Exact results in 2D percolation

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Abstract

This article presents and explains some exact results in 2D percolation. Self-duality, self-matching, and star–triangle transformations are used in deriving the exact percolation thresholds of some lattices. The exact probability of having a percolating cluster in finite systems is computationally intense to calculate, but they are useful in estimating the percolation threshold. Conformal invariance, though only rigorously proven in few cases, is extensively used to derive exact critical exponents.

1 Introduction

In general, a site/bond percolation model is described by the following [21, p. 4]: each site/bond (vertex/edge) of a lattice (infinte graph) is occupied with probability p independently; what are the properties of the resulting clusters, consisting of connected occupied sites/bonds, and is there an infinite cluster (called the percolating cluster)?

Percolation theory arises from different areas of studies. For example, in condensed matter physics, the critical behavior of metal–insulator transitions in disordered systems can be modeled by percolation theory [16]. Other applications include the study of the fragmentation of biological virus shells [3], the spread of plagues [7], etc. The list of applications is extensive.

Percolation is interesting in that it is (in many aspects) the simplest model that is not exactly solved and has a critical phase transition [21, p. 1]. People have tried different ways to study percolation. Two popular approximation methods are numerical calculations using Monte Carlo simulations [13, p. 58] and using finite-size scaling and the renormalization group [4, p. 174].

Although many percolation problems are not exactly solved, some of the 2-dimensional ones are. In this article, we present some exact results in 2D percolation. The purpose of this article is to explain some exact results in 2D percolation in a way that is easy to understand for non-mathematicians.

2 Thresholds

By Kolmogorov's zero–one law, the probability of having a percolating cluster in a percolation model is either zero or one, so there must exist a critical probability p_{c} such that there almost surely exist a percolating cluster if $p > p_{\mathrm{c}}$ and that it is almost impossible to have a percolating cluster if $p < p_{\mathrm{c}}$ [24]. It is then interesting to find the exact value of p_{c}.

Kesten [10] proved some theorems that are useful in finding the exact value of p_{c} for some 2D percolation models:

- The bond percolation model on a graph G is equivalent to the site percolation model on its covering graph $\tilde{G}$ [10, p. 42].
- If periodic graphs G and G^* are a matching pair, then their site percolation thresholds sum to one [10, p. 52] [22].

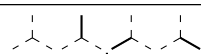
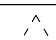
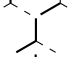
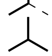
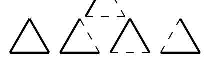
The definition of the covering graph $\tilde{G}$ of any graph G is as follows: the vertices of $\tilde{G}$ are the edges of G, two of which are connected with an edge iff they share a vertex as edges in G [10, p. 36]. The definition of a matching pair is as follows: two graphs G and G^* are a mathcing pair, if there is a simple planar graph M and a subset F of M's faces such that G is obtained from M by close-packing all faces in F, and G^* is obtained from M by close-packing all faces not in F (close-packing a face means connecting every pair of non-adjacent vertices on that face with edges) [10, p. 18]. With these two theorems, one can prove that the bond percolation thresholds of a periodic planar simple graph G and its dual graph G_{d} sum to one because $\tilde{G}$ and $\tilde{G}_{\mathrm{d}}$ are a matching pair [10, p. 38].

These theorems then provide ways of finding the exact value of p_{c} for various cases:

- A graph's bond percolation threshold is its covering graph's site percolation threshold.
- If two periodic graphs are a matching pair, one minus the site percolation threshold of one of them is the site percolation threshold of the other. Specially, if a periodic graph is self-matching, then its site percolation threshold is 1/2.
- If two periodic graphs are dual to each other, one minus the bond percolation threshold of one of them is the bond percolation threshold of the other. Specially, if a periodic graph is self-dual, then its bond percolation threshold is 1/2.

Because the bond percolation threshold of a self-dual periodic graph is 1/2, we immediately get the famous result that $p_{\mathrm{c}} = 1/2$ for bond percolation on the square lattice [11], which was long being conjectured before it was rigorously proven. Also, because the site percolation threshold of a self-matching periodic graph is 1/2, we immediately get the result that $p_{\mathrm{c}} = 1/2$ for site percolation on the triangular lattice.

