第2章：贝叶斯决策理论

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统计模式识别方法

生成模型
(Density-based, Bayes decision) 近似$p(\mathbf{x}|\omega_i)$

Parametric
- ✓ Gaussian
- ✓ Dirichlet
- ✓ Bayesian network
- ✓ Hidden Markov model

Non-Parametric
- ✓ Histogram density
- ✓ Parzen window
- ✓ K-nearest neighbor

Semi-Parametric
- ✓ Gaussian mixture

判别模型
(discriminant/decision function) 近似$p(\omega_i|\mathbf{x})$

- ✓ Linear methods
- ✓ Neural network
- ✓ Logistic regression
- ✓ Decision tree
- ✓ Kernel (SVM)
- ✓ Boosting

注意discriminative和discriminant的区别！

Discriminative model/learning

Discriminant function/analysis

提纲

- 分类问题表示
- 最小错误率决策：2类的例子
- 最小风险决策
  - 扩展：开放集识别的贝叶斯决策
- 判别函数和决策面
- 高斯概率密度
  - 协方差矩阵的性质
  - 相关知识：特征提取/降维
- 高斯密度下的判别函数
  - 扩展：广义线性判别函数
- 分类错误率

导论：分类问题表示

- 类别：　$\omega_i, i=1,\ldots,c$
- 特征矢量　$\mathbf{x}=[x_1,\ldots,x_d]\in R^d$
- 先验概率　$P(\omega_i)$　$\sum_{i=1}^{c} P(\omega_i)=1$
- 概率密度函数(条件概率)　$p(\mathbf{x}|\omega_i)$
- 后验概率

$$P(\omega_i|\mathbf{x})=\frac{p(\mathbf{x}|\omega_i)P(\omega_i)}{p(\mathbf{x})}=\frac{p(\mathbf{x}|\omega_i)P(\omega_i)}{\sum_{j=1}^{c} p(\mathbf{x}|\omega_j)P(\omega_j)}$$

$$\sum_{i=1}^{c} P(\omega_i|\mathbf{x})=1$$

最小错误率决策：2类的例子

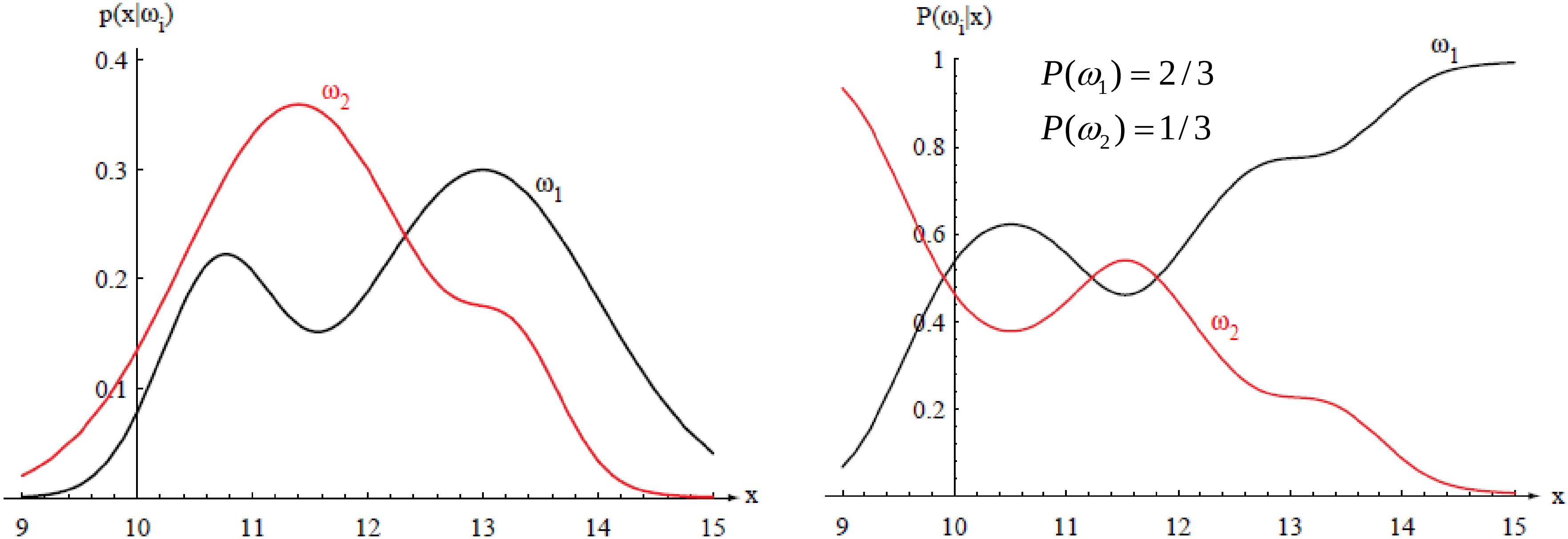
- Salmon (ω_1) and sea bass (ω_2)
- If we have only prior probability
  - 例如，教室门口判断进来的是男生还是女生，没有任何传感器
  - Decide ω_1 if $P(\omega_1) > P(\omega_2)$, otherwise ω_2
  - Minimum error decision

$$P(error) = \begin{cases} P(\omega_2) & \text{if we decide } \omega_1 \\ P(\omega_1) & \text{if we decide } \omega_2 \end{cases}$$

  - 教室门口判断性别的例子：错误率？

2类的例子

- 有传感器(特征)的情况
  Decision based on posterior probabilities



x轴：一维特征空间

$$P(\omega_i \mid \mathbf{x}) = \frac{p(\mathbf{x} \mid \omega_i) P(\omega_i)}{p(\mathbf{x})} = \frac{p(\mathbf{x} \mid \omega_i) P(\omega_i)}{\sum_{j=1}^{c} p(\mathbf{x} \mid \omega_j) P(\omega_j)}$$

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- Decision based on posterior probabilities

$$P(error|x) = \begin{cases} P(\omega_1|x) & \text{if we decide } \omega_2 \\ P(\omega_2|x) & \text{if we decide } \omega_1. \end{cases}$$

$$\text{Decide } \omega_1 \text{ if } \underline{P(\omega_1|x) \;\; > P(\omega_2|x);} \quad \text{otherwise decide } \omega_2$$

$$P(error|x) = \min\left[P(\omega_1|x), P(\omega_2|x)\right].$$

- Evidence (a.k.a. likelihood)

$$\text{Decide } \omega_1 \text{ if } \underline{p(x|\omega_1)P(\omega_1) > p(x|\omega_2)P(\omega_2);} \quad \text{otherwise decide } \omega_2$$

  – see $$P(\omega_i \mid \mathbf{x}) = \frac{p(\mathbf{x} \mid \omega_i)P(\omega_i)}{p(\mathbf{x})}$$

教室门口判断性别的例子：用什么传感器(**x**)?

最小风险决策：贝叶斯决策的一般形式

- 决策代价(loss, cost)
  - True class ω_j, decided as α_i $\lambda_{ij} = \lambda(\alpha_i \mid \omega_j)$
    - 有时λ_{ij}和λ_{ji}相差很大，比如医疗诊断的场合、工业检测、自动商店判断性别
- Condition risk

$$R(\alpha_i|\mathbf{x}) = \sum_{j=1}^{c} \lambda(\alpha_i|\omega_j) P(\omega_j|\mathbf{x})$$

- Overall (expected) risk

$$R = \int R(\alpha(\mathbf{x})|\mathbf{x}) p(\mathbf{x}) \, d\mathbf{x} \qquad \alpha(\mathbf{x}) \in \{\alpha_1, \cdots, \alpha_c\}$$

- Minimum risk decision (Bayes decision)

$$\arg\min_i R(\alpha_i \mid x)$$

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- Minimum risk decision: 2-class case
  - Condition risk

$$\begin{aligned} R(\alpha_1|\mathbf{x}) &= \lambda_{11}P(\omega_1|\mathbf{x}) + \lambda_{12}P(\omega_2|\mathbf{x}) \\ R(\alpha_2|\mathbf{x}) &= \lambda_{21}P(\omega_1|\mathbf{x}) + \lambda_{22}P(\omega_2|\mathbf{x}) \end{aligned}$$

  - Decision rule

$$R(\alpha_1 \mid x) < R(\alpha_2 \mid x) \leftrightarrow (\lambda_{21} - \lambda_{11})P(\omega_1|\mathbf{x}) > (\lambda_{12} - \lambda_{22})P(\omega_2|\mathbf{x})$$

    - Equivalently, decide ω_1 if

$$(\lambda_{21} - \lambda_{11})\underline{p(\mathbf{x}|\omega_1)P(\omega_1)} > (\lambda_{12} - \lambda_{22})p(\mathbf{x}|\omega_2)P(\omega_2)$$

$$\frac{p(\mathbf{x}|\omega_1)}{p(\mathbf{x}|\omega_2)} > \frac{\lambda_{12} - \lambda_{22}}{\lambda_{21} - \lambda_{11}} \frac{P(\omega_2)}{P(\omega_1)}$$

(Likelihood ratio)

最小错误率分类

- Zero-one loss

$$\lambda(\alpha_i|\omega_j) = \begin{cases} 0 & i = j \\ 1 & i \neq j \end{cases} \qquad i, j = 1, ..., c$$

$$\begin{aligned} R(\alpha_i|\mathbf{x}) &= \sum_{j=1}^{c} \lambda(\alpha_i|\omega_j) P(\omega_j|\mathbf{x}) \\ &= \sum_{j \neq i} P(\omega_j|\mathbf{x}) \\ &= 1 - P(\omega_i|\mathbf{x}) \end{aligned}$$

- Minimum error decision: Maximum a posteriori (MAP)

