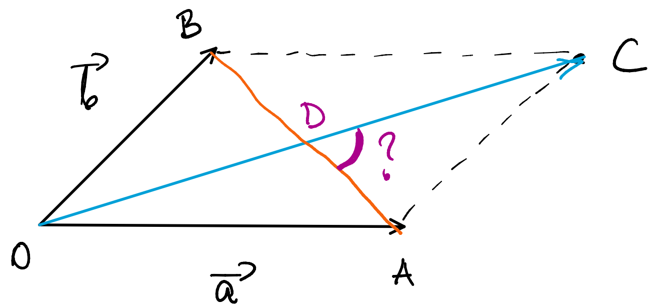
3. szeminárium

Thursday, March 11, 2021 9:51 AM

1. Adottak az $\overrightarrow{OA} = (4, -2, -4)$, $\overrightarrow{OB} = (2, 4, 3)$ vektorok. Határozzuk meg az $OACB$ paralelogramma átlóinak hosszát és közrezárt szögüket!



$$\boxed{\vec{a}(a_1, a_2, a_3) \Downarrow \|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}}$$

OACB paralelogramma átlói ?

$\overrightarrow{OC} = ?$

$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$ (OACB paral.)

- $\overrightarrow{OC} = (4, -2, -4) + (2, 4, 3) = (6, 2, -1)$

$\|\overrightarrow{OC}\| = \sqrt{36 + 4 + 1} = \sqrt{41}$

- $\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{OB} + \overrightarrow{AO} =$

$= (2, 4, 3) + (-4, 2, 4) = (-2, 6, 7)$

$\|\overrightarrow{AB}\| = \sqrt{4 + 36 + 49} = \sqrt{89}$

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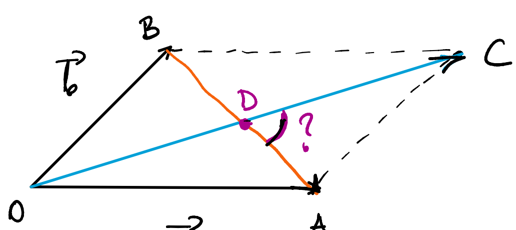


ADC_Δ :

$m(\widehat{ADC}) = (\widehat{\overrightarrow{DC}, \overrightarrow{DA}})$

$$\boxed{\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos(\widehat{\vec{a}, \vec{b}})}$$

$\Rightarrow \cos(\widehat{ADC}) = \cos(\widehat{\overrightarrow{DC}, \overrightarrow{DA}}) =$

$\overrightarrow{DC} \cdot \overrightarrow{DA}$

$$= \frac{\vec{DC}\cdot\vec{DA}}{\|\vec{DC}\|\cdot\|\vec{DA}\|}$$

$$\|\vec{DC}\| = DC = \frac{OC}{2} = \frac{\sqrt{41}}{2}$$

$$\|\vec{DA}\| = \frac{\sqrt{89}}{2}$$

$$\left.\begin{array}{l} \vec{DC} = \frac{1}{2}\vec{OC} = \frac{1}{2}(6,2,-1) \\ \vec{DA} = \frac{1}{2}\vec{BA} = \frac{1}{2}\cdot(2,-6,-7) \end{array}\right| \Rightarrow$$

$$\Rightarrow \vec{DC}\cdot\vec{DA} = \frac{1}{4}\cdot(12-12+7) = \frac{7}{4}$$

$$\Rightarrow \cos(\widehat{ADC}) = \frac{\frac{7}{4}}{\frac{\sqrt{41}}{\cancel{2}}\cdot\frac{\sqrt{89}}{\cancel{2}}} = \frac{7}{\sqrt{41}\cdot\sqrt{89}}.$$

2. Adottak az $\overrightarrow{AB}(1,2,-2)$, $\overrightarrow{BC}(2,1,2)$, $\overrightarrow{CD}(-1,-2,2)$ vektorok. Igazoljuk, hogy $ABCD$ négyzet!

