7

Steering Dynamics

To maneuver a vehicle we need a steering mechanism to turn wheels. Steering dynamics which we review in this chapter, introduces new requirements and challenges.

7.1 Kinematic Steering

Consider a front-wheel-steering $4WS$ vehicle that is turning to the left, as shown in Figure 7.1. When the vehicle is moving very slowly, there is a kinematic condition between the inner and outer wheels that allows them to turn slip-free. The condition is called the *Ackerman condition* and is expressed by

$$\cot\delta_o - \cot\delta_i = \frac{w}{l} \tag{7.1}$$

where, δ_i is the steer angle of the *inner wheel*, and δ_o is the steer angle of the *outer wheel*. The inner and outer wheels are defined based on the turning center O.

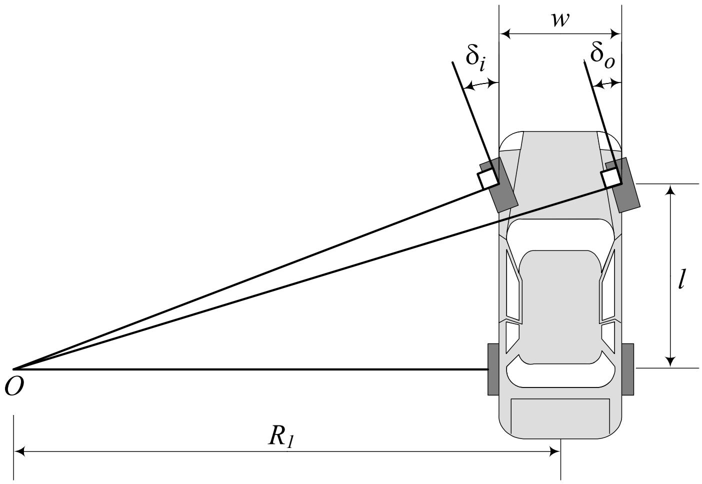

FIGURE 7.1. A front-wheel-steering vehicle and the Ackerman condition.

The distance between the steer axes of the steerable wheels is called the *track* and is shown by w. The distance between the front and real axles

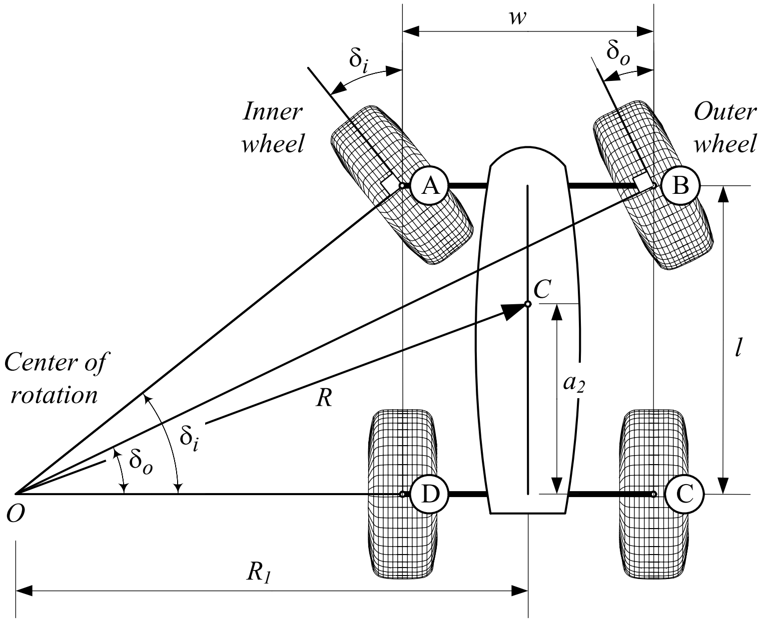

FIGURE 7.2. A front-wheel-steering vehicle and steer angles of the inner and outer wheels.

is called the *wheelbase* and is shown by l. Track w and wheelbase l are considered as kinematic width and length of the vehicle.

The mass center of a steered vehicle will turn on a circle with radius R,

$$R = \sqrt{a_2^2 + l^2 \cot^2 \delta} \tag{7.2}$$

where δ is the cot-average of the inner and outer steer angles.

$$\cot \delta = \frac{\cot \delta_o + \cot \delta_i}{2}. \tag{7.3}$$

The angle δ is the equivalent steer angle of a bicycle having the same wheelbase l and radius of rotation R.

Proof. To have all wheels turning freely on a curved road, the normal line to the center of each tire-plane must intersect at a common point. This is the Ackerman condition.

Figure 7.2 illustrates a vehicle turning left. So, the turning center O is on the left, and the inner wheels are the left wheels that are closer to the center of rotation. The inner and outer steer angles δ_i and δ_o may be calculated from the triangles $\triangle OAD$ and $\triangle OBC$ as follows:

$$\tan\delta_i = \frac{l}{R_1 - \frac{w}{2}} \tag{7.4}$$

$$\tan\delta_o = \frac{l}{R_1 + \frac{w}{2}} \tag{7.5}$$

Eliminating R_1

$$\begin{aligned} R_1 &= \frac{1}{2}w + \frac{l}{\tan\delta_i} \\ &= -\frac{1}{2}w + \frac{l}{\tan\delta_o} \end{aligned} \tag{7.6}$$

provides the Ackerman condition (7.1), which is a direct relationship between δ_i and δ_o.

$$\cot\delta_o - \cot\delta_i = \frac{w}{l} \tag{7.7}$$

To find the vehicle's turning radius R, we define an equivalent bicycle model, as shown in Figure 7.3. The radius of rotation R is perpendicular to the vehicle's velocity vector $\mathbf{v}$ at the mass center C. Using the geometry shown in the bicycle model, we have

$$R^2 = a_2^2 + R_1^2 \tag{7.8}$$

$$\begin{aligned} \cot\delta &= \frac{R_1}{l} \\ &= \frac{1}{2}\left(\cot\delta_i + \cot\delta_o\right) \end{aligned} \tag{7.9}$$

and therefore,

$$R = \sqrt{a_2^2 + l^2\cot^2\delta}. \tag{7.10}$$

The Ackerman condition is needed when the speed of the vehicle is too small, and slip angles are zero. There is no lateral force and no centrifugal force to balance each other. The Ackerman steering condition is also called the *kinematic steering condition*, because it is a static condition at zero velocity.

A device that provides steering according to the Ackerman condition (7.1) is called *Ackerman steering*, *Ackerman mechanism*, or *Ackerman geometry.* There is no four-bar linkage steering mechanism that can provide the Ackerman condition perfectly. However, we may design a multi-bar linkages to work close to the condition and be exact at a few angles.

Figure 7.4 illustrates the Ackerman condition for different values of w/l. The inner and outer steer angles get closer to each other by decreasing w/l.
■

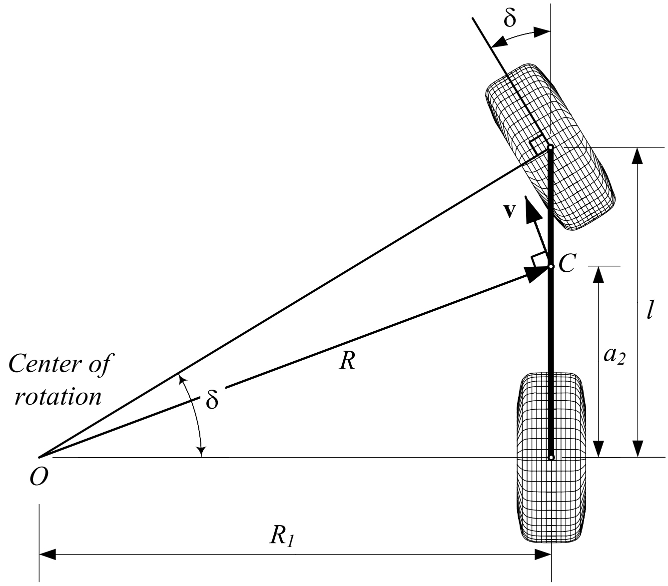

FIGURE 7.3. Equivalent bicycle model for a front-wheel-steering vehicle.

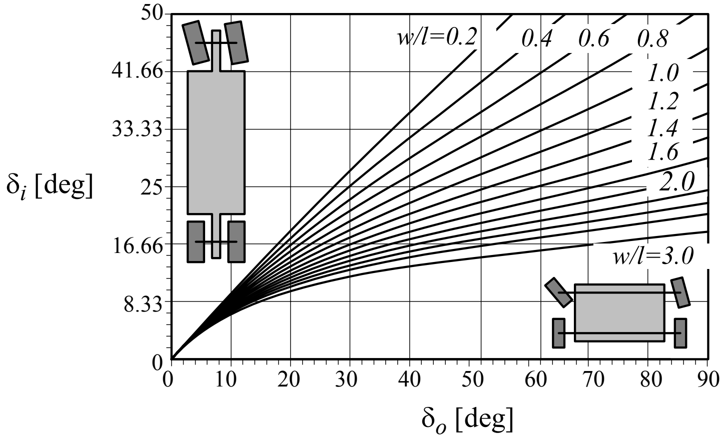

FIGURE 7.4. Effect of w/l on the Ackerman condition for front-wheel-steering vehicles.

Example 257 *Turning radius, or radius of rotation.*
Consider a vehicle with the following dimensions and steer angle:

$$\begin{aligned} l &= 103.1\,\text{in} \approx 2.619\,\text{m} \\ w &= 61.6\,\text{in} \approx 1.565\,\text{m} \\ a_2 &= 60\,\text{in} \approx 1.524\,\text{m} \\ \delta_i &= 12\,\text{deg} \approx 0.209\,\text{rad} \end{aligned} \tag{7.11}$$

The kinematic steering characteristics of the vehicle would be

$$\begin{aligned} \delta_o &= \cot^{-1}\left(\frac{w}{l} + \cot\delta_i\right) \\ &= 0.186\,\text{rad} \approx 10.661\,\text{deg} \end{aligned} \tag{7.12}$$

$$\begin{aligned} R_1 &= l\cot\delta_i + \frac{1}{2}w \\ &= 516.9\,\text{in} \approx 13.129\,\text{m} \end{aligned} \tag{7.13}$$

$$\begin{aligned} \delta &= \cot^{-1}\left(\frac{\cot\delta_o + \cot\delta_i}{2}\right) \\ &= 0.19684\,\text{rad} \approx 11.278\,\text{deg} \end{aligned} \tag{7.14}$$

$$\begin{aligned} R &= \sqrt{a_2^2 + l^2\cot^2\delta} \\ &= 520.46\,\text{in} \approx 13.219\,\text{m}. \end{aligned} \tag{7.15}$$

Example 258 *w is the front track.*
Most cars have different tracks in front and rear. The track w in the kinematic condition (7.1) refers to the front track w_f. The rear track has no effect on the kinematic condition of a front-wheel-steering vehicle. The rear track w_r of a FWS vehicle can be zero with the same kinematic steering condition (7.1).

Example 259 *Space requirement.*
The kinematic steering condition can be used to calculate the space requirement of a vehicle during a turn. Consider the front wheels of a two-axle vehicle, steered according to the Ackerman geometry as shown in Figure 7.5.

The outer point of the front of the vehicle will run on the maximum radius R_{Max}, whereas a point on the inner side of the vehicle at the location of the rear axle will run on the minimum radius R_{min}. The front outer point has an overhang distance g from the front axle. The maximum radius R_{Max} is

$$R_{Max} = \sqrt{\left(R_{min} + w\right)^2 + \left(l + g\right)^2}. \tag{7.16}$$

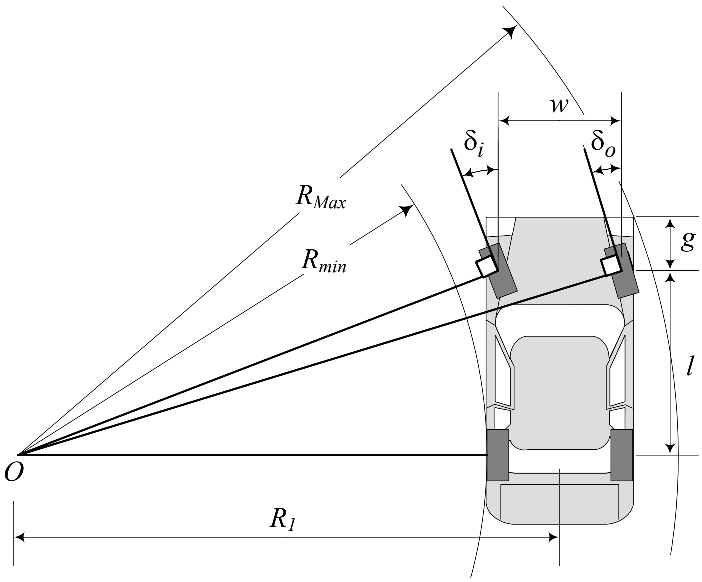

FIGURE 7.5. The required space for a turning two-axle vehicle.

Therefore, the required space for turning is a ring with a width $\triangle R$, which is a function of the vehicle's geometry.

$$\begin{aligned} \triangle R &= R_{Max} - R_{min} \\ &= \sqrt{(R_{min} + w)^2 + (l + g)^2} - R_{min} \end{aligned} \tag{7.17}$$

The required space $\triangle R$ can be calculated based on the steer angle by substituting R_{min}

$$R_{min} = R_1 - \frac{1}{2}w$$
$$= \frac{l}{\tan\delta_i} \tag{7.18}$$
$$= \frac{l}{\tan\delta_o} - w \tag{7.19}$$

and getting

$$\triangle R = \sqrt{\left(\frac{l}{\tan\delta_i} + 2w\right)^2 + (l+g)^2} - \frac{l}{\tan\delta_i} \tag{7.20}$$
$$= \sqrt{\left(\frac{l}{\tan\delta_o} + w\right)^2 + (l+g)^2} - \frac{l}{\tan\delta_o} + w. \tag{7.21}$$

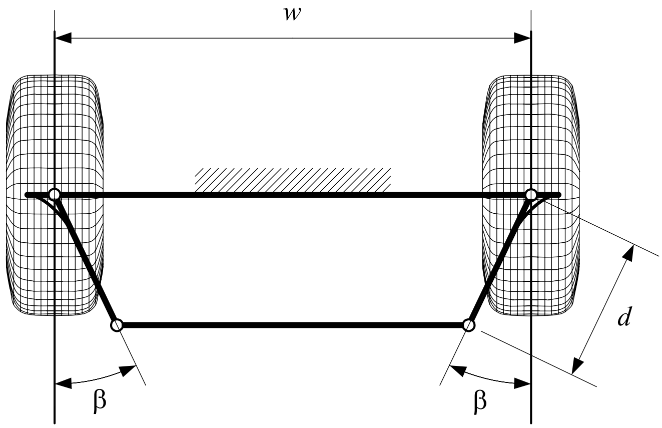

FIGURE 7.6. A trapezoidal steering mechanism.

In this example the width of the car w_v and the track w are assumed to be equal. The width of vehicles are always greater than their track.

$$w_v > w \tag{7.22}$$

Example 260 *Trapezoidal steering mechanism.*

Figure 7.6 illustrates a symmetric four-bar linkage, called a ***trapezoidal steering mechanism****, that has been used for more than* 100 *years. The mechanism has two characteristic parameters: angle β and offset arm length d. A steered position of the trapezoidal mechanism is shown in Figure 7.7 to illustrate the inner and outer steer angles δ_i and δ_o.*

The relationship between the inner and outer steer angles of a trapezoidal steering mechanism is

$$\begin{aligned} & \sin(\beta+\delta_i)+\sin(\beta-\delta_o) \\ = \; & \frac{w}{d}+\sqrt{\left(\frac{w}{d}-2\sin\beta\right)^2-\left(\cos(\beta-\delta_o)-\cos(\beta+\delta_i)\right)^2}. \end{aligned} \tag{7.23}$$

To prove this equation, we examine Figure 7.8. In the triangle $\triangle ABC$ we can write

$$\begin{aligned} (w-2d\sin\beta)^2 &= (w-d\sin(\beta+\delta_i)-d\sin(\beta-\delta_o))^2 \\ &\quad +(d\cos(\beta-\delta_o)-d\cos(\beta+\delta_i))^2 \end{aligned} \tag{7.24}$$

and derive Equation (7.23) with some manipulation.

The functionality of a trapezoidal steering mechanism, compared to the associated Ackerman condition, is shown in Figure 7.9 for $x = 2.4\,\mathrm{m} \approx 7.87\,\mathrm{ft}$ and $d = 0.4\,\mathrm{m} \approx 1.3\,\mathrm{ft}$. The horizontal axis shows the inner steer angle and the vertical axis shows the outer steer angle. It depicts that for

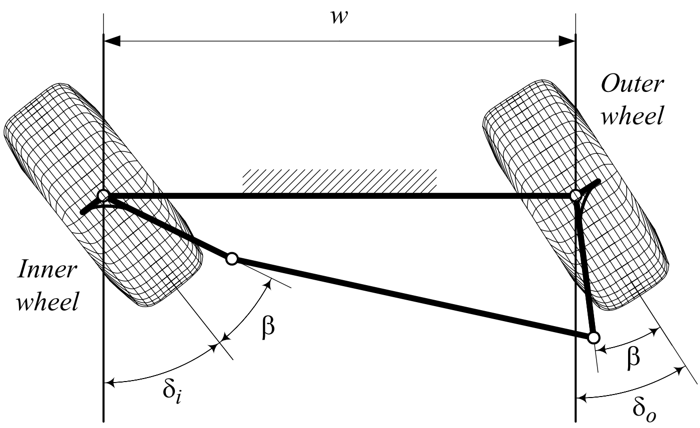

FIGURE 7.7. Steered configuration of a trapezoidal steering mechanism.

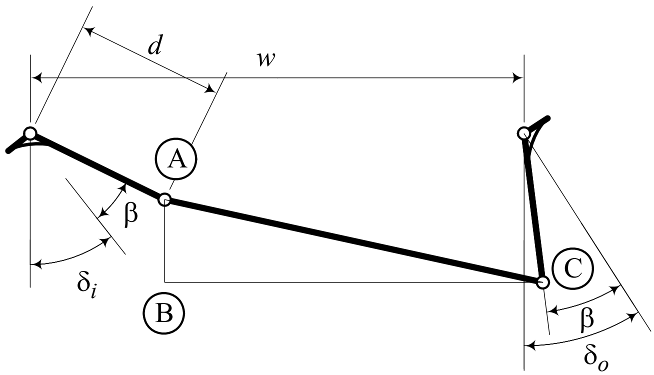

FIGURE 7.8. Trapezoidal steering triangle ABC.

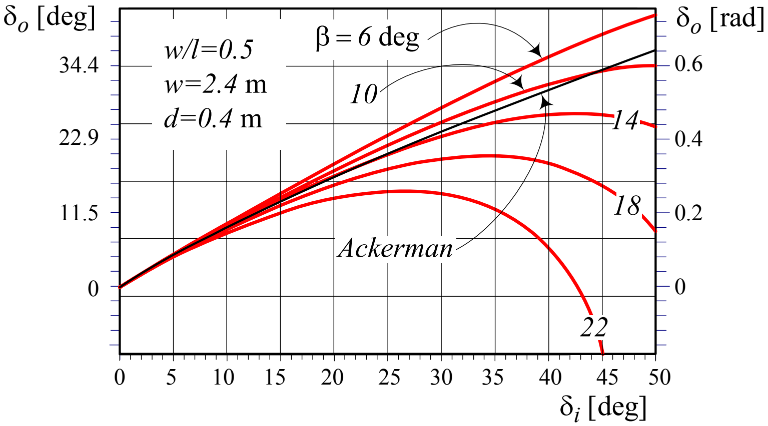

FIGURE 7.9. Behavior of a trapezoidal steering mechanism, compared to the associated Ackerman mechanism.

a given l and w, a mechanism with $\beta \approx 10\,\text{deg}$ is the best simulator of an Ackerman mechanism if $\delta_i < 50\,\text{deg}$.

To examine the trapezoidal steering mechanism and compare it with the Ackerman condition, we define an error parameter $e = \delta_{D_o} - \delta_{A_o}$. The error e is the difference between the outer steer angles calculated by the trapezoidal mechanism and the Ackerman condition at the same inner steer angle δ_i.

$$\begin{aligned} e &= \Delta\delta_o \\ &= \delta_{D_o} - \delta_{A_o} \end{aligned} \tag{7.25}$$

Figure 7.10 depicts the error e for a sample steering mechanism using the angle β as a parameter.

Example 261 ★ *Locked rear axle.*

Sometimes in a simple design of vehicles, we eliminate the differential and use a locked rear axle in which no relative rotation between the left and right wheels is possible. Such a simple design is usually used in toy cars, or small off-road vehicles such as a mini Baja.

Consider the vehicle shown in Figure 7.2. In a slow left turn, the speed of the inner rear wheel should be

$$v_{ri} = \left(R_1 - \frac{w}{2}\right) r = R_w \omega_{ri} \tag{7.26}$$

and the speed of the outer rear wheel should be

$$v_{ro} = \left(R_1 + \frac{w}{2}\right) r = R_w \omega_{ro} \tag{7.27}$$

where, r is the yaw velocity of the vehicle, R_w is rear wheels radius, and ω_{ri}, ω_{ro} should be the angular velocity of the rear inner and outer wheels

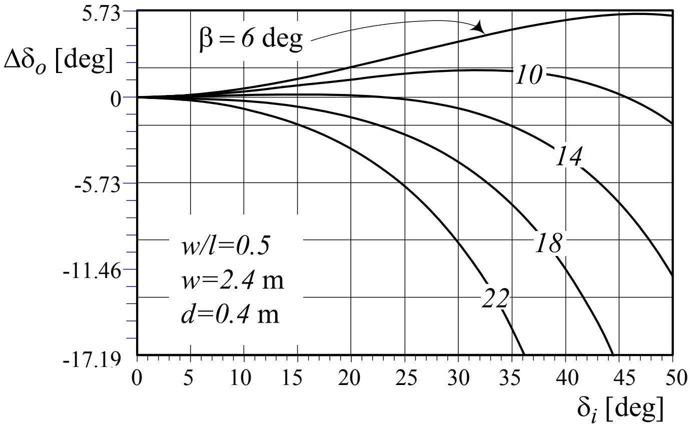

FIGURE 7.10. The error parameter $e = \delta_{D_o} - \delta_{A_o}$ for a sample trapezoidal steering mechanism.

about their common axle. If the rear axle is locked, we have

$$\omega_{ri} = \omega_{ro} = \omega \tag{7.28}$$

however,

$$\left(R_1 - \frac{w}{2}\right) \neq \left(R_1 + \frac{w}{2}\right) \tag{7.29}$$

which shows it is impossible to have a locked axle for a nonzero w.

