Spinor fields in curved spacetime

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This article presents a comprehensive account of spinor fields in curved spacetime, motivated by their role in unifying general relativity and quantum field theory. Beginning with the necessity of a spin structure on the spacetime manifold, it develops the formal machinery needed to define spinor bundles, covariant derivatives, and spin connections. The transformation properties of spinors are analyzed via the representation theory of the spin group, and their compatibility with Lorentzian geometry is established through the tetrad formalism. Wave equations for spinor fields are then derived, including both massive and massless cases, and their generalization to curved backgrounds is discussed along with limitations. The article also covers the Petrov classification using spinor techniques and concludes with an explicit analysis of the superradiance problem for Dirac fields in the Kerr black hole background, illustrating the absence of fermionic superradiance and connecting the result to the mechanism of Hawking radiation.

I. Introduction

In general relativity, the dynamics of matter and energy are governed by the geometry of spacetime, described mathematically by a pseudo-Riemannian manifold [1, p. 73]. Incorporating spinor fields into this framework introduces additional structure beyond the usual tensorial quantities. Spinors are essential for describing half-integer spin particles, such as electrons, and play a foundational role in the formulation of quantum field theory. However, their transformation properties under Lorentz transformations do not extend naturally to general coordinate transformations. To define spinor fields in curved spacetime, one must work with local frames and the associated spin structure of the manifold.

A curved spacetime admits spinor fields only if it possesses a spin structure, which requires the second Stiefel–Whitney class of the manifold to vanish [2]. Given such a structure, one introduces a tetrad (vierbein) field that locally relates the curved spacetime metric to the flat Minkowski metric. This frame field allows spinors to be defined at each point by anchoring them to the tangent space, where the usual representation theory of the Lorentz group applies. Covariant derivatives of spinor fields are then constructed using the spin connection, derived from the Levi-Civita connection and the tetrads [3, p. 52].

Spinor fields in curved spacetime appear prominently in efforts to unify general relativity with quantum field theory, especially in semiclassical [4] and quantum gravity [5] contexts. The Dirac equation, when coupled to a curved background, serves as the simplest example of a spinor field theory in this setting. Its solutions reveal insights into the behavior of fermionic matter in the presence of strong gravitational fields, such as near black holes or in the early universe.

Spinor fields in curved spacetime also have applications where fermions are not even relevant. In Section VII, we will see how spinor fields can be used to classify the Weyl tensor of a spacetime, where there may not be physical spinors involved.

In Section VIII, we will actually do some calculations for a problem involving spinor fields in curved spacetime, the superradiance problem of a Kerr black hole, and then see some implications about Hawking radiation from this simple example.

This article outlines the formalism required to define spinor fields in curved spacetime, including the mathematical prerequisites, the construction of spin connections, and the covariant Dirac equation, providing the groundwork for further study in both classical and quantum contexts.

II. Motivation

The space of quantum states is the projective unit sphere $\mathrm{PS}_{\mathcal{H}}(1)$ of a Hilbert space $\mathcal{H}$. The exact definition of a projective sphere with radius r on a normed space V over $\mathbb{C}$ is

$$\mathrm{PS}_V(r) \coloneqq \mathrm{S}_V(r) \,/ \sim,$$

where

$$\mathrm{S}_V(r) \coloneqq \{x \in V \mid \|x\| = r\}$$

denotes the the sphere with radius r, and the equivalence relation $\sim$ is defined by

$$x \sim y \iff \exists \varphi \in \mathbb{R} : y = \mathrm{e}^{\mathrm{i}\varphi} x.$$

In other words, the projective sphere is just the sphere, but different vectors that differ only by a phase factor are identified.

For any group G and any $\rho : G \to \mathrm{U}(\mathcal{H})$, where $\mathrm{U}(\mathcal{H})$ is unitary operators on $\mathcal{H}$, if it satisfies

$$\forall g, h \in G : \exists c \in \mathbb{C} : \rho(g)\,\rho(h) = c\rho(gh)$$

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(i.e., if ρ is a projective unitary representation of G on $\mathcal{H}$; notice that we necessarily have $|c|=1$), then it naturally gives rise to a left action of G on $\mathrm{PS}_{\mathcal{H}}(1)$ defined by

$$\forall\psi\in\mathrm{S}_{\mathcal{H}}(1):g\left[\psi\right]\coloneqq\left[\rho(g)\,\psi\right],$$

where $[\psi]\in\mathrm{PS}_{\mathcal{H}}(1)$ is the equivalence class represented by ψ under the equivalence relation $\sim$.

Bargmann's theorem [6] states that, if G is a connected Lie group, and the cohomology group $H^2(\mathfrak{g};\mathbb{R})$ of the Lie algebra $\mathfrak{g}$ of G is trivial, then every projective unitary representation ρ of G can be lifted to an (ordinary) unitary representation $\hat\rho$ of the universal cover $\hat G$ of G. This means that $\hat\rho$ maps each element of the kernel of the covering map to a scalar multiple of the identity so that, at the projective level, $\hat\rho$ descends to G.

By the principle of special relativity, any physical state needs to transform under the (orthocronous) Poincaré group $\mathrm{ISO}^+(1,3)$[1] in some way. Therefore, we can assume that $\mathrm{ISO}^+(1,3)$ has a left action on the space of physical states. Now, the space of physical states is $\mathrm{PS}_{\mathcal{H}}(1)$. For no apparent reason, we only consider those left actions of $\mathrm{ISO}^+(1,3)$ on it that are naturally given by a projective unitary representation $\rho:\mathrm{ISO}^+(1,3)\to\mathrm{U}(\mathcal{H})$ in the way described above. Because the Lie algebra of $\mathrm{ISO}^+(1,3)$ satisfies Bargmann's condition [8], we only need to consider the unitary representations of the universal cover $\mathrm{ISpin}(1,3)$ of $\mathrm{ISO}^+(1,3)$.