$$\|\vec{AB}\| = \sqrt{1+4+4} = \sqrt{9} = 3$$

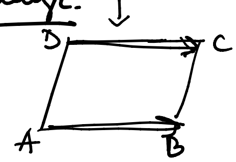
$$\|\vec{CD}\| = \sqrt{1+4+4} = 3$$

Megjegyzés: $\vec{AB} = -\vec{CD}$ $\Leftrightarrow \vec{AB} = \vec{DC} \Rightarrow ABCD$ paral.

$$\left.\begin{array}{l} \vec{a}\parallel\vec{b} \Leftrightarrow \exists\lambda\in\mathbb{R}: \vec{b} = \lambda\cdot\vec{a} \\ \vec{b}(b_1,b_2,b_3) \\ \vec{a}(a_1,a_2,a_3) \end{array}\right/ \Rightarrow \boxed{\vec{a}\parallel\vec{b} \Leftrightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}}$$

- $\vec{AB}\parallel\vec{CD}$? $\Leftrightarrow \frac{1}{-1} = \frac{2}{-2} = \frac{-2}{2}$ igaz ✓

$$\left.\begin{array}{l} \Rightarrow \vec{AB}\parallel\vec{CD} \\ \text{de } \|\vec{AB}\| = \|\vec{CD}\| \end{array}\right\} \Rightarrow ABCD \text{ ▱ paralelogr.}$$



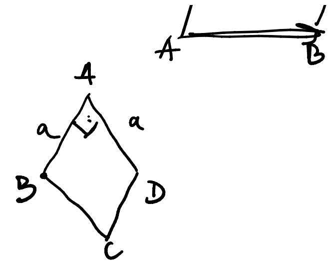
- $\|\vec{BC}\| = \sqrt{4+1+4} = 3.$

$\Downarrow$

• $\|\vec{BC}\| = \sqrt{4+1+4} = 3.$

$\Downarrow$

ABCD □ rombusz.



• $m(\widehat{ABC}) = 90°$?

$\Updownarrow$

$\vec{AB} \cdot \vec{BC} = 0$?

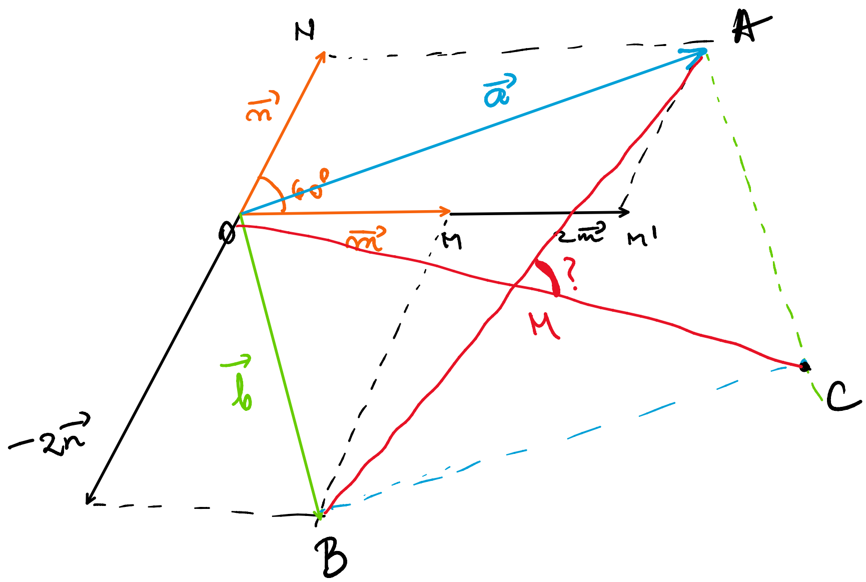
$\boxed{\vec{a} \cdot \vec{b} = 0 \iff \vec{a} \perp \vec{b} \text{ v. } \vec{a} = \vec{0} \text{ v. } \vec{b} = \vec{0}}$!