Turning with a locked rear axle reduces the load on the inner wheels and makes the rear inner wheel overcome the friction force and spin. Hence, the traction of the inner wheel drops to the maximum friction force under a reduced load. However, the load on the outer wheels increases and hence, the friction limit of the outer wheel helps to have higher traction force on the outer rear wheel.

Eliminating the differential and using a locked drive axle is an impractical design for street cars. However, it can be an acceptable design for small and light cars moving on dirt or slippery surfaces. It reduces the cost and simplifies the design significantly.

In a conventional two-wheel-drive motor vehicle, the rear wheels are driven using a differential, and the vehicle is steered by changing the direction of the front wheels. With an ideal differential, equal torque is delivered to each drive wheel. The rotational speed of the drive wheels are determined by the differential and the tire-road characteristics. However, a vehicle using a differential has disadvantages when one wheel has lower traction. Differences in traction characteristics of each of the drive wheels may come from different tire-road characteristics or weight distribution. Because a differential delivers equal torque, the wheel with greater tractive

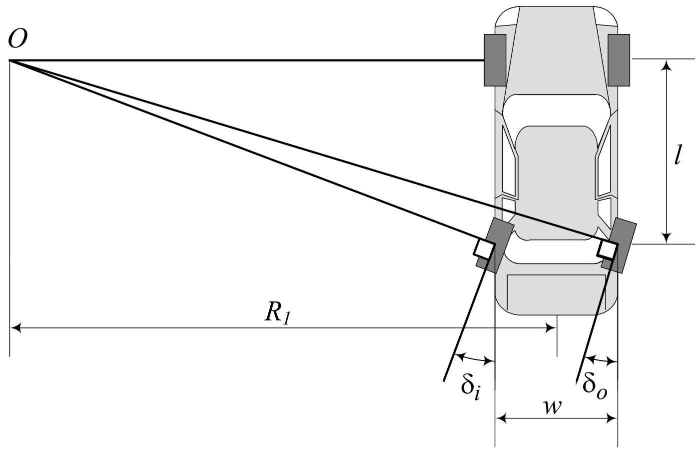

FIGURE 7.11. A rear-wheel-steering vehicle.

ability can deliver only the same amount of torque as the wheel with the lower traction. The steering behavior of a vehicle with a differential is relatively stable under changing tire-road conditions. However, the total thrust may be reduced when the traction conditions are different for each drive wheel.

Example 262 ★ *Rear-wheel-steering.*

Rear-wheel-steering is used where high maneuverability is a necessity on a low-speed vehicle, such as forklifts. Rear-wheel-steering is not used on street vehicles because it is unstable at high speeds. The center of rotation for a rear-wheel-steeringe vehicle is always a point on the front axle.

Figure 7.11 illustrates a rear-wheel-steering vehicle. The kinematic steering condition (7.1) remains the same for a rear-wheel steering vehicle.

$$\cot\delta_o - \cot\delta_i = \frac{w}{l} \tag{7.30}$$

Example 263 ★ *Alternative kinematic steer angles equation.*

Consider a rear-wheel-drive vehicle with front steerable wheels as shown in Figure 7.12. Assume that the front and rear tracks of the vehicle are equal and the drive wheels are turning without slip. If we show the angular velocities of the inner and outer drive wheels by ω_i and ω_o, respectively, the kinematical steer angles of the front wheels can be expressed by

$$\delta_i = \tan^{-1}\left(\frac{l}{w}\left(\frac{\omega_o}{\omega_i} - 1\right)\right) \tag{7.31}$$

$$\delta_o = \tan^{-1}\left(\frac{l}{w}\left(1 - \frac{\omega_i}{\omega_o}\right)\right). \tag{7.32}$$

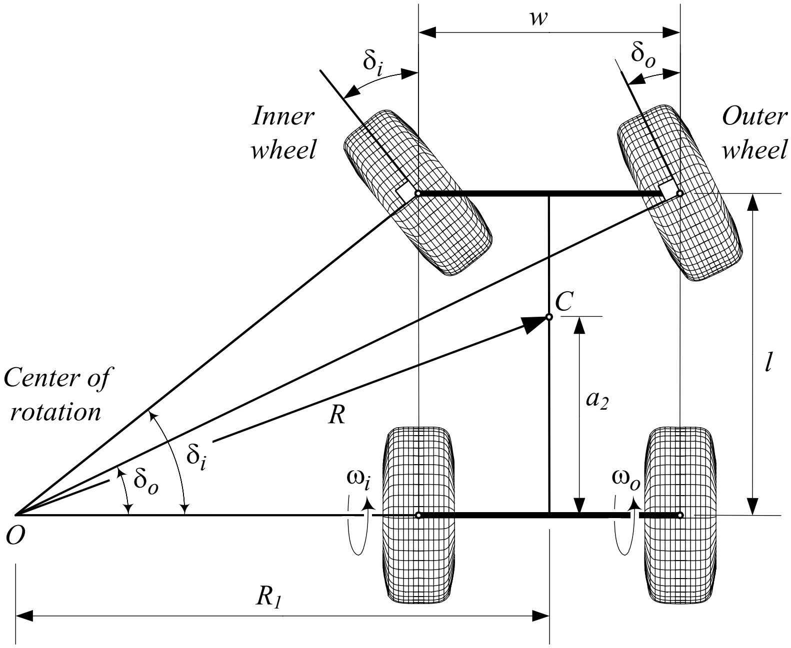

FIGURE 7.12. Kinematic condition of a FWS vehicle using the angular velocity of the inner and outer wheels.

To prove these equations, we may start from the following equation, which is the non-slipping condition for the drive wheels:

$$\frac{R_w\,\omega_o}{R_1+\dfrac{w}{2}}=\frac{R_w\,\omega_i}{R_1-\dfrac{w}{2}}. \tag{7.33}$$

Equation (7.33) can be rearranged to

$$\frac{\omega_o}{\omega_i}=\frac{R_1+\dfrac{w}{2}}{R_1-\dfrac{w}{2}} \tag{7.34}$$

and substituted in Equations (7.31) and (7.32) to reduce them to Equations (7.4) and (7.5).

The equality (7.33) is the yaw rate of the vehicle, which is the vehicle's angular velocity about the center of rotation.

$$r=\frac{R_w\,\omega_o}{R_1+\dfrac{w}{2}}=\frac{R_w\,\omega_i}{R_1-\dfrac{w}{2}} \tag{7.35}$$

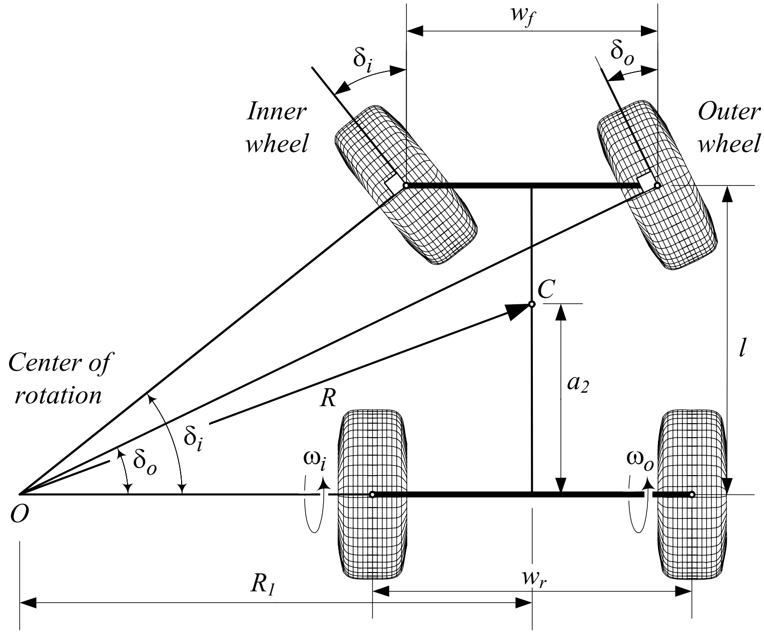

FIGURE 7.13. Kinematic steering condition for a vehicle with different tracks in the front and in the back.

Example 264 ★ *Unequal front and rear tracks.*

It is possible to design a vehicle with different tracks in the front and rear. It is a common design for race cars, which are usually equipped with wider and larger rear tires to increase traction and stability. For street cars we use the same tires in the front and rear, however, it is common to have a few centimeters of larger track in the back. Such a vehicle is illustrated in Figure 7.13.

The angular velocity of the vehicle is

$$r = \frac{R_w\, \omega_o}{R_1 + \dfrac{w_r}{2}} = \frac{R_w\, \omega_i}{R_1 - \dfrac{w_r}{2}} \tag{7.36}$$

and the kinematic steer angles of the front wheels are

$$\delta_i = \tan^{-1} \frac{2l\,(\omega_o + \omega_i)}{w_f\,(\omega_o - \omega_i) + w_r\,(\omega_o + \omega_i)} \tag{7.37}$$

$$\delta_o = \tan^{-1} \frac{2l\,(\omega_o - \omega_i)}{w_f\,(\omega_o - \omega_i) + w_r\,(\omega_o + \omega_i)}. \tag{7.38}$$

To show these equations, we should find R_1 from Equation (7.36)

$$R_1 = \frac{w_r}{2} \frac{\omega_o + \omega_i}{\omega_o - \omega_i} \tag{7.39}$$

and substitute it in the following equations.

$$\tan\delta_i = \frac{l}{R_1 - \frac{w_f}{2}} \tag{7.40}$$

$$\tan\delta_o = \frac{l}{R_1 + \frac{w_f}{2}} \tag{7.41}$$

In the above equations, w_f is the front track, w_r is the rear track, and R_w is the wheel radius.

Example 265 ★ *Independent rear-wheel-drive.*

For some special-purpose vehicles, such as moon rovers and autonomous mobile robots, we may attach each drive wheel to an independently controlled motor to apply any desired angular velocity. Furthermore, the steerable wheels of such vehicles are able to turn more than 90 deg *to the left and right. Such a vehicle is highly maneuverable at a low speed.*

Figure 7.14 illustrates the advantages of such a steerable vehicle and its possible turnings. Figures 7.14 (a)-(c) illustrate forward maneuvering. The arrows by the rear wheels, illustrate the magnitude of the angular velocity of the wheel, and the arrows on the front wheels illustrate the direction of their motion. The maneuvering in backward motion is illustrated in Figures 7.14(d)-(f). Having such a vehicle allows us to turn the vehicle about any point on the rear axle including the inner points. In Figure 7.14(g) the vehicle is turning about the center of the rear right wheel, and in Figure 7.14(h) about the center of the rear left wheel. Figure 7.14(i) illustrates a rotation about the center point of the rear axle.

In any of the above scenarios, the steer angle of the front wheels should be determined using a proper equation, such as (7.40) and (7.41). The ratio of the outer to inner angular velocities of the drive wheels ω_o/ω_i may be determined using either the outer or inner steer angles.

$$\frac{\omega_o}{\omega_i} = \frac{\delta_o\,(w_f - w_r) - 2l}{\delta_o\,(w_f + w_r) - 2l} \tag{7.42}$$

$$\frac{\omega_o}{\omega_i} = \frac{\delta_i\,(w_f + w_r) + 2l}{\delta_i\,(w_f - w_r) + 2l} \tag{7.43}$$

Example 266 ★ *Race car steering.*

The Ackerman or kinematic steering is a correct condition when the turning speed of the vehicle is slow. When the vehicle turns fast, significant lateral acceleration is needed, and therefore, the wheels operate at high slip angles. Furthermore, the loads on the inner wheels will be much lower than the outer wheels. Tire performance curves show that by increasing the wheel load, less slip angle is required to reach the peak of the lateral force. Under these conditions the inner front wheel of a kinematic steering vehicle would be at a higher slip angle than required for maximum lateral force.

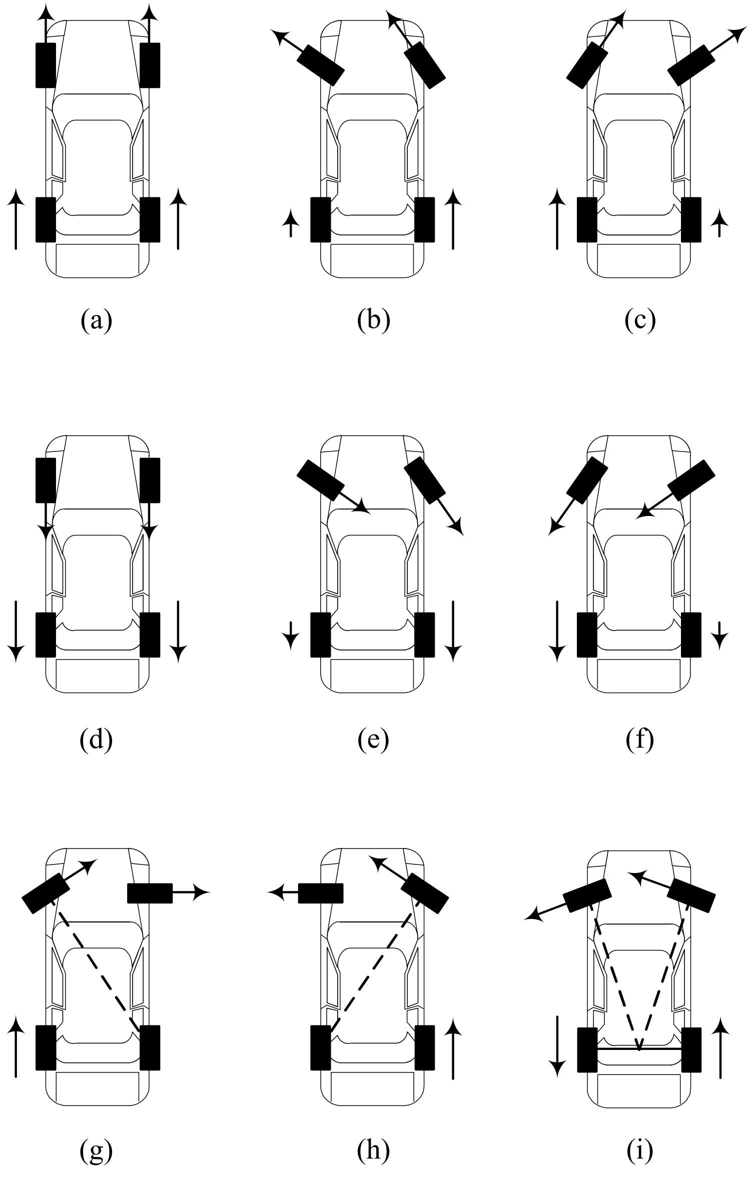

FIGURE 7.14. A highly steerable vehicle.

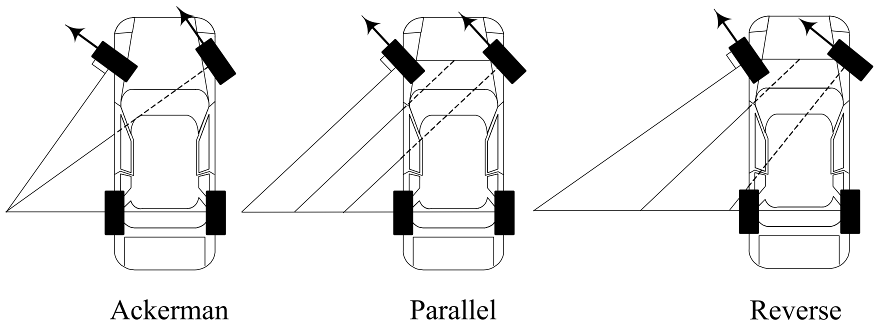

FIGURE 7.15. By increasing the speed at a turn, parallel or reverse steering is needed instead of Ackerman steering.

Therefore, the inner wheel of a vehicle in a high speed turn must operate at a lower steer angle than kinematic steering. Reducing the steer angle of the inner wheel reduces the difference between steer angles of the inner and outer wheels.

For race cars, it is common to use parallel or reverse steering. Ackerman, parallel, and reverse Ackerman steering are illustrated in Figure 7.15.

The correct steer angle is a function of the instant wheel load, road condition, speed, and tire characteristics. Furthermore, the vehicle must also be able to turn at a low speed under an Ackerman steering condition. Hence, there is no ideal steering mechanism unless we control the steer angle of each steerable wheel independently using a smart system.

Example 267 ★ *Speed dependent steering system.*

There is a speed adjustment idea that says it is better to have a harder steering system at high speeds. This idea can be applied in power steering systems to make them speed dependent, such that the steering be heavily assisted at low speeds and lightly assisted at high speeds. The idea is supported by this fact that the drivers might need large steering for parking, and small steering when traveling at high speeds.

Example 268 ★ *Ackerman condition history.*

Correct steering geometry was a major problem in the early days of carriages, horse-drawn vehicles, and cars. Four- or six-wheel cars and carriages always left rubber marks behind. This is why there were so many three-wheeled cars and carriages in the past. The problem was making a mechanism to give the inner wheel a smaller turning radius than the outside wheel when the vehicle was driven in a circle.

The required geometric condition for a front-wheel-steering four-wheel-carriage was introduced in 1816 *by George Langensperger in Munich, Germany. Langensperger's mechanism is illustrated in Figure 7.16.*

Rudolf Ackerman met Langensperger and saw his invention. Ackerman

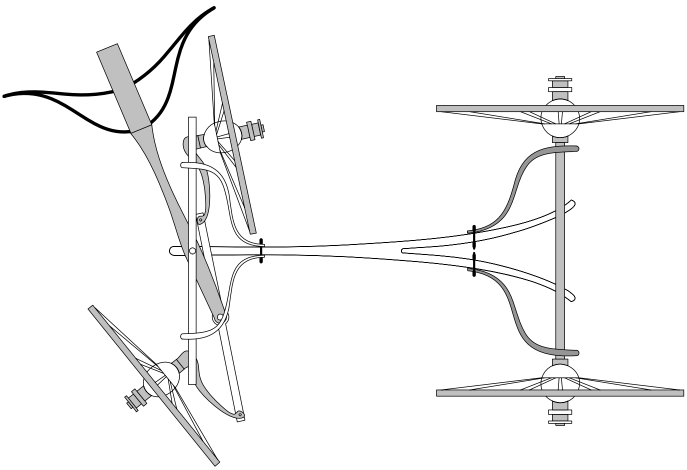

FIGURE 7.16. Langensperger invention for the steering geometry condition.

acted as Langensperger's patent agent in London and introduced the invention to British carriage builders. Car manufacturers have been adopting and improving the Ackerman geometry for their steering mechanisms since 1881*.*

The basic design of vehicle steering systems has changed little since the invention of the steering mechanism. The driver's steering input is transmitted by a shaft through some type of gear reduction mechanism to generate steering motion at the front wheels.

7.2 Vehicles with More Than Two Axles

If a vehicle has more than two axles, all the axles, except one, must be steerable to provide slip-free turning at zero velocity. When an n-axle vehicle has only one non-steerable axle, there are $n-1$ geometric steering conditions. A three-axle vehicle with two steerable axles is shown in Figure 7.17.

To indicate the geometry of a multi-axle vehicle, we start from the front axle and measure the longitudinal distance a_i between axle i and the mass center C. Hence, a_1 is the distance between the front axle and C, and a_2 is the distance between the second axle and C. Furthermore, we number the wheels in a clockwise rotation starting from the driver's wheel as number 1.

For the three-axle vehicle shown in Figure 7.17, there are two indepen-

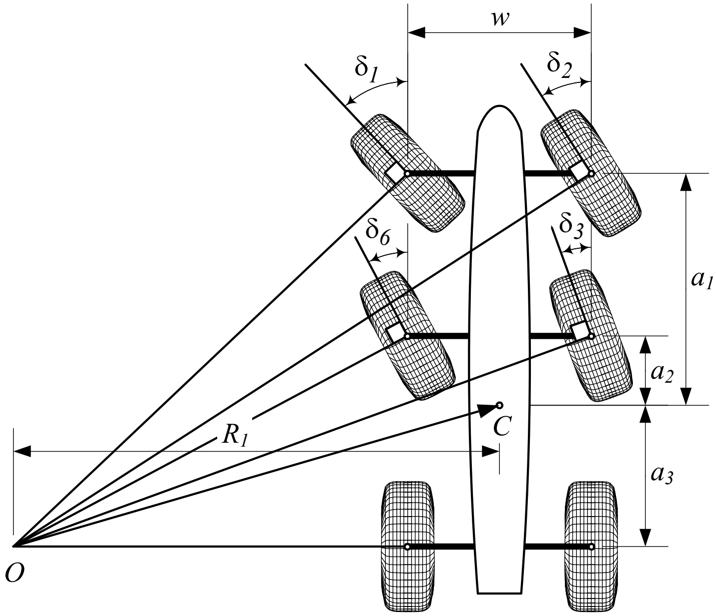

FIGURE 7.17. Steering of a three-axle vehicle.

dent Ackerman conditions:

$$\cot\delta_2 - \cot\delta_1 = \frac{w}{a_1 + a_3} \tag{7.44}$$

$$\cot\delta_3 - \cot\delta_6 = \frac{w}{a_2 + a_3}. \tag{7.45}$$

Example 269 *A six-wheel vehicle with one steerable axle.*

When a multi-axle vehicle has only one steerable axle, slip-free rotation is impossible for the non-steering wheels. The kinematic length or wheelbase of the vehicle is not clear, and it is not possible to define an Ackerman condition. Strong wear occurs for the tires, especially at low speeds and large steer angles. Hence, such a combination is not recommended. However, in case of a long three-axle vehicle with two nonsteerable axles close to each other, an approximated analysis is possible for low-speed steering.