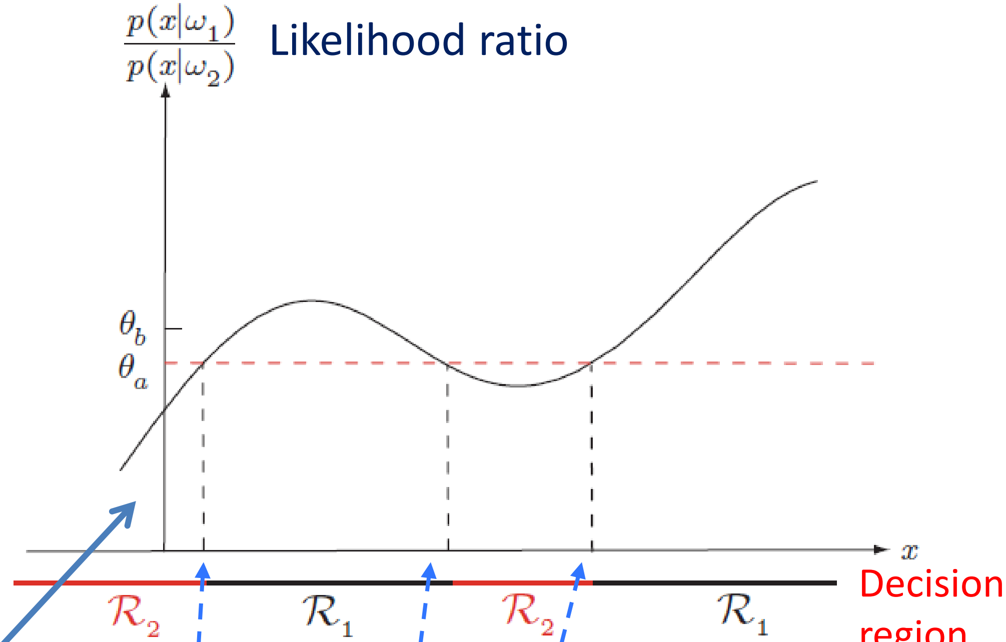
$$\text{Decide } \omega_i \text{ if } P(\omega_i|\mathbf{x}) > P(\omega_j|\mathbf{x}) \qquad \text{for all } j \neq i$$

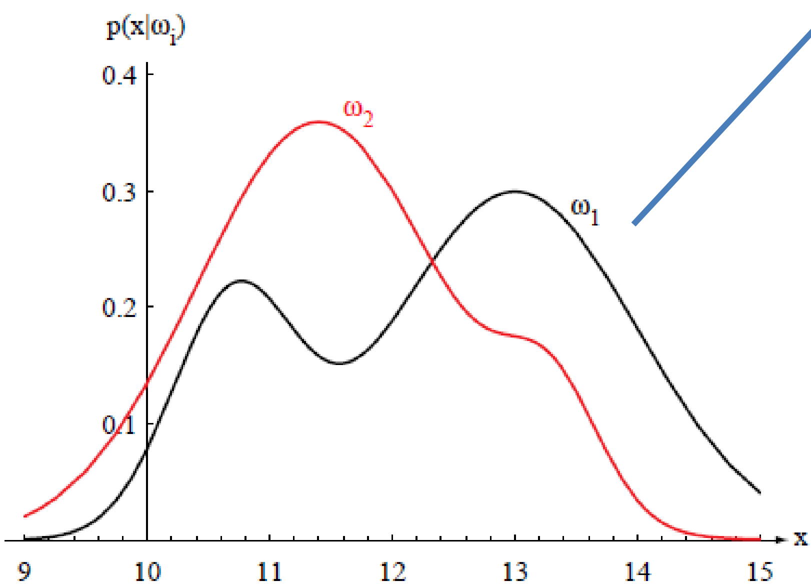
- 2-class case
  - decide ω_1 if

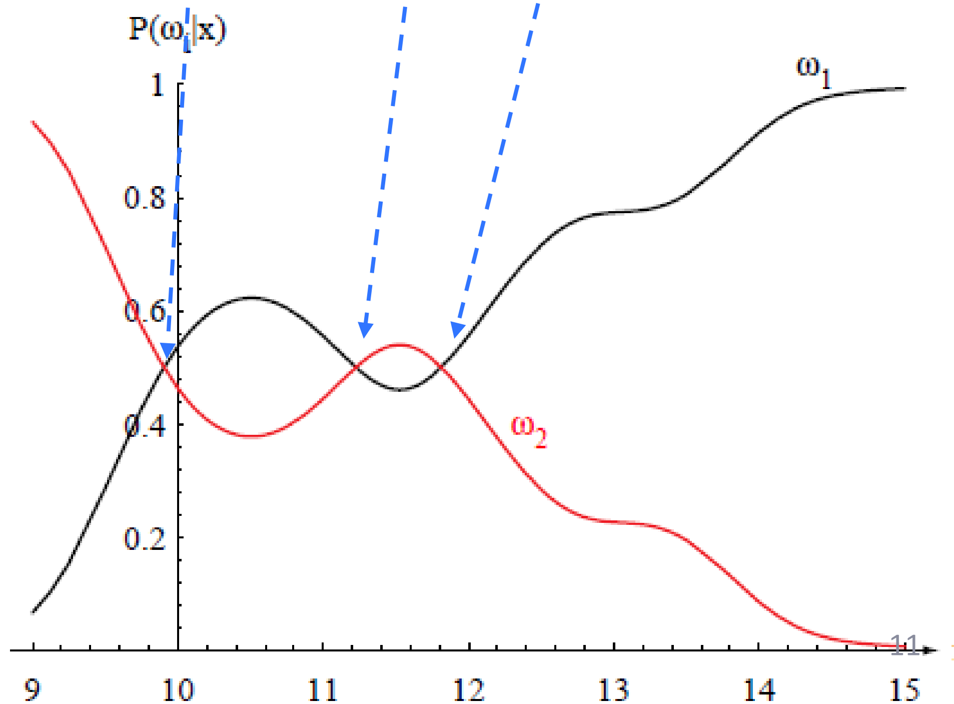
$$\frac{p(\mathbf{x}|\omega_1)}{p(\mathbf{x}|\omega_2)} > \frac{\lambda_{12}-\lambda_{22}}{\lambda_{21}-\lambda_{11}}\frac{P(\omega_2)}{P(\omega_1)}$$

0-1 loss

$$\frac{p(\mathbf{x}|\omega_1)}{p(\mathbf{x}|\omega_2)} > \frac{P(\omega_2)}{P(\omega_1)}$$



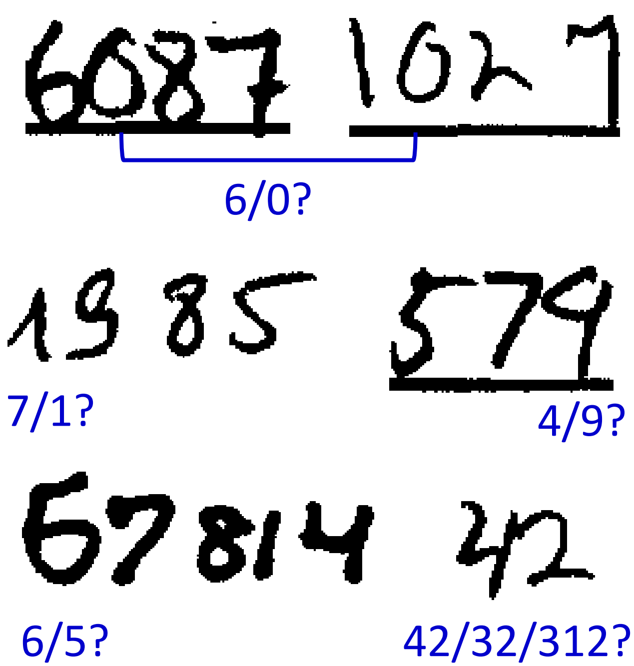




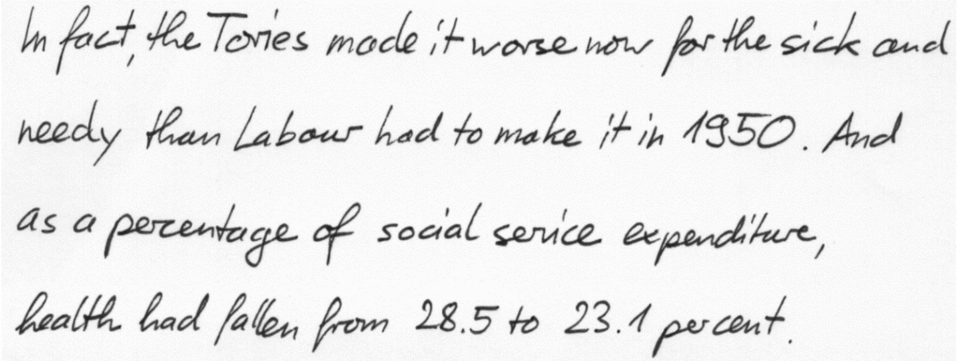
带拒识的决策

- 为什么要拒识？错误识别可能带来严重后果
  – 比如医疗诊断，金额识别

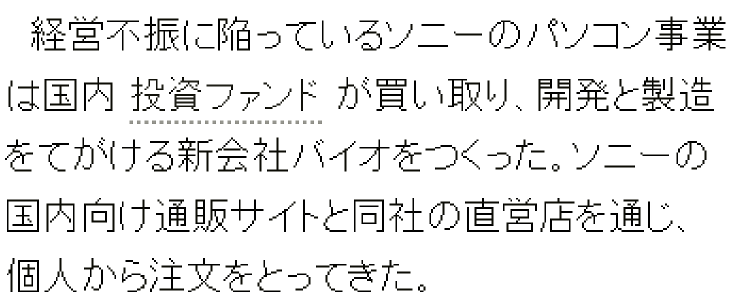
Ambiguity Rejection



Distance Rejection


In fact, the Tories made it worse now for the sick and needy than Labour had to make it in 1950. And as a percentage of social service expenditure, health had fallen from 28.5 to 23.1 percent.

