1、 $Ae_j = A_{*j}$ $e_i^T A e_j = A_{i*} e_j = a_{ij}$

2. $\text{trace}(AB) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} b_{ji}$ $\text{trace}(BA) = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij} a_{ji} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ji} b_{ij}$

$\therefore \text{trace}(AB) = \text{trace}(BA)$

$\therefore \text{trace}(ABC) = \text{trace}[A(BC)] = \text{trace}[(BC)A] = \text{trace}(BCA)$

3. 对于上三角矩阵 $a_{ij}=0$ ，$j<i$

设A，B为上三角矩阵，$C=AB$，

$i<j$ 时

$$C_{ij} = \sum_p a_{ip} \cdot b_{pj} = \begin{cases} 0 \cdot b_{pj} & p \le j < i \\ 0 \cdot b_{pj} & j < p < i \\ a_{ip} \cdot 0 & j < i \le p \end{cases} = 0$$

$\therefore$ $C=AB$ 为上三角矩阵，下三角矩阵同理

4. $B = A + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} = 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} = 2e_3 e_2^T$

$B = A + \alpha e_i e_j^T$

$$B^{-1} = A^{-1} - \alpha \frac{A^{-1} e_i e_j^T A^{-1}}{1 + \alpha e_j^T A^{-1} e_i} = A^{-1} - \alpha \frac{[A^{-1}]_{*i} [A^{-1}]_{j*}}{1 + \alpha [A^{-1}]_{ji}}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix} - 2 \cdot \frac{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}}{1 + 2 \cdot (-1)}$$

$$= \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$$

$$C = B + e_3 e_3^T$$

$$C^{-1} = B^{-1} - \frac{[B^{-1}]_{*3} [B^{-1}]_{3*}}{1 + [B^{-1}]_{33}}$$

$$= \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix} - \frac{\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} [1 \ 4 \ -2]}{1 - 2}$$

$$= \begin{bmatrix} 0 & -2 & 1 \\ 1 & 3 & -1 \\ -1 & -4 & 2 \end{bmatrix}$$

5. $$A = \begin{bmatrix} 1 & 2 & 4 & 17 \\ 3 & 6 & -12 & 3 \\ 2 & 3 & -3 & 2 \\ 0 & 2 & -2 & 6 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 3 & 6 & -12 & 3 \\ 1 & 2 & 4 & 17 \\ 2 & 3 & -3 & 2 \\ 0 & 2 & -2 & 6 \end{bmatrix} \xrightarrow[R_3 - \frac{2}{3}R_1]{R_2 - \frac{1}{3}R_1} \begin{bmatrix} 3 & 6 & -12 & 3 \\ \frac{1}{3} & 0 & 8 & 16 \\ \frac{2}{3} & -1 & 5 & 0 \\ 0 & 2 & -2 & 6 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_4} \begin{bmatrix} 3 & 6 & -12 & 3 \\ 0 & 2 & -2 & 6 \\ \frac{2}{3} & -1 & 5 & 0 \\ \frac{1}{3} & 0 & 8 & 16 \end{bmatrix} \xrightarrow{R_3 + \frac{1}{2}R_2} \begin{bmatrix} 3 & 6 & -12 & 3 \\ 0 & 2 & -2 & 6 \\ \frac{2}{3} & -\frac{1}{2} & 4 & 3 \\ \frac{1}{3} & 0 & 8 & 16 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 3 & 6 & -12 & 3 \\ 0 & 2 & -2 & 6 \\ \frac{1}{3} & 0 & 8 & 16 \\ \frac{2}{3} & -\frac{1}{2} & 4 & 3 \end{bmatrix} \xrightarrow{R_4 - \frac{1}{2}R_3} \begin{bmatrix} 3 & 6 & -12 & 3 \\ 0 & 2 & -2 & 6 \\ \frac{1}{3} & 0 & 8 & 16 \\ \frac{2}{3} & -\frac{1}{2} & \frac{1}{2} & -5 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & 1 & 0 \\ \frac{2}{3} & -\frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 3 & 6 & -12 & 3 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 8 & 16 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$Ax = b \qquad PAx = Pb \qquad LUx = Pb$$

$$\Downarrow$$

$$Ly = Pb \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & 1 & 0 \\ \frac{2}{3} & -\frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} y = \begin{bmatrix} 3 \\ 4 \\ 17 \\ 3 \end{bmatrix} \qquad y = \begin{bmatrix} 3 \\ 4 \\ 16 \\ -5 \end{bmatrix}$$

$$\Downarrow$$

$$Ux = y \quad \begin{bmatrix} 3 & 6 & -12 & 3 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 8 & 16 \\ 0 & 0 & 0 & -5 \end{bmatrix} x = \begin{bmatrix} 3 \\ 4 \\ 16 \\ -5 \end{bmatrix} \qquad x = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \ x = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