Figure 7.18 illustrates a six-wheel vehicle with only one steerable axle in front. We design the steering mechanism such that the center of rotation O is on a lateral line, called the ***midline****, between the couple rear axles. The kinematic length of the vehicle, l, is the distance between the front axle and the midline. For this design we have*

$$\cot\delta_o - \cot\delta_i = \frac{w}{l} \tag{7.46}$$

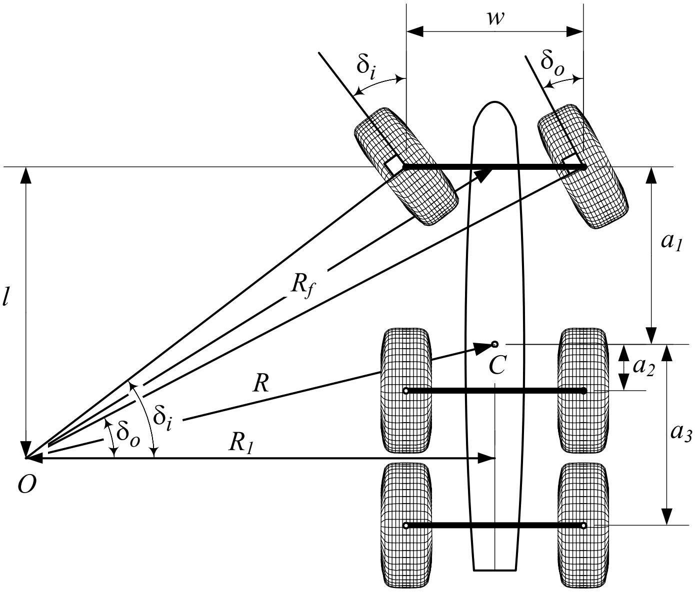

FIGURE 7.18. A six-wheel vehicle with one steerable axle in front.

and

$$\begin{aligned} R_1 &= l\cot\delta_o - \frac{w}{2} \\ &= l\cot\delta_i + \frac{w}{2}. \end{aligned} \tag{7.47}$$

The center of the front axle and the mass center of the vehicle are turning about O by radii R_f and R.

$$R_f = \frac{R_1}{\cos\left(\tan^{-1}\dfrac{l}{R_1}\right)} \tag{7.48}$$

$$R = \frac{R_1}{\cos\left(\tan^{-1}\dfrac{a_3 - a_2}{2R_1}\right)} \tag{7.49}$$

If the radius of rotation is large compared to the wheelbase, we may approximate Equations (7.48) and (7.49).

$$R_f \approx \frac{R_1}{\cos\left(\dfrac{l}{R_1}\right)} \tag{7.50}$$

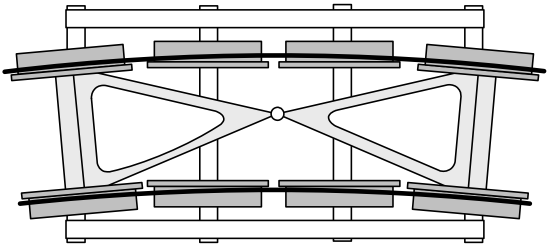

FIGURE 7.19. A self-steering axle mechanism for locomotive wagons.

$$R \approx \frac{R_1}{\cos\left(\frac{a_3 - a_2}{2R_1}\right)} \tag{7.51}$$

$$R_1 = \frac{l}{2}\left(\cot\delta_o + \cot\delta_i\right) \tag{7.52}$$

To avoid strong wear, it is possible to lift an axle when the vehicle is not carrying heavy loads. For such a vehicle, we may design the steering mechanism to follow an Ackerman condition based on a wheelbase for the non-lifted axle. However, when this vehicle is carrying a heavy load and using all the axles, the liftable axle encounters huge wear in large steer angles.

Another option for multi-axle vehicles is to use ***self-steering*** *wheels that can adjust themselves to minimize sideslip. Such wheels cannot provide lateral force, and hence, cannot help in maneuvers very much. Self-steering wheels may be installed on buggies and trailers. Such a self-steering axle mechanism for locomotive wagons is shown in Figure 7.19.*

7.3 ★ Vehicle with Trailer

If a four-wheel vehicle has a trailer with one axle, it is possible to derive a kinematic condition for slip-free steering. Figure 7.20 illustrates a vehicle with a one-axle trailer. The mass center of the vehicle is turning on a circle with radius R, while the trailer is turning on a circle with radius R_t.

$$R_t = \sqrt{\left(l\cot\delta_i + \frac{1}{2}w\right)^2 + b_1^2 - b_2^2} \tag{7.53}$$

$$R_t = \sqrt{\left(l\cot\delta_o - \frac{1}{2}w\right)^2 + b_1^2 - b_2^2} \tag{7.54}$$

At a steady-state condition, the angle between the trailer and the vehicle

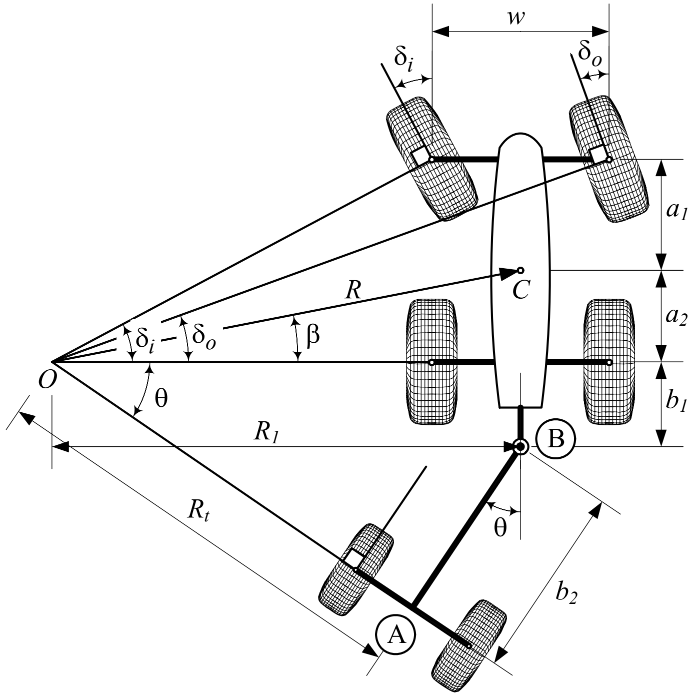

FIGURE 7.20. A vehicle with a one-axle trailer.

is

$$\theta = \begin{cases} 2\tan^{-1}\left[\dfrac{1}{b_1 - b_2}\left(R_t - \sqrt{R_t^2 - b_1^2 + b_2^2}\right)\right] & b_1 - b_2 \neq 0 \\ 2\tan^{-1}\dfrac{1}{2R_t}\left(b_1 + b_2\right) & b_1 - b_2 = 0 \end{cases} \tag{7.55}$$

Proof. Using the right triangle $\triangle OAB$ in Figure 7.20, we may write the trailer's radius of rotation as

$$R_t = \sqrt{R_1^2 + b_1^2 - b_2^2} \tag{7.56}$$

because the length $\overline{OB}$ is

$$\begin{aligned} \overline{OB}^2 &= R_t^2 + b_2^2 \\ &= R_1^2 + b_1^2. \end{aligned} \tag{7.57}$$

Substituting R_1 from Equation (7.6) shows that the trailer's radius of ro-

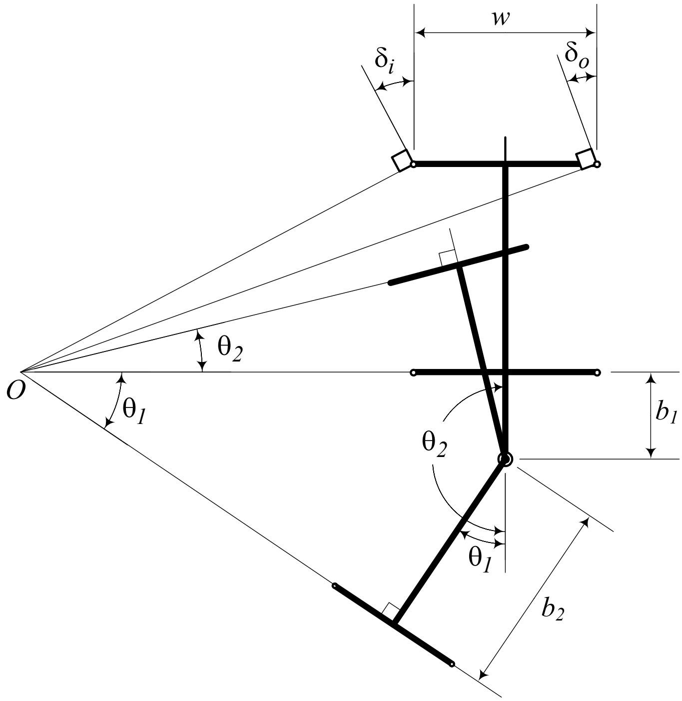

FIGURE 7.21. Two possible angle θ for a set of (R_t, b_1, b_2).

tation is related to the vehicle's geometry by

$$R_t = \sqrt{\left(l\cot\delta_i + \frac{1}{2}w\right)^2 + b_1^2 - b_2^2} \tag{7.58}$$

$$R_t = \sqrt{\left(l\cot\delta_o - \frac{1}{2}w\right)^2 + b_1^2 - b_2^2} \tag{7.59}$$

$$R_t = \sqrt{R^2 - a_2^2 + b_1^2 - b_2^2}. \tag{7.60}$$

Using the equation

$$R_t \sin\theta = b_1 + b_2 \cos\theta \tag{7.61}$$

and employing trigonometry, we may calculate the angle θ between the trailer and the vehicle as (7.55).

The minus sign, in case $b_1 - b_2 \neq 0$, is the usual case in forward motion, and the plus sign is a solution associated with a backward motion. Both possible configuration θ for a set of (R_t, b_1, b_2) are shown in Figure 7.21. The θ_2 is called a **jackknifing** configuration. ■

Example 270 ★ *Two possible trailer-vehicle angles.*

Consider a four-wheel vehicle that is pulling a one-axle trailer with the following dimensions:

$$\begin{aligned} l &= 103.1\,\text{in} \approx 2.619\,\text{m} \\ w &= 61.6\,\text{in} \approx 1.565\,\text{m} \\ b_1 &= 24\,\text{in} \approx 0.61\,\text{m} \\ b_2 &= 90\,\text{in} \approx 2.286\,\text{m} \\ \delta_i &= 12\,\text{deg} \approx 0.209\,\text{rad} \end{aligned} \tag{7.62}$$

The kinematic steering characteristics of the vehicle would be

$$\begin{aligned} \delta_o &= \cot^{-1}\left(\frac{w}{l} + \cot\delta_i\right) \\ &= 0.186\,\text{rad} \approx 10.661\,\text{deg} \end{aligned} \tag{7.63}$$

$$\begin{aligned} R_t &= \sqrt{\left(l\cot\delta_i + \frac{1}{2}w\right)^2 + b_1^2 - b_2^2} \\ &= 509.57\,\text{in} \approx 12.943\,\text{m} \end{aligned} \tag{7.64}$$

$$\begin{aligned} R_1 &= l\cot\delta_i + \frac{1}{2}w \\ &= 516.9\,\text{in} \approx 13.129\,\text{m} \end{aligned} \tag{7.65}$$

$$\begin{aligned} \delta &= \cot^{-1}\left(\frac{\cot\delta_o + \cot\delta_i}{2}\right) \\ &= 0.19684\,\text{rad} \approx 11.278\,\text{deg} \end{aligned} \tag{7.66}$$

$$\begin{aligned} R &= \sqrt{a_2^2 + l^2\cot^2\delta} \\ &= 520.46\,\text{in} \approx 13.219\,\text{m} \end{aligned} \tag{7.67}$$

$$\begin{aligned} \theta &= 2\tan^{-1}\left[\frac{1}{b_1 - b_2}\left(R_t \pm \sqrt{R_t^2 - b_1^2 + b_2^2}\right)\right] \\ &= \begin{cases} -3.0132\,\text{rad} \approx -172.64\,\text{deg} \\ 0.22121\,\text{rad} \approx 12.674\,\text{deg} \end{cases} \end{aligned} \tag{7.68}$$

Example 271 ★ *Space requirement.*

The kinematic steering condition can be used to calculate the space requirement of a vehicle with a trailer during a turn. Consider that the front wheels of a two-axle vehicle with a trailer are steered according to the Ackerman geometry, as shown in Figure 7.22.

The outer point of the front of the vehicle will run on the maximum radius R_{Max}*, whereas a point on the inner side of the wheel at the trailer's rear*

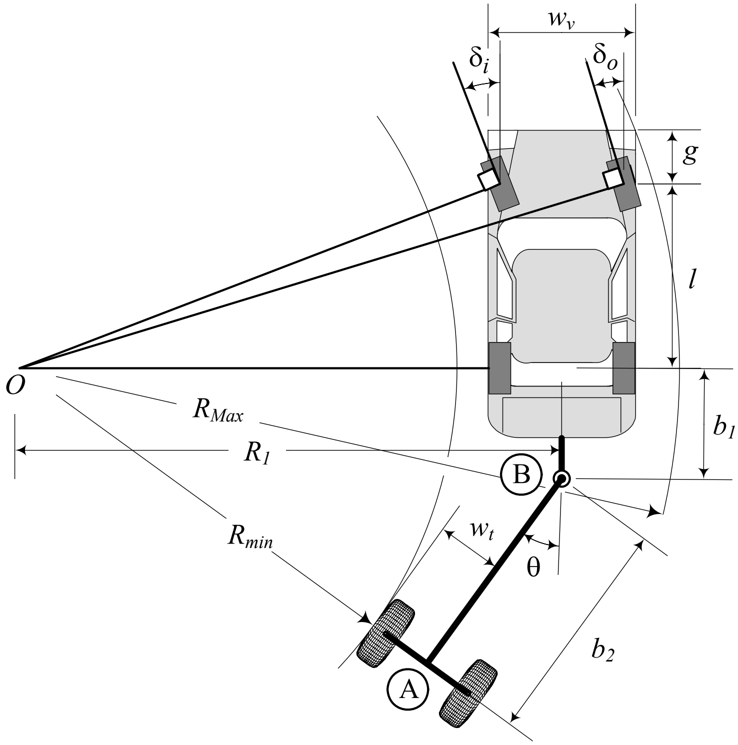

FIGURE 7.22. A two-axle vehicle with a trailer is steered according to the Ackerman condition.

axle will run on the minimum radius R_{min}. The maximum radius R_{Max} is

$$R_{Max} = \sqrt{\left(R_1 + \frac{w_v}{2}\right)^2 + (l+g)^2} \tag{7.69}$$

where

$$R_1 = \sqrt{(R_{min} + w_t)^2 + b_2^2 - b_1^2}. \tag{7.70}$$

and the width of the vehicle is shown by w_v.

The required space for turning the vehicle and trailer is a ring with a width $\triangle R$, which is a function of the vehicle and trailer geometry.

$$\triangle R = R_{Max} - R_{min} \tag{7.71}$$

The required space $\triangle R$ can be calculated based on the steer angle by *substituting R_{min}*

$$\begin{aligned} R_{min} &= R_t - \frac{1}{2}w_t \\ &= \sqrt{\left(l\cot\delta_i + \frac{1}{2}w\right)^2 + b_1^2 - b_2^2} - \frac{1}{2}w_t \\ &= \sqrt{\left(l\cot\delta_o - \frac{1}{2}w\right)^2 + b_1^2 - b_2^2} - \frac{1}{2}w_t \\ &= \sqrt{R^2 - a_2^2 + b_1^2 - b_2^2} - \frac{1}{2}w_t. \end{aligned} \tag{7.72}$$

7.4 Steering Mechanisms

A steering system begins with the *steering wheel* or *steering handle.* The driver's steering input is transmitted by a shaft through a gear reduction system, usually rack-and-pinion or recirculating ball bearings. The steering gear output goes to steerable wheels to generate motion through a *steering mechanism.* The lever, which transmits the steering force from the steering gear to the steering linkage, is called *Pitman arm.*

The direction of each wheel is controlled by one steering arm. The steering arm is attached to the steerable wheel hub by a keyway, locking taper, and a hub. In some vehicles, it is an integral part of a one-piece hub and steering knuckle.

To achieve good maneuverability, a minimum steering angle of approximately 35 deg must be provided at the front wheels of passenger cars.

A sample parallelogram steering mechanism and its components are shown in Figure 7.23. The parallelogram steering linkage is common on independent front-wheel vehicles. There are many varieties of steering mechanisms each with some advantages and disadvantages.

Example 272 *Steering ratio.*

The ***Steering ratio*** *is the rotation angle of a steering wheel divided by the steer angle of the front wheels. The steering ratio of street cars is around* $10:1$ *steering ratio of race cars varies between* $5:1$ *to* $20:1$.

The steering ratio of Ackerman steering is different for inner and outer wheels. Furthermore, it has a nonlinear behavior and is a function of the wheel angle.

Example 273 *Rack-and-pinion steering.*

Rack-and-pinion is the most common steering system of passenger cars. Figure 7.24 illustrates a sample rack-and-pinion steering system. The rack is either in front or behind the steering axle. The driver's rotary steering command δ_S *is transformed by a steering box to translation* $u_R = u_R(\delta_S)$

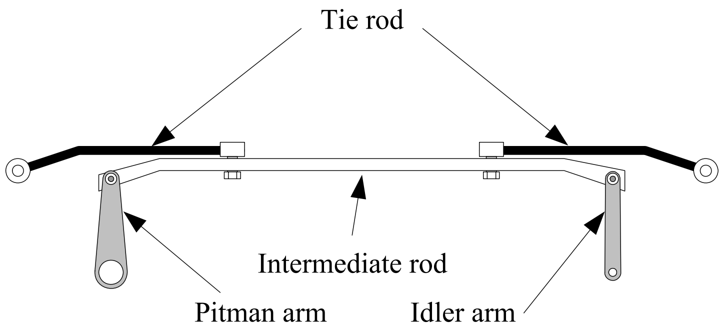

FIGURE 7.23. A sample parallelogram steering linkage and its components.

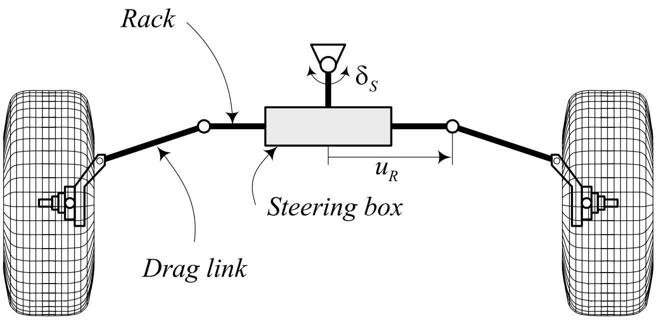

FIGURE 7.24. A rack-and-pinion steering system.

of the racks, and then by the ***drag links*** *to the wheel steering* $\delta_i = \delta_i(u_R)$, $\delta_o = \delta_o(u_R)$. *The drag link is also called the* ***tie rod****.*

The overall steering ratio depends on the ratio of the steering box and on the kinematics of the steering linkage.

Example 274 *Lever arm steering system.*

Figure 7.25 illustrates a steering linkage that sometimes is called a lever arm steering system. Using a lever arm steering system, large steering angles at the wheels are possible. This steering system is used on trucks with large wheel bases and independent wheel suspension at the front axle. The steering box and triangle can also be placed outside of the axle's center.

Example 275 *Drag link steering system.*

It is sometimes better to send the steering command to only one wheel and connect the other one to the first wheel by a drag link, as shown in Figure 7.26. Such steering linkages are usually used for trucks and busses with a front solid axle. The rotations of the steering wheel are transformed by a steering box to the rotation of the steering arm and then to the rotation

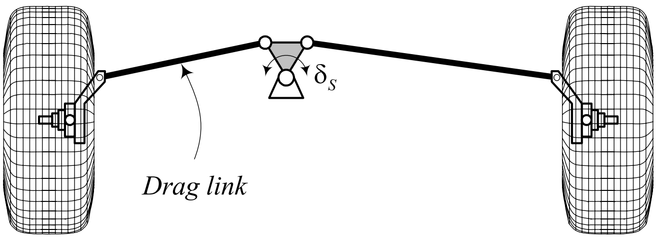

FIGURE 7.25. A lever arm steering system.

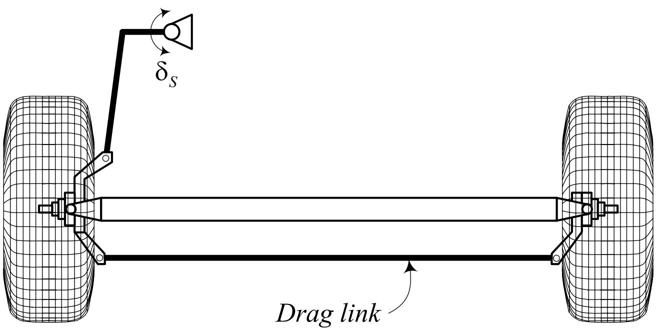

FIGURE 7.26. A drag link steering system.

of the left wheel. A drag link transmits the rotation of the left wheel to the right wheel.

Figure 7.27 shows a sample for connecting a steering mechanism to the Pitman arm of the left wheel and using a trapezoidal linkage to connect the right wheel to the left wheel.

Example 276 *Multi-link steering mechanism.*

In busses and big trucks, the driver may sit more than $2\,\mathrm{m} \approx 7\,\mathrm{ft}$ *in front of the front axle. These vehicles need large steering angles at the front wheels to achieve good maneuverability. So a more sophisticated multi-link steering mechanism needed. A sample multi-link steering mechanism is shown in Figure 7.28.*

The rotations of the steering wheel are transformed by the steering box to a steering lever arm. The lever arm is connected to a distributing linkage, which turns the left and right wheels by a long tire rod.

Example 277 ★ *Reverse efficiency.*

The ability of the steering mechanism to feedback the road inputs to the driver is called ***reverse efficiency****. Feeling the applied steering torque or aligning moment helps the driver to make smoother turn.*

Rack-and-pinion and recirculating ball steering gears have a feedback of

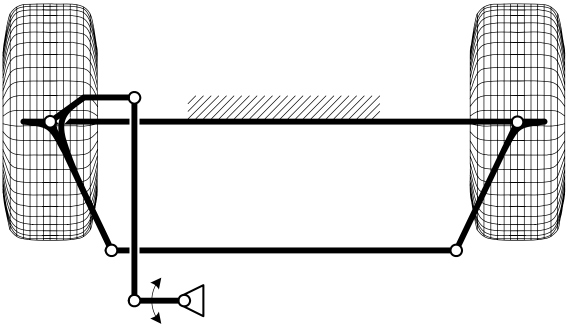

FIGURE 7.27. Connection of the Pitman arm to a trapezoidal steering mechanism.