$\left.\begin{array}{l} \vec{AB}(1, 2, -2) \\ \vec{BC}(2, 1, 2) \end{array}\right| \Rightarrow \vec{AB} \cdot \vec{BC} = 2 + 2 - 4 = \underline{\underline{0}}$

$\Downarrow$

$\left.\begin{array}{l} AB \perp BC \\ ABCD \text{ rombusz} \end{array}\right| \Rightarrow$ ABCD négyzet!

3. Határozzuk meg az $\vec{a} = 2\vec{m} + \vec{n}$ és $\vec{b} = \vec{m} - 2\vec{n}$ vektorokra épített paralelogramma átlóinak hosszát, ahol az $\vec{m}$ és $\vec{n}$ vektorok hossza 1 és a közrezárt szögük mértéke 60°.



OACB paral.

$AB = ?$

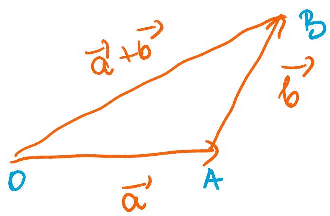
$OC = ?$

$m(\widehat{AMC}) = ?$

$$\boxed{\begin{array}{l} \vec{AB} = \vec{AO} + \vec{OB} = \\ \quad = -\vec{a} + \vec{b} \end{array}}$$

$$\vec{OC} = \vec{OA} + \vec{OB} = (2\vec{m} + \vec{n}) + (\vec{m} - 2\vec{n}) = 3\vec{m} - \vec{n}$$

! $\|\vec{a}+\vec{b}\| \neq \|\vec{a}\| + \|\vec{b}\| \iff OA + AB \neq OB$



! nem igaz!

$$\boxed{\vec{a}\cdot\vec{b} = \|\vec{a}\|\cdot\|\vec{b}\|\cdot\cos(\widehat{\vec{a},\vec{b}})}$$

$$\Rightarrow \quad \vec{a}\cdot\vec{a} = \|\vec{a}\|\cdot\|\vec{a}\|\cdot\cos 0^\circ = \|\vec{a}\|^2$$

$$\Rightarrow \quad \boxed{\|\vec{a}\|^2 = \vec{a}\cdot\vec{a}}$$

$$\vec{OC} = 3\vec{m} - \vec{n} \quad \Rightarrow \quad \|\vec{OC}\|^2 = \vec{OC}\cdot\vec{OC} = (3\vec{m}-\vec{n})(3\vec{m}-\vec{n}) =$$

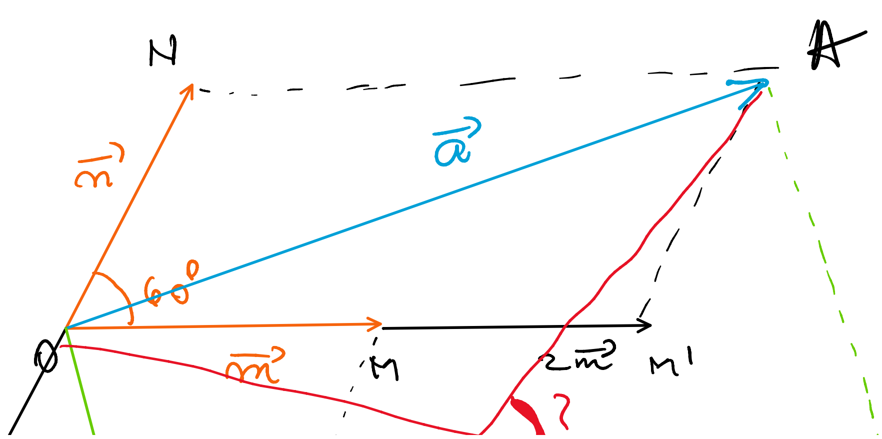
$$= 9\,\vec{m}\cdot\vec{m} - 3\,\vec{m}\cdot\vec{n} - 3\,\vec{n}\cdot\vec{m} + \vec{n}\cdot\vec{n} =$$

