|\cdot\cos(\widehat{BMO}) =$

$= \frac{AB}{2}\cdot\frac{CO}{2}\cdot\cos(\widehat{BMO})$

$= \frac{\sqrt{13}}{2}\cdot\frac{\sqrt{7}}{2}\cdot\cos(\widehat{BMO})$ (**)



$\vec{MB} = \frac{1}{2}\cdot\vec{AB} = -\frac{1}{2}\cdot\vec{BA} = -\frac{1}{2}(\vec{m} + 3\vec{n})$

$\vec{MO} = \frac{1}{2}\vec{CO} = -\frac{1}{2}\vec{OC} = -\frac{1}{2}(3\vec{m} - \vec{n})$ $\Rightarrow$

$\Rightarrow \vec{MB}\cdot\vec{MO} = \frac{1}{4}(\vec{m} + 3\vec{n})(3\vec{m} - \vec{n}) =$

$$\Rightarrow \vec{MB}\cdot\vec{MO} = \frac{1}{4}(\vec{m}+3\vec{n})(3\vec{m}-\vec{n}) =$$

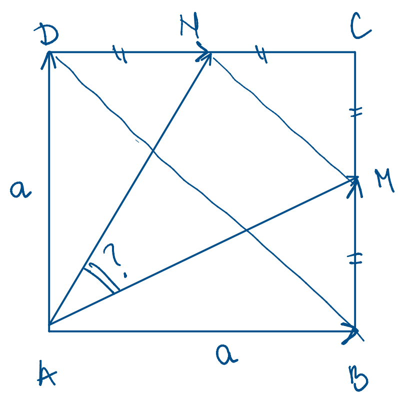
$$= \frac{1}{4}\cdot\Big[3\underbrace{\vec{m}\cdot\vec{m}}_{1}\underbrace{-\vec{m}\cdot\vec{n}+9\vec{m}\cdot\vec{n}}_{+8\vec{m}\cdot\vec{n}}-3\underbrace{\vec{n}\cdot\vec{n}}_{1}\Big)$$

$$= \frac{1}{4}\cdot\Big[\cancel{3}+8\cdot 1\cdot 1\cdot\underbrace{\cos 60^\circ}_{\frac{1}{2}}\cancel{-3}\Big] = \frac{1}{2}. \quad (\cancel{***})$$

$$\left.\begin{matrix}(***)\\(**)\end{matrix}\right\} \Rightarrow 1 = \frac{\sqrt{13}}{2}\cdot\frac{\sqrt{7}}{2}\cdot\cos(\widehat{BMO}) \Rightarrow \cos\widehat{BMO} = \frac{4}{\sqrt{13\cdot 7}} = \frac{4}{\sqrt{91}}$$

$$m(\widehat{BMO}) = \arccos\frac{4}{\sqrt{91}}.$$

5. Az $ABCD$ négyzet A csúcspontját összekötjük a $[BC]$ oldal M felezőpontjával és a $[DC]$ oldal N felezőpontjával. Számítsuk ki az MAN szög mértékét (vektoriálisan!).



$$m(\widehat{MAN}) = m(\widehat{\vec{AM},\vec{AN}})$$

$$\cos(\widehat{\vec{AM},\vec{AN}}) = \frac{\vec{AM}\cdot\vec{AN}}{\|\vec{AM}\|\cdot\|\vec{AN}\|} = \frac{(\vec{AB}+\vec{BM})\cdot(\vec{AD}+\vec{DN})}{\frac{a\sqrt{5}}{2}\cdot\frac{a\sqrt{5}}{2}} \Rightarrow$$

$AMB_\triangle$-ben: $AM^2 = AB^2+BM^2 = a^2+\left(\frac{a}{2}\right)^2 = \frac{5a^2}{4} \Rightarrow AM = \frac{a\sqrt{5}}{2} = AN$

$$\Rightarrow \cos(\widehat{\vec{AM},\vec{AN}}) = \frac{4}{5a^2}\cdot\Big[\underbrace{\vec{AB}\cdot\vec{AD}}_{=0\,(\perp)}+\vec{AB}\cdot\vec{DN}+\vec{BM}\cdot\vec{AD}+\underbrace{\vec{BM}\cdot\vec{DN}}_{=0\,(\text{mert } \perp)}\Big] =$$