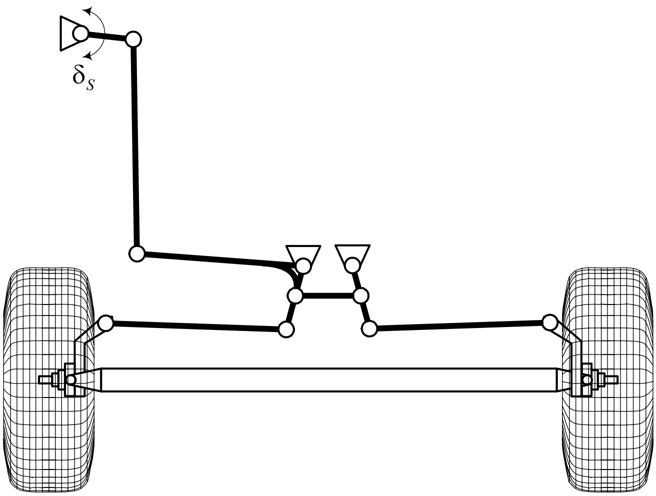

FIGURE 7.28. A multi-link steering mechanism.

the wheels steering torque to the driver. However, worm and sector steering gears have very weak feedback. Low feedback may be desirable for off-road vehicles, to reduce the driver's fatigue.

Because of safety, the steering torque feedback should be proportional to the speed of the vehicle. In this way, the required torque to steer the vehicle is higher at higher speeds. Such steering prevents a sharp and high steer angle. A steering damper with a damping coefficient increasing with speed is the mechanism that provides such behavior. A steering damper can also reduce shimmy vibrations.

Example 278 ★ *Power steering.*

Power steering has been developed in the 1950*s when a hydraulic power steering assist was first introduced. Since then, power assist has become a standard component in automotive steering systems. Using hydraulic pressure, supplied by an engine-driven pump, amplifies the driver-applied torque at the steering wheel. As a result, the steering effort is reduced.*

In recent years, electric torque amplifiers were introduced in automotive steering systems as a substitute for hydraulic amplifiers. Electrical steering eliminates the need for the hydraulic pump. Electric power steering is more efficient than conventional power steering, because the electric power steering motor needs to provide assistance when only the steering wheel is turned, whereas the hydraulic pump runs constantly. The assist level is also tunable by vehicle type, road speed, and driver preference.

Example 279 *Bump steering.*

*The steer angle generated by the vertical motion of the wheel with respect to the body is called **bump steering**. Bump steering is usually an undesirable phenomenon and is a function of the suspension and steering mechanisms. If the vehicle has a bump steering character, then the wheel steers when it runs over a bump or when the vehicle rolls in a turn. As a result, the vehicle will travel in a path not selected by the driver.*

Bump steering occurs when the end of the tie rod is not at the instant center of the suspension mechanism. Hence, in a suspension deflection, the suspension and steering mechanisms will rotate about different centers.

Example 280 ★ *Offset steering axis.*

Theoretically, the steering axis of each steerable wheel must vertically go through the center of the wheel at the tire-plane to minimize the required steering torque. Figure 7.27 is an example of matching the center of a wheel with the steering axis. However, it is possible to attach the wheels to the steering mechanism, using an offset design, as shown in Figure 7.29.

Figure 7.30 depicts a steered trapezoidal mechanism with an offset wheel attachment. The path of motion for the center of the tireprint for an offset design is a circle with radius e equal to the value of the offset arm. Such a design is not recommended for street vehicles, especially because of the huge steering torque in stationary vehicle. However, the steering torque

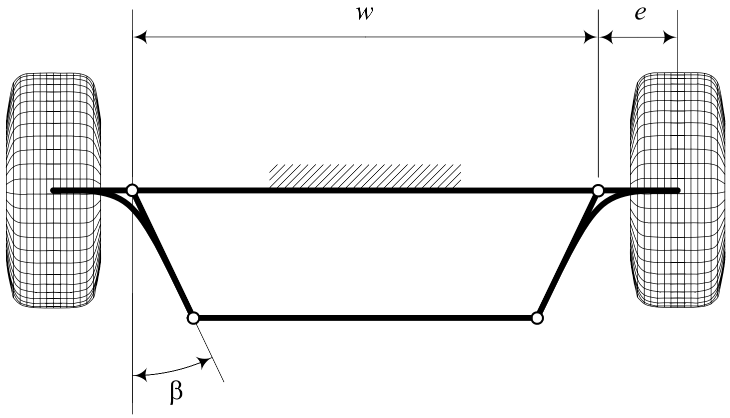

FIGURE 7.29. An offset design for wheel attachment to an steering mechanism.

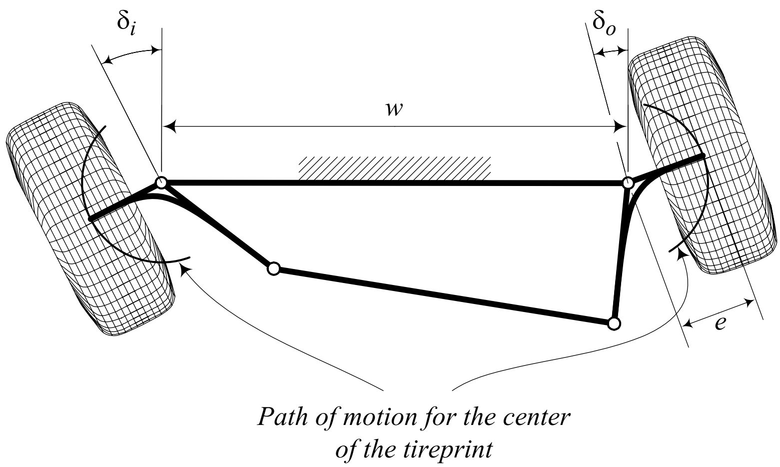

FIGURE 7.30. Offset attachment of steerable wheels to a trapezoidal steering mechanism.

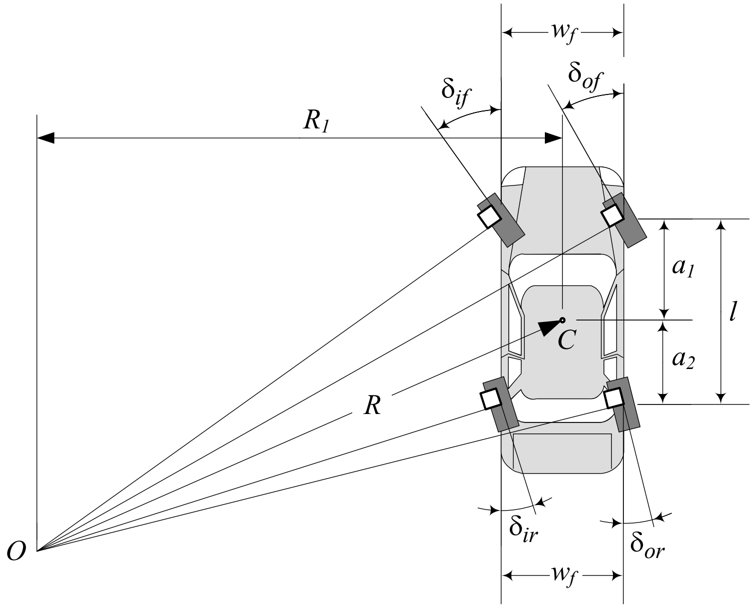

FIGURE 7.31. A positive four-wheel steering vehicle.

reduces dramatically to an acceptable value when the vehicle is moving. Furthermore, an offset design sometimes makes more room to attach the other devices, and simplifies manufacturing. So, it may be used for small off-road vehicles, such as a mini Baja, and toy vehicles.

7.5 ★ Four wheel steering.

At very low speeds, the kinematic steering condition that the perpendicular lines to each tire meet at one point, must be applied. The intersection point is the *turning center* of the vehicle.

Figure 7.31 illustrates a *positive* four-wheel steering vehicle, and Figure 7.32 illustrates a *negative* $4WS$ vehicle. In a *positive* $4WS$ situation the front and rear wheels steer in the same direction, and in a *negative* $4WS$ situation the front and rear wheels steer opposite to each other. The kinematic condition between the steer angles of a $4WS$ vehicle is

$$\cot \delta_{of} - \cot \delta_{if} = \frac{w_f}{l} - \frac{w_r}{l} \frac{\cot \delta_{of} - \cot \delta_{if}}{\cot \delta_{or} - \cot \delta_{ir}} \tag{7.73}$$

where, w_f and w_r are the front and rear tracks, δ_{if} and δ_{of} are the steer angles of the front inner and outer wheels, δ_{ir} and δ_{or} are the steer angles of the rear inner and outer wheels, and l is the wheelbase of the vehicle.

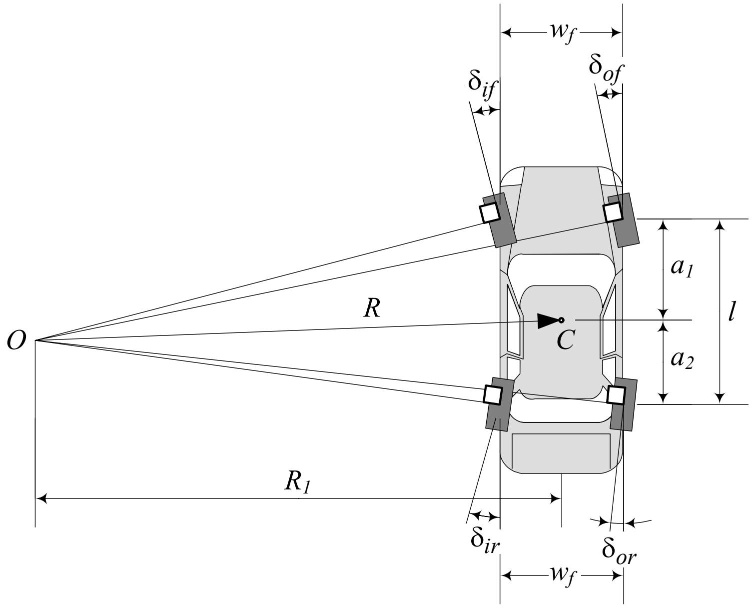

FIGURE 7.32. A negative four-wheel steering vehicle.

We may also use the following more general equation for the kinematic condition between the steer angles of a $4WS$ vehicle

$$\cot\delta_{fr} - \cot\delta_{fl} = \frac{w_f}{l} - \frac{w_r}{l}\frac{\cot\delta_{fr} - \cot\delta_{fl}}{\cot\delta_{rr} - \cot\delta_{rl}} \tag{7.74}$$

where, δ_{fl} and δ_{fr} are the steer angles of the front left and front right wheels, and δ_{rl} and δ_{rr} are the steer angles of the rear left and rear right wheels.

If we define the steer angles according to the sign convention shown in Figure 7.33 then, Equation (7.73) expresses the kinematic condition for both, positive and negative $4WS$ systems. Employing the wheel coordinate frame (x_w, y_w, z_w), we define the steer angle as the angle between the vehicle x-axis and the wheel x_w-axis, measured about the z-axis. Therefore, a steer angle is positive when the wheel is turned to the left, and it is negative when the wheel is turned to the right.

Proof. The slip-free condition for wheels of a $4WS$ in a turn requires that the normal lines to the center of each tire-plane intersect at a common point. This is the kinematic steering condition.

Figure 7.34 illustrates a positive $4WS$ vehicle in a left turn. The turning center O is on the left, and the inner wheels are the left wheels that are closer to the turning center. The longitudinal distance between point O and the axles of the car are indicated by c_1, and c_2 measured in the body

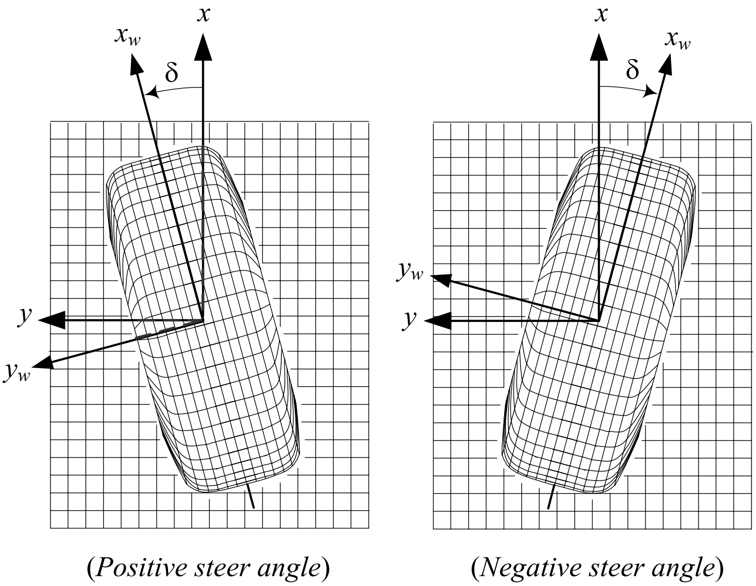

FIGURE 7.33. Sign convention for steer angles.

coordinate frame.

The front inner and outer steer angles δ_{if}, δ_{of} may be calculated from the triangles $\triangle OAE$ and $\triangle OBF$, while the rear inner and outer steer angles δ_{ir}, δ_{or} may be calculated from the triangles $\triangle ODG$ and $\triangle OCH$ as follows.

$$\tan\delta_{if} = \frac{c_1}{R_1 - \frac{w_f}{2}} \tag{7.75}$$

$$\tan\delta_{of} = \frac{c_1}{R_1 + \frac{w_f}{2}} \tag{7.76}$$

$$\tan\delta_{ir} = \frac{c_2}{R_1 - \frac{w_r}{2}} \tag{7.77}$$

$$\tan\delta_{or} = \frac{c_2}{R_1 + \frac{w_r}{2}} \tag{7.78}$$

Eliminating R_1

$$R_1 = \frac{1}{2}w_f + \frac{c_1}{\tan\delta_{if}} \tag{7.79}$$

$$= -\frac{1}{2}w_f + \frac{c_1}{\tan\delta_{of}} \tag{7.80}$$

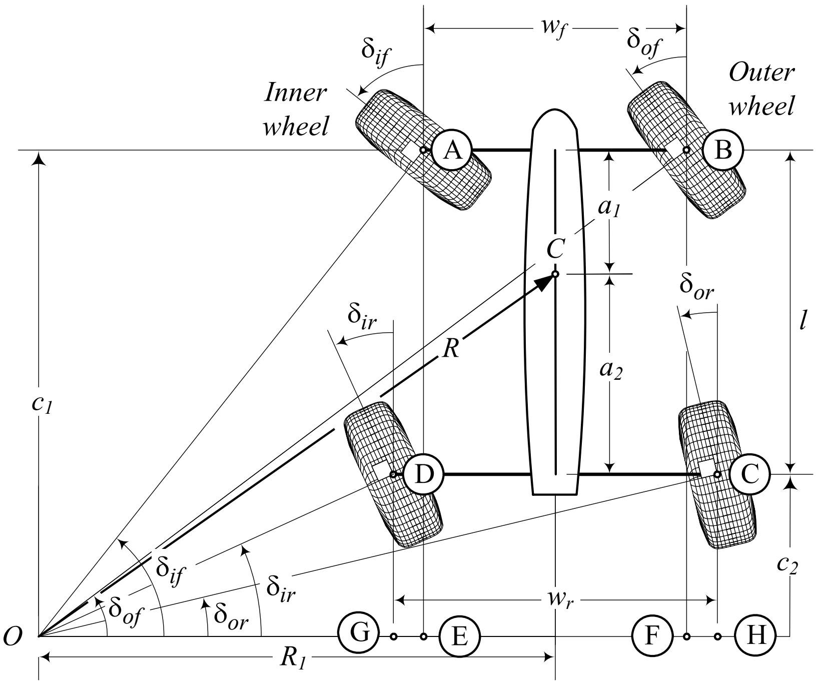

FIGURE 7.34. Illustration of a negative four-wheel steering vehicle in a left turn.

between (7.75) and (7.76) provides the kinematic condition between the front steering angles δ_{if} and δ_{of}.

$$\cot\delta_{of} - \cot\delta_{if} = \frac{w_f}{c_1} \tag{7.81}$$

Similarly, we may eliminate R_1

$$\begin{aligned} R_1 &= \frac{1}{2}w_r + \frac{c_2}{\tan\delta_{ir}} && (7.82) \\ &= -\frac{1}{2}w_r + \frac{c_2}{\tan\delta_{or}} && (7.83) \end{aligned}$$

between (7.77) and (7.78) to provide the kinematic condition between the rear steering angles δ_{ir} and δ_{or}.

$$\cot\delta_{or} - \cot\delta_{ir} = \frac{w_r}{c_2} \tag{7.84}$$

Using the following constraint

$$c_1 - c_2 = l \tag{7.85}$$

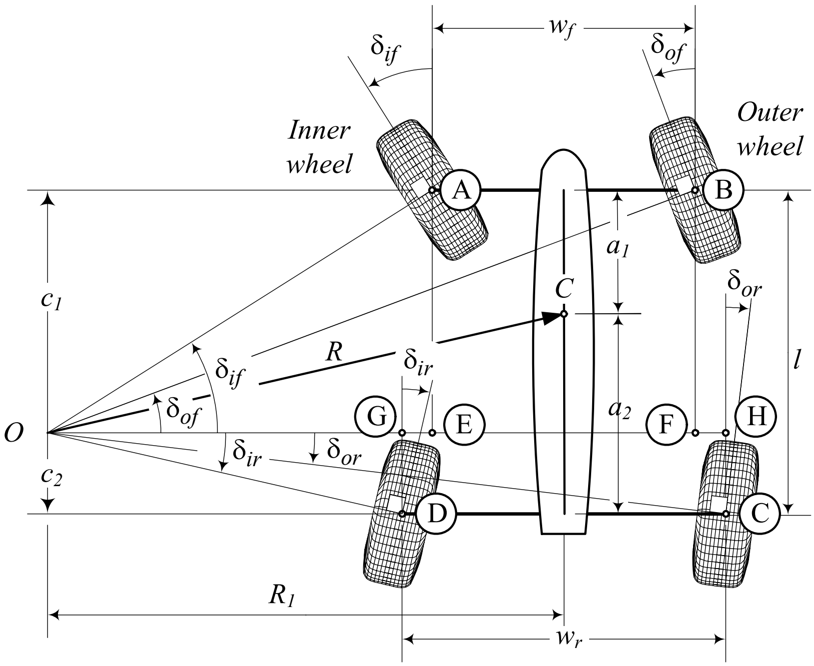

FIGURE 7.35. Illustration of a positive four-wheel steering vehicle in a left turn.

we may combine Equations (7.81) and (7.84)

$$\frac{w_f}{\cot\delta_{of} - \cot\delta_{if}} - \frac{w_r}{\cot\delta_{or} - \cot\delta_{ir}} = l \tag{7.86}$$

to find the kinematic condition (7.73) between the steer angles of the front and rear wheels for a positive $4WS$ vehicle.

Figure 7.35 illustrates a negative $4WS$ vehicle in a left turn. The turning center O is on the left, and the inner wheels are the left wheels that are closer to the turning center. The front inner and outer steer angles δ_{if}, δ_{of} may be calculated from the triangles $\triangle OAE$ and $\triangle OBF$, while the rear inner and outer steer angles δ_{ir}, δ_{or} may be calculated from the triangles $\triangle ODG$ and $\triangle OCH$ as follows.

$$\tan\delta_{if} = \frac{c_1}{R_1 - \frac{w_f}{2}} \tag{7.87}$$

$$\tan\delta_{of} = \frac{c_1}{R_1 + \frac{w_f}{2}} \tag{7.88}$$

$$-\tan\delta_{ir} = \frac{-c_2}{R_1 - \frac{w_r}{2}} \tag{7.89}$$

$$-\tan\delta_{or} = \frac{-c_2}{R_1 + \frac{w_r}{2}} \tag{7.90}$$

Eliminating R_1

$$R_1 = \frac{1}{2}w_f + \frac{c_1}{\tan\delta_{if}} \tag{7.91}$$

$$= -\frac{1}{2}w_f + \frac{c_1}{\tan\delta_{of}} \tag{7.92}$$

between (7.87) and (7.88) provides the kinematic condition between the front steering angles δ_{if} and δ_{of}.

$$\cot\delta_{of} - \cot\delta_{if} = \frac{w_f}{c_1} \tag{7.93}$$

Similarly, we may eliminate R_1

$$R_1 = \frac{1}{2}w_r + \frac{c_2}{\tan\delta_{ir}} \tag{7.94}$$

$$= -\frac{1}{2}w_r + \frac{c_2}{\tan\delta_{or}} \tag{7.95}$$

between (7.89) and (7.90) to provide the kinematic condition between the rear steering angles δ_{ir} and δ_{or}.

$$\cot\delta_{or} - \cot\delta_{ir} = \frac{w_r}{c_2} \tag{7.96}$$

Using the following constraint

$$c_1 - c_2 = l \tag{7.97}$$

we may combine Equations (7.93) and (7.96)

$$\frac{w_f}{\cot\delta_{of} - \cot\delta_{if}} - \frac{w_r}{\cot\delta_{or} - \cot\delta_{ir}} = l \tag{7.98}$$

to find the kinematic condition (7.73) between the steer angles of the front and rear wheels for a negative $4WS$ vehicle.

Using the sign convention shown in Figure 7.33, we may re-examine Figures 7.35 and 7.34. When the steer angle of the front wheels are positive then, the steer angle of the rear wheels are negative in a negative $4WS$ system, and are positive in a positive $4WS$ system. Therefore, Equation (7.74)

$$\cot\delta_{fr} - \cot\delta_{fl} = \frac{w_f}{l} - \frac{w_r}{l}\frac{\cot\delta_{fr} - \cot\delta_{fl}}{\cot\delta_{rr} - \cot\delta_{rl}} \tag{7.99}$$

can express the kinematic condition for both, positive and negative $4WS$ systems. Similarly, the following equations can uniquely determine c_1 and c_2 regardless of the positive or negative $4WS$ system.

$$c_1 = \frac{w_f}{\cot \delta_{fr} - \cot \delta_{fl}} \tag{7.100}$$

$$c_2 = \frac{w_r}{\cot \delta_{rr} - \cot \delta_{rl}} \tag{7.101}$$

Four-wheel steering or all wheel steering AWS may be applied on vehicles to improve steering response, increase the stability at high speeds maneuvering, or decrease turning radius at low speeds. A negative $4WS$ has shorter turning radius R than a front-wheel steering FWS vehicle.