Table 1: The star–triangle transformation [22]. In the equations, p, p' are the bond percolation thresholds of the honeycomb and triangular lattices, respectively. One matches connectivity cases for the star (honeycomb) and the triangle (triangular) to derive the relation between p, p' for the two cases be equivalent. By duality, $p + p' = 1$, and then this set of equations reduces to $1 - 3p^2 + p^3 = 0$. This gives the famous result $p' = 2\sin(\pi/18)$.

Case	Honeycomb	Triangular	Equation
no connection			$(1-p)^3 + 3p\,(1-p)^2 = (1-p')^3$
connect two			$p^2\,(1-p) = p'\,(1-p')^2$
connect all			$p^3 = p'^3 + 3p'^2\,(1-p')$

Combining these statements and the star–triangle transformation, one can get exact results for bond percolation on the triangular and honeycomb lattices [22], which are dual to each other. By translating a triangle, one can tile the plane by forming a triangular lattice; if we transform the triangles into stars, the lattice becomes a honeycomb lattice. To make the percolation problem before and after the transformation equivalent, one needs to match the different connectivity cases of the star and the triangle. The cases are listed in Table 1. By solving those equations under the condition imposed by the duality of the two lattices, one get the famous result of $p_c = 2\sin(\pi/18)$ for bond percolation on the triangular lattice and $p_c = 1 - 2\sin(\pi/18)$ for bond percolation on the honeycomb lattice. This approach is used to find the exact value of p_c for other lattices [23, 29].

3 Finite systems

In principle, for any finite graph with N sites, the site percolation problem on that graph can be fully solved by enumerating all possible 2^N states and sum up terms of the form $p^k (1-p)^{N-k}$ to get the probablity $\Pi(p)$ that there is a percolating cluster as a degree-N polynomial of p.

For a square square lattice with linear size L, we have $N = L^2$, so the number of enumerations is 2^{L^2}, which quickly becomes infeasible as L grows. An algorithm based on dynamical programming is proposed and used to get the exact expression of Π for up to $L = 24$ [17]. Results for up to $L = 16$ with periodic boundary condition in one direction (cylinder) and up to $L = 12$ with periodic boundary condition in both directions (torus) are also obtained [1].

The reason why we are interested in the exact expression of Π for a finite system is that it can be used to give an estimate of p_c, and there are different methods to do that [26]:

- The estimate p^* is defined as the solution to $\Pi(p^*) = \Pi^* \in (0,1)$, where $\Pi^* := \lim_{L\to\infty} \Pi(p_c)$ [28] (equal to $1/2$ for square lattice, whether site [18] or bond [2] percolation), or simply any other value in $(0,1)$ but with a slower convergence [21, p. 71].
- The estimate p^* is defined as the inflection point of Π [18].
- The estimate p^* is defined as the solution to $\Pi_{L_1}(p^*) = \Pi_{L_2}(p^*)$ for different L_1, L_2 (e.g., L and $L-1$, or L and $L/2$) [18].

Mixture of those methods can also be used because sometimes their linear combination converges faster [17].

Although, in general, those methods work in any dimension, they are especially useful in 2D. In fact, those methods perform better than the Monte Carlo methods in estimating p_c for some 2D cases [17][1]. For example, the best estimate of p_c for site percolation on the square lattice based on the said method is 0.592 746 050 792 10(2) [9] while the best estimate based on Monte Carlo method is 0.592 746 21(13) [26].

4 Conformal invariance

At the critical point, the percolation system is scale invariant [21, p. 71]. Moreover, most QFTs obtained from the scaling limit of 2D lattice models satisfy the conditions to enhance the scale invariance to conformal invariance, and conformal invariance is especially useful in 2D problems because conformal transformations corresponds to complex analytic functions [19]. Therefore, it is

[1] The referenced paper [17] also says R. M. Ziff said “Monte Carlo is dead” for 2D percolation for this reason in his talk at *34th M. Smoluchowski Symposium on Statistical Physics*. I cannot find the original source of this quote, though.

tempting to conjecture that 2D percolation is conformally invariant in the scaling limit and that one can study percolation at the critical point as a conformal field theory [14].