To study the representations of $\mathrm{ISpin}(1,3)$, we can study the representations of $\mathrm{Spin}(1,3)$ instead. Suppose that we have a representation of $\mathrm{Spin}(1,3)$ on a vector space V, then we can build a representation of $\mathrm{ISpin}(1,3)$ by considering V-valued functions (or fields) on $M^{1,3}$ (the Minkowski spacetime):

$$\psi:M^{1,3}\to V:x\mapsto\psi(x)\,.$$

The action $\psi\mapsto\psi'$ of $g\in\mathrm{ISpin}(1,3)$ on ψ can then be given by

$$\psi'(x)\coloneqq L\psi\big(P^{-1}x\big)\,,$$

where $P\in\mathrm{ISO}^+(1,3)$ is the image of g under the covering map, and $L\in\mathrm{GL}(V)$ is the representation of the homogeneous part $\in\mathrm{Spin}(1,3)$ of g on V.

III. Spinor space for Lorentzian metric

This spin group is very special because of the accidental isomorphisms

$$\mathrm{Spin}(1,3)\cong\mathrm{SL}(2,\mathbb{C})\cong\mathrm{Sp}(2,\mathbb{C})\,.$$

[1] I have only seen this notation in [7] but not in other literature. Here, "I" means "inhomogeneous". Later in this article, the same prefix will be used for other groups as well. They mean the same thing, which is to take the semidirect product with the translation group $\mathbb{R}^4$.

The first isomorphism is a well-known fact [9, p. 72]. The second isomorphism can be easily proven (see Appendix B).

First, generally when the spacetime dimension $n=2m$ is even, we can employ the methods in Appendix C to construct a complex vector space W with m complex dimensions, and then construct irreducible and faithful representations of the spin group on either the even-grade subspace $S^+\coloneqq\bigwedge^{\text{even}}W$ or the odd-grade subspace $S^-\coloneqq\bigwedge^{\text{odd}}W$ of the exterior algebra $S\coloneqq\bigwedge W$. In the case of Lorentzian metric, we have $m=2$. In this case, $\bigwedge^{\text{odd}}W$ only has the grade-1 subspace, which is just W itself, and this representation is also the defining representation of $\mathrm{SL}(2,\mathbb{C})$ (some linear transformations on a vector space with two complex dimensions). For this reason, some authors (e.g. [7, p. 347]) call W instead of S the spinor space.

When working in the Weyl basis of the gamma matrices (see Equation C3) specifically where the Pauli matrices (Equation C2) recover the usual Pauli matrices, we can choose the charge conjugation (Equation C4) to be $C\coloneqq\mathrm{i}\gamma_2\gamma_0$. It then just so happens that the invariant bilinear form (Equation C6) of $\mathrm{Spin}(1,3)$ is a symplectic form if we identify $S^+=S^-$, and thus we have the accidental isomorphism $\mathrm{Spin}(1,3)\cong\mathrm{Sp}(2,\mathbb{C})$ recovered. Therefore, it is useful to view W as a symplectic vector space, with the symplectic form denoted as ε_{AB} (see Appendix B). Because it is an invariant bilinear form, we have Equation B1 recovered for any $L\in\mathrm{Spin}(1,3)$.

Let $\left\{o^A,\iota^A\right\}$ be a symplectic basis of W. Then, we can define four tensors of type $(1,0,1,0)$ as follows, recovering the Pauli matrices defined in Equation C2[2]:

$$\begin{aligned}
t^{AA'}&\coloneqq\frac{1}{\sqrt2}\left(o^A\bar o^{A'}+\iota^A\bar\iota^{A'}\right),\\
x^{AA'}&\coloneqq\frac{1}{\sqrt2}\left(o^A\bar\iota^{A'}+\iota^A\bar o^{A'}\right),\\
y^{AA'}&\coloneqq\frac{1}{\sqrt2}\left(o^A\bar\iota^{A'}-\iota^A\bar o^{A'}\right),\\
z^{AA'}&\coloneqq\frac{1}{\sqrt2}\left(o^A\bar o^{A'}-\iota^A\bar\iota^{A'}\right).
\end{aligned}\tag{1}$$

They span the real vector space V consisting of all real tensors of type $(1,0,1,0)$. The tensor

$$g_{AA'BB'}\coloneqq\varepsilon_{AB}\bar\varepsilon_{A'B'},\tag{2}$$

[2] The index structure of these objects is actually different from the Pauli matrices defined in Equation C2, and this is for the discrepancy between Equation C5 and Equation B1. Namely, the former has a complex conjugate, while the latter does not. The Pauli matrices defined in Equation 1 resolve this by absorbing the complex conjugate into W^* (identified with S^+ through the symplectic form) to get $\overline{W}^*$, and thus there is a primed index. Another thing to notice is that their components are actually $1/\sqrt2$ times the Pauli matrices, and this is because of the absence of a normalization factor of $1/2$ in Equation 2. However, we cannot introduce this normalization in $g_{AA'BB'}$ because otherwise we cannot directly use it to lower a pair of indices.

when viewed as a multilinear map $V \times V \to \mathbb{R}$, defines a Lorentzian metric on V [7, p. 349]. Any symplectic map $L^A{}_B \in \mathrm{Sp}(W)$ on W can be mapped to an orthogonal transformation $\lambda^{AA'}{}_{BB'} : V \to V$ defined by

$$\lambda^{AA'}{}_{BB'} = L^A{}_B \overline{L}^{A'}{}_{B'}.$$

It is "orthogonal" in the sense that [7, p. 350]

$$\lambda^{AA'}{}_{CC'} \lambda^{BB'}{}_{DD'} g_{AA'BB'} = g_{CC'DD'}.$$

The map $L \mapsto \lambda$ is in fact the universal covering map from $\mathrm{Sp}(W)$ to $\mathrm{SO}(V)$. We then would naturally identify V with $\mathbb{R}^4$, using a linear isometry

$$\sigma^a{}_{AA'} \coloneqq t^a t_{AA'} - x^a x_{AA'} - y^a y_{AA'} - z^a z_{AA'},$$

where t^a, x^a, y^a, z^a are the orthonormal basis vectors of the Lorentzian metric on $\mathbb{R}^4$. The components of $\sigma^a{}_{AA'}$ are then the Pauli matrices, hence the notation. Under this identification, one may say that a pair of an unprimed index and a primed index combine to form a 4-vector index. These results can all be considered consequences of Equation C5.