For a FWS vehicle, the perpendicular to the front wheels meet at a point on the extension of the rear axle. However, for a $4WS$ vehicle, the intersection point can be any point in the xy plane. The point is the *turning center* of the car and its position depends on the steer angles of the wheels. Positive steering is also called *same steer*, and a negative steering is also called *counter steer*. ■

Example 281 ★ *Steering angles relationship.*

Consider a car with the following dimensions.

$$\begin{aligned} l &= 2.8\,\mathrm{m} \\ w_f &= 1.35\,\mathrm{m} \\ w_r &= 1.4\,\mathrm{m} \end{aligned} \tag{7.102}$$

The set of equations (7.75)-(7.78) which are the same as (7.87)-(7.90) must be used to find the kinematic steer angles of the tires. Assume one of the angles, such as

$$\delta_{if} = 15\,\mathrm{deg} \tag{7.103}$$

is a known input steer angle. To find the other steer angles, we need to know the position of the turning center O. The position of the turning center can be determined if we have one of the three parameters c_1, c_2, R_1. To clarify this fact, let's assume that the car is turning left and we know the value of δ_{if}. Therefore, the perpendicular line to the front left wheel is known. The turning center can be any point on this line. When we pick a point, the other wheels can be adjusted accordingly.

The steer angles for a $4WS$ system is a set of four equations, each with two variables.

$$\delta_{if} = \delta_{if}(c_1, R_1) \tag{7.104}$$

$$\delta_{of} = \delta_{of}(c_1, R_1) \tag{7.105}$$

$$\delta_{ir} = \delta_{ir}(c_2, R_1) \tag{7.106}$$

$$\delta_{or} = \delta_{or}(c_2, R_1) \tag{7.107}$$

If c_1 and R_1 are known, we will be able to determine the steer angles δ_{if}, δ_{of}, δ_{ir}, and δ_{or} uniquely. However, a practical situation is when we have one of the steer angles, such as δ_{if}, and we need to determine the required steer angle of the other wheels, δ_{of}, δ_{ir}, δ_{or}. It can be done if we know c_1 or R_1.

The turning center is the curvature center of the path of motion. If the path of motion is known, then at any point of the road, the turning center can be found in the vehicle coordinate frame.

In this example, let's assume

$$R_1 = 50\,\mathrm{m} \tag{7.108}$$

therefore, from Equation (7.75), we have

$$\begin{aligned} c_1 &= \left(R_1 - \frac{w_f}{2}\right)\tan\delta_{if} \\ &= \left(50 - \frac{1.35}{2}\right)\tan\frac{\pi}{12} = 13.217\,\mathrm{m} \end{aligned} \tag{7.109}$$

Because $c_1 > l$ and $\delta_{if} > 0$ the vehicle is in a positive $4WS$ configuration and the turning center is behind the car.

$$\begin{aligned} c_2 &= c_1 - l \\ &= 13.217 - 2.8 = 10.417\,\mathrm{m}. \end{aligned} \tag{7.110}$$

Now, employing Equations (7.76)-(7.78) provides the other steer angles.

$$\begin{aligned} \delta_{of} &= \tan^{-1}\frac{c_1}{R_1 + \frac{w_f}{2}} = \tan^{-1}\frac{13.217}{50 + \frac{1.35}{2}} \\ &= 0.25513\,\mathrm{rad} \approx 14.618\,\mathrm{deg} \end{aligned} \tag{7.111}$$

$$\begin{aligned} \delta_{ir} &= \tan^{-1}\frac{c_2}{R_1 - \frac{w_r}{2}} = \tan^{-1}\frac{10.417}{50 - \frac{1.4}{2}} \\ &= 0.20824\,\mathrm{rad} \approx 11.931\,\mathrm{deg} \end{aligned} \tag{7.112}$$

$$\begin{aligned} \delta_{or} &= \tan^{-1}\frac{c_2}{R_1 + \frac{w_r}{2}} = \tan^{-1}\frac{10.417}{50 + \frac{1.4}{2}} \\ &= 0.202\,64\,\mathrm{rad} \approx 11.61\,\mathrm{deg} \end{aligned} \tag{7.113}$$

Example 282 ★ *Position of the turning center.*

The turning center of a vehicle, in the vehicle body coordinate frame, is at a point with coordinates (x_O, y_O). The coordinates of the turning center *are*

$$\begin{aligned} x_O &= -a_2 - c_2 \\ &= -a_2 - \frac{w_r}{\cot\delta_{or} - \cot\delta_{ir}} \end{aligned} \tag{7.114}$$

$$\begin{aligned} y_O &= R_1 \\ &= \frac{l + \frac{1}{2}\left(w_f \tan\delta_{if} - w_r \tan\delta_{ir}\right)}{\tan\delta_{if} - \tan\delta_{ir}}. \end{aligned} \tag{7.115}$$

Equation (7.115) is found by substituting c_1 *and* c_2 *from (7.91) and (7.94) in (7.97), and define* y_O *in terms of* δ_{if} *and* δ_{ir}*. It is also possible to define* y_O *in terms of* δ_{of} *and* δ_{or}*.*

Equations (7.114) and (7.115) can be used to define the coordinates of the turning center for both positive and negative $4WS$ *systems.*

As an example, let's examine a car with the following data.

$$\begin{aligned} l &= 2.8\,\mathrm{m} \\ w_f &= 1.35\,\mathrm{m} \\ w_r &= 1.4\,\mathrm{m} \\ a_1 &= a_2 \end{aligned} \tag{7.116}$$

$$\begin{aligned} \delta_{if} &= 0.26180\,\mathrm{rad} \approx 15\,\mathrm{deg} \\ \delta_{of} &= 0.25513\,\mathrm{rad} \approx 14.618\,\mathrm{deg} \\ \delta_{ir} &= 0.20824\,\mathrm{rad} \approx 11.931\,\mathrm{deg} \\ \delta_{or} &= 0.20264\,\mathrm{rad} \approx 11.61\,\mathrm{deg} \end{aligned} \tag{7.117}$$

and find the position of the turning center.

$$\begin{aligned} x_O &= -a_2 - \frac{w_r}{\cot\delta_{or} - \cot\delta_{ir}} \\ &= -\frac{2.8}{2} - \frac{1.4}{\cot 0.20264 - \cot 0.20824} = -11.802\,\mathrm{m} \end{aligned} \tag{7.118}$$

$$\begin{aligned} y_O &= \frac{l + \frac{1}{2}\left(w_f \tan\delta_{if} - w_r \tan\delta_{ir}\right)}{\tan\delta_{if} - \tan\delta_{ir}} \\ &= \frac{2.8 + \frac{1}{2}\left(1.35 \tan 0.26180 - 1.4 \tan 0.20824\right)}{\tan 0.26180 - \tan 0.20824} = 50.011\,\mathrm{m} \end{aligned} \tag{7.119}$$

The position of turning center for a FWS *vehicle is at*

$$\begin{aligned} x_O &= -a_2 \\ y_O &= \frac{1}{2} w_f + \frac{l}{\tan\delta_{if}} \end{aligned} \tag{7.120}$$

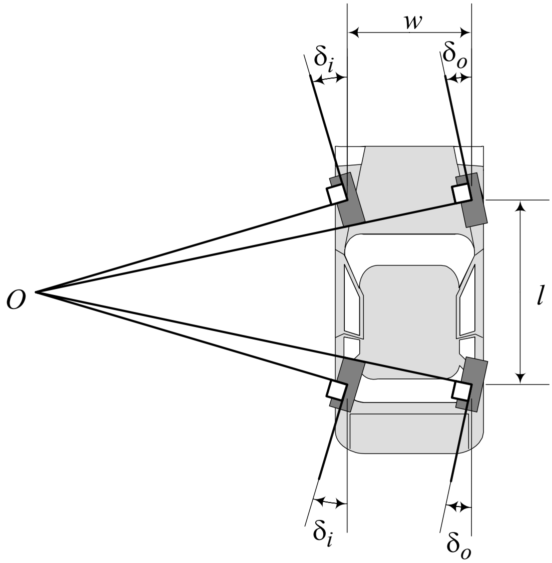

FIGURE 7.36. A symmetric four-wheel steering vehicle.

and for a RWS vehicle is at

$$\begin{aligned} x_O &= a_1 \\ y_O &= \frac{1}{2}w_r + \frac{l}{\tan\delta_{ir}}. \end{aligned} \tag{7.121}$$

Example 283 ★ *Curvature.*

Consider a road as a path of motion that is expressed mathematically by a function $Y = f(X)$, in a global coordinate frame. The radius of curvature R_κ of such a road at point X is

$$R_\kappa = \frac{\left(1 + Y'^2\right)^{3/2}}{Y''} \tag{7.122}$$

where

$$Y' = \frac{dY}{dX} \tag{7.123}$$

$$Y'' = \frac{d^2Y}{dX^2}. \tag{7.124}$$

Example 284 ★ *Symmetric four-wheel steering system.*

Figure 7.36 illustrates a symmetric 4WS vehicle that the front and rear wheels steer opposite to each other equally. The kinematic steering condition for a symmetric steering is simplified to

$$\cot\delta_o - \cot\delta_i = \frac{w_f}{l} + \frac{w_r}{l} \tag{7.125}$$

and c_1 and c_2 are reduced to

$$c_1 = \frac{1}{2}l \tag{7.126}$$
$$c_2 = -\frac{1}{2}l. \tag{7.127}$$

Example 285 ★ *c_2/c_1 ratio.*

*Longitudinal distance of the turning center of a vehicle from the front axle is c_1 and from the rear axle is c_2. We show the ratio of these distances by c_s and call it the $4WS$ **factor**.*

$$\begin{aligned} c_s &= \frac{c_2}{c_1} \\ &= \frac{w_r}{w_f} \frac{\cot\delta_{fr} - \cot\delta_{fl}}{\cot\delta_{rr} - \cot\delta_{rl}} \end{aligned} \tag{7.128}$$

c_s is negative for a negative $4WS$ vehicle and is positive for a positive $4WS$ vehicle. When $c_s = 0$, the car is FWS, and when $c_s = -\infty$, the car is RWS. A symmetric $4WS$ system has $c_s = -\frac{1}{2}$.

Example 286 ★ *Steering length l_s.*

*For a $4WS$ vehicle, we may define a **steering length** l_s as*

$$\begin{aligned} l_s &= \frac{c_1 + c_2}{l} = \frac{l}{c_1} + 2c_s \\ &= \frac{1}{l}\left(\frac{w_f}{\cot\delta_{fr} - \cot\delta_{fl}} + \frac{w_r}{\cot\delta_{rr} - \cot\delta_{rl}}\right) \end{aligned} \tag{7.129}$$

Steering length l_s is 1 for a FWS car, zero for a symmetric car, and -1 for a RWS car. When a car has a negative $4WS$ system then, $-1 < l_s < 1$, and when the car has a positive $4WS$ system then, $1 < l_s$ or $l_s < -1$. The case $1 < l_s$ happens when the turning center is behind the car, and the case $l_s < -1$ happens when the turning center is ahead of the car.

Example 287 ★ *FWS and Ackerman condition.*

When a car is FWS vehicle, then the Ackerman condition (7.1) can be written as the following equation.

$$\cot\delta_{fr} - \cot\delta_{fl} = \frac{w}{l} \tag{7.130}$$

Writing the Ackerman condition as this equation frees us from checking the inner and outer wheels.

Example 288 ★ *Turning radius.*

To find the vehicle's turning radius R, we may define equivalent bicycle models as shown in Figure 7.37 and 7.38 for positive and negative $4WS$

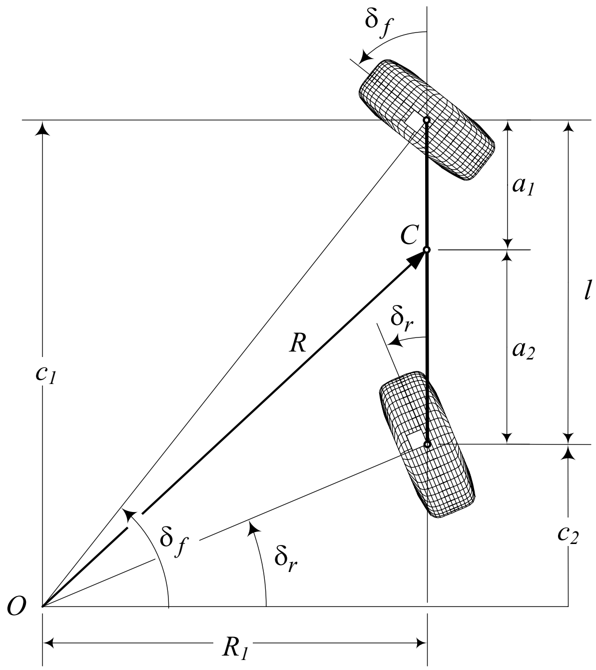

FIGURE 7.37. Bicycle model for a positive $4WS$ vehicle.

vehicles. The radius of rotation R is perpendicular to the vehicle's velocity vector $\mathbf{v}$ at the mass center C.

Let's examine the positive $4WS$ situation in Figure 7.37. Using the geometry shown in the bicycle model, we have

$$R^2 = (a_2 + c_2)^2 + R_1^2 \tag{7.131}$$

$$\begin{aligned} \cot \delta_f &= \frac{R_1}{c_1} \\ &= \frac{1}{2} (\cot \delta_{if} + \cot \delta_{of}) \end{aligned} \tag{7.132}$$

and therefore,

$$R = \sqrt{(a_2 + c_2)^2 + c_1^2 \cot^2 \delta_f}. \tag{7.133}$$

Examining Figure 7.38 shows that the turning radius of a negative $4WS$ vehicle can be determined from the same equation (7.133).

Example 289 ★ *FWS and $4WS$ comparison.*

The turning center of a FWS car is always on the extension of the rear axel, and its steering length l_s is always equal to 1. However, the turning center of a $4WS$ car can be:

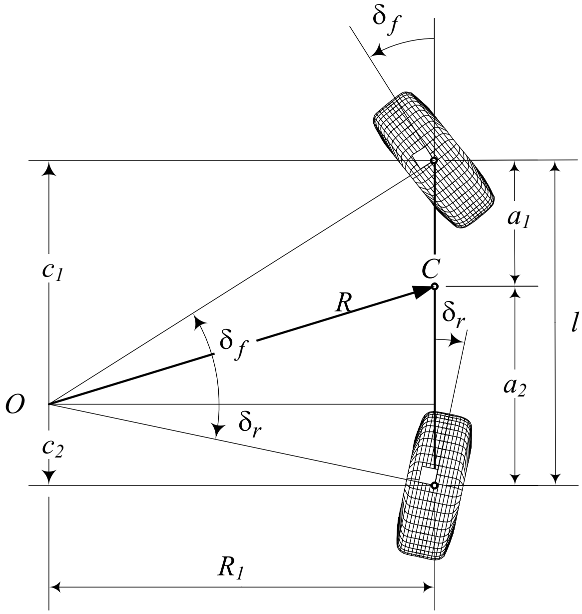

FIGURE 7.38. Bicycle model for a negative $4WS$ vehicle.

1− *ahead of the front axle, if* $l_s < -1$
2− *for a* FWS *car, if* $-1 < l_s < 1$ *or*
3− *behind the rear axle, if* $1 < l_s$

A comparison among the different steering lengths is illustrated in Figure 7.39. A FWS *car is shown in Figure 7.39(a), while the* $4WS$ *systems with* $l_s < -1$, $-1 < l_s < 1$, *and* $1 < l_s$ *are shown in Figures 7.39(b)-(d) respectively.*

Example 290 ★ *Passive and active four-wheel steering.*

The negative $4WS$ *is not recommended at high speeds because of high yaw rates, and the positive steering is not recommended at low speeds because of increasing radius of turning. Therefore, to maximize the advantages of a* $4WS$ *system, we need a smart system to allow the wheels to change the mode of steering depending on the speed of the vehicle and adjust the steer angles for different purposes. A* ***smart steering*** *is also called* ***active steering*** *system.*

An active system may provide a negative steering at low speeds and a positive steering at high speeds. In a negative steering, the rear wheels are steered in the opposite direction as the front wheels to turn in a significantly smaller radius, while in positive steering, the rear wheels are steered in the same direction as the front wheels to increase the lateral force.

When the $4WS$ *system is passive, there is a constant proportional ratio*

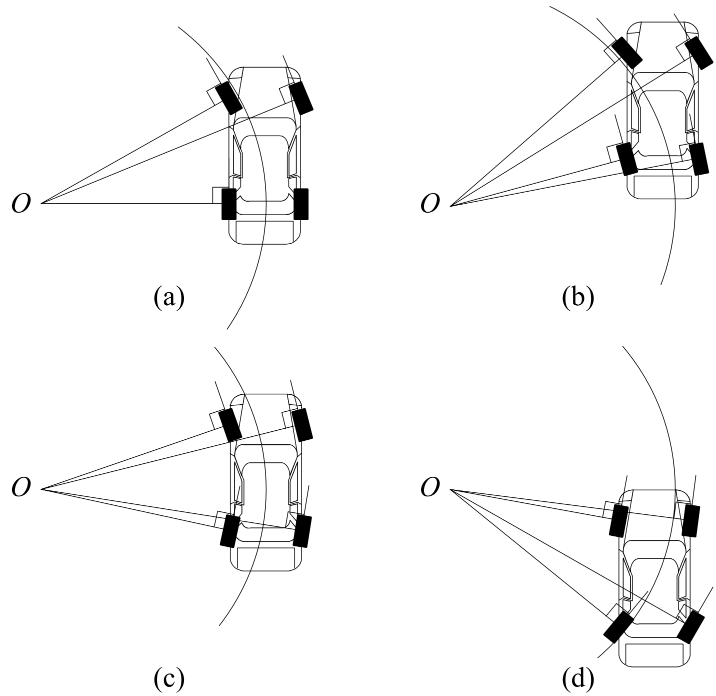

FIGURE 7.39. A comparison among the different steering lengths.

between the front and rear steer angles which is equivalent to have a constant c_s.

A passive steering may be applied in vehicles to compensate some vehicle tendencies. As an example, in a FWS *system, the rear wheels tend to steer slightly to the outside of a turn. Such tendency can reduce stability.*

Example 291 ★ *Autodriver.*

Consider a car that is moving on a road, as shown in Figure 7.40. Point O *indicates the center of curvature of the road at the car's position. Center of curvature of the road is supposed to be the turning center of the car at the instant of consideration.*

There is a global coordinate frame G *attached to the ground, and a vehicle coordinate frame* B *attached to the car at its mass center* C*. The* z *and* Z *axes are parallel and the angle* ψ *indicates the angle between* X *and* x *axes. If* (X_O, Y_O) *are the coordinates of* O *in the global coordinate frame* G *then,*

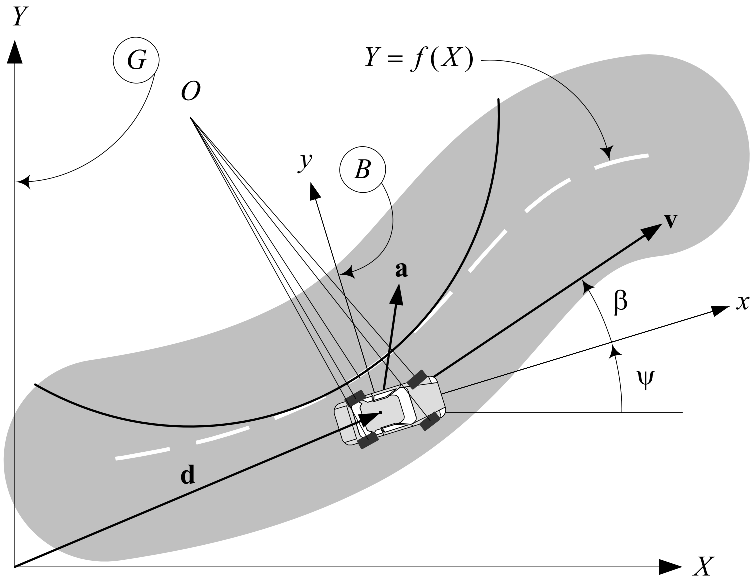

FIGURE 7.40. Illustration of a car that is moving on a road at the point that O is the center of curvature.

the coordinates of O in B would be

$$
\begin{aligned}
{}^{B}\mathbf{r}_{O} &= R_{z,\psi}\,{}^{G}\mathbf{r}_{O} \\
\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} &= \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} X\cos\psi + Y\sin\psi \\ Y\cos\psi - X\sin\psi \\ 0 \end{bmatrix}.
\end{aligned}
\tag{7.134}
$$

Having coordinates of O in the vehicle coordinate frame is enough to determine R_1, c_1, and c_2.

$$
\begin{aligned}
R_1 &= y_O \\
&= Y\cos\psi - X\sin\psi
\end{aligned}
\tag{7.135}
$$

$$
\begin{aligned}
c_2 &= -a_2 - x_O \\
&= X\cos\psi + Y\sin\psi - a_2
\end{aligned}
\tag{7.136}
$$

$$
\begin{aligned}
c_1 &= c_2 + l \\
&= X\cos\psi + Y\sin\psi + a_1
\end{aligned}
\tag{7.137}
$$

Then, the required steer angles of the wheels can be uniquely determined by Equations (7.75)-(7.78).