English is outlier for a digit recognizer


経営不振に陥っているソニーのパソコン事業は国内投資ファンドが買い取り、開発と製造をてがける新会社バイオをつくった。ソニーの国内向け通販サイトと同社の直営店を通じ、個人から注文をとってきた。

These are outlier for an English recognizer

带拒识的决策

- Formulation (Problem 13, Chapter 2)
  - C+1 classes

$$\lambda(\alpha_i \mid \omega_j) = \begin{cases} 0, & i = j \\ \lambda_s, & i \neq j \\ \lambda_r, & \text{reject} \end{cases} \qquad \lambda_r < \lambda_s$$

$$R(\alpha_i|\mathrm{x}) = \sum_{j=1}^{c} \lambda(\alpha_i|\omega_j) P(\omega_j|\mathrm{x})$$

$$\Rightarrow \quad R_i(\mathbf{x}) = \begin{cases} \lambda_s[1 - P(\omega_i \mid \mathbf{x})], & i = 1, \ldots, c \\ \lambda_r, & \text{reject} \end{cases}$$

$$\arg\min_i R_i(\mathbf{x}) = \begin{cases} \arg\max_i P(\omega_i \mid \mathbf{x}), & \text{if } \max_i P(\omega_i \mid \mathbf{x}) > 1 - \lambda_r / \lambda_s \\ \text{reject}, & \text{otherwise} \end{cases}$$

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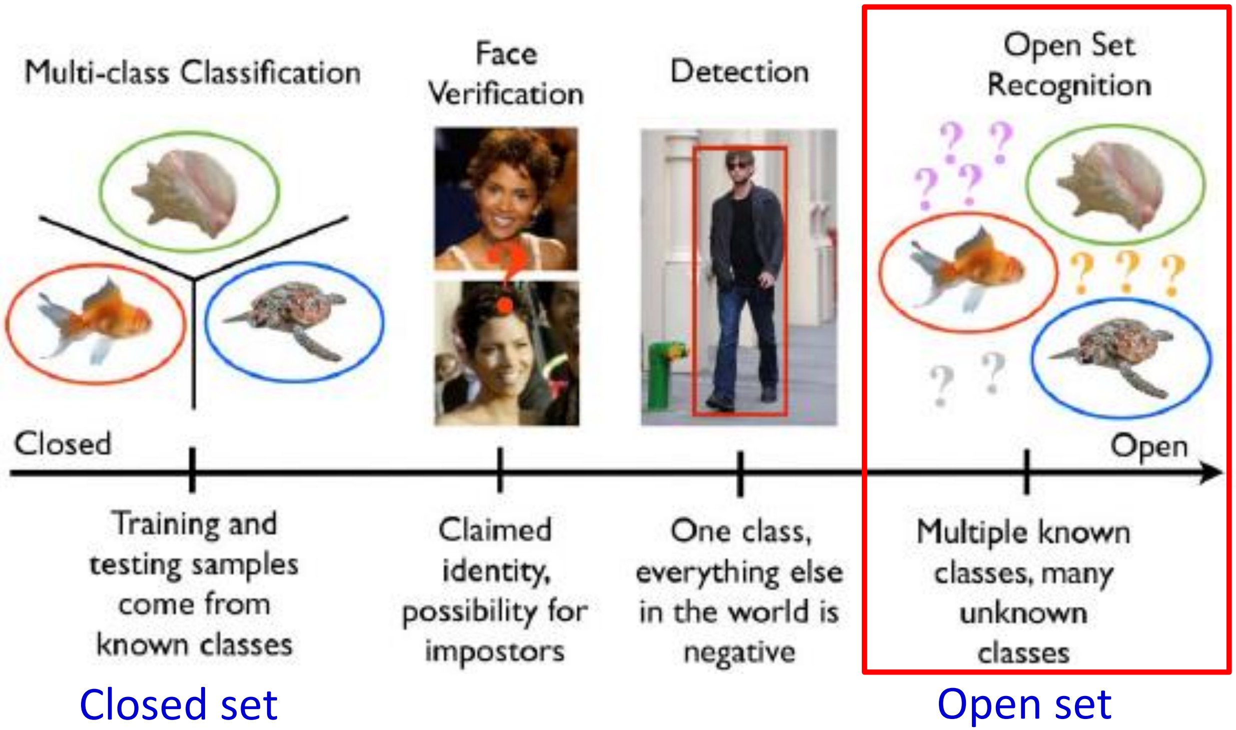
扩展：开放集识别的贝叶斯决策

- 传统的分类器假设训练样本和测试样本都来自预设的C个类别（闭合集, Closed set）。
- 实际环境中测试样本可能不属于预设的C个类别（异常样本, outlier），这种情况称为开放集(Open set)。
- 开放集的难点是异常样本没有训练集，只能训练已知C类的分类器。



W.J. Scheirer, et al, Toward open set recognition, *IEEE Trans. PAMI*, 35(7): 1757-1772, 2013.

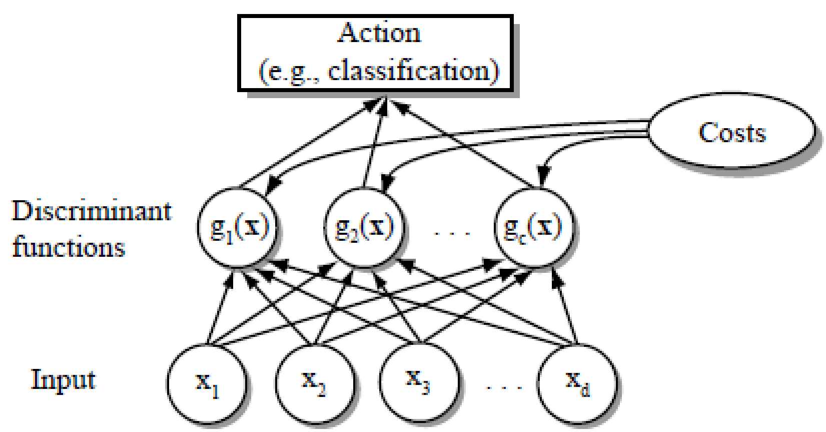
开放集分类贝叶斯决策

- 问题表示
  - 已知类别：$\omega_i, i=1,\ldots,c$
  - 先验概率 $\sum_{i=1}^{c} P(\omega_i) \le 1$
  - 后验概率 $\sum_{i=1}^{c} P(\omega_i \mid \mathbf{x}) \le 1 \qquad \sum_{i=1}^{c+1} P(\omega_i \mid \mathbf{x}) = 1$
  - 条件概率密度 $p(\mathbf{x} \mid \omega_i), i=1,\ldots,c, \quad p(\mathbf{x} \mid \omega_{c+1}) = ?$
- 分类决策
  - 假设 $p(\mathbf{x} \mid \omega_{c+1}) = \rho$ ρ为很小的常数
  - 后验概率 $P(\omega_i \mid \mathbf{x}) = \dfrac{p(\mathbf{x} \mid \omega_i) P(\omega_i)}{p(\mathbf{x})} = \dfrac{p(\mathbf{x} \mid \omega_i) P(\omega_i)}{\sum_{j=1}^{c+1} p(\mathbf{x} \mid \omega_j) P(\omega_j)}$
  - 最大后验概率决策 $\begin{cases} \text{in-class}, & \text{if } \max\limits_{i=1,\ldots,c} p(\mathbf{x} \mid \omega_i) P(\omega_i) > \rho P(\omega_{c+1}) \\ \text{outlier}, & \text{otherwise} \end{cases}$

判别函数、决策面

- 判别函数(Discriminant Function)
  - 表征模式属于每一类的广义似然度$g_i(\mathbf{x})$, i=1,…,c
  - 分类决策 $\arg\max_i g_i(\mathbf{x})$
  - E.g., conditional risk $g_i(\mathbf{x}) = -R(\alpha_i \mid \mathbf{x})$
  - Posterior probability $g_i(\mathbf{x}) = P(\omega_i \mid \mathbf{x})$
  - Likelihood $g_i(\mathbf{x}) = p(\mathbf{x} \mid \omega_i)P(\omega_i)$

$$g_i(\mathbf{x}) = \log p(\mathbf{x} \mid \omega_i) + \log P(\omega_i)$$



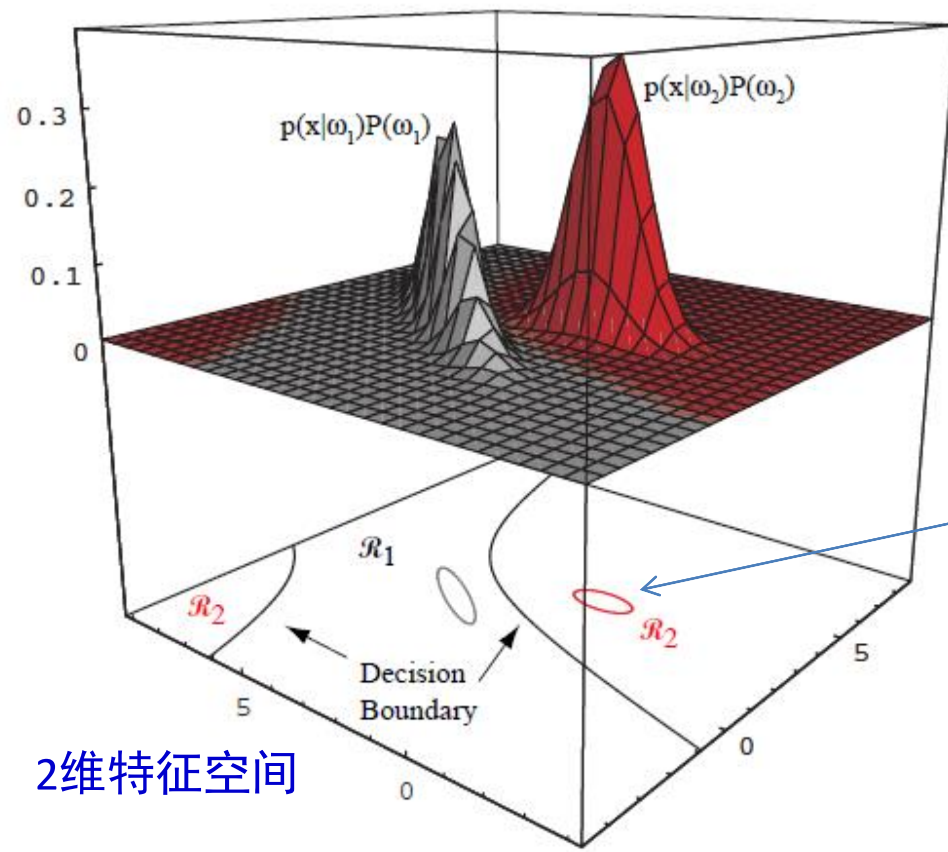
- 决策面(Decision surface)
  - 特征空间中二类判别函数相等的点的集合

$$g(\mathbf{x}) \equiv g_1(\mathbf{x}) - g_2(\mathbf{x}) \qquad g(\mathbf{x}) = P(\omega_1|\mathbf{x}) - P(\omega_2|\mathbf{x}) = 0$$

$$g(\mathbf{x}) = \ln\frac{p(\mathbf{x}|\omega_1)}{p(\mathbf{x}|\omega_2)} + \ln\frac{P(\omega_1)}{P(\omega_2)}$$