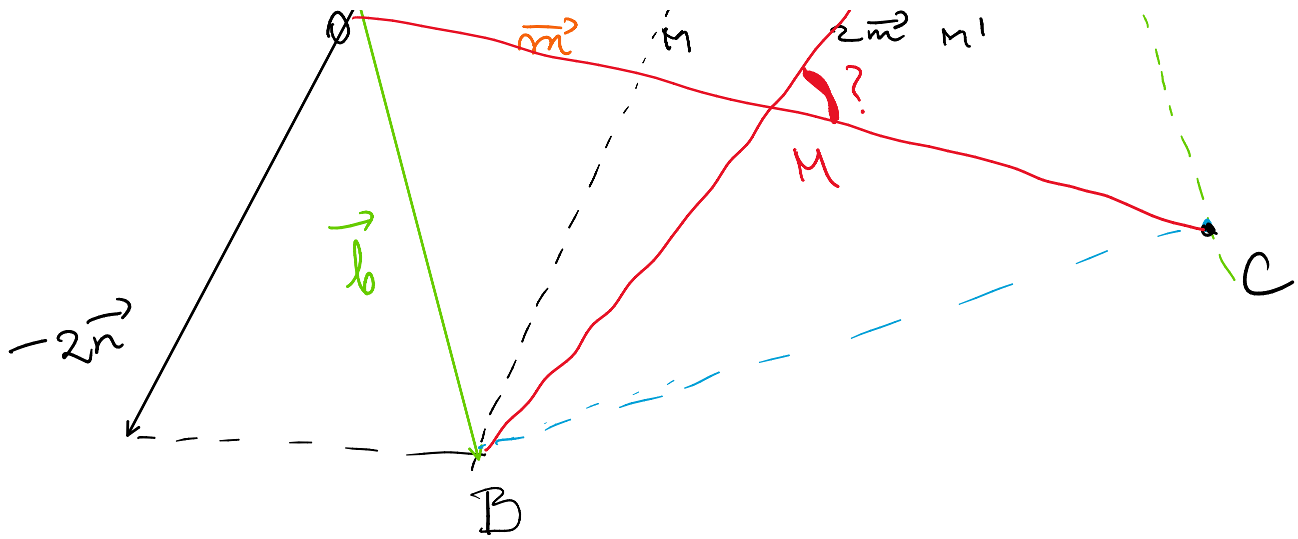
$$= 9\cdot 1\cdot 1\cdot\cos 0^\circ - 3\cdot 1\cdot 1\cdot\underbrace{\cos 60^\circ}_{\frac{1}{2}} - 3\cdot 1\cdot 1\cdot\underbrace{\cos 60^\circ}_{\frac{1}{2}} + 1\cdot 1 =$$

$$= 9 - \frac{3}{2} - \frac{3}{2} + 1 = 7 \quad \Rightarrow \quad \boxed{OC = \sqrt{7}}$$

$$\vec{AB} = -\vec{m} - 3\vec{n}$$

$$\|\vec{AB}\|^2 = \vec{AB}\cdot\vec{AB} = (-\vec{m} - 3\vec{n})^2 = 1 + \frac{3}{2} + \frac{3}{2} + 9 = 13$$





- $\cos(\widehat{AMC}) = \cos(\widehat{\vec{MA}, \vec{MC}})$

- $\cos(\widehat{AMC}) = \cos(\widehat{\vec{BA}, \vec{OC}}) = \dfrac{\vec{BA} \cdot \vec{OC}}{\|\vec{BA}\| \cdot \|\vec{OC}\|} =$