It is possible to define a road by a mathematical function $Y = f(X)$ in a global coordinate frame. At any point X of the road, the position of the vehicle and the position of the turning center in the vehicle coordinate frame can be determined. The required steer angles can accordingly be set to keep the vehicle on the road and run the vehicle in the correct direction. This principle may be used to design an ***autodriver****.*

Example 292 ★ *Curvature equation.*

Consider a vehicle that is moving on a path $Y = f(X)$ with velocity $\mathbf{v}$ and acceleration $\mathbf{a}$. The curvature $\kappa = 1/R$ of the path that the vehicle is moving on is

$$\kappa = \frac{1}{R} = \frac{a_n}{v^2} \tag{7.138}$$

where, a_n is the normal component of the acceleration $\mathbf{a}$. The normal component a_n is toward the rotation center and is equal to

$$\begin{aligned} a_n &= \left|\frac{\mathbf{v}}{v} \times \mathbf{a}\right| = \frac{1}{v}\left|\mathbf{v} \times \mathbf{a}\right| \\ &= \frac{1}{v}\left(a_Y v_X - a_X v_Y\right) = \frac{\ddot{Y}\dot{X} - \ddot{X}\dot{Y}}{\sqrt{\dot{X}^2 + \dot{Y}^2}} \end{aligned} \tag{7.139}$$

and therefore,

$$\kappa = \frac{\ddot{Y}\dot{X} - \ddot{X}\dot{Y}}{\left(\dot{X}^2 + \dot{Y}^2\right)^{3/2}} = \frac{\ddot{Y}\dot{X} - \ddot{X}\dot{Y}}{\dot{X}^3} \frac{1}{\left(1 + \dfrac{\dot{Y}^2}{\dot{Y}^2}\right)^{3/2}}. \tag{7.140}$$

However,

$$Y' = \frac{dY}{dX} = \frac{\dot{Y}}{\dot{X}} \tag{7.141}$$

$$Y'' = \frac{d^2Y}{dX^2} = \frac{d}{dx}\left(\frac{\dot{Y}}{\dot{X}}\right) = \frac{d}{dt}\left(\frac{\dot{Y}}{\dot{X}}\right)\frac{1}{\dot{X}} = \frac{\ddot{Y}\dot{X} - \ddot{X}\dot{Y}}{\dot{X}^3} \tag{7.142}$$

and we find the following equation for the curvature of the path based on the equation of the path.

$$\kappa = \frac{Y''}{\left(1 + Y'^2\right)^{3/2}} \tag{7.143}$$

7.6 ★ Steering Mechanism Optimization

Optimization means steering mechanism is the design of a system that works as closely as possible to a desired function. Assume the Ackerman kinematic condition is the desired function for a steering system. Comparing the function of the designed steering mechanism to the Ackerman condition, we may define an error function e to compare the two functions. An example for the e function can be the difference between the outer steer angles of the designed mechanism δ_{D_o} and the Ackerman δ_{A_o} for the same inner angle δ_i.

The error function may be the absolute value of the maximum difference,

$$e = \max \left| \delta_{D_o} - \delta_{A_o} \right| \tag{7.144}$$

or the root mean square (RMS) of the difference between the two functions

$$e = \sqrt{\int \left(\delta_{D_o} - \delta_{A_o} \right)^2 \, d\delta_i} \tag{7.145}$$

in a specific range of the inner steer angle δ_i.

The error e, would be a function of a set of parameters. Minimization of the error function for a parameter, over the working range of the steer angle δ_i, generates the optimized value of the parameter.

The RMS function (7.145) is defined for continuous variables δ_{D_o} and δ_{A_o}. However, depending on the designed mechanism, it is not always possible to find a closed-form equation for e. In this case, the error function cannot be defined explicitly, and hence, the error function should be evaluated for n different values of the inner steer angle δ_i numerically. The error function for a set of discrete values of e is define by

$$e = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \left(\delta_{D_o} - \delta_{A_o} \right)^2}. \tag{7.146}$$

The error function (7.145) or (7.146) must be evaluated for different values of a parameter. Then a plot for $e = e(parameter)$ can show the trend of variation of e as a function of the parameter. If there is a minimum for e, then the optimal value for the parameter can be found. Otherwise, the trend of the e function can show the direction for minimum searching.

Example 293 ★ *Optimized trapezoidal steering mechanism.*

The inner-outer angles relationship for a trapezoidal steering mechanism, shown in Figure 7.6, is

$$\begin{aligned} & \sin(\beta + \delta_i) + \sin(\beta - \delta_o) \\ = \quad & \frac{w}{d} + \sqrt{\left(\frac{w}{d} - 2\sin\beta \right)^2 - \left(\cos(\beta - \delta_o) - \cos(\beta + \delta_i) \right)^2}. \end{aligned} \tag{7.147}$$

Comparing Equation (7.147) with the Ackerman condition,

$$\cot\delta_o - \cot\delta_i = \frac{w}{l} \tag{7.148}$$

we may define an error function

$$e = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\delta_{D_o} - \delta_{A_o}\right)^2} \tag{7.149}$$

and search for its minimum to optimize the trapezoidal steering mechanism.

Consider a vehicle with the dimensions

$$\begin{aligned} w &= 2.4\,\mathrm{m} \\ l &= 4.8\,\mathrm{m}. \end{aligned} \tag{7.150}$$

Let's optimize a trapezoidal steering mechanism for

$$d = 0.4\,\mathrm{m} \tag{7.151}$$

and use β *as a parameter.*

A plot of comparison between such a mechanism and the Ackerman condition, for a set of different β*, is shown in Figure 7.9, and their difference* $\Delta\delta_o = \delta_{D_o} - \delta_{A_o}$ *is shown in Figure 7.10.*

We may set a value for β*, say* $\beta = 6\,\mathrm{deg}$*, and evaluate* δ_{D_o} *and* δ_{A_o} *at* $n = 100$ *different values of* δ_i *for a working range such as* $-40\,\mathrm{deg} \leq \delta_i \leq 40\,\mathrm{deg}$*. Then, we calculate the associated error function* e

$$e = \sqrt{\frac{1}{100}\sum_{i=1}^{100}\left(\delta_{D_o} - \delta_{A_o}\right)^2} \tag{7.152}$$

for the specific β*. Now we conduct our calculation again for new values of* β*, such as* $\beta = 8\,\mathrm{deg}, 9\,\mathrm{deg}, \cdots$*. Figure 7.41 depicts the function* $e = e(\beta)$ *with a minimum at* $\beta \approx 12\,\mathrm{deg}$*.*

The geometry of the optimal trapezoidal steering mechanism is shown in Figure 7.42(a). The two side arms intersect at point G on their extensions. For an optimal mechanism, the intersection of point G is at the outer side of the rear axle. However, it is recommended to put the intersection point at the center of the rear axle and design a near optimal trapezoidal steering mechanism. Using the recommendation, it is possible to eliminate the optimization process and get a good enough design. Such an estimated design is shown in Figure 7.42(b). The angle β *for the optimal design is* $\beta = 12.6\,\mathrm{deg}$*, and for the estimated design is* $\beta = 13.9\,\mathrm{deg}$*.*

Example 294 ★ *There is no exact Ackerman mechanism.*

It is not possible to make a simple steering linkage to work exactly based on the Ackerman steering condition. However, it is possible to optimize various steering linkages for a working range, to work closely to the Ackerman condition, and be exact at a few points. An isosceles trapezoidal linkage is

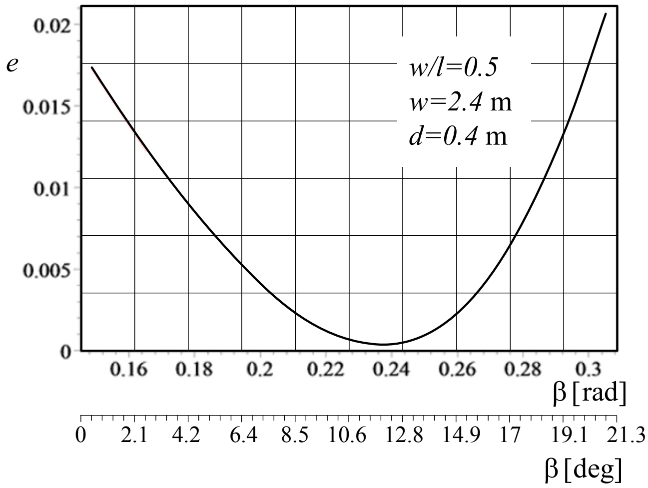

FIGURE 7.41. Error function $e = e(\beta)$ for a specific trapezoidal steering mechanism, with a minimum at $\beta \approx 12\,\text{deg}$.

not as exact as the Ackerman steering at every arbitrary turning radius, however, it is simple enough to be mass produced, and exact enough work in street cars.

Example 295 ★ *Optimization of a multi-link steering mechanism.*

Assume that we want to design a multi-link steering mechanism for a vehicle with the following dimensions.

$$\begin{aligned} w &= 2.4\,\text{m} \\ l &= 4.8\,\text{m} \\ a_2 &= 0.45l \end{aligned} \tag{7.153}$$

Due to space constraints, the position of some joints of the mechanism are determined as shown in Figure 7.43. However, we may vary the length x to design the best mechanism according to the Ackerman condition.

$$\cot\delta_2 - \cot\delta_1 = \frac{w}{l} = \frac{1}{2} \tag{7.154}$$

The steering wheel input δ_s turns the triangle PBC which turns both the left and the right wheels.

The vehicle must be able to turn in a circle with radius R_m.

$$R_m = 10\,\text{m} \tag{7.155}$$

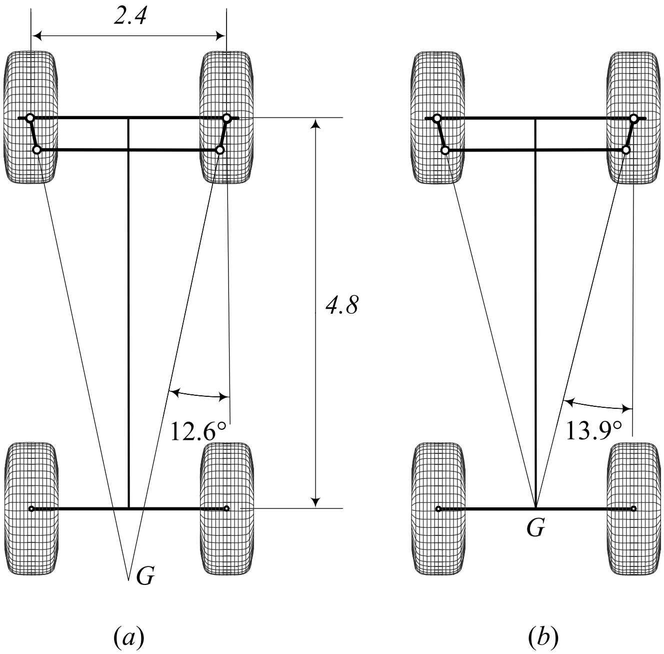

FIGURE 7.42. The geometry of the optimal trapezoidal steering mechanism and the estimated design.

The minimum turning radius determines the maximum steer angle δ

$$\begin{aligned} R_m &= \sqrt{a_2^2 + l^2 \cot^2 \delta_M} \\ 10 &= \sqrt{(0.45 \times 4.8)^2 + 4.8^2 \cot^2 \delta_M} \\ \delta_M &= 0.23713\,\mathrm{rad} \approx 13.587\,\mathrm{deg} \end{aligned} \tag{7.156}$$

where δ *is the* cot*-average of the inner and outer steer angles. Having* R *and* δ *is enough to determine* δ_o *and* δ_i*.*

$$\begin{aligned} R_1 &= l \cot \delta_M \\ &= 4.8 \cot 0.23713 = 19.861\,\mathrm{m} \end{aligned} \tag{7.157}$$

$$\delta_i = \tan^{-1} \frac{l}{R_1 - \frac{w}{2}} = 0.25176\,\mathrm{rad} \approx 14.425\,\mathrm{deg} \tag{7.158}$$

$$\delta_o = \tan^{-1} \frac{l}{R_1 + \frac{w}{2}} = 0.22408\,\mathrm{rad} \approx 12.839\,\mathrm{deg} \tag{7.159}$$

Because the mechanism is symmetric, each wheel of the steering mechanism

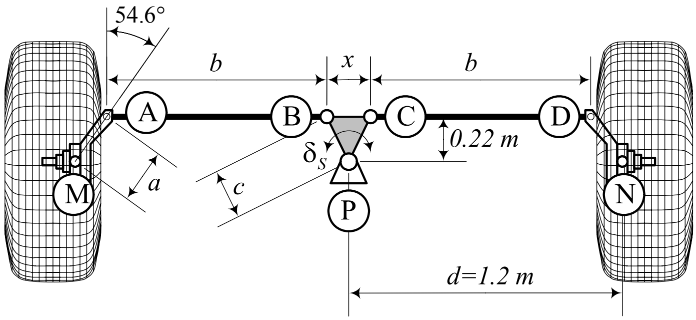

FIGURE 7.43. A multi-link steering mechanism that must be optimized by varying x.

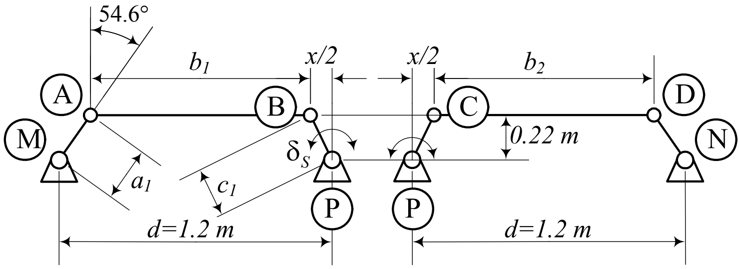

FIGURE 7.44. The multi-link steering is a 6-link mechanism that may be treeted as two combined 4-bar linkages.

in Figure 7.43 must be able to turn at least 14.425 deg*. To be safe, we try to optimize the mechanism for* $\delta = \pm 15$ deg*.*

The multi-link steering mechanism is a six-link Watt linkage. Let us divide the mechanism into two four-bar linkages. The linkage 1 *is on the left and the linkage* 2 *is on the right, as shown in Figure 7.44. We may assume that* MA *is the input link of the left linkage and* PB *is its output link. Link* PB *is rigidly attached to* PC*, which is the input of the right linkage. The output of the right linkage is* ND*. To find the inner-outer steer angles relationship, we need to find the angle of* ND *as a function of the angle of* MA*. The steer angles can be calculated based on the angle of these two links.*

$$\delta_1 = \theta_2 - (90 - 54.6)\,\mathrm{deg} \tag{7.160}$$

$$\delta_2 = \varphi_4 - (90 + 54.6)\,\mathrm{deg} \tag{7.161}$$

Figure 7.44 illustrates the link numbers, and the input-output angles of the four-bar linkages. The length of the links for the mechanisms are collected in Table 7.1*.*

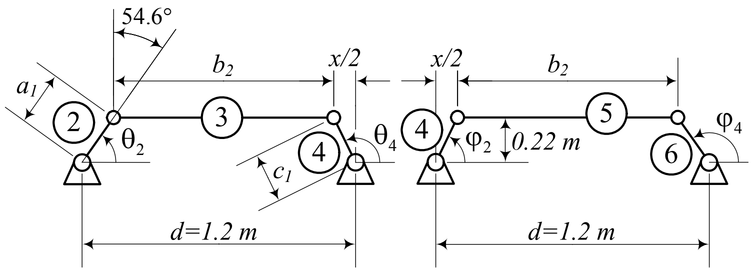

FIGURE 7.45. The input and output angles of the two 4-bar linkages.

Table 7.1 - Link numbers, and the input-output angles for the multi-link steering mechanism

Left linkage		
Link	*Length*	*angle*
1	$d_1 = 1.2$	180
2	$a_1 = 0.22/\cos 54.6 = 0.37978$	θ_2
3	$b_1 = 1.2 - 0.22 \tan 54.6 - \frac{x}{2}$ $= 0.89043 - \frac{x}{2}$	θ_3
4	$c_1 = \sqrt{0.22^2 + x^2/4}$	θ_4
Right linkage		
Link	*Length*	*angle*
1	$d_1 = 1.2$	180
4	$a_1 = \sqrt{0.22^2 + x^2/4}$	$\varphi_2 = \theta_4 - 2\tan^{-1}\frac{x}{0.44}$
5	$b_1 = 0.89043 - \frac{x}{2}$	φ_3
6	$c_1 = 0.22/\cos 54.6 = 0.37978$	φ_4

Equation (6.1) that is repeated below, provides the angle θ_4 as a function of θ_2.

$$\theta_4 = 2\tan^{-1}\left(\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}\right) \tag{7.162}$$

$$A = J_3 - J_1 + (1 - J_2)\cos\theta_2 \tag{7.163}$$

$$B = -2\sin\theta_2 \tag{7.164}$$

$$C = J_1 + J_3 - (1 + J_2)\cos\theta_2 \tag{7.165}$$

$$J_1 = \frac{d_1}{a_1} \tag{7.166}$$

$$J_2 = \frac{d_1}{c_1} \tag{7.167}$$

$$J_3 = \frac{a_1^2 - b_1^2 + c_1^2 + d_1^2}{2a_1c_1} \tag{7.168}$$

$$J_4 = \frac{d_1}{b_1} \tag{7.169}$$

$$J_5 = \frac{c_1^2 - d_1^2 - a_1^2 - b_1^2}{2a_1b_1} \tag{7.170}$$

The same equation (7.162) can be used to connect the input-output angles of the right four-bar linkage.

$$\varphi_4 = 2\tan^{-1}\left(\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}\right) \tag{7.171}$$

$$A = J_3 - J_1 + (1 - J_2)\cos\varphi_2 \tag{7.172}$$

$$B = -2\sin\varphi_2 \tag{7.173}$$

$$C = J_1 + J_3 - (1 + J_2)\cos\varphi_2 \tag{7.174}$$

$$J_1 = \frac{d_2}{a_2} \tag{7.175}$$

$$J_2 = \frac{d_2}{c_2} \tag{7.176}$$

$$J_3 = \frac{a_2^2 - b_2^2 + c_2^2 + d_2^2}{2a_2c_2} \tag{7.177}$$

$$J_4 = \frac{d_2}{b_2} \tag{7.178}$$

$$J_5 = \frac{c_2^2 - d_2^2 - a_2^2 - b_2^2}{2a_2b_2} \tag{7.179}$$

Starting with a guess value for x, we are able to calculate the length of the links. Using Equations (7.162) and (7.171), along with (7.160) and (7.161), we calculate δ_2 for a given value of δ_1.

Let's start with $x = 0$, then

$$\begin{aligned} a_1 &= 0.37978\,\mathrm{m} \\ b_1 &= 0.89043\,\mathrm{m} \\ c_1 &= 0.22\,\mathrm{m} \end{aligned} \tag{7.180}$$

$$\begin{aligned} a_2 &= 0.22\,\mathrm{m} \\ b_2 &= 0.89043\,\mathrm{m} \\ c_2 &= 0.37978\,\mathrm{m}. \end{aligned} \tag{7.181}$$

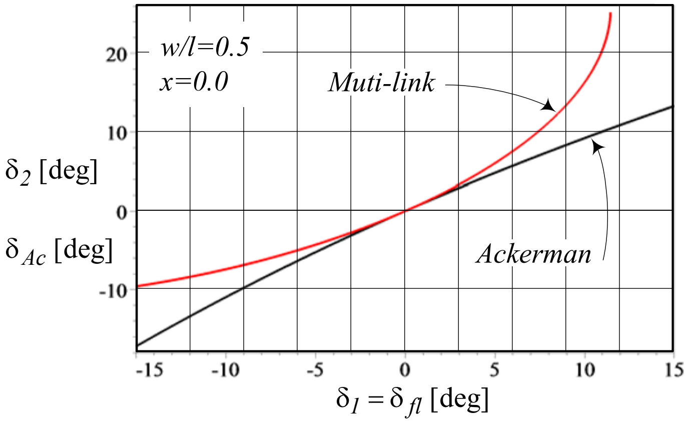

FIGURE 7.46. Steer angles δ_2 and δ_{Ac} versus δ_1.

Using Equations (7.160) and (7.162), we may calculate the output of the first four-bar linkage, θ_4, for a range of the left steer angle $-15\deg < \delta_1 < 15\deg$. The following constraint, provides the numerical values for φ_2 to be used as the input of the right four-bar linkage.

$$\varphi_2 = \theta_4 - 2\tan^{-1}\frac{x}{0.44} \tag{7.182}$$

Then, using Equations (7.171) and (7.162), we can calculate the steer angle δ_2 for the right wheel.

Figure 7.46 depicts the numerical values of the steer angles δ_2 and δ_{Ac} versus δ_1. The angle δ_{Ac} is the steer angle of the right wheel based on the Ackerman equation (7.154).

Having δ_2 and δ_{Ac}, we calculate the difference Δ

$$\Delta = \delta_2 - \delta_{Ac} \tag{7.183}$$

for n different values of δ_1 in the working range angle $-15\deg < \delta_1 < 15\deg$. Based on the n numbers for Δ, we may find the error e.

$$e = \sqrt{\frac{\Delta^2}{n}} \tag{7.184}$$

Changing the value of x and recalculating e, results an error function $e = e(x)$.

Figure 7.47 illustrates the result of the calculation. It shows that the error is minimum for $x = -0.824\,\mathrm{m}$, which is the best length for the base of the triangle PBC.

The behavior of the multi-link steering mechanism for different values of x, is shown in Figure 7.48. The Ackerman condition is also plotted to

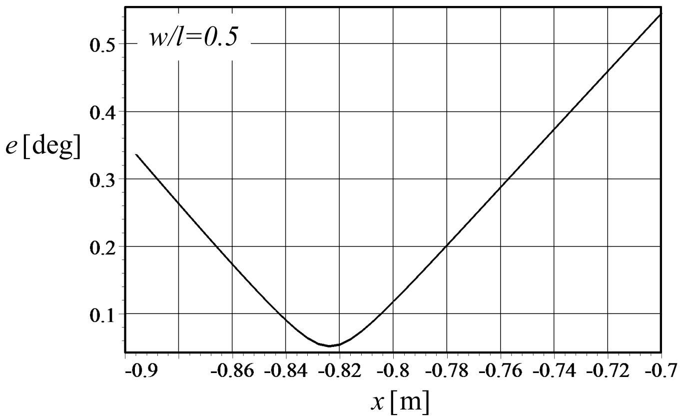

FIGURE 7.47. Illustration of the error function $e = e(x)$.