Table I summarizes the comparison between the spinor space W and the 4-vector space $\mathbb{R}^4$.

IV. Spinor bundle

In this section, we will discuss how to define a spinor field on a curved spacetime M. For readers not familiar with fiber bundles, Appendix D provides a brief introduction and the formal definitions of concepts used in this section.

Before heading right into spinor spaces, first consider the space of usual tangent vectors. In the latter half of Appendix D, we considered the vector fields as sections of the tangent bundle $\mathrm{T}M$, which is in turn the associated bundle of the oriented orthonormal frame bundle $\mathrm{F}_{\mathrm{SO}}M$. In this way, we replace the structure group in the principal bundle with its representation space, and then the group action orbits contained in the vector bundle then describes how the vectors in the fiber transform under the group action (while leaving the element in the vector bundle unchanged). To help understanding, it may be beneficial to think of the tangent bundle as a $\mathrm{SO}(n)$ (for Riemannian manifold; pseudo-Riemannian manifolds are similar) gauge theory for $\mathbb{R}^n$-valued fields on M. A gauge transformation is then a local symmetry transformation where each point is assigned a group element in $\mathrm{SO}(n)$ to transform the field-value at that point, but the underlying structure of the field (the section) is gauge-invariant. The covariant derivative on the vector bundle is then precisely the gauge covariant derivative, and the gauge field strength is then precisely (the pullback of) the curvature 2-form [10, p. 39].

If we want to define a spinor field on orientable time-orientable[3] pseudo-Riemannian manifold M with Lorentzian metric, we can then try to devise a similar construction, but with the structure group $\mathrm{SO}^+(1,3)$ replaced by $\mathrm{Spin}(1,3)$ and the representation space replaced by $\mathbb{C}^2$. The difficulty lie in devising the principal bundle with structure group $\mathrm{Spin}(1,3)$ and base manifold M.

It turns out that not any M admits such a construction of a principal bundle. The argument is from [7, p. 365], and Figure 1 illustrates the idea. Consider the case where M is simply connected. There is no difficulty in constructing the oriented time-oriented orthonormal frame bundle $\mathrm{F}_{\mathrm{SO}^+}M$. Consider, now, a closed curve γ in $\mathrm{F}_{\mathrm{SO}^+}M$. Since M is simply connected, the curve $\pi \circ \gamma$ in M is homotopic to the trivial curve through $x \in M$. This implies that in $\mathrm{F}_{\mathrm{SO}^+}M$, the curve γ is homotopic to a curve γ' that lies entirely in the fiber $\pi^{-1}(x)$, Because this fiber is diffeomorphic to $\mathrm{SO}^+(1,3)$, it has two homotopy classes. A spinor at x continuously transformed along γ' will either pick up a sign or not depending on which homotopy class γ' is in. However, in the case where $\mathrm{F}_{\mathrm{SO}^+}M$ is simply connected, this situation creates a sign problem for the spinor, where "has no sign change" can be continuously deformed to "has sign change", which makes it impossible for $\mathrm{Spin}(1,3)$ to be the structure group.

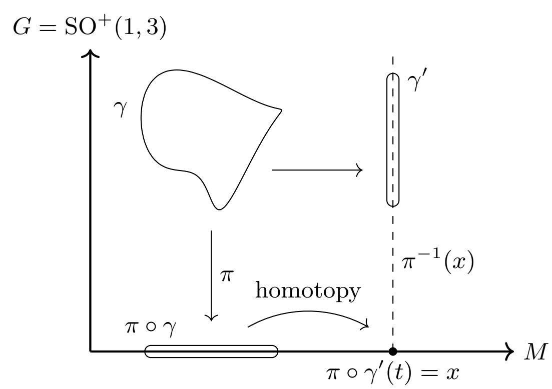

Figure 1. An illustration for why $\mathrm{F}_{\mathrm{SO}}M$ may not permit the spin group as its structure group. The key point is that the two homotopy classes of $\pi^{-1}(x)$ that γ' may be in must not be the same homotopy class in $\mathrm{F}_{\mathrm{SO}}M$.

An example where both M and $\mathrm{F}_{\mathrm{SO}^+}M$ are simply connected is given by [11]. Consider two manifolds A_1 and A_2, which are both diffeomorphic to $D_2 \times D_2$, where D_2 is the unit 2-disk. Define the polar coordinates on each disk so that A_1 has coordinates $(\theta_1, r_1, \varphi_1, \rho_1)$ and A_2 has coordinates $(\theta_2, r_2, \varphi_2, \rho_2)$. Identify the boundaries of A_1

[3] The requirement of time-orientability is used so that the oriented time-oriented orthonormal frame bundle $\mathrm{F}_{\mathrm{SO}^+}M$ can be defined, where the structure group is $\mathrm{SO}^+(1,3)$ instead of $\mathrm{SO}(1,3)$.

Table I. Comparison between the spinor space and the 4-vector space.