Cardy [5] explicitly employed this idea and found some exact results for 2D percolation. One main result is his formula for the so-called crossing probability at critical point, which is presumably universal, i.e., valid for any lattice type and whether it is site or bond percolation. The crossing probability P_s is the probability that there is a path connecting two segments on the boundary of the system [19]. By Riemann mapping theorem, the (compact) domain of the system can be conformally mapped to the upper half plane, and the boundary is mapped onto the real axis. Therefore, the two segments can then be written as two intervals (x_1, x_2) and (x_3, x_4), where the four real numbers are in ascending order. Cardy's formula is then

$$P_\mathrm{s}(x_1, x_2, x_3, x_4) = \frac{\Gamma(2/3)}{\Gamma(4/3)\,\Gamma(1/3)} \eta^{1/3} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{4}{3}; \eta\right), \quad \eta \coloneqq \frac{(x_1 - x_2)\,(x_3 - x_4)}{(x_1 - x_3)\,(x_2 - x_4)},$$

where η is called the cross ratio, of which a function must P_s be due to the conformal invariance. Cardy's ideas and results are used and generalized to obtain further exact results in 2D percolation, such as a universal and exact formula for the average number of clusters on a cluster of N sites [12] and the universal and exact formula for the distribution of the area of clusters [6].

Despite the extensive use of Cardy's ideas and the presumable universality, there is a rigorous proof of Cardy's formula only for site percolation on the triangular lattice by Smirnov [20]. He also proved that the scaling limit of site percolation on the triangular lattice at critical point exists and converges to SLE_6 (stochastic Löwner evolution with parameter $\kappa = 6$) (it was conjectured that this is true for any 2D percolation model [15]). This result can then be used to prove the existence and derive the exact values of several well-known universal critical exponents [19], listed in Table 2.

Table 2: Some exact values of critical exponents in 2D percolation [19]. They are universal but only rigorously proven for site percolation on the triangular lattice as a corollary of the convergence to SLE_6 in the scaling limit. The list goes on by relating exponents with scaling relations.

Quantity	Proportional to	Exact value of exponent
Percolation strength P	$(p - p_\mathrm{c})^\beta$	$\beta = 5/36$
Mean cluster size χ	$\lvert p - p_\mathrm{c}\rvert^{-\gamma}$	$\gamma = 43/18$
Cluster moments ratio	$\lvert p - p_\mathrm{c}\rvert^{-\Delta}$	$\Delta = 91/36$
Correlation length ξ	$\lvert p - p_\mathrm{c}\rvert^{-\nu}$	$\nu = 4/3$
Mean cluster number per site	$\lvert p - p_\mathrm{c}\rvert^{2-\alpha}$	$\alpha = 2 - \gamma - 2\beta = -2/3$
Percolating cluster size	L^D	$D = d - \beta/\nu = 91/48$
...		

There are still critical exponents in 2D percolation problems that are not derived (except conjectures from numerical evidence) despite the success in deriving other critical exponents using CFT. For example, the exact value of the shortest-path fractal dimension D_min (defined as in the power law $\langle l\rangle \propto r^{D_\mathrm{min}}$, where l is the shortest-path length between two points separated by distance r [10, p. 97]) is unknown though numerical results strongly support $D_\mathrm{min} = 217/192$ [25], a corollary of a conjecture [8]. However, a closely related exponent g_1 (defined as in $\rho \propto r^{g_1}$, where ρ is the probability that two clusters growing from two seed points separated by distance r coalesce after they grow for a fixed amount of time) is exactly known to be $g_1 = 25/24$, derived from Cardy's results [27].

5 Conclusions

In this article, we have presented and explained some exact results in 2D percolation. Most of the methods used to derive exact results in percolation problems are specific to 2D, and this explains why there are much less exact results for $d \geq 3$. For example, we can define dual graphs and matching pairs of graphs only with planar graphs, which are the basis of various exact results in the percolation thresholds of 2D lattices. Conformal invariance is also specific and especially useful to 2D problems. They are used to derive the exact values of several critical exponents and to find other universal properties of 2D percolation systems. Though, there are still some critical exponents that are not derived, except some conjectures. The method of estimating the percolation threshold using the exact probability of having a percolating cluster in finite systems is especially useful in 2D, where it sometimes performs better than the Monte Carlo method.

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