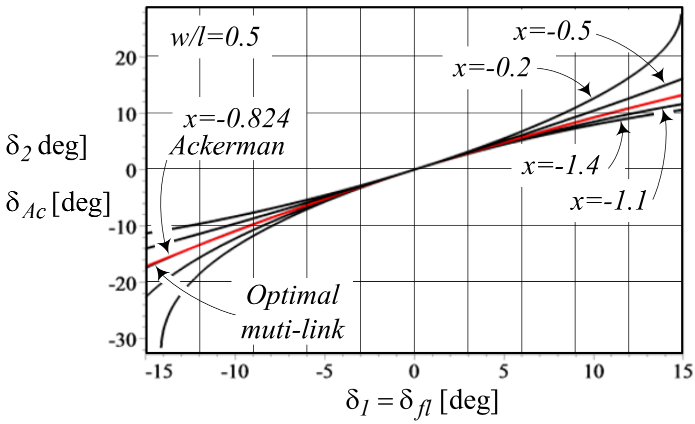

FIGURE 7.48. The behavior of the multi-link steering mechanism for different values of x.

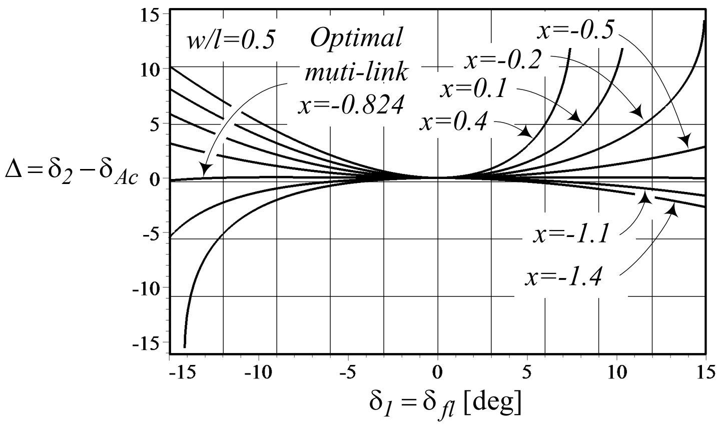

FIGURE 7.49. Illustration of the difference $\Delta = \delta_2 - \delta_{Ac}$ for different values of x.

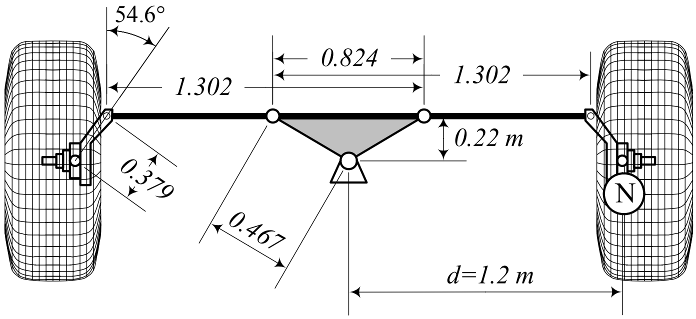

FIGURE 7.50. The optimal multi-link steering mechanism along with the length of its links.

compare with the optimal multi-link mechanism. The optimality of $x = -0.824\,\mathrm{m}$ *may be more clear in Figure 7.49 that shows the difference* $\Delta = \delta_2 - \delta_{Ac}$ *for different values of* x.

The optimal multi-link steering mechanism along with the length of its links is shown in Figure 7.50. The mechanism and the meaning of negative value for x *are shown in Figure 7.51 where the mechanism is in a positive turning position.*

7.7 ★ Trailer-Truck Kinematics

Consider a car pulling a one-axle trailer, as shown in Figure 7.52. We may normalize the dimensions such that the length of the trailer is 1. The

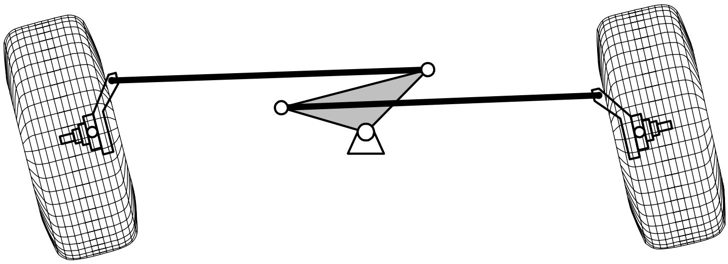

FIGURE 7.51. The optimal multi-link steering mechanism in a positive turning position.

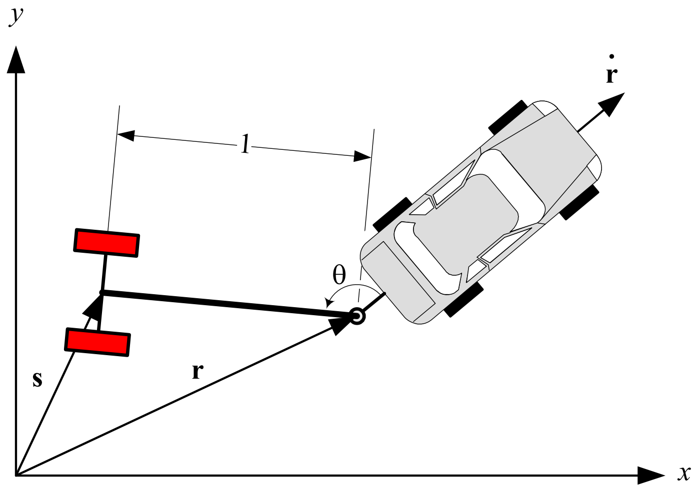

FIGURE 7.52. A car pulling a one-axle trailer.

positions of the car at the hinge point and the trailer at the center of its axle are shown by vectors $\mathbf{r}$ and $\mathbf{s}$.

Assuming $\mathbf{r}$ is a given differentiable function of time t, we would like to examine the behavior of the trailer by calculating $\mathbf{s}$, and predict jackknifing. When the car is moving forward, we say the car and trailer are *jackknifed* if

$$\dot{\mathbf{r}} \cdot \mathbf{z} < 0 \tag{7.185}$$

where

$$\mathbf{z} = \mathbf{r} - \mathbf{s}. \tag{7.186}$$

A jackknifed configuration is shown in Figure 7.53, while Figure 7.52 is showing an *unjackknifed* configuration.

Mathematically, we want to know if the truck-trailer will jackknife for a

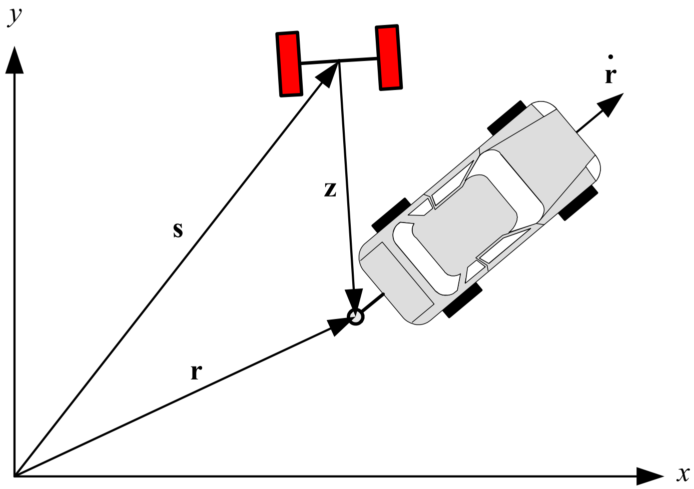

FIGURE 7.53. A jackknifed configuration of a car pulling a one-axle trailer.

given path of motion $\mathbf{r} = \mathbf{r}(t)$ and what conditions we must impose on $\mathbf{r}(t)$ to prevent jackknifing.

The velocity of the trailer can be expressed by

$$\dot{\mathbf{s}} = c\,(\mathbf{r} - \mathbf{s}) \tag{7.187}$$

where

$$c = \dot{\mathbf{r}} \cdot \mathbf{z} \tag{7.188}$$

and the unjackknifing condition is

$$c > 0. \tag{7.189}$$

Assume the twice continuously differentiable function $\mathbf{r}$ is the path of car motion. If $|\mathbf{z}| = 1$, and $\mathbf{r}$ has a radius of curvature $R(t) > 1$, and

$$\dot{\mathbf{r}}(0) \cdot \mathbf{z}(0) > 0 \tag{7.190}$$

then

$$\dot{\mathbf{r}}(t) \cdot \mathbf{z}(t) > 0 \tag{7.191}$$

for all $t > 0$.

Therefore, if the car is moving forward and the car-trailer combination is not originally jackknifed, then it will remain unjackknifed.

Proof. The normalized trailer length is 1 and is constant, therefore, $\mathbf{z}$ is a unit vector

$$\begin{aligned} |\mathbf{z}| &= |\mathbf{r} - \mathbf{s}| \\ &= 1 \end{aligned} \tag{7.192}$$

and

$$(\mathbf{r} - \mathbf{s}) \cdot (\mathbf{r} - \mathbf{s}) = 1. \tag{7.193}$$

The nonslip wheels of the trailer constrain the vector $\mathbf{s}$ such that its velocity vector $\dot{\mathbf{s}}$ must be directed along the trailer axis indicated by $\mathbf{z}$.

$$\begin{aligned} \dot{\mathbf{s}} &= c\,(\mathbf{r} - \mathbf{s}) \\ &= c\mathbf{z} \end{aligned} \tag{7.194}$$

Differentiating (7.193) yields

$$2\,(\dot{\mathbf{r}} - \dot{\mathbf{s}}) \cdot (\mathbf{r} - \mathbf{s}) = 0 \tag{7.195}$$

$$\dot{\mathbf{r}} \cdot (\mathbf{r} - \mathbf{s}) = \dot{\mathbf{s}} \cdot (\mathbf{r} - \mathbf{s}) \tag{7.196}$$

and therefore,

$$\dot{\mathbf{r}} \cdot (\mathbf{r} - \mathbf{s}) = c\,(\mathbf{r} - \mathbf{s}) \cdot (\mathbf{r} - \mathbf{s}) \tag{7.197}$$

$$\begin{aligned} c &= \dot{\mathbf{r}} \cdot (\mathbf{r} - \mathbf{s}) \\ &= \dot{\mathbf{r}} \cdot \mathbf{z}. \end{aligned} \tag{7.198}$$

Having c enables us to write Equation (7.194) as

$$\begin{aligned} \dot{\mathbf{s}} &= [\dot{\mathbf{r}} \cdot (\mathbf{r} - \mathbf{s})]\,(\mathbf{r} - \mathbf{s}) \\ &= (\dot{\mathbf{r}} \cdot \mathbf{z})\,\mathbf{z}. \end{aligned} \tag{7.199}$$

There are three situations

1. When $c > 0$, the velocity vector of the trailer $\dot{\mathbf{s}}$ is along the trailer axis $\mathbf{z}$. The trailer follows the car and the system is stable.

2. When $c = 0$, the velocity vector of the trailer $\dot{\mathbf{s}}$ is zero. In this case, the trailer spins about the center of its axle and the system is neutral-stable.

3. When $c < 0$, the velocity vector of the trailer $\dot{\mathbf{s}}$ is along the trailer axis $-\mathbf{z}$. The trailer does not follow the car and the system is unstable.

Using a Cartesian coordinate expression, we may show the car and trailer position vectors by

$$\mathbf{r} = \begin{bmatrix} x_c \\ y_c \end{bmatrix} \tag{7.200}$$

$$\mathbf{s} = \begin{bmatrix} x_t \\ y_t \end{bmatrix} \tag{7.201}$$

and therefore,

$$\begin{aligned}
\dot{\mathbf{s}} &= \begin{bmatrix} \dot{x}_t \\ \dot{y}_t \end{bmatrix} = [\dot{\mathbf{r}} \cdot (\mathbf{r} - \mathbf{s})] (\mathbf{r} - \mathbf{s}) \\
&= \begin{bmatrix} \dot{x}_c (x_c - x_t)^2 + (x_c - x_t)(y_c - y_t) \dot{y}_c \\ \dot{x}_c (x_c - x_t)(y_c - y_t) + (y_c - y_t)^2 \dot{y}_c \end{bmatrix} \qquad (7.202) \\
c &= (x_c - x_t) \dot{x}_c + (y_c - y_t) \dot{y}_c \\
&= \dot{x}_c x_c + \dot{y}_c y_c - (\dot{x}_c x_t + \dot{y}_c y_t). \qquad (7.203)
\end{aligned}$$

Let's define a function $f(t) = \dot{\mathbf{r}} \cdot \mathbf{z}$ and assume that conclusion (7.191) is wrong while assumption (7.190) is correct. Then there exists a time $t_1 > 0$ such that $f(t_1) = 0$ and $f'(t_1) \leq 0$. Using $|\mathbf{z}| = 1$ and $\dot{\mathbf{r}} \neq 0$, we have $\dot{\mathbf{r}}(t_1) \cdot \mathbf{z}(t_1) = 0$ and therefore, $\dot{\mathbf{r}}(t_1)$ is perpendicular to $\mathbf{z}(t_1)$. The derivative $f'(t)$ would be

$$\begin{aligned}
f'(t) &= \ddot{\mathbf{r}} \cdot \mathbf{z} + \dot{\mathbf{r}} \cdot \dot{\mathbf{z}} \\
&= \ddot{\mathbf{r}} \cdot \mathbf{z} + \dot{\mathbf{r}} \cdot (\dot{\mathbf{r}} - \dot{\mathbf{s}}) \\
&= \ddot{\mathbf{r}} \cdot \mathbf{z} + |\dot{\mathbf{r}}|^2 - \dot{\mathbf{r}} \cdot \dot{\mathbf{s}} \\
&= \ddot{\mathbf{r}} \cdot \mathbf{z} + |\dot{\mathbf{r}}|^2 - \dot{\mathbf{r}} \cdot ((\dot{\mathbf{r}} \cdot \mathbf{z}) \mathbf{z}) \\
&= \ddot{\mathbf{r}} \cdot \mathbf{z} + |\dot{\mathbf{r}}|^2 - (\dot{\mathbf{r}} \cdot \mathbf{z})^2 \\
&= \ddot{\mathbf{r}} \cdot \mathbf{z} + |\dot{\mathbf{r}}|^2 - f^2(t) \qquad (7.204)
\end{aligned}$$

and therefore,

$$f'(t_1) = \ddot{\mathbf{r}} \cdot \mathbf{z} + |\dot{\mathbf{r}}|^2 . \qquad (7.205)$$

The acceleration $\ddot{\mathbf{r}}$ in a normal-tangential coordinate frame $(\hat{e}_n, \hat{e}_t)$ is

$$\begin{aligned}
\ddot{\mathbf{r}} &= \frac{d\,|\dot{\mathbf{r}}|}{dt} \hat{e}_t + \kappa\, |\dot{\mathbf{r}}|^2\, \hat{e}_n \qquad (7.206) \\
\kappa &= \frac{1}{R} \qquad (7.207)
\end{aligned}$$

where $\hat{e}_n$ and $\hat{e}_t$ are the unit normal and tangential vectors. $\hat{e}_t$ is parallel to $\dot{\mathbf{r}}(t_1)$, and $\hat{e}_n$ is parallel to $\mathbf{z}(t_1)$. Hence,

$$\ddot{\mathbf{r}} \cdot \mathbf{z} = \pm\kappa(t_1)\ |\dot{\mathbf{r}}(t_1)|^2 \qquad (7.208)$$

and

$$\begin{aligned}
f'(t_1) &= |\dot{\mathbf{r}}(t_1)|^2 \pm \kappa(t_1)\ |\dot{\mathbf{r}}(t_1)|^2 \\
&= [1 \pm \kappa(t_1)]\ |\dot{\mathbf{r}}(t_1)|^2 . \qquad (7.209)
\end{aligned}$$

Because $\kappa(t_1) = 1/R(t) > 0$, we conclude that $f'(t_1) > 0$, and it is not possible to have $f'(t_1) \leq 0$. ■

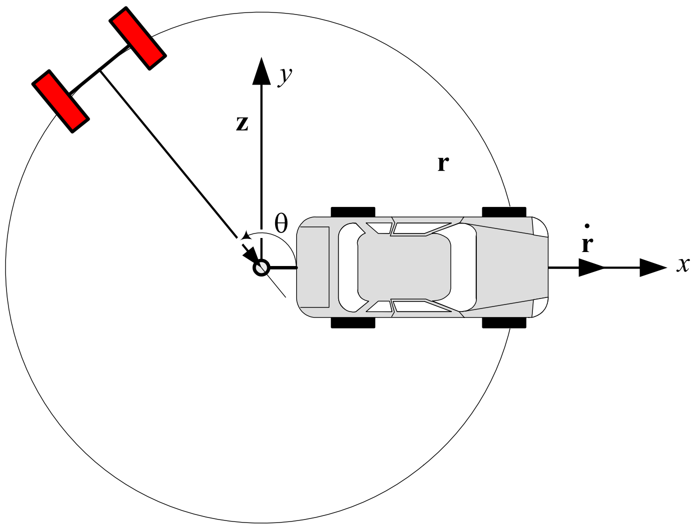

FIGURE 7.54. The initial position of a one-axle trailer pulled by a car moving forward in a straight line with a constant velocity is a circle about the hinge point.

Example 296 ★ *Straight motion of the car with constant velocity.*

Consider a car moving forward in a straight line with a constant velocity. We may use a normalization and set the speed of the car as 1 *moving in positive* x *direction starting from* $x = 0$. *Using a two-dimensional vector expression we have*

$$\mathbf{r} = \begin{bmatrix} x_c \\ y_c \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}. \tag{7.210}$$

Because of (7.192), we get

$$\begin{aligned} \mathbf{z}(0) &= \mathbf{r}(0) - \mathbf{s}(0) \\ &= -\mathbf{s}(0) \end{aligned} \tag{7.211}$$

and therefore, the initial position of the trailer must lie on a unit circle as shown in Figure 7.54.

Using two dimensional vectors, we may express $\mathbf{z}(0)$ *as a function of* θ

$$\begin{aligned} \mathbf{z}(0) &= -\mathbf{s}(0) \\ &= \begin{bmatrix} x_t(0) \\ y_t(0) \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \end{aligned} \tag{7.212}$$

and simplify Equation (7.202) as

$$\dot{\mathbf{s}} = \begin{bmatrix} \dot{x}_t \\ \dot{y}_t \end{bmatrix} = \begin{bmatrix} (t - x_t)^2 \\ -y_t\,(t - x_t) \end{bmatrix}. \tag{7.213}$$

Equation (7.213) is a set of two coupled first-order ordinary differential equations with the solution

$$\mathbf{s} = \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} t + \dfrac{e^{-2t} - C_1}{e^{-2t} + C_1} \\ \dfrac{C_2 e^{-t}}{e^{-2t} + C_1} \end{bmatrix}. \tag{7.214}$$

Applying the initial conditions (7.212) we find

$$C_1 = \frac{\cos\theta - 1}{\cos\theta + 1} \tag{7.215}$$

$$C_2 = \frac{2\sin\theta}{\cos\theta + 1}. \tag{7.216}$$

If $\theta \neq k\pi$, then the solution depends on time, and when time goes to infinity, the solution leads to the following limits asymptotically:

$$\begin{aligned} \lim_{t\to\infty} x_t &= t - 1 \\ \lim_{t\to\infty} y_t &= 0 \end{aligned} \tag{7.217}$$

When the car is moving with a constant velocity, this solution shows that the trailer will approach the position of straight forward moving, following the car.

We may also consider that the car is backing up. In this situation, the solution shows that, except for the unstable initial condition $\theta = \pi$, all solutions ultimately approach the jackknifed position.

If $\theta = 0$, then

$$\begin{aligned} C_1 &= 0 \\ C_2 &= 0 \\ x_t &= t + 1 \\ y_t &= 0 \end{aligned} \tag{7.218}$$

and the trailer moves in an unstable configuration. Any deviation from $\theta = 0$ ends up to change the situation and leads to the stable limiting solution (7.217).

If $\theta = \pi$, then

$$\begin{aligned} C_1 &= \infty \\ C_2 &= \infty \\ x_t &= t - 1 \\ y_t &= 0 \end{aligned} \tag{7.219}$$

and the trailer follows the car in an stable configuration. Any deviation from $\theta = 0$ will disappear after a while.

Example 297 ★ *Straight car motion with different initial θ.*

Consider a car moving on an x-axis with constant speed. The car is pulling a trailer, which is initially at θ such as shown in Figure 7.52. Using a normalized length, we assume the distance between the center of the trailer axle and the hinge is the length of trailer, and is equal to 1.

If we show the absolute position of the car at hinge by $\mathbf{r} = \begin{bmatrix} x_c & y_c \end{bmatrix}^T$ and the absolute position of the trailer by $\mathbf{s} = \begin{bmatrix} x_t & y_t \end{bmatrix}^T$ then the position of the trailer is a function of the car's motion. When the position of the car is given by a time-dependent vector function

$$\mathbf{r} = \begin{bmatrix} x_c(t) \\ y_c(t) \end{bmatrix} \tag{7.220}$$

the trailer position can be found by solving two coupled differential equation.

$$\dot{x}_t = (x_c - x_t)^2 \dot{x}_c + (x_c - x_t)(y_c - y_t)\dot{y}_c \tag{7.221}$$

$$\dot{y}_t = (x_c - x_t)(y_c - y_t)\dot{x}_c + (y_c - y_t)^2 \dot{y}_c \tag{7.222}$$

For a constantly uniform car motion $\mathbf{r} = \begin{bmatrix} t & 0 \end{bmatrix}^T$, Equations (7.221) and (7.222) reduce to

$$\dot{x}_t = (t - x_t)^2 \tag{7.223}$$

$$\dot{y}_t = -y_t (t - x_t) \tag{7.224}$$

The first equation (7.223) is independent of the second equation (7.224) and can be solved independently.

$$\begin{aligned} x_t &= \frac{C_1 e^{2t}(t-1) - t - 1}{C_1 e^{2t} - 1} \\ &= t + \frac{e^{-2t} - C_1}{e^{-2t} + C_1} \end{aligned} \tag{7.225}$$

Substituting Equation (7.225) in (7.224) generates the following differential equation:

$$\dot{y}_t = \frac{e^{-2t} - C_1}{e^{-2t} + C_1} y_t \tag{7.226}$$

with the solution

$$y_t = \frac{C_2 e^{-t}}{e^{-2t} + C_1}. \tag{7.227}$$

When the trailer starts from $\mathbf{s} = \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix}^T$, the constants of integral would be equal to Equations (7.215) and (7.216) and therefore,

$$x_t = t + \frac{e^{-2t}(\cos\theta + 1) - \cos\theta + 1}{e^{-2t}(\cos\theta + 1) + \cos\theta - 1} \tag{7.228}$$

$$y_t = \frac{2e^{-t}\sin\theta}{e^{-2t}(\cos\theta + 1) + \cos\theta - 1}. \tag{7.229}$$

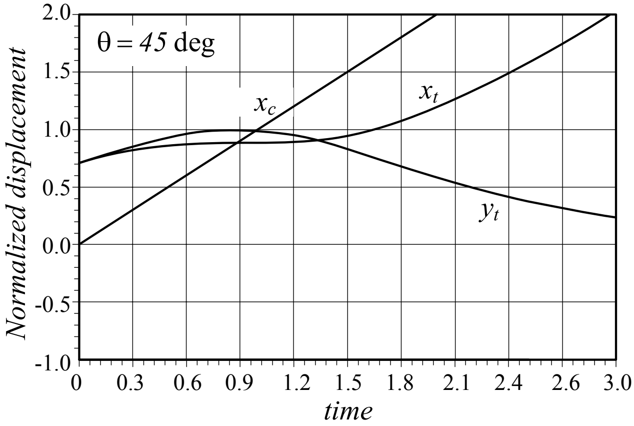

FIGURE 7.55. Trailer kinematics for a $\theta = 45\,\text{deg}$ starting position.