有判别函数就可以分类了，为什么还来求决策面？

答：加深对特征空间和分类器性质的理解



正态分布下的一个例子

Density 1/e ellipse

2维特征空间

贝叶斯决策用于模式分类

- Bayes决策的关键
  - 类条件概率密度估计
  - 先验概率：从训练样本估计或假设等概率
  - 决策代价[λ_{ij}]，一般为0-1代价
- 分类器设计
  - 收集训练样本
  - 用每一类的样本估计类条件概率密度 $p(\mathbf{x}|\omega_i)$
  - 估计类先验概率
  - 模型参数集：$\{p(\mathbf{x}|\omega_i,\theta_i), P(\omega_i)\}, i=1,\ldots,c$
- 分类过程
  - 计算测试样本x属于每一类的后验概率
  - 最大后验概率/最小风险决策

概率密度估计方法

- 参数法：假定概率密度函数形式

$$p(\mathbf{x}\,|\,\omega_i) = p(\mathbf{x}\,|\,\theta_i)$$

  - Distribution: Gaussian, Gamma, Bernouli
  - Parameter estimation: maximum-likelihood (ML), Bayesian estimation
- 非参数法：可以表示任意概率分布，无函数形式
  - Parzen window, k-NN
  - 需要保存所有或大部分样本
- Semi-parametric, 近似任意概率分布，有函数形式
  - Distribution: Gaussian mixture (GM)
  - Estimation: expectation-maximization (EM)

高斯密度函数

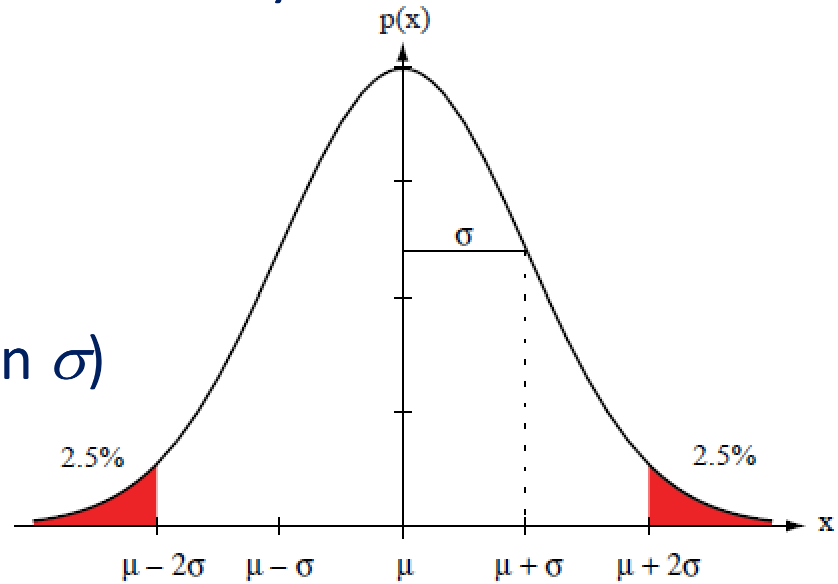
- Gaussian density (normal distribution)

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

  - Mean μ
  - Variance σ^2 (standard deviation σ)

$$\mu \equiv \mathcal{E}[x] = \int_{-\infty}^{\infty} x p(x)\, dx$$

$$\sigma^2 \equiv \mathcal{E}[(x-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 p(x)\, dx$$



$$H(p(x)) = -\int p(x) \ln p(x)\, dx$$

- 在给定均值和方差的所有分布中，正态分布的熵最大(Problem 20, Chapter 2)
- 根据Central Limit Theorem，大量独立随机变量之和趋近正态分布
- 实际环境中，很多类别的特征分布趋近正态分布

- Multivariate normal density $p(\mathbf{x}) \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$
  - 公式要牢记

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^t \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]$$

  - Mean $\boldsymbol{\mu} \equiv \mathcal{E}[\mathbf{x}] = \int \mathbf{x} p(\mathbf{x})\, d\mathbf{x} \qquad \mu_i = \mathcal{E}[x_i]$
  - Covariance matrix

$$\boldsymbol{\Sigma} \equiv \mathcal{E}[(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^t] = \int (\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^t p(\mathbf{x})\, d\mathbf{x}$$

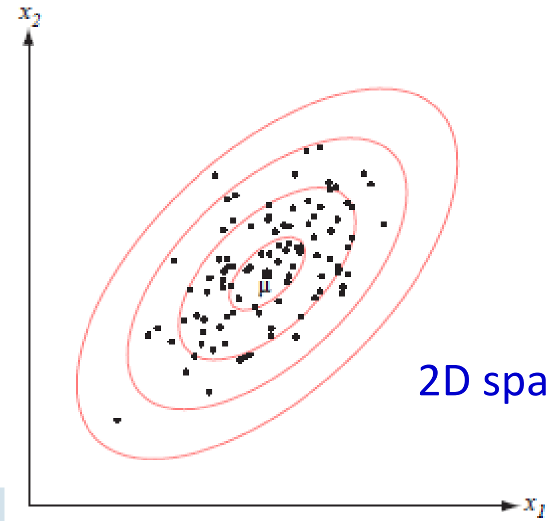
$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1d} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{d1} & \sigma_{d2} & \cdots & \sigma_{dd} \end{bmatrix}$$

$$\sigma_{ij} = \mathcal{E}[(x_i - \mu_i)(x_j - \mu_j)] \qquad \sigma_{ii} = \sigma_i^2$$

If x_i and x_j are *statistically independent*, $\sigma_{ij} = 0$

  - 等密度点轨迹：hyperellipsoid
    - 特殊情况下为圆形或超球面
  - Mahalanobis distance

$$r^2 = (\mathbf{x}-\boldsymbol{\mu})^t \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})$$



2D space

Break

协方差矩阵的性质

- 实对称矩阵
- Eigenvalues & eigenvecters (本征值，本征向量)

$$\Sigma\phi_i = \lambda_i\phi_i \qquad \Phi = [\phi_1\phi_2\cdots\phi_d] \qquad \Lambda = diag[\lambda_1,\lambda_2,\cdots,\lambda_d]$$

  – Orthonormal

$$\Phi^T\Phi = I \qquad \Phi^T = \Phi^{-1} \quad \Longleftrightarrow \quad \phi_i^t\phi_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

  – 矩阵表示

$$\Sigma\Phi = \Phi\Lambda \leftrightarrow \Sigma = \Phi\Lambda\Phi^T \leftrightarrow \Sigma = \sum_{i=1}^{d}\lambda_i\phi_i\phi_i^T$$

- 矩阵对角化

$$\Phi^T\Sigma\Phi = \Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_d \end{bmatrix} \Longleftrightarrow \phi_i^T\Sigma\phi_i = \lambda_i$$

- 应用：Principal component analysis (PCA)
  - 一种降维(特征提取)方法
  - 将随机矢量投影到低维子空间，使子空间投影的重建误差最小
  - 选择本征值最大的m (m<d)个本征向量作为子空间的基(basis)

线性空间中正交变换($\Phi^T\Phi = I$) 不影响欧氏距离

$$\sum_{j=1}^{d}[(\mathbf{x}-\mu)^T\phi_j]^2 = \|\mathbf{x}-\mu\|^2$$

子空间投影 $\mu+\sum_{j=1}^{m}[(\mathbf{x}-\mu)^T\phi_j]\phi_j$

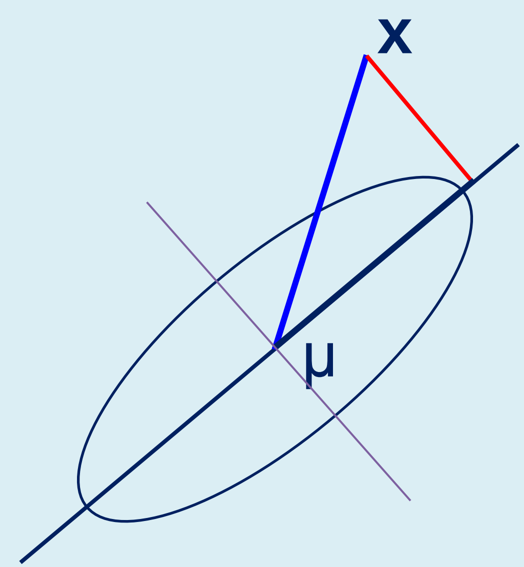
投影重建误差

$$r_E = \|\mathbf{x}-\mu\|^2 - \sum_{j=1}^{m}[(\mathbf{x}-\mu)^T\phi_j]^2 = \sum_{j=m+1}^{d}[(\mathbf{x}-\mu)^T\phi_j]^2$$

期望

$$\mathcal{E}(r_E) = \mathcal{E}\left\{\sum_{j=m+1}^{d}[(\mathbf{x}-\mu)^T\phi_j]^2\right\} = \mathcal{E}\left\{\sum_{j=m+1}^{d}\phi_j^T(\mathbf{x}-\mu)(\mathbf{x}-\mu)^T\phi_j\right\}$$

$$= \sum_{j=m+1}^{d}\phi_j^T\mathcal{E}[(\mathbf{x}-\mu)(\mathbf{x}-\mu)^T]\phi_j = \sum_{j=m+1}^{d}\phi_j^T\Sigma\phi_j = \sum_{j=m+1}^{d}\lambda_j$$



$\min \sum_{j=m+1}^{d}\lambda_j$ or $\max \sum_{j=1}^{m}\lambda_j$

意味着取λ_j最大的m个本征向量作为子空间基(basis)

线性变换的高斯分布

- 线性变换 $\mathbf{y} = \mathbf{A}^t\mathbf{x}$
  - $A^tA=1$: 正交变换(坐标轴旋转)
  - 变换后的分布仍为正态分布

    $$p(\mathbf{y}) \sim N(\mathbf{A}^t\boldsymbol{\mu}, \mathbf{A}^t\boldsymbol{\Sigma}\mathbf{A})$$
  - Diagonalization