	4-vectors	spinors
space	$V := \mathbb{R}^4$ over $\mathbb{R}$	$W := \mathbb{C}^2$ over $\mathbb{C}$
related spaces	V, V^*	$W, W^*, \overline{W}, \overline{W}^*$
representation of	$\mathrm{O}(1,3)$	$\mathrm{SL}(2,\mathbb{C})$
invariant bilinear forms	η_{ab}, η^{ab}	$\varepsilon_{AB}, \varepsilon^{AB}, \overline{\varepsilon}_{A'B'}, \overline{\varepsilon}^{A'B'}$
basis	$(e_\mu)^a = t^a, x^a, y^a, z^a$	$(\xi_\Gamma)^A = o^A, \iota^A$
basis normalization	$(e_\mu)_a (e_\nu)^a = \eta_{\mu\nu}$	$(\xi_\Gamma)_A (\xi_\Sigma)^A = \varepsilon_{\Gamma\Sigma}$

and A_2 by the following conditions to get the manifold M_m, where $m \in \mathbb{N}$ is fixed:

$$\theta_1 = \theta_2, \quad \varphi_1 = \varphi_2 + m\varphi_1, \quad r_1 = r_2 = 1, \quad \rho_1 = \rho_2.$$

Then, M_m is a 4-dimensional non-compact manifold without boundary, which can be assigned a Lorentzian metric[4]. Then, if m is odd, no matter what Lorentzian metric was defined on M_m, it has no spin structure. To see this, choose a closed curve γ which lies entirely in the fiber over the point $\rho_1 = r_1 = 0$ and is not homotopically trivial in the fiber. When m is odd, then γ may be deformed to a single point in $\mathrm{F}_{\mathrm{SO}^+}M_m$ by sliding it over the 2-sphere $\rho_1 = \rho_2 = 0$.

Therefore, when M is simply connected, we need to require that $\pi_1(\mathrm{F}_{\mathrm{SO}^+}M) = \mathbb{Z}_2$. then, the universal covering manifold of $\mathrm{F}_{\mathrm{SO}^+}M$ will have the natural structure of a principal $\mathrm{Spin}(1,3)$-bundle, denoted as $\mathrm{F}_{\mathrm{Spin}}M$, which can then be used to construct the spinor bundle as its associated bundle with the action of $\mathrm{Spin}(1,3) \cong \mathrm{Sp}(2,\mathbb{C})$ on $\mathbb{C}^2$. More generally, when M is not simply connected, we need to require that

$$\pi_1(\mathrm{F}_{\mathrm{SO}^+}M) \cong \mathbb{Z}_2 \times \pi_1(M)\,,$$

and then define $\mathrm{F}_{\mathrm{Spin}}M$ to be a double cover of $\mathrm{F}_{\mathrm{SO}^+}M$ so that

$$\pi_1(\mathrm{F}_{\mathrm{Spin}}M) \cong \pi_1(M)\,.$$

Then, it can have the structure of a principal $\mathrm{Spin}(1,3)$-bundle over M, which can then be used to construct the spinor bundle as its associated bundle. Such a construct is called a spin structure. The spin structure on M, if exists, may not be unique, due to different possible group isomorphisms from $\pi_1(\mathrm{F}_{\mathrm{SO}^+}M)$ to $\mathbb{Z}_2 \times \pi_1(M)$. In fact, M does not have a unique spin structure unless M is simply connected [11].

The spinorial tensor bundles can similarly be defined if the spinor bundle can be defined. The bundle of spinorial tensors with k indices can be defined as the associated bundle of $\mathrm{F}_{\mathrm{Spin}}M$ under the action of $\mathrm{Spin}(1,3)$ on $\mathbb{C}^{2^k}$ (properly according to the index structure of the tensor).

[4] Every non-compact 4-dimensional manifold can be given a time-oriented Lorentzian metric with no closed timelike curves [12].

Then, a spinorial tensor field can then be defined as a section of this bundle. Specially, a spinor field is a section of the spinor bundle.

The concept of the spinor bundle can be generalized to any oriented vector bundle B over the manifold M with fiber F (as long as it satisfies the topological requirements for it to avoid the sign problem) [13, p. 90], where M is a paracompact[5] topological manifold, and F is an inner product space[6]. First, one constructs the oriented orthonormal frame bundle[7] $\mathrm{F}_{\mathrm{SO}}B$, which is a principal $\mathrm{SO}(n)$-bundle. Then, find a principal $\mathrm{Spin}(n)$-bundle $\mathrm{F}_{\mathrm{Spin}}B$ such that there exists a bundle map $\phi : \mathrm{F}_{\mathrm{Spin}}B \to \mathrm{F}_{\mathrm{SO}}B$ such that

$$\forall y \in \mathrm{F}_{\mathrm{Spin}}B, g \in \mathrm{Spin}(n) : \phi(gy) = \rho(g)\,\phi(y)\,,$$

where $\rho : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is the covering map. Finally, take the associated bundle of $\mathrm{F}_{\mathrm{Spin}}B$ under the action of $\mathrm{Spin}(n)$ to get the spinor bundle.

The sufficient and necessary condition for the oriented vector bundle B to permit a spin structure is that the second Stiefel–Whitney class of B vanishes [2].

V. Connections and curvatures

This section mainly presents results from [7, pp. 370–373].

Similar to how a connection on $\mathrm{F}_{\mathrm{SO}^+}M$ defines a covariant derivative on the tangent vector fields (and the tensor fields) of M, a connection on $\mathrm{F}_{\mathrm{Spin}}M$ defines a covariant derivative on the spinor fields and the spinorial tensor fields. Similar to how the covariant derivative on $(0,2)$-tensor fields annihilates the metric tensor g_{ab}, the covariant derivative on $(0,2,0,2)$-tensor fields annihilates the metric tensor $g_{AA'BB'}$ defined in Equation 2 and thus also ε_{AB}, though one must be careful when stating this

[5] Any connected spacetime (Hausdorff and smooth, with Lorentzian metric) is paracompact [11].

[6] Which means that the metric on F is positive-definite, so this means that the generalization is not exactly a generalization. I did not find a good reference for the pseudo-Riemannian case.