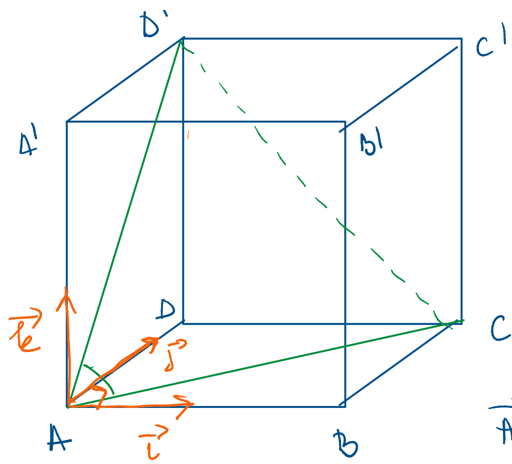
$$= \frac{4}{5a^2}\cdot\Big[a\cdot\frac{a}{2}\cdot\underbrace{\cos 0}_{1}+\frac{a}{2}\cdot a\cdot\underbrace{\cos 0}_{1}\Big]$$

$$= \frac{4}{5a^2}\cdot a^2 = \frac{4}{5} \Rightarrow m(\widehat{MAN}) = \arccos\frac{4}{5}$$

6. Legyen $ABCDA'B'C'D'$ egy kocka és N a $CDD'C'$lap középpontja, M pedig az $A'B'C'D'$ lap középpontja. Számítsuk ki:

a) az AC és AD' hajlásszögének mértékét;

b) az AN és AM hajlásszögének mértékét.



a) $m(\widehat{D'AC}) = 60^\circ$, mert ACD'_{Δ} egyenlő oldalú.
($AC = AD' = D'C = a\sqrt{2}$)

Másképp

$$m(\widehat{D'AC}) = m(\widehat{\overrightarrow{AD'}, \overrightarrow{AC}}) = ?$$

$$\cos(\widehat{D'AC}) = \frac{\overrightarrow{AD'} \cdot \overrightarrow{AC}}{\|\overrightarrow{AD'}\| \cdot \|\overrightarrow{AC}\|} = \frac{a^2}{a\sqrt{2} \cdot a\sqrt{2}} = \frac{1}{2} \Rightarrow m(\widehat{D'AC}) = 60^\circ$$

$$\overrightarrow{AD'} \cdot \overrightarrow{AC} = (\overrightarrow{AA'} + \overrightarrow{A'D'}) \cdot (\overrightarrow{AD} + \overrightarrow{DC}) =$$

$$= \underbrace{\overrightarrow{AA'} \cdot \overrightarrow{AD}}_{0(\perp)} + \underbrace{\overrightarrow{AA'} \cdot \overrightarrow{DC}}_{0(\perp)} + \underbrace{\overrightarrow{A'D'} \cdot \overrightarrow{AD}} + \underbrace{\overrightarrow{A'D'} \cdot \overrightarrow{DC}}_{0(\perp)}$$

$$= a \cdot a \cdot \cos 0 = a^2$$

Másképp:

$$\cos(\widehat{D'AC}) = \frac{\overrightarrow{AD'} \cdot \overrightarrow{AC}}{\|\overrightarrow{AD'}\| \cdot \|\overrightarrow{AC}\|}$$

$AB = a \Rightarrow \overrightarrow{AB} = a \cdot \vec{i} \Rightarrow \overrightarrow{AB}(a, 0, 0)$

$\overrightarrow{AD} = a \cdot \vec{j} \Rightarrow \overrightarrow{AD}(0, a, 0)$

$\overrightarrow{AA'} = a \cdot \vec{k} \Rightarrow \overrightarrow{AA'}(0, 0, a)$