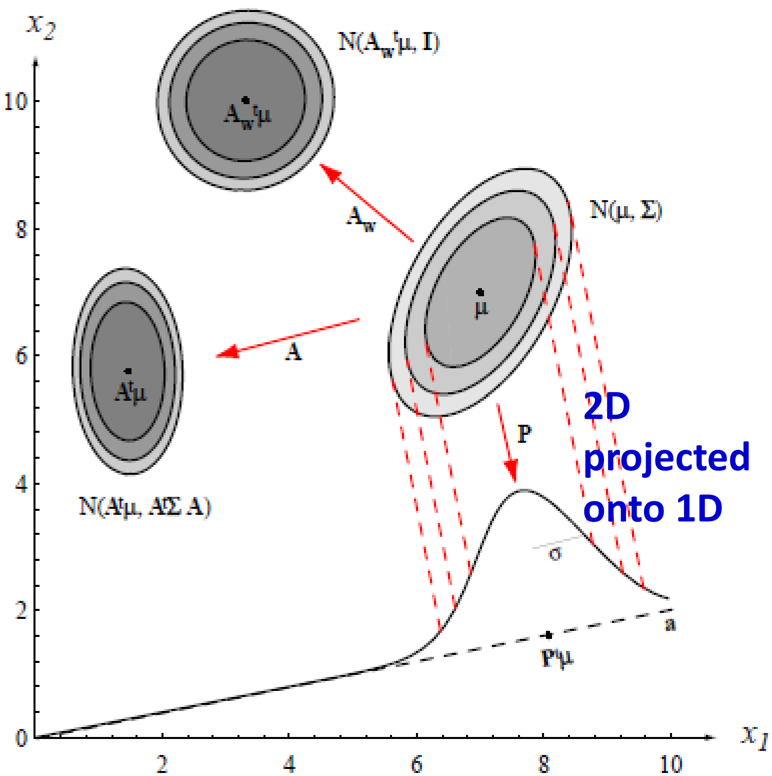
    $$A = \Phi$$

    $$A^t\Sigma A = \Lambda$$
  - Whitening transform

    $$\mathbf{A}_w = \boldsymbol{\Phi}\boldsymbol{\Lambda}^{-1/2}$$

    $$A_w^t\Sigma A_w = \Lambda^{-1/2}\Phi^t\Sigma\Phi\Lambda^{-1/2}$$

    $$= \Lambda^{-1/2}\Lambda\Lambda^{-1/2} = I$$



相关知识：特征提取/降维

- Dimensionality reduction
- Feature extraction
  - Feature generation: original data **d**→**x**
  - Linear feature extraction $\mathbf{x} = \mathbf{A}^T \mathbf{d}$
- Feature selection (for reduction and performance)
  - Feature subset selection: a learning/optimization problem
- Feature transform (for extraction or reduction)
  - Linear transform $\mathbf{y} = \mathbf{A}^T \mathbf{x}$
  - Nonlinear transform: may increase dimensionality, e.g. kernel PCA, kernel LDA
- Handcrafted feature
- Feature learning
  - Automatic feature generation, e.g. convolutional neural network (CNN)

高斯密度下的判别函数

- 判别函数 $g_i(\mathrm{x}) = \ln p(\mathrm{x}|\omega_i) + \ln P(\omega_i)$

$$p(\mathbf{x} \mid \omega_i) = \frac{1}{(2\pi)^{d/2} \mid \Sigma_i \mid^{1/2}} \exp\left[-\frac{1}{2} (\mathbf{x} - \mu_i) \Sigma_i^{-1} (\mathbf{x} - \mu_i) \right]$$

$$g_i(\mathrm{x}) = -\frac{1}{2} (\mathrm{x} - \boldsymbol{\mu}_i)^t \boldsymbol{\Sigma}_i^{-1} (\mathrm{x} - \boldsymbol{\mu}_i) - \frac{d}{2} \ln 2\pi - \frac{1}{2} \ln |\boldsymbol{\Sigma}_i| + \ln P(\omega_i)$$

  - Quadratic discriminant function (QDF)
  - 在不同covariance假设条件下得到一些特殊形式

- Case 1: $\Sigma_i=\sigma^2 I$

  (去掉与类别无关项)

$$g_i(\mathbf{x}) = -\frac{\|\mathbf{x}-\boldsymbol{\mu}_i\|^2}{2\sigma^2} + \ln P(\omega_i)$$

  – Euclidean distance $\|\mathbf{x}-\boldsymbol{\mu}_i\|^2 = (\mathbf{x}-\boldsymbol{\mu}_i)^t(\mathbf{x}-\boldsymbol{\mu}_i)$

    - Nearest mean/distance classifier

  – 展开二次式 $(\mathbf{x}-\boldsymbol{\mu}_i)^t(\mathbf{x}-\boldsymbol{\mu}_i)$

$$g_i(\mathbf{x}) = -\frac{1}{2\sigma^2}[\mathbf{x}^t\mathbf{x} - 2\boldsymbol{\mu}_i^t\mathbf{x} + \boldsymbol{\mu}_i^t\boldsymbol{\mu}_i] + \ln P(\omega_i)$$

  – 忽略与类别无关项，得到线性判别函数

$$g_i(\mathbf{x}) = \mathbf{w}_i^t\mathbf{x} + w_{i0}$$

$$\mathbf{w}_i = \frac{1}{\sigma^2}\boldsymbol{\mu}_i \qquad w_{i0} = \frac{-1}{2\sigma^2}\boldsymbol{\mu}_i^t\boldsymbol{\mu}_i + \ln P(\omega_i)$$

- Case 1: $\Sigma_i=\sigma^2 I$ (continued)
  - 二类决策面(判别函数相等的点构成)

$$g_i(\mathbf{x}) = g_j(\mathbf{x})$$

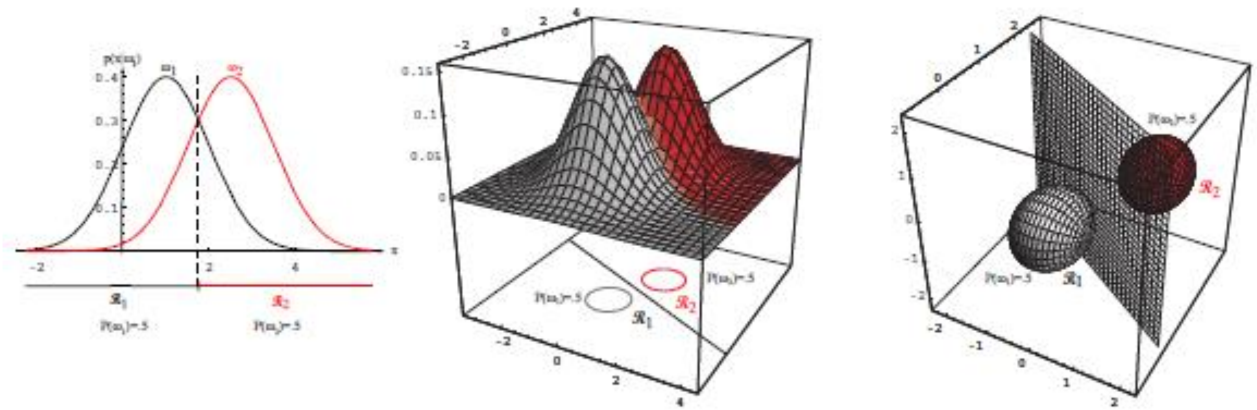
$$\Rightarrow \mathbf{w}^t(\mathbf{x}-\mathbf{x}_0)=0 \qquad \mathbf{w}=\boldsymbol{\mu}_i-\boldsymbol{\mu}_j$$

W: 法向量
平面与μ_1-μ_2垂直

$$\mathbf{x}_0=\frac{1}{2}(\boldsymbol{\mu}_i+\boldsymbol{\mu}_j)-\frac{\sigma^2}{\|\boldsymbol{\mu}_i-\boldsymbol{\mu}_j\|^2}\ln\frac{P(\omega_i)}{P(\omega_j)}(\boldsymbol{\mu}_i-\boldsymbol{\mu}_j)$$