[7] The notation $\mathrm{F}_{\mathrm{SO}}B$ does not mean the disjoint union of orthonormal frames of tangent spaces on B, but actually means the disjoint union of orthonormal frames of F on each point of M.

because the notation ∇_a no longer makes sense in the spin structure (because a spinor field cannot carry the 4-vector index a). Therefore, we need to rewrite Equation D1 by using the Pauli matrices to convert a tangent vector field on M to a spinorial tensor field on M:

$$\nabla_T \psi = T^a \sigma_a{}^{AA'} \nabla_{AA'} \psi^{\cdots}{}_{\ldots},$$

where ψ can be a spinorial tensor field of any type. Therefore, we see that the correct index structure of the derivative operator should be type $(0,1,0,1)$. In other words, it maps a spinorial tensor field of type (k,l,k',l') to a spinorial tensor field of type $(k,l+1,k',l'+1)$.

Because in general relativity, usually only the Levi–Civita connection is used in $\mathrm{F}_{\mathrm{SO}^+}M$ so that the induced covariant derivative in $\mathrm{T}M$ is torsion-free, we will also demand that the covariant derivative for spinors is torsion-free. A connection that induces such a covariant derivative is called a **spin connection**. Another feature of $\nabla_{AA'}$ is that it is real. In other words, $\overline{\nabla\psi} = \nabla\overline{\psi}$ for any spinorial tensor field ψ.

The pullback of the spin connection by the bundle projection gives an object whose components on a symplectic basis are called the spin coefficients, and they are analog to the Ricci rotation coefficients (mentioned in the end of Appendix D). The pullback of the connection 1-form would eat one tangent vector and give a linear endomorphism on the spinor space, so in abstract indices it should look like $\gamma_{AA'B}{}^C$. Similar to the Ricci rotation coefficients, given a symplectic basis $\left\{(\xi_\Gamma)^A\right\}$ at each point, the spin coefficients are given by

$$\gamma_{\Gamma\Delta'\Sigma\Lambda} = (\xi_\Gamma)^A \left(\overline{\xi}_{\Delta'}\right)^{A'} (\xi_\Sigma)_B \nabla_{AA'} (\xi_\Lambda)^B .$$

These are quantities that one can actually compute by using the formula

$$\gamma_{AA'\Sigma\Lambda} = \frac{1}{2}\overline{\varepsilon}^{\Gamma'\Delta'} \left(\overline{\xi}_{\Gamma'}\right)_{B'} (\xi_\Sigma)_B \nabla_{AA'} \left((\xi_\Lambda)^B \left(\overline{\xi}_{\Delta'}\right)^{B'}\right). \tag{3}$$

Notice that the object $(\xi_\Lambda)^B \left(\overline{\xi}_{\Delta'}\right)^{B'}$ can be converted to a tangent vector field by using the isometry $\sigma^b{}_{BB'}$. Therefore, we can use the usual way of computing the covariant derivative for tangent vector fields to compute its covariant derivative, and then plug it into Equation 3 to get the spin coefficients. The spin coefficients can in turn be used to compute the covariant derivative for spinorial tensor fields and the various curvature tensors that we will discuss below.

The (spinor analog of) Riemann curvature tensor would then be an object that eats two tangent vectors and gives a linear endomorphism on the spinor space, so in abstract indices it should look like $\chi_{AA'BB'C}{}^D$. Naturally it would satisfy

$$\left(\nabla_{AA'}\nabla_{BB'} - \nabla_{BB'}\nabla_{AA'}\right)\alpha_C = \chi_{AA'BB'C}{}^D \alpha_D .$$

However, one may then ask how the Riemann curvature tensor $R_{abc}{}^d$ in the tangent bundle (which now should really be written as $R_{AA'BB'CC'}{}^{DD'}$ under the isometry from the spinor space to the 4-vector space given by the Pauli matrices) relates to the spinor analog $\chi_{AA'BB'C}{}^D$. With a simple calculation of the derivative commutator on a 4-covector-like object such as $\alpha_C\overline{\alpha}_{C'}$, one sees that

$$R_{AA'BB'CC'}{}^{DD'} = \chi_{AA'BB'C}{}^D \overline{\varepsilon}_{C'}{}^{D'} + \overline{\chi}_{AA'BB'C'}{}^{D'} \varepsilon_C{}^D .$$

The (spinor analog of) Ricci decomposition of $\chi_{AA'BB'C}{}^D$ takes the form

$$\begin{aligned}\chi_{AA'BB'CD} = {}& \Psi_{ABCD}\overline{\varepsilon}_{A'B'} + \Phi_{A'B'CD}\varepsilon_{AB} \\ & + \Lambda\left(\varepsilon_{AC}\varepsilon_{BD} + \varepsilon_{BC}\varepsilon_{AD}\right)\overline{\varepsilon}_{A'B'},\end{aligned} \tag{4}$$

where

$$\Psi_{ABCD} = \Psi_{(ABCD)}, \quad \Phi_{A'B'CD} = \Phi_{(A'B')(CD)},$$

and $\Phi_{A'B'CD}$ and Λ are real. We can see that Ψ_{ABCD} can be regarded as the analog of the Weyl tensor, that $\Phi_{A'B'CD}$ can be regarded as the analog of the traceless[8] Ricci tensor, and that Λ can be regarded as the analog of the Ricci scalar. This decomposition leads to the decomposition of the Riemann curvature tensor

$$\begin{aligned}R_{AA'BB'CC'}{}^{DD'} = {}& \Psi_{ABC}{}^D \overline{\varepsilon}_{A'B'}\overline{\varepsilon}_{C'}{}^{D'} \\ & + \Phi_{A'B'C}{}^D \varepsilon_{AB}\overline{\varepsilon}_{C'}{}^{D'} \\ & + \Lambda\left(\varepsilon_{AC}\varepsilon_B{}^D + \varepsilon_{BC}\varepsilon_A{}^D\right)\overline{\varepsilon}_{A'B'}\overline{\varepsilon}_{C'}{}^{D'} \\ & + \text{c.c.}.\end{aligned}$$