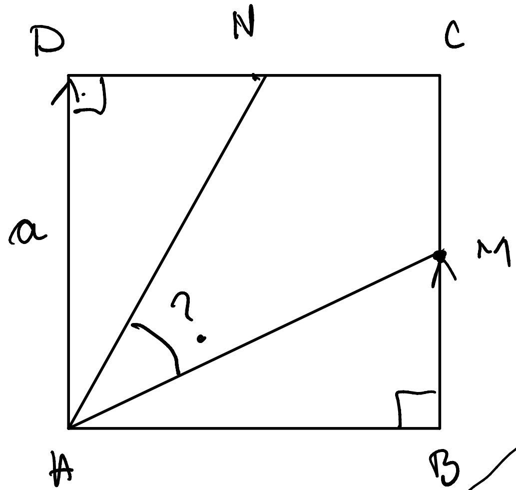
$$= \frac{(\vec{m} + 3\vec{n}) \cdot (3\vec{m} - \vec{n})}{\sqrt{7} \cdot \sqrt{13}} =$$

$$= \frac{3\vec{m} \cdot \vec{m} - \vec{m} \cdot \vec{n} + 9\vec{m} \cdot \vec{n} - 3\vec{n} \cdot \vec{n}}{\sqrt{7} \cdot \sqrt{13}} =$$

$$= \frac{\cancel{3 \cdot 1 \cdot 1} - 1 \cdot 1 \cdot \frac{1}{2} + 9 \cdot 1 \cdot 1 \cdot \frac{1}{2} - \cancel{3 \cdot 1 \cdot 1}}{\sqrt{7} \cdot \sqrt{13}} = \frac{4}{\sqrt{91}}$$

$$\cos(\widehat{AMC}) = \frac{4}{\sqrt{91}}$$

5. Az $ABCD$ négyzet A csúcspontját összekötjük a $[BC]$ oldal M felezőpontjával és a $[DC]$ oldal N felezőpontjával. Számítsuk ki az MAN szög mértékét (vektoriálisan!).



$$m(\widehat{MAN}) = m(\widehat{\overrightarrow{AN}, \overrightarrow{AM}})$$

$$\begin{cases} \overrightarrow{AN} = \overrightarrow{AD} + \overrightarrow{DN} \\ \overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BM} \end{cases}$$

$$\cos(\widehat{\overrightarrow{AN}, \overrightarrow{AM}}) = ?$$

$$\cos(\overrightarrow{AN}, \overrightarrow{AM}) = \frac{\overrightarrow{AN} \cdot \overrightarrow{AM}}{\|\overrightarrow{AN}\| \cdot \|\overrightarrow{AM}\|} = \frac{\underbrace{\overrightarrow{AD} \cdot \overrightarrow{AB}}_{=0} + \overrightarrow{AD} \cdot \overrightarrow{BM} + \overrightarrow{DN} \cdot \overrightarrow{AB} + \underbrace{\overrightarrow{DN} \cdot \overrightarrow{BM}}_{=0}}{\frac{5a^2}{4}} \overset{*}{=}$$

$$AN^2 = AD^2 + DN^2 = a^2 + \left(\frac{a}{2}\right)^2 = a^2 + \frac{a^2}{4} = \frac{5a^2}{4} = AM^2$$

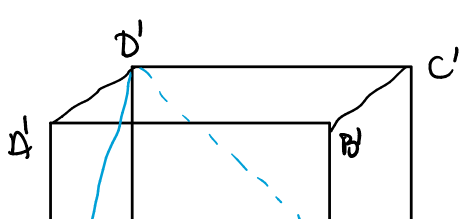
$$\overset{*}{=} \frac{0 + a \cdot \frac{a}{2} \cdot \underbrace{\cos 0^\circ}_{1} + \frac{a}{2} \cdot a \cdot \underbrace{\cos 0^\circ}_{1} + 0}{\frac{5a^2}{4}} = \frac{a^2}{\frac{5a^2}{4}} = \frac{4}{5}$$

$$\Rightarrow \boxed{m(\widehat{MAN}) = \arccos\frac{4}{5}}$$

6. Legyen $ABCDA'B'C'D'$ egy kocka és N a $CDD'C'$ lap középpontja, M pedig az $A'B'C'D'$ lap középpontja. Számítsuk ki:

a) az AC és AD' hajlásszögének mértékét;