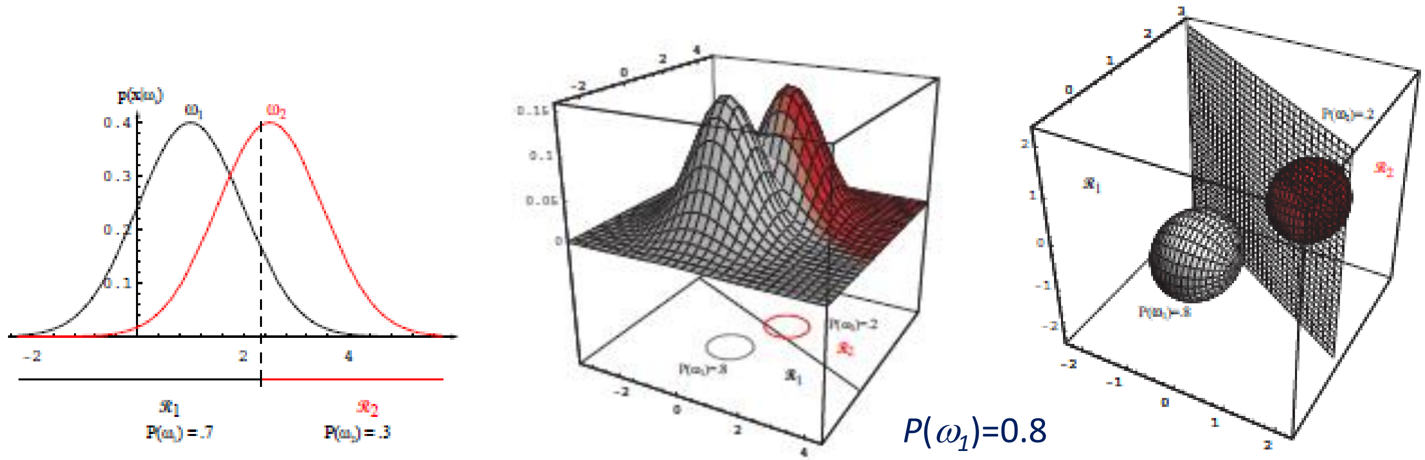
位置移向先验概率小的类别

  - 1D, 2D, 3D的情况
    - 当$P(\omega_1)=P(\omega_2)$，决策面为二类均值的等分面

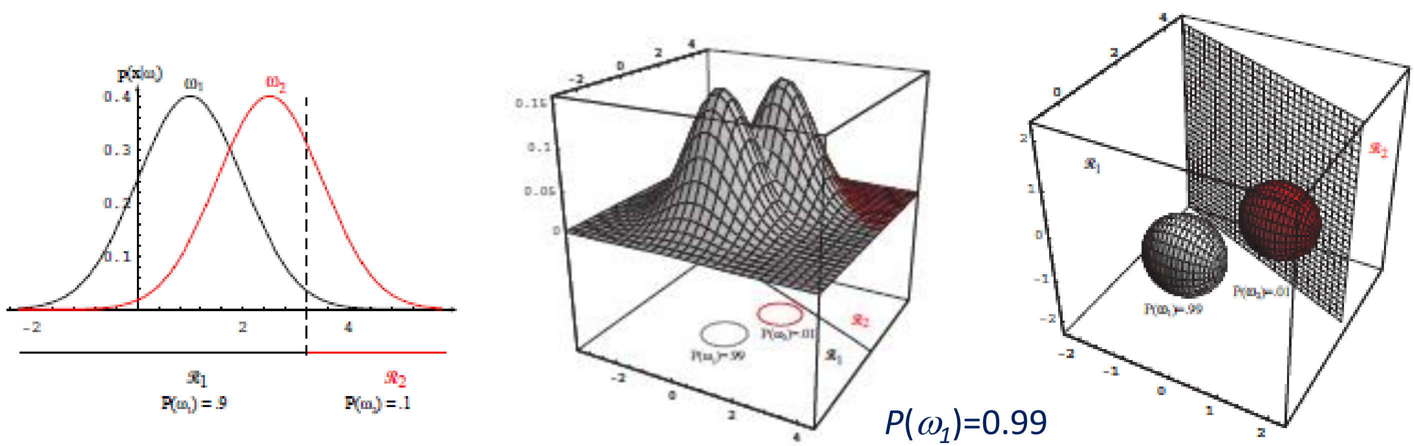


– 当先验概率变化，决策面发生平移

移向概率
小的类别



$P(\omega_1)$=0.8



$P(\omega_1)$=0.99

- Case 2: $\Sigma_i = \Sigma$ (所有类别共享协方差矩阵)

$$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_i)^t\boldsymbol{\Sigma}_i^{-1}(\mathbf{x}-\boldsymbol{\mu}_i) - \frac{d}{2}\ln 2\pi - \frac{1}{2}\ln|\boldsymbol{\Sigma}_i| + \ln P(\omega_i)$$

$$\Rightarrow \quad g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_i)^t\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}_i) + \ln P(\omega_i)$$

  – 展开二次式 $(\mathbf{x}-\boldsymbol{\mu}_i)^t\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}_i)$

线性判别函数！ $g_i(\mathbf{x}) = \mathbf{w}_i^t\mathbf{x} + w_{i0}$

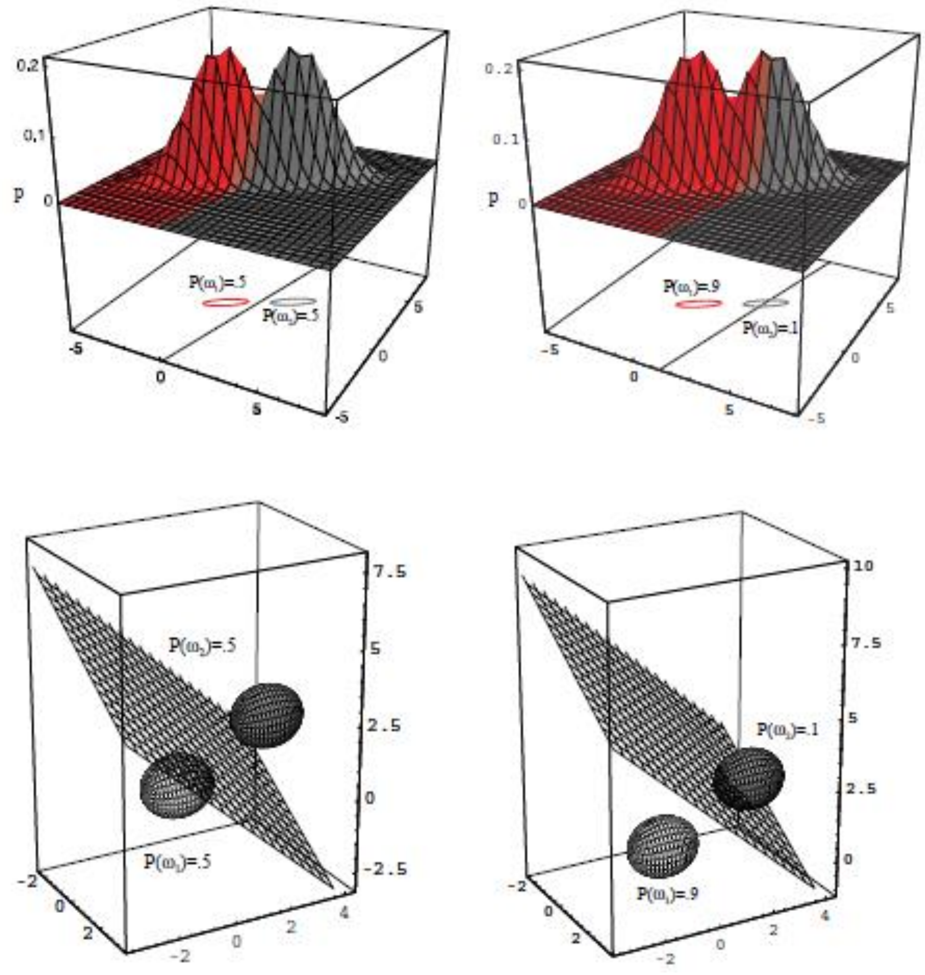
$$\mathbf{w}_i = \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_i \qquad w_{i0} = -\frac{1}{2}\boldsymbol{\mu}_i^t\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_i + \ln P(\omega_i)$$

  – 二类决策面 $g_i(\mathbf{x}) = g_j(\mathbf{x})$

$$\Rightarrow \quad \mathbf{w}^t(\mathbf{x}-\mathbf{x}_0) = 0 \qquad \mathbf{w} = \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)$$

$$\mathbf{x}_0 = \frac{1}{2}(\boldsymbol{\mu}_i + \boldsymbol{\mu}_j) - \frac{\ln\left[P(\omega_i)/P(\omega_j)\right]}{(\boldsymbol{\mu}_i-\boldsymbol{\mu}_j)^t\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_i-\boldsymbol{\mu}_j)}(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)$$

    - 注意跟μ_1-μ_2的关系，决策面不一定与之垂直
    - 当$P(\omega_1)= P(\omega_2)$，决策面经过$(\mu_1+\mu_2)/2$



$P(\omega_1) = P(\omega_2)$ $\qquad$ $P(\omega_1) \neq P(\omega_2)$

- Case 3: Σ_i= arbitrary

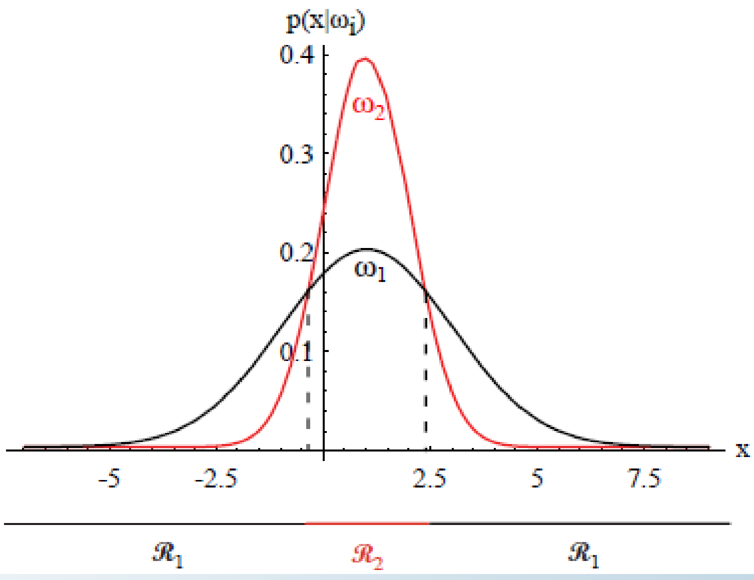
$$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_i)^t\boldsymbol{\Sigma}_i^{-1}(\mathbf{x}-\boldsymbol{\mu}_i) - \frac{d}{2}\ln 2\pi - \frac{1}{2}\ln|\boldsymbol{\Sigma}_i| + \ln P(\omega_i)$$

$$g_i(\mathbf{x}) = \mathbf{x}^t\mathbf{W}_i\mathbf{x} + \mathbf{w}_i^t\mathbf{x} + w_{i0}$$

$$\mathbf{W}_i = -\frac{1}{2}\boldsymbol{\Sigma}_i^{-1} \qquad \mathbf{w}_i = \boldsymbol{\Sigma}_i^{-1}\boldsymbol{\mu}_i$$