One may then get some nice correspondence between the vectorial objects and spinorial objects by comparing this to the usual Ricci decomposition. The Ricci tensor is

$$R_{AA'CC'} = -2\Phi_{A'C'AC} + 6\Lambda\overline{\varepsilon}_{A'C'}\varepsilon_{AC}; \tag{5}$$

the Weyl tensor is

$$C_{AA'BB'CC'DD'} = \Psi_{ABCD}\overline{\varepsilon}_{A'B'}\overline{\varepsilon}_{C'D'} + \text{c.c.}; \tag{6}$$

and the Ricci scalar is

$$R = -24\Lambda.$$

VI. Wave equations

This section is mainly presenting results from [7, pp. 357–359] and [8].

In flat spacetime, the wave equations arise from the study of irreducible representations (irreps) of $\mathrm{ISpin}(1,3)$. Because the set of all spinor fields (or spinorial tensor

[8] The reason that it is analog to the traceless Ricci tensor instead of the usual Ricci tensor is that a symmetric spinorial tensor is automatically traceless. Another way to see this is through Equation 5.

fields) on $M^{1,3}$ is a representation space of $\mathrm{ISpin}(1,3)$, an irrep is thus a subspace of this space. To define a subspace, people usually impose a linear operator on the space of spinor fields so that a subspace is given by the kernel of the operator. The homogeneous linear equation satisfied by this kernel often takes the form of a wave equation.

The irreps of $\mathrm{ISpin}(1,3)$ can be fully classified by two parameters: m^2 (mass squared) and s (spin/helicity). The current article will only focus on the irreps with $m^2 \neq 0$.

For $m^2 > 0$, the label s is the spin parameter. It can take any non-negative half-integer value. Irreps are spinorial tensor fields that satisfy

$$\left(\partial_{AA'}\,\partial^{AA'} + m^2\right)\varphi^{A_1\cdots A_n} = 0, \quad n = 2s,$$

where φ is a totally symmetric[9] $(n,0,0,0)$-tensor field. When $s = 0$, the equation reduces to the Klein–Gordon equation. When $s = 1$, the equation reduces to the Proca equation. By using some spinor properties, one can reduce the second-order PDE to a first-order PDE but with twice the degrees of freedom. Introduce another spinorial tensor field $\sigma_{A_1'}{}^{A_2\cdots A_n}$ as the extra degrees of freedom, and then we get an equivalent first-order system[10]:

$$\begin{cases} -\mathrm{i}\sqrt{2}\,\partial_{A_1'A_1}\varphi^{A_1\cdots A_n} = m\sigma_{A_1'}{}^{A_2\cdots A_n}, \\ -\mathrm{i}\sqrt{2}\,\partial^{A_1'A_1}\sigma_{A_1'}{}^{A_2\cdots A_n} = m\varphi^{A_1\cdots A_n}. \end{cases}$$

Specially, when $s = 1/2$, the equation reduces to the Dirac equation. To show that, pack together the 4-component objects

$$\Psi \coloneqq \begin{pmatrix} \varphi^A \\ \sigma_{A'} \end{pmatrix}, \quad \overline{\Psi} \coloneqq \begin{pmatrix} \overline{\sigma}_A & \overline{\varphi}^{A'} \end{pmatrix},$$

and then the wave equation can be written as

$$\begin{cases} (\mathrm{i}\gamma^a\,\partial_a + m)\,\Psi = 0, \\ \overline{\Psi}\left(-\mathrm{i}\gamma^a\overleftarrow{\partial}_a + m\right) = 0, \end{cases}$$

where

$$\gamma^a \coloneqq \sqrt{2}\begin{pmatrix} & \sigma^{aAA'} \\ \sigma^a{}_{AA'} & \end{pmatrix}$$

is the Dirac gamma matrix in the chiral representation.

For $m^2 = 0$, some irreps have translations all represented by the identity operator, and they are not discussed in this article. The other irreps are either labeled by the helicity parameter s, which can take any half-integer value, or are the so-called continuous spin representations, which this article will not discuss. For the irreps labeled by helicity, when $s = 0$, the equation is exactly the same as the $m^2 > 0$ case. When $s > 0$, the wave equation is

$$\partial_{A_1'A_1}\varphi^{A_1\cdots A_n} = 0, \quad n = 2s,$$

where φ is a totally symmetric $(n,0,0,0)$-tensor field. When $s < 0$, the equation is the complex conjugate of the $s > 0$ case. When $s = 1/2$, the equation reduces to the Weyl equation. When $s = 1$, the equation reduces to the Maxwell equation. When $s = 2$, the equation reduces to the linearized Einstein equation.

In Section II, it was mentioned that we want the representations of $\mathrm{ISpin}(1,3)$ to be unitary representations. Therefore, we want to define inner products on the representation space of the irreps so that the representations are unitary. One can do this by first construct a current $j^{AA'}(\varphi,\psi)$ for a pair of solutions φ,ψ such that it is conserved ($\partial_{AA'}j^{AA'} = 0$) by the wave equation, and then define the inner product as

$$\langle\psi,\varphi\rangle \coloneqq \int_\Sigma j^{AA'}n_{AA'}\,\mathrm{d}V,$$

where Σ is a Cauchy surface in $M^{1,3}$, and $\mathrm{d}V$ is the volume element on Σ.