6.

$$T = \begin{bmatrix} \beta_1 & \gamma_1 & 0 & 0 \\ \alpha_1 & \beta_2 & \gamma_2 & 0 \\ 0 & \alpha_2 & \beta_3 & \gamma_3 \\ 0 & 0 & \alpha_3 & \beta_4 \end{bmatrix} \qquad \pi_1 = \beta_1 \quad \pi_i = \beta_i - \frac{\alpha_{i-1}\gamma_{i-1}}{\pi_{i-1}} \quad i = 2,3,4$$

$$\xrightarrow{R_2 - \frac{\alpha_1}{\pi_1}R_1} \begin{bmatrix} \pi_1 & \gamma_1 & 0 & 0 \\ \frac{\alpha_1}{\pi_1} & \beta_2 - \frac{\alpha_1\gamma_1}{\pi_1} & \gamma_2 & 0 \\ 0 & \alpha_2 & \beta_3 & \gamma_3 \\ 0 & 0 & \alpha_3 & \beta_4 \end{bmatrix} = \begin{bmatrix} \pi_1 & \gamma_1 & 0 & 0 \\ \frac{\alpha_1}{\pi_1} & \pi_2 & \gamma_2 & 0 \\ 0 & \alpha_2 & \beta_3 & \gamma_3 \\ 0 & 0 & \alpha_3 & \beta_4 \end{bmatrix}$$

$$\xrightarrow{R_3 - \frac{\alpha_2}{\pi_2}R_2} \begin{bmatrix} \pi_1 & \gamma_1 & 0 & 0 \\ \frac{\alpha_1}{\pi_1} & \pi_2 & \gamma_2 & 0 \\ 0 & \frac{\alpha_2}{\pi_2} & \beta_3 - \frac{\alpha_2\gamma_2}{\pi_2} & \gamma_3 \\ 0 & 0 & \alpha_3 & \beta_4 \end{bmatrix} = \begin{bmatrix} \pi_1 & \gamma_1 & 0 & 0 \\ \frac{\alpha_1}{\pi_1} & \pi_2 & \gamma_2 & 0 \\ 0 & \frac{\alpha_2}{\pi_2} & \pi_3 & \gamma_3 \\ 0 & 0 & \alpha_3 & \beta_4 \end{bmatrix}$$

$$\xrightarrow{R_4 - \frac{\alpha_3}{\pi_3}R_3} \begin{bmatrix} \pi_1 & \gamma_1 & 0 & 0 & 0 \\ \frac{\alpha_1}{\pi_1} & \pi_2 & \gamma_2 & 0 & 0 \\ 0 & \frac{\alpha_2}{\pi_2} & \pi_3 & \gamma_3 & 0 \\ 0 & 0 & \frac{\alpha_3}{\pi_3} & \beta_4 - \frac{\alpha_3\gamma_3}{\pi_3} \end{bmatrix} = \begin{bmatrix} \pi_1 & \gamma_1 & 0 & 0 \\ \frac{\alpha_1}{\pi_1} & \pi_2 & \gamma_2 & 0 \\ 0 & \frac{\alpha_2}{\pi_2} & \pi_3 & \gamma_3 \\ 0 & 0 & \frac{\alpha_3}{\pi_3} & \pi_4 \end{bmatrix}$$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{\alpha_1}{\pi_1} & 1 & 0 & 0 \\ 0 & \frac{\alpha_2}{\pi_2} & 1 & 0 \\ 0 & 0 & \frac{\alpha_3}{\pi_3} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} \pi_1 & \gamma_1 & 0 & 0 \\ 0 & \pi_2 & \gamma_2 & 0 \\ 0 & 0 & \pi_3 & \gamma_3 \\ 0 & 0 & 0 & \pi_4 \end{bmatrix}$$