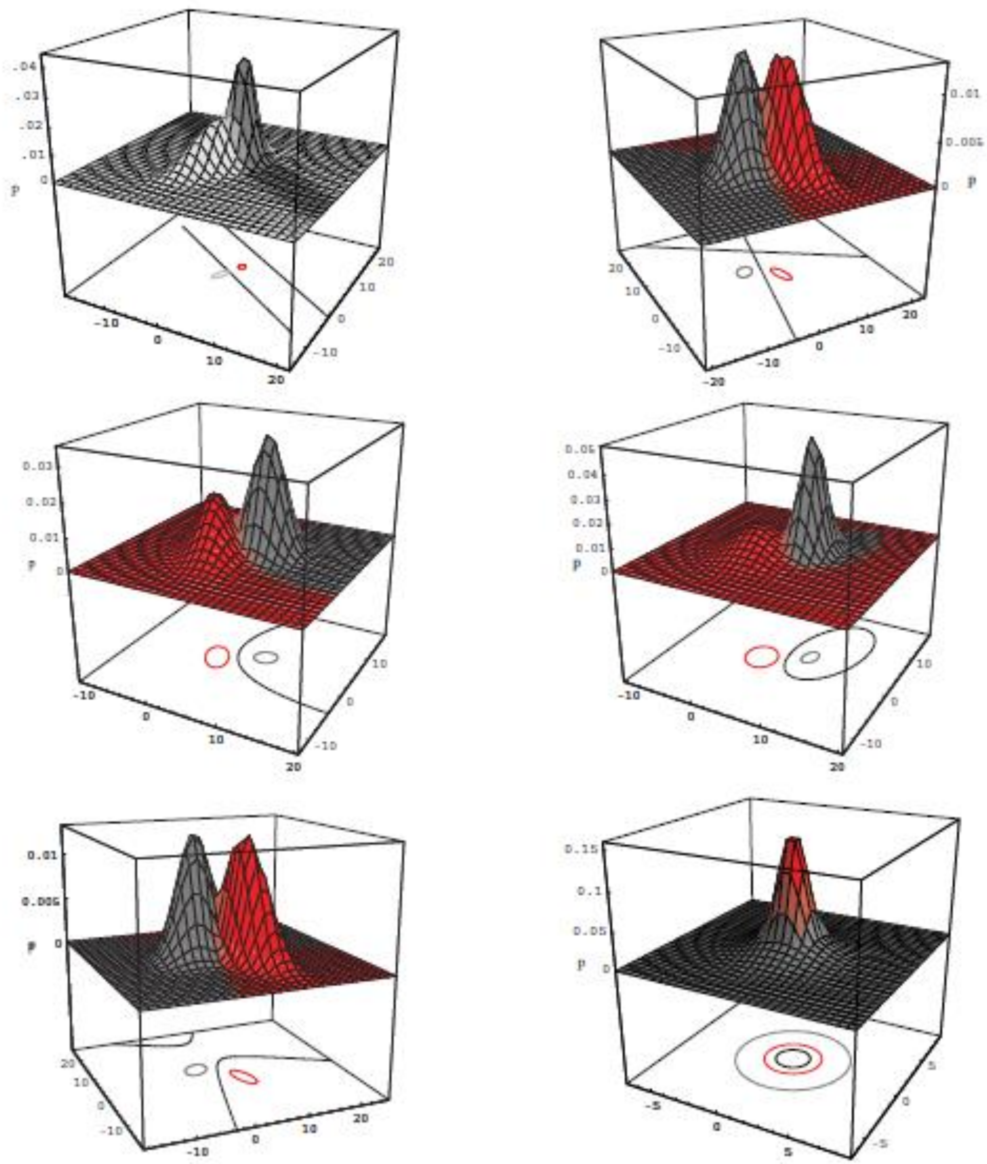
$$w_{i0} = -\frac{1}{2}\boldsymbol{\mu}_i^t\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\mu}_i - \frac{1}{2}\ln|\boldsymbol{\Sigma}_i| + \ln P(\omega_i)$$

  - 二类决策面：$g_1(\mathbf{x})=g_2(\mathbf{x})$, hyperquadratics
    - 等均值的情况下，1D的例子

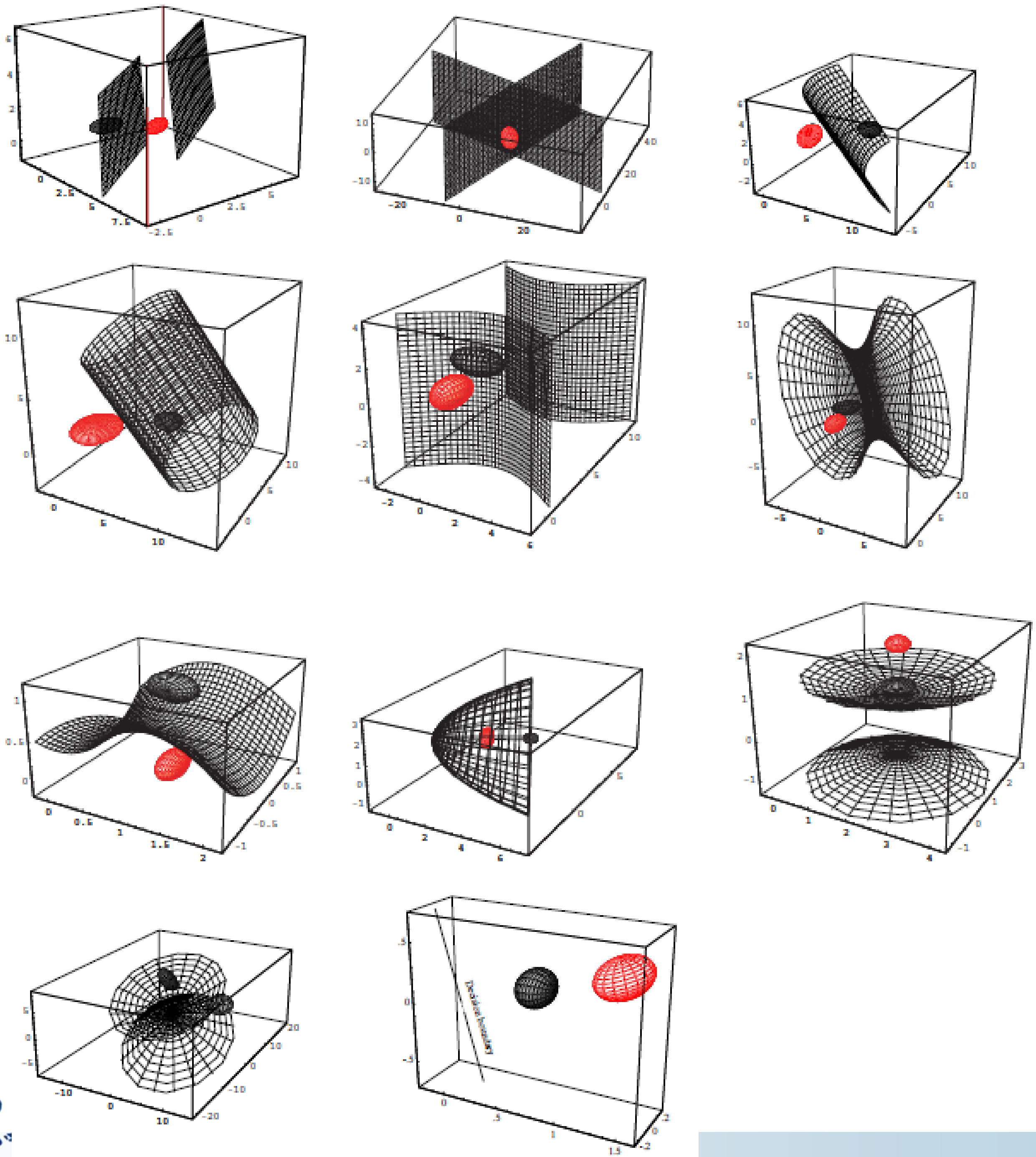


2D的例子

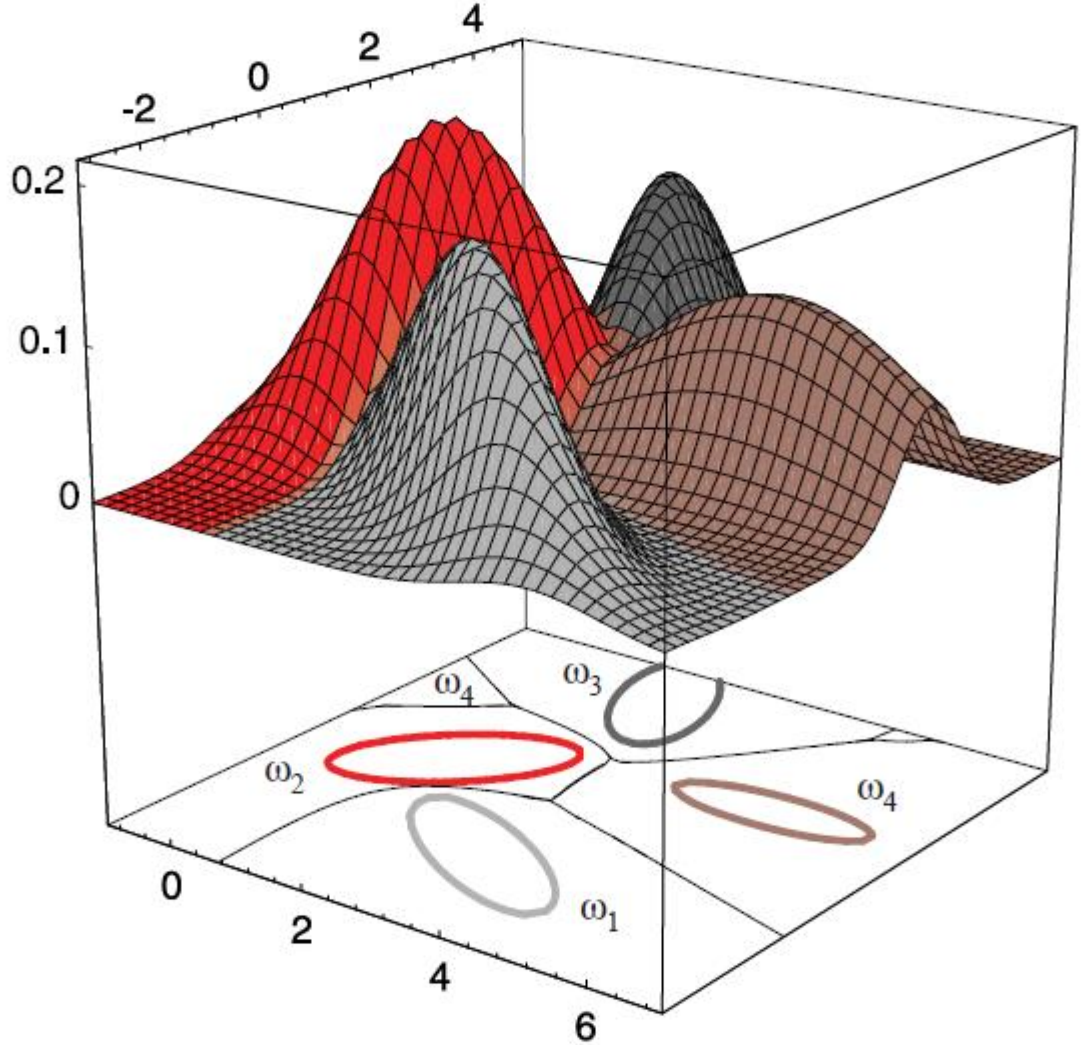
（z轴是概率密度）



3D的例子



2D，4类的例子



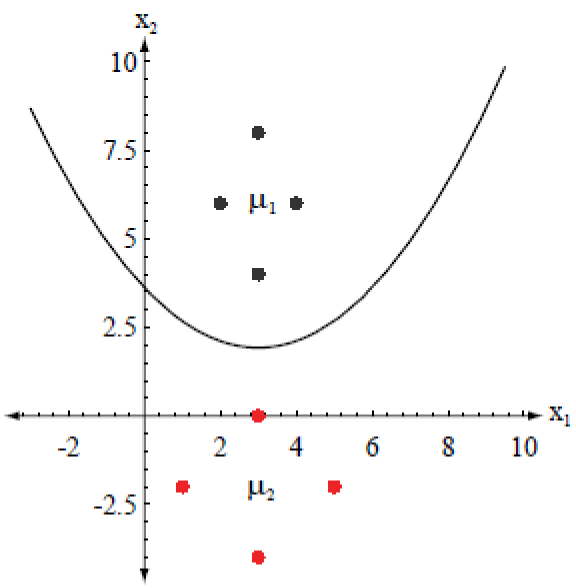
- 一个具体例子
  - 2类，2D $P(\omega_1) = P(\omega_2) = 0.5$

$$\boldsymbol{\mu}_1 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}; \quad \boldsymbol{\Sigma}_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \boldsymbol{\Sigma}_1^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

$$\boldsymbol{\mu}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}; \quad \boldsymbol{\Sigma}_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \boldsymbol{\Sigma}_2^{-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