For the case when $m^2 > 0$ or $s = 0$, for a pair of solutions $(\varphi,\sigma), (\psi,\rho)$, the current can be explicitly constructed as

$$\begin{aligned} j^{AA'} \coloneqq (-\mathrm{i})^{n-1}\Big(&\overline{\varphi}^{A'A_2'\cdots A_n'}\,\partial_{A_2'A_2}\cdots\partial_{A_n'A_n}\psi^{AA_2\cdots A_n} \\ &+ \overline{\sigma}^{AA_2'\cdots A_n'}\,\partial_{A_2'A_2}\cdots\partial_{A_n'A_n}\rho^{A'A_2\cdots A_n}\Big), \end{aligned}$$

and the inner product is positive definite for any solutions when s is odd, and is positive definite for solutions with positive frequency when s is even. Specially, when $s = 1/2$, the current can be expressed in terms of the Dirac spinor field Ψ as

$$j^a = \overline{\Phi}\gamma^a\Psi,$$

where $\Phi \coloneqq (\varphi,\sigma)$ and $\Psi \coloneqq (\psi,\rho)$ are the Dirac spinors, and $j^a \coloneqq \sigma^a{}_{AA'}j^{AA'}$. This is the same as the Noether current for the $\mathrm{U}(1)$ charge when $\Psi = \Phi$.

For the case when $m^2 = 0$ and $s = 1/2$, the current can be constructed as

$$j^{AA'} \coloneqq \overline{\varphi}^{A'}\psi^A,$$

and the inner product is positive definite. For $s > 1/2$, the expression is complicated, but one may consult [8] for an expression in the momentum space.

These wave equations cannot be easily generalized to curved spacetime, though. With the minimal replacement $\partial_{AA'} \to \nabla_{AA'}$, they do not generally constitute a well-posed initial value problem, and if they do, the constructed currents may not still be conserved. Table II summarizes the situation for different cases of m^2 and s.

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9 Tracelessness is implied.

10 Here I use a factor of $-\mathrm{i}$ in both equations instead of a minus sign in just one of them because this form reduces to the Dirac equation more readily.

Table II. Under minimal replacement $\partial_{AA'} \to \nabla_{AA'}$, whether the wave equations are well-posed initial value problems and whether the currents are conserved. Consult [7, p. 375] for details.

m^2	s	well-posed IVP	current conservation
>0	$0, \frac{1}{2}, 1$	✓	✓
>0	>1	✓	✗
0	$0, \frac{1}{2}, 1$	✓	✓
0	>1	✗	

VII. Petrov classification

A complex null tetrad is four vector fields $\{l^a, n^a, m^a, \overline{m}^a\}$ satisfying

$$\begin{cases} l_a l^a = n_a n^a = m_a m^a = \overline{m}_a \overline{m}^a = 0, \\ l_a n^a = -m_a \overline{m}^a = 1, \\ l_a m^a = l_a \overline{m}^a = n_a m^a = n_a \overline{m}^a = 0. \end{cases} \tag{7}$$

The Newman–Penrose formalism [14] is a set of notation for general relativity by reexpressing all the covariant derivatives and curvature tensors in terms of the complex null tetrad and the four corresponding directional covariant derivative operators

$$D \coloneqq l^a \nabla_a, \quad \Delta \coloneqq n^a \nabla_a, \quad \delta \coloneqq m^a \nabla_a, \quad \overline{\delta} \coloneqq \overline{m}^a \nabla_a.$$

Equation 3 inspires the Newman–Penrose formalism: for a symplectic basis $\left\{(\xi_\Gamma)^A\right\} = \left\{o^A, \iota^A\right\}$, all combinations of the form $(\xi_\Lambda)^A \left(\overline{\xi}_{\Delta'}\right)^{A'}$ give exactly a complex null tetrad. Namely,

$$\begin{aligned} l^{AA'} &= \iota^A \overline{o}^{A'}, \quad n^{AA'} = o^A \overline{o}^{A'}, \\ m^{AA'} &= \iota^A \overline{o}^{A'}, \quad \overline{m}^{AA'} = o^A \overline{\iota}^{A'}. \end{aligned}$$

One can verify that they satisfy Equation 7. Because their covariant derivatives give the spin coefficients, which can in turn be used to express the curvature tensors, we can thus use express them using the complex null tetrad.

This method is especially useful for the Petrov classification. The Petrov classification classifies the spacetime curvature at some point in terms of the principal null directions (PND), which are defined as those null directions k^a that satisfy the relation [7, p. 179].

$$k^b k^c k_{[e} C_{a]bc[d} k_{f]} = 0,$$

where C_{abcd} is the Weyl tensor. In general, there are four PNDs at each point, but sometimes some of them may coincide (and the coincident PNDs satisfy a stronger condition; see Table III). Spacetimes that have fewer than four PNDs at every point are called to be algebraically special, and they are classified according to how the PNDs coincide. The full classification is shown in Table IV.

The importance of Petrov classification is that, it is very hard to determine whether two metrics are related by a diffeomorphism and thus the same, but the Petrov classification is a feature that is invariant under diffeomorphisms [15, p. 70]. Therefore, having different Petrov types is a sufficient condition of being fundamentally different spacetime metrics.

In practice, the Newman–Penrose formalism is often used to determine the Petrov type. First, one defines the five Weyl–NP scalars [14]:

$$\begin{aligned} \Psi_0 &\coloneqq C_{abcd} l^a m^b l^c m^d, \\ \Psi_1 &\coloneqq C_{abcd} l^a n^b l^c m^d, \\ \Psi_2 &\coloneqq C_{abcd} l^a m^b \overline{m}^c n^d, \\ \Psi_3 &\coloneqq C_{abcd} l^a n^b \overline{m}^c n^d, \\ \Psi_4 &\coloneqq C_{abcd} n^a \overline{m}^b n^c \overline{m}^d. \end{aligned}$$

Then, the Petrov type can be determined by whether some of them are zero in a frame where $\Psi_0 = 0$ [16]. Table IV summarizes the conditions satisfied by the Weyl–NP scalars for the different Petrov types.

