1. $A = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}$

$\|A\|_F = \sqrt{\Sigma |a_{ij}|^2} = \sqrt{10}$

$\|A\|_1 = 4$

$\|A\|_2 = \sqrt{\lambda_{max}}$ $\quad A^TA = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix}$

$\det(A^TA - \lambda I) = \lambda^2 - 10\lambda = 0$

$\|A\|_2 = \sqrt{10}$

$\|A\|_\infty = 3$

$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$\|B\|_F = \sqrt{3}$

$\|B\|_1 = 1$

$B^TB = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\|B\|_2 = \sqrt{\lambda_{max}} = 1$

$\|B\|_\infty = 1$

$C = \begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix}$

$\|C\|_F = 9$

$\|C\|_1 = 10$

$$C^TC = \begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix}\begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix} = \begin{pmatrix} 36 & -18 & 36 \\ -18 & 9 & -18 \\ 36 & -18 & 36 \end{pmatrix}$$

$$\det\begin{pmatrix} 36-\lambda & -18 & 36 \\ -18 & 9-\lambda & -18 \\ 36 & -18 & 36-\lambda \end{pmatrix} = 0$$

$$-\lambda^3 + 81\lambda^2 = 0$$

$\|C\|_2 = \sqrt{\lambda_{max}} = 9$

$\|C\|_\infty = 10$

2. (1) $\langle A, B\rangle = trace(A^TB)$

① $\langle A, A\rangle = trace(A^TA) = \sum |a_{ij}|^2 \geq 0$ $=0$

当且仅当 $a_{ij}=0$ 即 $A=0$ 时 $\langle A, A\rangle = 0$

② $\langle A, \alpha B\rangle = trace(A^T\alpha B) = \alpha\, trace(A^TB) = \alpha\langle A, B\rangle$

③ $\langle A, B+C\rangle = trace[A^T(B+C)] = trace[A^TB + A^TC]$

$= trace(A^TB) + trace(A^TC) = \langle A, B\rangle + \langle A, C\rangle$

④ $\langle A, B\rangle = trace(A^TB) = trace(B^TA) = \langle B, A\rangle$

∴ 是内积

(2) $B = \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \right\}$

$\dim(R^{2\times 2}) = 4$

$\|B_i\| = \sqrt{\langle B_i, B_i\rangle} = \sqrt{trace(B_i^TB_i)} = 1$

$\langle B_i, B_j\rangle = trace(B_i^TB_j) = 0$

∴ B 为 $R^{2\times 2}$ 的一组标准正交基

$\langle A, B_1\rangle = trace(A^TB_1) = \sqrt{2}$

$\langle A, B_2 \rangle = trace(A^T B_2) = 0$

$\langle A, B_3 \rangle = trace(A^T B_3) = 1$

$\langle A, B_4 \rangle = trace(A^T B_4) = 1$

$\therefore A = \sqrt{2} B_1 + B_3 + B_4$

(3) $B' = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$