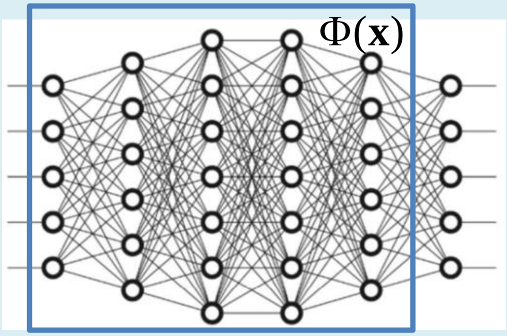
  - 决策面 $g_1(\mathbf{x})=g_2(\mathbf{x})$

$$x_2 = 3.514 - 1.125x_1 + 0.1875x_1^2$$



线性判别函数、广义线性判别函数、神经网络

- 线性判别函数(LDF) $g_i(\mathbf{x}) = \mathbf{w}_i^T \mathbf{x} + w_{i0}$
  - 参数估计方法：Gaussian density, logistic regression, single-layer neural network, linear SVM, etc
- 广义线性判别函数 $g_i(\mathbf{x}) = \mathbf{w}_i^T \Phi(\mathbf{x}) + w_{i0}$
  - 扩充特征向量 $\Phi(\mathbf{x})$，如多项式特征，特征函数等
  - Nonlinear SVM→kernel function $\Phi(\mathbf{x}_i) \cdot \Phi(\mathbf{x}_i) = k(\mathbf{x}_i, \mathbf{x}_j)$
  - 多层神经网络的输出层可看作是广义线性判别函数



- LDF跟soft-max的关系
  - 由Gaussian density based LDF可知，

$$g_i(\mathbf{x}) \propto \log p(\mathbf{x} \mid \omega_i) P(\omega_i) \Rightarrow p(\omega_i \mid \mathbf{x}) = \frac{e^{g_i(\mathbf{x})}}{\sum_{j=1}^{c} e^{g_j(\mathbf{x})}}$$

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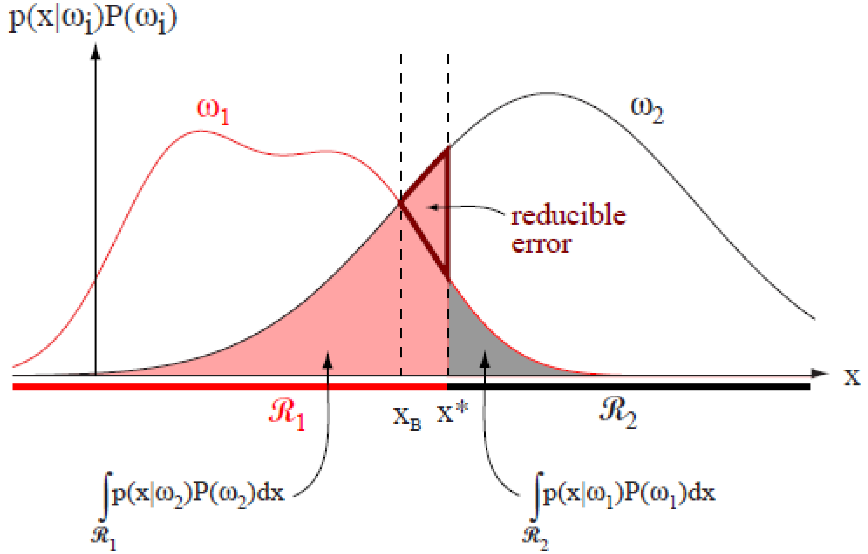
分类错误率

- 2类的情况

$$\begin{aligned} P(error) &= P(\mathrm{x} \in \mathcal{R}_2, \omega_1) + P(\mathrm{x} \in \mathcal{R}_1, \omega_2) \\ &= P(\mathrm{x} \in \mathcal{R}_2 | \omega_1) P(\omega_1) + P(\mathrm{x} \in \mathcal{R}_1 | \omega_2) P(\omega_2) \\ &= \int_{\mathcal{R}_2} p(\mathrm{x}|\omega_1) P(\omega_1)\ d\mathrm{x} + \int_{\mathcal{R}_1} p(\mathrm{x}|\omega_2) P(\omega_2)\ d\mathrm{x}. \end{aligned}$$

- 一般情况

$$\begin{aligned} P(correct) &= \sum_{i=1}^{c} P(\mathrm{x} \in \mathcal{R}_i, \omega_i) \\ &= \sum_{i=1}^{c} P(\mathrm{x} \in \mathcal{R}_i | \omega_i) P(\omega_i) \\ &= \sum_{i=1}^{c} \int_{\mathcal{R}_i} p(\mathrm{x}|\omega_i) P(\omega_i)\ d\mathrm{x} \end{aligned}$$



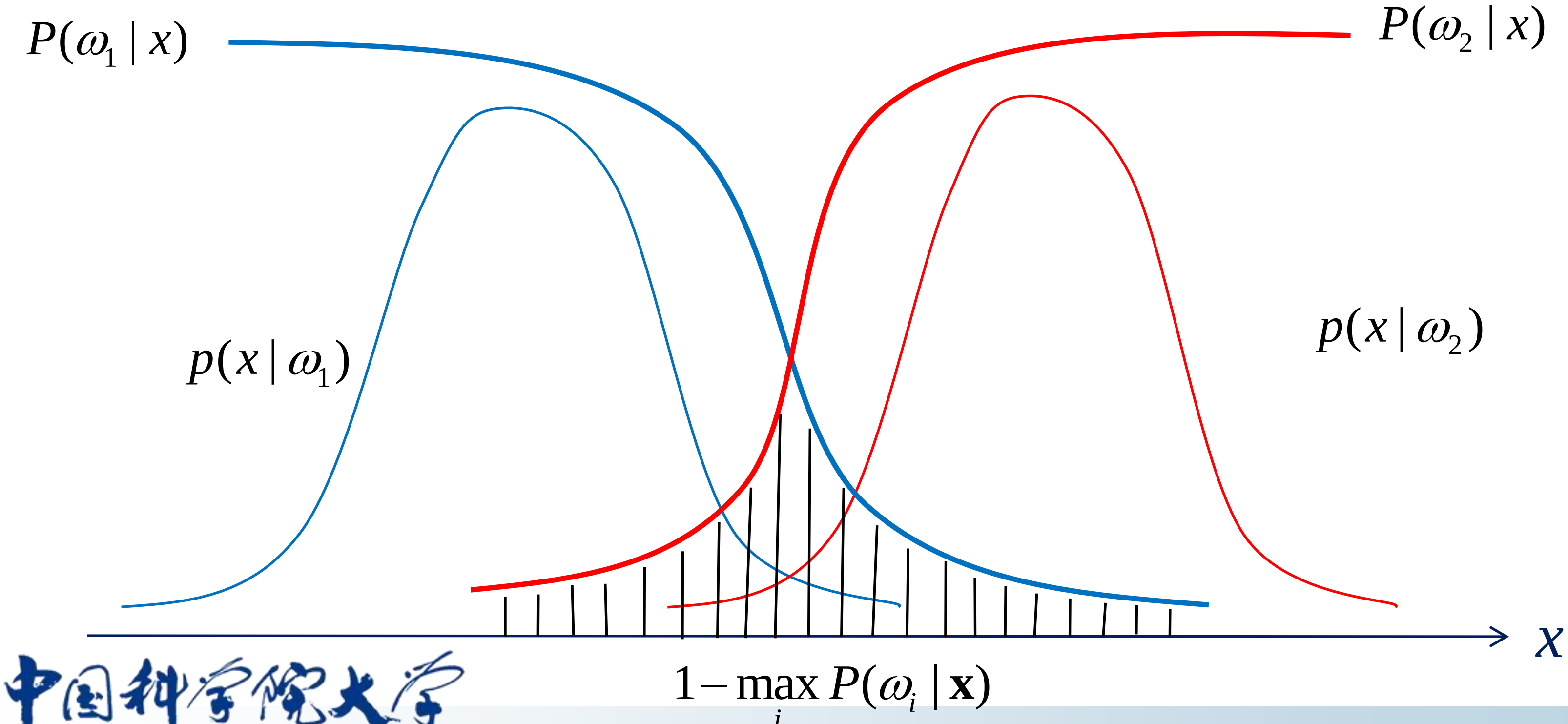
决策面为x_B时为最小错误率分类

- 最大后验概率决策(0-1 loss)的情况

$$P(correct) = \int_{\mathbf{x}} \max_i p(\mathbf{x} \mid \omega_i) P(\omega_i) d\mathbf{x}$$

$$= \int_{\mathbf{x}} \max_i P(\omega_i \mid \mathbf{x}) p(\mathbf{x}) d\mathbf{x}$$

$$P(error) = \int_{\mathbf{x}} \left[1 - \max_i P(\omega_i \mid \mathbf{x}) \right] p(\mathbf{x}) d\mathbf{x}$$



讨 论

- 贝叶斯分类器(基于贝叶斯决策的分类器)是最优的吗？
  - 最小风险、最大后验概率决策
  - 最优的条件：概率密度、风险能准确估计
  - 具体的参数法、非参数法是贝叶斯分类器的近似，实际中难以达到最优
  - 判别模型：回避了概率密度估计，以较小复杂度估计后验概率或判别函数
  - 什么方法能胜过贝叶斯分类器：在不同的特征空间才有可能！

下次课内容

- 第2章
  - 离散变量的贝叶斯决策
  - 复合模式分类
- 第3章
  - 最大似然参数估计
  - 贝叶斯估计