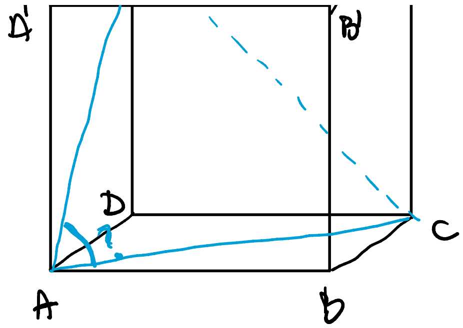
b) az AN és AM hajlásszögének mértékét.



a) $AC = AD' = D'C = a\sqrt{2}$ (lapátló)

$\Downarrow$

$AD'C_\triangle$ egy. oldalú

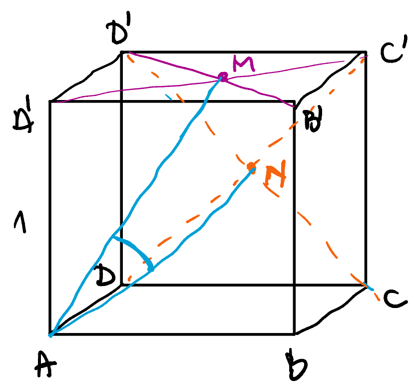


$AD'C_\Delta$ egy. oldalú

$\Downarrow$

$m(\widehat{D'AC}) = 60°$ ✓

b)



$m(\widehat{MAN}) = ?$

M a fedő, N a hátsó lap középp.

$\cos(\widehat{MAN}) = \cos(\widehat{\overrightarrow{AM}, \overrightarrow{AN}}) = ?$

$\overrightarrow{AM} = \overrightarrow{AA'} + \overrightarrow{A'M}$

$\overrightarrow{AN} = \overrightarrow{AD} + \overrightarrow{DN}$

$AB = 1 \Rightarrow A'M = \frac{\sqrt{2}}{2} = DN$

$$\cos(\widehat{\overrightarrow{AM}, \overrightarrow{AN}}) = \frac{\overrightarrow{AM} \cdot \overrightarrow{AN}}{\|\overrightarrow{AM}\| \cdot \|\overrightarrow{AN}\|} = \frac{\underbrace{\overrightarrow{AA'} \cdot \overrightarrow{AD}}_{=0} + \overrightarrow{AA'} \cdot \overrightarrow{DN} + \overrightarrow{A'M} \cdot \overrightarrow{AD} + \overrightarrow{A'M} \cdot \overrightarrow{DN}}{3/2}$$

$AM = ?$ $AA'M_\Delta$-ben: $AM^2 = AA'^2 + A'M^2 = 1 + \frac{1}{2} = \frac{3}{2}$

$AM = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$

AND_Δ-ben hasonlóan $AN = \sqrt{\frac{3}{2}}$ $\Rightarrow AM \cdot AN = \sqrt{\frac{3}{2}} \cdot \sqrt{\frac{3}{2}} = \frac{3}{2}$

($AC'D_\Delta$ egy. old.)