Figures 7.55, 7.56, and 7.57 illustrate the behavior of the trailer starting from $\theta = 45\,\text{deg}$*,* $\theta = 90\,\text{deg}$*,* $\theta = 135\,\text{deg}$*.*

Example 298 ★ *Circular motion of the car with constant velocity.*

Consider a car pulling a trailer such as Figure 7.52 shows. The car is traveling along a circle of radius $R > 1$*, based on a normalized length in which the length of the trailer is* 1*. In a circular motion with a normalized angular velocity* $\omega = 1$ *and period* $T = 2\pi$*, the position of the car is given by the following time-dependent vector function:*

$$\begin{aligned} \mathbf{r} &= \begin{bmatrix} x_c(t) \\ y_c(t) \end{bmatrix} \\ &= \begin{bmatrix} R\cos(t) \\ R\sin(t) \end{bmatrix}. \end{aligned} \tag{7.230}$$

The initial position of the trailer must lie on a unit circle with a center at $\mathbf{r}(0) = \begin{bmatrix} x_c(0) & y_c(0) \end{bmatrix}^T$*.*

$$\begin{aligned} \mathbf{s}(0) &= \begin{bmatrix} x_t(0) \\ y_t(0) \end{bmatrix} \\ &= \begin{bmatrix} x_c(0) \\ y_c(0) \end{bmatrix} + \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \end{aligned} \tag{7.231}$$

The car-trailer combination approaches a steady-state configuration as shown in Figure 7.58.

Substituting

$$\dot{\mathbf{r}} = \begin{bmatrix} -R\sin(t) \\ R\cos(t) \end{bmatrix}. \tag{7.232}$$

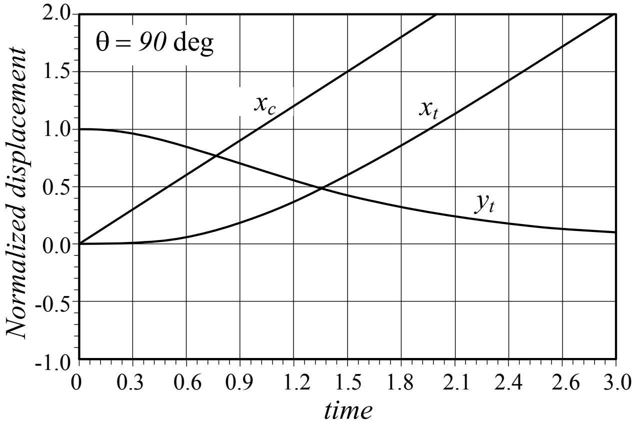

FIGURE 7.56. Trailer kinematics for a $\theta = 90\,\deg$ starting position.

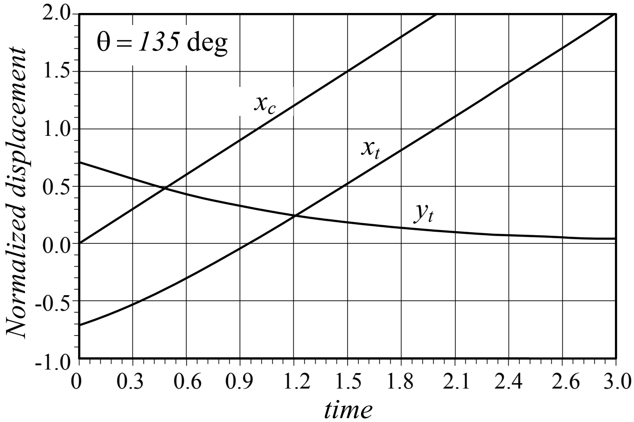

FIGURE 7.57. Trailer kinematics for a $\theta = 135\,\deg$ starting position.

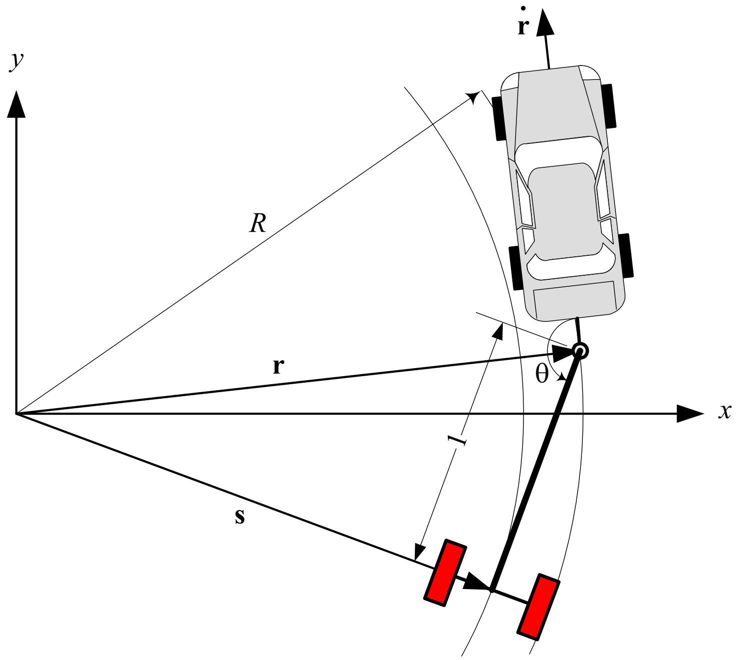

FIGURE 7.58. Steady state configuration of a car-trailer combination.

and the initial conditions (7.231) in (7.202) will generate two differential equations for trailer position.

$$\dot{x}_t = R\,(R\cos t - x_t)\,(x_t \sin t - y_t \cos t) \tag{7.233}$$

$$\dot{y}_t = R\,(R\sin t - y_t)\,(x_t \sin t - y_t \cos t) \tag{7.234}$$

Assuming $\mathbf{r}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ *the steady-state solutions of these equations are*

$$x_t = c\cos(t-\alpha) \tag{7.235}$$

$$y_t = c\sin(t-\alpha) \tag{7.236}$$

where c *is the trailer's radius of rotation, and* α *is the angular position of the trailer behind the car.*

$$c = \sqrt{R^2 - 1} \tag{7.237}$$

$$\sin\alpha = \frac{1}{R} \tag{7.238}$$

$$\cos\alpha = \frac{c}{R} \tag{7.239}$$

We may check the solution by employing two new variables, u and v, such that

$$u = x_t \sin t - y_t \cos t \tag{7.240}$$
$$v = x_t \cos t + y_t \sin t \tag{7.241}$$

and

$$\dot{u} = v. \tag{7.242}$$

Using the new variables we find

$$x_t = u \sin t + v \cos t \tag{7.243}$$
$$y_t = -u \cos t + v \sin t \tag{7.244}$$

$$\dot{x}_t = Ru(R \cos t - u \sin t - v \cos t) \tag{7.245}$$
$$\dot{y}_t = Ru(R \sin t + u \cos(t) - v \sin t). \tag{7.246}$$

Direct differentiating from (7.240), (7.241), (7.243), and (7.244) shows that

$$\dot{u} = x_t \cos t + y_t \sin t + \dot{x}_t (\sin t) - \dot{y}_t (\cos t) \tag{7.247}$$
$$\dot{v} = -x_t \sin t + y_t \cos t + \dot{x}_t \cos t + \dot{y}_t \sin t \tag{7.248}$$

$$\dot{x}_t = \dot{u} \sin t + \dot{v} \cos t + u \cos t - v \sin t \tag{7.249}$$
$$\dot{y}_t = \dot{u} \cos t - \dot{v} \sin t - u \sin t - v \cos t \tag{7.250}$$

and therefore, the problem can be expressed in a new set of equations.

$$\dot{u} = v - Ru^2 \tag{7.251}$$
$$\dot{v} = u \left(R^2 - Rv - 1\right) \tag{7.252}$$

At the steady-state condition the time differentials must be zero, and therefore, the steady-state solutions would be the answers to the following algebraic equations:

$$v - Ru^2 = 0 \tag{7.253}$$
$$u \left(R^2 - Rv - 1\right) = 0 \tag{7.254}$$

There are three sets of solutions.

$$\{u = 0, v = 0\} \tag{7.255}$$
$$\{u = \frac{c}{R}, v = \frac{c^2}{R}\} \tag{7.256}$$
$$\{u = -\frac{c}{R}, v = \frac{c^2}{R}\} \tag{7.257}$$

The first solution is associated with $\mathbf{s} = 0$*,*

$$x_t = 0 \tag{7.258}$$
$$y_t = 0 \tag{7.259}$$

which shows that the center of the trailer's axle remains at the origin and the car is turning on a circle $R = 1$*. This is a stable motion.*

The second solution is associated with

$$x_t = \frac{c}{R}\sin t + \frac{c^2}{R}\cos t \tag{7.260}$$
$$y_t = -\frac{c}{R}\cos t + \frac{c^2}{R}\sin t \tag{7.261}$$

which are equivalent to (7.235) and (7.236).

To examine the stability of the second solution, we may substitute a perturbed solution

$$u = \frac{c}{R} + p \tag{7.262}$$
$$v = \frac{c^2}{R} + q \tag{7.263}$$

in the linearized equations of motion (7.251) and (7.252) at the second set of solutions

$$\dot{u} = v - 2cu \tag{7.264}$$
$$\dot{v} = -cv \tag{7.265}$$

to get two equations for the perturbed functions p *and* q*.*

$$\dot{p} = q - 2cp \tag{7.266}$$
$$\dot{q} = cq \tag{7.267}$$

The set of linear perturbed equations can be set in a matrix form

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} -2c & 1 \\ 0 & -c \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}. \tag{7.268}$$

The stability of Equation (7.268) is determined by the eigenvalues λ_i *of the coefficient matrix, which are*

$$\lambda_1 = -c$$
$$\lambda_2 = -2c.$$

Because both eigenvalues λ_1 *and* λ_2 *are negative, the solution of the perturbed equations symptomatically goes to zero. Therefore, the second set of* *solutions (7.256) is stable and it absorbs any near path that starts close to it.*

The third solution is associated with

$$x_t = -\frac{c}{R}\sin t + \frac{c^2}{R}\cos t \tag{7.269}$$

$$y_t = \frac{c}{R}\cos t + \frac{c^2}{R}\sin t. \tag{7.270}$$

The linearized equations of motion at the third set of solutions (7.257) are

$$\dot{u} = v + 2cu \tag{7.271}$$

$$\dot{v} = cv. \tag{7.272}$$

The perturbed equations would then be

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 2c & 1 \\ 0 & c \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \tag{7.273}$$

with two positive eigenvalues

$$\lambda_1 = c$$
$$\lambda_2 = 2c$$

Positive eigenvalues show that the solution of the perturbed equations diverges and goes to infinity. Therefore, the third set of solutions (7.257) is unstable and repels any near path that starts close to it.

7.8 Summary

Steering is required to guide a vehicle in a desired direction. When a vehicle turns, the wheels closer to the center of rotation are called the inner wheels, and the wheels further from the center of rotation are called the outer wheels. If the speed of a vehicle is very slow, there is a kinematic condition between the inner and outer steerable wheels, called the Ackerman condition.

Street cars are four-wheel vehicles and usually have front-wheel-steering. The kinematic condition between the inner and outer steered wheels is

$$\cot\delta_o - \cot\delta_i = \frac{w}{l} \tag{7.274}$$

where δ_i is the steer angle of the inner wheel, δ_o is the steer angle of the outer wheel, w is the track, and l is the wheelbase of the vehicle. Track w and wheel base l are considered the kinematic width and length of the vehicle.

The mass center of a steered vehicle will turn on a circle with radius R,

$$R = \sqrt{a_2^2 + l^2 \cot^2 \delta} \tag{7.275}$$

where δ is the cot-average of the inner and outer steer angles.

$$\cot \delta = \frac{\cot \delta_o + \cot \delta_i}{2}. \tag{7.276}$$

The angle δ is the equivalent steer angle of a bicycle having the same wheelbase l and radius of rotation R.

7.9 Key Symbols

$4WS$	four-wheel-steering
a, b, c, d	lengths of the links of a four-bar linkage
a_i	distance of the axle number i from the mass center
A, B, C	input angle parameters of a four-bar linkage
AWS	all-wheel-steering
b_1	distance of the hinge point from rear axle
b_2	distance of trailer axle from the hinge point
c	stability index of a trailer motion
c	trailer's radius of rotation
c_1	longitudinal distance of turn center and front axle of a $4WS$ car
c_2	longitudinal distance of turn center and rear axle of a $4WS$ car
C	mass center
$C_1, C_2, \cdots$	constants of integration
d	arm length in trapezoidal steering mechanism
e	error
e	length of the offset arm
FWS	front-wheel-steering
g	overhang distance
J	link parameters of a four-bar linkage
l	wheelbase
n	number of increments
O	center of rotation in a turn
r	yaw velocity of a turning vehicle
$\mathbf{r}$	position vector of a car at the hinge
R	radius of rotation at mass center
R_1	radius of rotation at the center of the rear axle for FWS
R_1	horizontal distance of O and the center of axles
R_t	radius of rotation at the center of the trailer axle
R_w	radius of the rear wheel
RWS	rear wheel steering
$\mathbf{s}$	position vector of a trailer at the axle center
t	time
u_R	steering rack translation
$v \equiv \dot{x},\ \mathbf{v}$	vehicle velocity
v_{ri}	speed of the inner rear wheel
v_{ro}	speed of the outer rear wheel
w	track
w_f	front track
w_r	rear track
$x, y, z,\ \mathbf{x}$	displacement
$\mathbf{z} = \mathbf{r} - \mathbf{s}$	position vector of a trailer relative to the car

β	arm angle in trapezoidal steering mechanism
δ	cot-average of the inner and outer steer angles
$\delta_1 = \delta_{fl}$	front left wheel steer angle
$\delta_2 = \delta_{fr}$	front right wheel steer angle
δ_{Ac}	steer angle based on Ackerman condition
δ_{fl}	front left wheel steer angle
δ_{fr}	front right wheel steer angle
δ_i	inner wheel
δ_{rl}	rear left wheel steer angle
δ_{rr}	rear right wheel steer angle
δ_o	outer wheel
δ_S	steer command
$\Delta = \delta_2 - \delta_{Ac}$	steer angle difference
θ	angle between trailer and vehicle longitudinal axes
ω	angular velocity
$\omega_i = \omega_{ri}$	angular velocity of the rear inner wheel
$\omega_o = \omega_{ro}$	angular velocity of the rear outer wheel

Exercises

1. Bicycle model and radius of rotation.

   Mercedes-Benz $GL450^{TM}$ has the following dimensions.

$$
\begin{aligned}
l &= 121.1\,\mathrm{in} \\
w_f &= 65.0\,\mathrm{in} \\
w_r &= 65.1\,\mathrm{in} \\
R &= 39.7\,\mathrm{ft}
\end{aligned}
$$

   Assume $a_1 = a_2$ and use an average track to determine the maximum steer angle δ for a bicycle model of the car.

2. Radius of rotation.

   Consider a two-axle truck that is offered in different wheelbases.

$$
\begin{aligned}
l &= 109\,\mathrm{in} \\
l &= 132.5\,\mathrm{in} \\
l &= 150.0\,\mathrm{in} \\
l &= 176.0\,\mathrm{in}
\end{aligned}
$$

   If the front track of the vehicles is

$$w = 70\,\mathrm{in}$$

   and $a_1 = a_2$, calculate the radius of rotations if $\delta = 30\,\mathrm{deg}$.

3. Required space.

   Consider a two-axle vehicle with the following dimensions.

$$
\begin{aligned}
l &= 4\,\mathrm{m} \\
w &= 1.3\,\mathrm{m} \\
g &= 1.2\,\mathrm{m}
\end{aligned}
$$

   Determine R_{min}, R_{Max}, and ΔR for $\delta = 30\,\mathrm{deg}$.

4. Rear wheel steering lift truck.

   A battery powered lift truck has the following dimensions.

$$
\begin{aligned}
l &= 55\,\mathrm{in} \\
w &= 30\,\mathrm{in}
\end{aligned}
$$

   Calculate the radius of rotations if $\delta = 55\,\mathrm{deg}$ for $a_1 = a_2$.

5. Wheel angular velocity.

   Consider a two-axle vehicle with the following dimensions.

$$\begin{aligned} l &= 2.7\,\mathrm{m} \\ w &= 1.36\,\mathrm{m} \end{aligned}$$

   What is the angular velocity ratio of ω_o/ω_i?

6. A three-axle vehicle.

   A three-axle vehicle is shown in Figure 7.17. Find the relationship between δ_2 and δ_3, and also between δ_1 and δ_6.

7. A three-axle truck.

   Consider a three-axle truck that has only one steerable axle in front. The dimensions of the truck are

$$\begin{aligned} a_1 &= 5300\,\mathrm{mm} \\ a_2 &= 300\,\mathrm{mm} \\ a_3 &= 1500\,\mathrm{mm} \\ w &= 1800\,\mathrm{mm}. \end{aligned}$$

   Determine maximum steer angles of the front wheels if the truck is supposed to be able to turn with $R = 11\,\mathrm{m}$.

8. A vehicle with a one-axle trailer.

   Determine the angle between the trailer and vehicle with the following dimensions.

$$\begin{aligned} a_1 &= 1000\,\mathrm{mm} \\ a_2 &= 1300\,\mathrm{mm} \\ w_v &= 1500\,\mathrm{mm} \\ b_1 &= 1200\,\mathrm{mm} \\ b_2 &= 1800\,\mathrm{mm} \\ w_t &= 1100\,\mathrm{mm} \\ g &= 800\,\mathrm{mm} \\ \delta_i &= 12\,\mathrm{deg}\,. \end{aligned}$$

   What is the rotation radius of the trailer R_t, and the vehicle R? Determine minimum radius $R_{\min}$, maximum R_{Max}, and difference radius ΔR?

9. ★ Turning radius of a $4WS$ vehicle.

   Consider a FWS vehicle with the following dimensions.

$$\begin{aligned} l &= 2300\,\text{mm} \\ w_f &= 1457\,\text{mm} \\ w_r &= 1607\,\text{mm} \\ \frac{a_1}{a_2} &= \frac{38}{62} \end{aligned}$$

   Determine the turning radius of the vehicle for $\delta_{fl} = 5\,\text{deg}$.

   What should be the steer angles of the front and rear wheels to decrease 10% of the turning radius, if we make the vehicle $4WS$?

10. ★ Coordinates of the turning center.

    Determine the coordinates of the turning center for the vehicle in Exercise 9 if $\delta_{fl} = 5\,\text{deg}$ and $c_1 = 1300\,\text{mm}$.

11. ★ Different front and rear tracks.

    Lotus 2-Eleven^{TM} is a RWD sportscar with the following specifications.

$$\begin{aligned} l &= 2300\,\text{mm} \\ w_f &= 1457\,\text{mm} \\ w_r &= 1607\,\text{mm} \\ Front\ tire &= 195/50R16 \\ Rear\ tire &= 225/45R17 \\ \frac{F_{z_1}}{F_{z_2}} &= \frac{38}{62} \end{aligned}$$

    Determine the angular velocity ratio of ω_o/ω_i, R, δ_i, and δ_o for $\delta = 5\,\text{deg}$.

12. ★ Coordinates of turning center.

    Determine the coordinates of turning center of a $4WS$ vehicle in terms of outer steer angles δ_{of} and δ_{or}.

13. ★ Turning radius.

    Determine the turning radius of a $4WS$ vehicle in terms of δ_r.

$$\cot\delta_r = \frac{1}{2}\left(\cot\delta_{ir} + \cot\delta_{or}\right)$$

14. ★ A three-axle car.

Consider a three-axle off-road pick-up car. Assume

$$\begin{aligned} a_1 &= 1100\,\text{mm} \\ a_2 &= 1240\,\text{mm} \\ a_3 &= 1500\,\text{mm} \\ w &= 1457\,\text{mm} \end{aligned}$$

and determine δ_o, R_1, R_f, and R if $\delta_i = 10\,\text{deg}$.

15. ★ Steering mechanism optimization.

Find the optimum length x for the multi-link steering mechanism shown in Figure 7.59 to operate as close as possible to the kinematic steering condition. The vehicle has a track $w = 2.64\,\text{m}$ and a wheelbase $l = 3.84\,\text{m}$, and must be able to turn the front wheels within a working range equal to $-22\,\text{deg} \le \delta \le 22\,\text{deg}$.

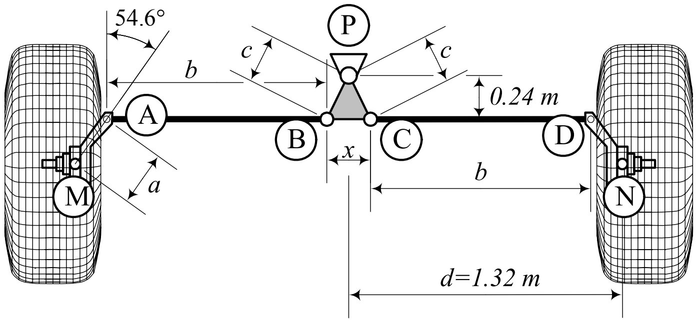

FIGURE 7.59. A multi-link steering mechanism tha must be optimized by varying x.