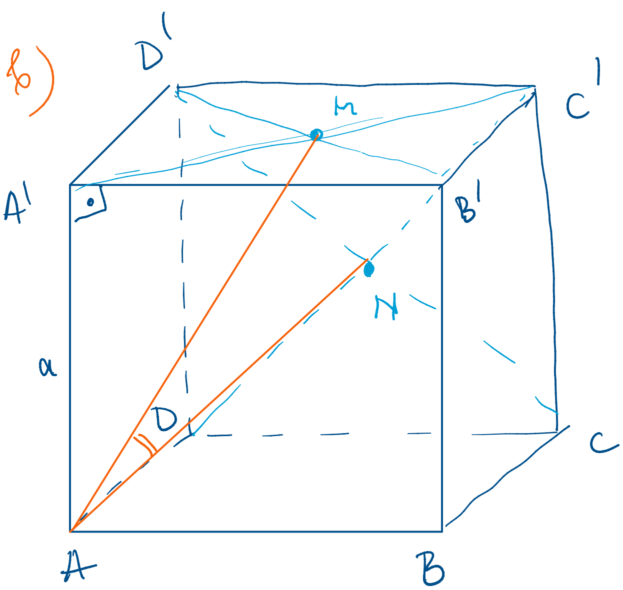
$\overrightarrow{AD'} = \overrightarrow{AD} + \overrightarrow{AA'} \Rightarrow \overrightarrow{AD'}(0, a, a)$

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD} \Rightarrow \overrightarrow{AC}(a, a, 0)$

$$\Rightarrow \cos(\widehat{D'AC}) = \frac{0 + a^2 + 0}{\sqrt{0 + a^2 + a^2} \cdot \sqrt{a^2 + a^2 + 0}} = \frac{a^2}{a\sqrt{2} \cdot a\sqrt{2}} = \frac{1}{2}$$

$$\Rightarrow m(\widehat{D'AC}) = 60^\circ.$$

b)



N – $CDD'C'$ lap k.p.
M – $A'B'C'D'$ — n —

$$m(\widehat{MAN}) = ? = m(\widehat{\overrightarrow{AM}, \overrightarrow{AN}})$$

$AA'M_{\Delta}$: $AM^2 = a^2 + \left(\frac{a\sqrt{2}}{2}\right)^2 = a^2 + \frac{2a^2}{4} = a^2 + \frac{a^2}{2} = \frac{3a^2}{2}$

ADN_{Δ}: $AN^2 = AD^2 + DN^2 = \frac{3a^2}{2}$

$$\cos(\widehat{MAN}) = \frac{\overrightarrow{AM} \cdot \overrightarrow{AN}}{\|\overrightarrow{AM}\| \cdot \|\overrightarrow{AN}\|} = \frac{\left(\overrightarrow{AA'} + \frac{1}{2}\overrightarrow{A'C'}\right) \cdot \left(\overrightarrow{AD} + \frac{1}{2}\overrightarrow{DC'}\right)}{\frac{3a^2}{2}} =$$

$$= \frac{2}{3a^2} \cdot \left[\underbrace{\overrightarrow{AA'} \cdot \overrightarrow{AD}}_{=0\,(\perp)} + \frac{1}{2} \cdot \underbrace{\overrightarrow{AA'} \cdot \overrightarrow{DC'}} + \frac{1}{2}\, \underbrace{\overrightarrow{A'C'} \cdot \overrightarrow{AD}} + \frac{1}{4}\, \underbrace{\overrightarrow{A'C'} \cdot \overrightarrow{DC'}} \right] =$$

$A'C'D_D$ egyenlő.

$$= \frac{2}{3a^2} \cdot \left[0 + \frac{1}{2} \cdot a \cdot a\sqrt{2} \cdot \cos 45^\circ + \frac{1}{2} \cdot a\sqrt{2} \cdot a \cdot \cos 45^\circ + \right.$$

$$\left. + \frac{1}{4} \cdot a\sqrt{2} \cdot a\sqrt{2} \cdot \cos 60^\circ \right] =$$

$$= \frac{2}{3a^2} \cdot \left[\underbrace{\frac{a^2\cancel{\sqrt{2}}}{2} \cdot \frac{\sqrt{2}}{\cancel{2}} + \frac{a^2\cancel{\sqrt{2}}}{2} \cdot \frac{\sqrt{2}}{\cancel{2}}}_{a^2 + a^2/4} + \frac{1}{4} \cdot a^2 \cdot \cancel{2} \cdot \frac{1}{\cancel{2}} \right]$$

$$= \frac{2}{3a^2} \cdot \frac{5a^2}{4} = \frac{10}{12} = \frac{5}{6}$$

$$\Rightarrow \quad m(\widehat{MAN}) = \arccos \frac{5}{6}.$$