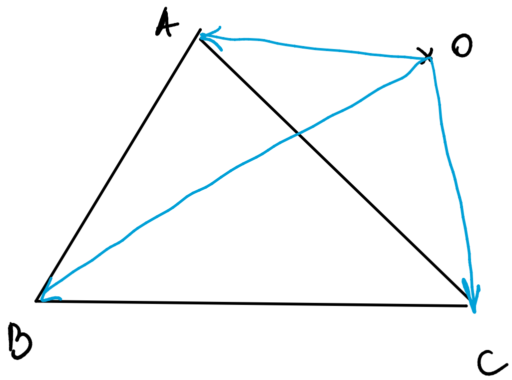
$$\Rightarrow \cos(\widehat{\overrightarrow{AM}, \overrightarrow{AN}}) = \frac{0 + 1 \cdot \frac{\sqrt{2}}{2} \cdot \cos 45° + 1 \cdot \frac{\sqrt{2}}{2} \cdot \cos 45° + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} \cdot \cos 60°}{\frac{3}{2}} =$$

$$= \left(\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{1}{4}\right) \cdot \frac{2}{3} = \frac{5}{4} \cdot \frac{2}{3} = \frac{5}{6}$$

$$\Rightarrow \boxed{m(\widehat{MAN}) = \arccos \frac{5}{6}}$$

7. Legyen ABC egy háromszög és O egy pont a térben. Mutassuk ki, hogy

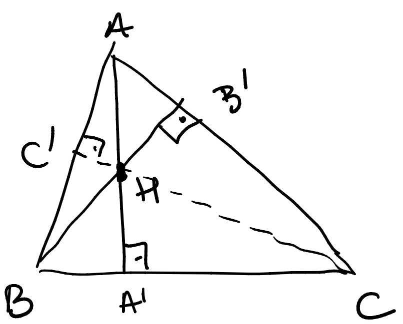
$$\overrightarrow{OA}\cdot\overrightarrow{BC}+\overrightarrow{OB}\cdot\overrightarrow{CA}+\overrightarrow{OC}\cdot\overrightarrow{AB}=0.$$



$$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} \quad \Big| \cdot \overrightarrow{OA}$$
$$\overrightarrow{CA} = \overrightarrow{CO} + \overrightarrow{OA} \quad \Big| \cdot \overrightarrow{OB}$$
$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} \quad \Big| \cdot \overrightarrow{OC}$$
(+)

$$\Rightarrow \overrightarrow{OA}\cdot\overrightarrow{BC} + \overrightarrow{OB}\cdot\overrightarrow{CA} + \overrightarrow{AB}\cdot\overrightarrow{OC} = \underline{\underline{\overrightarrow{BO}\cdot\overrightarrow{OA}}} + \overrightarrow{OC}\cdot\overrightarrow{OA} +$$
$$+ \underline{\overrightarrow{CO}\cdot\overrightarrow{OB}} + \underline{\underline{\overrightarrow{OA}\cdot\overrightarrow{OB}}} + \overrightarrow{AO}\cdot\overrightarrow{OC} + \underline{\overrightarrow{OB}\cdot\overrightarrow{OC}} = 0$$

12. Mutassuk ki vektoriálisan, hogy egy háromszög magasságai egy pontban metszik egymást!



F: $AA' \perp BC$

$BB' \perp AC$

$AA' \cap BB' = \{H\}$

K: $CH \perp AB$.

? $\overrightarrow{CH}\cdot\overrightarrow{AB} = 0$?

$$\overrightarrow{CH}\cdot\overrightarrow{AB} = (\overrightarrow{CA}+\overrightarrow{AH})\cdot(\overrightarrow{AC}+\overrightarrow{CB}) =$$
$$= \overrightarrow{CA}\cdot\overrightarrow{AC} + \overrightarrow{AH}\cdot\overrightarrow{AC} + \overrightarrow{CA}\cdot\overrightarrow{CB} + \underbrace{\overrightarrow{AH}\cdot\overrightarrow{CB}}_{=0} =$$
$$= \overrightarrow{AC}\,(\underbrace{\overrightarrow{CA}+\overrightarrow{AH}}+\overrightarrow{BC}) =$$
$$= \overrightarrow{AC}\cdot(\overrightarrow{CH}+\overrightarrow{BC}) = \underbrace{\overrightarrow{AC}\cdot\overrightarrow{BH}}_{=0} = 0$$