The classification is remarkably simple in the spinorial formalism, and the full derivation can be written down in just one page [7, p. 374]. Consider a symplectic basis $\left\{o^A, \iota^A\right\}$ such that $\Psi_{ABCD} \iota^A \iota^B \iota^C \iota^D = 1$ (where Ψ_{ABCD} is defined in Equation 4), and then define $\alpha^A \coloneqq z\iota^A + o^A$ for some $z \in \mathbb{C}$. Then,

$$f(z) \coloneqq \Psi_{ABCD} \alpha^A \alpha^B \alpha^C \alpha^D$$

is a polynomial of degree 4 in z and thus have in general 4 roots c_i $(i = 1, \ldots, 4)$. Define $(\kappa_i)^A \coloneqq o^A + c_i \iota^A$, which are called the principal spinors. Therefore,

$$f(z) = \prod_i (z - c_i) = \prod_i (\kappa_i)_A \alpha^A.$$

Compare this with the definition of $f(z)$, also considering that Ψ_{ABCD} is totally symmetric, we have the general decomposition

$$\Psi_{ABCD} = (\kappa_1)_{(A} (\kappa_2)_B (\kappa_3)_C (\kappa_4)_{D)}.$$

We can also now get the third column of Table III, which displays the conditions satisfied by a principal spinor if it has a certain multiplicity. Translating them to the principal null direction $k^{AA'} \coloneqq \kappa^A \overline{\kappa}^{A'}$ and using Equation 6 then give the second column of Table III.

VIII. Absence of fermion superradiance

In this section, an actual problem involving fermion fields that can actually be solved with fairly simple mathematics is presented, which is to determine whether superradiance occurs for a Dirac field incident on the horizon of a Kerr black hole. This section is mainly following [17].

Table III. The condition satisfied by the principal null direction k^a or principal spinor κ^A versus its multiplicity (how many times it is repeated). The higher the multiplicity, the stronger the condition is. Adapted from [7, pp. 180, 374].

M	Principal null direction	Principal spinor
1	$k^b k^c k_{[e} C_{a]bc[d} k_{f]} = 0$	$\Psi_{ABCD}\kappa^A\kappa^B\kappa^C\kappa^D = 0$
2	$k^b k^c C_{abc[d} k_{e]} = 0$	$\Psi_{ABCD}\kappa^A\kappa^B\kappa^C = 0$
3	$k^c C_{abc[d} k_{e]} = 0$	$\Psi_{ABCD}\kappa^A\kappa^B = 0$
4	$C_{abcd} = 0$	$\Psi_{ABCD}\kappa^A = 0$

Table IV. The definition of different Petrov types by the multiplicities of the PNDs, along with the conditions satisfied by the Weyl–NP scalars in a frame where $\Psi_0 = 0$. Adapted from [16].

Type	Definition	Conditions
I	4 single	
II	2 single, 1 double	$\Psi_1 = 0$
D	2 double	$\Psi_1 = \Psi_3 = \Psi_4 = 0$
III	1 single, 1 triple	$\Psi_1 = \Psi_2 = 0$
N	1 quadruple	$\Psi_1 = \Psi_2 = \Psi_3 = 0$
O	arbitrary	$\Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0$

The region K of the spacetime that is considered is between the two spacelike hypersurfaces Σ_1 at time t and Σ_2 at time $t + \delta t$ and bewteen the two timelike hypersurfaces H (horizon at $r = r_+$) and S_∞ (spatial infinity). The direction of the normal vector n^a on each boundary is chosen to be outward. The region is illustrated in Figure 2.

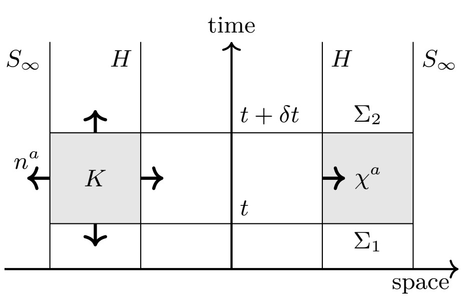

Figure 2. The region K considered in the superradiance problem.

We can either study the superradiance of energy by studing the energy current

$$J_a \coloneqq -T_{ab} t^b,$$

or study the superradiance of charge (particle number) by studying the charge current

$$j^a \coloneqq \frac{\delta \mathcal{L}}{\delta \left(\nabla_a \varphi^i\right)} \delta \varphi^i.$$

For illustration purpose, we will study j^a, and the procedure is very similar for J^a. Integrate $\nabla_a j^a = 0$ over K, and we have

$$0 = \int_{\Sigma_1} n_a j^a + \int_{\Sigma_2} n_a j^a + \int_H n_a j^a + \int_{S_\infty} n_a j^a.$$

The first two terms cancel for static solutions. Superradiance is present when there is net outgoing current at S_∞, which happens if

$$\int_H \chi_a j^a < 0,$$

where $\chi^a = t^a + \Omega \phi^a$ is a Killing vector field for the Kerr spacetime, and it is equal to $-n^a$ on the horizon H.

As a simple check, we can use this method to recover the well-known condition for the superradiance of a complex scalar field Φ. The solution to the equation of motion of a scalar field is

$$\Phi(x) = \Phi_0(t, \theta)\, \mathrm{e}^{\mathrm{i}(m\phi - \omega t)},$$

and the current (associated with the U(1) charge) is

$$j^a = -\mathrm{i}\left(\Phi^* \nabla^a \Phi - \Phi \nabla^a \Phi^*\right).$$

Compute

$$\chi^a j_a = 2\left(\omega - m\Omega\right) \left|\Phi_0\right|^2.$$

Therefore, there is superradiance if $0 < \omega < m\Omega$.

Now, to get the result for a Dirac field, we first need to obtain a representation of the gamma matrices. To do this, we need a tetrad $\{e_a\}$, which can be more easily obtained by first considering the ADM split of the metric. For the Kerr metric, the ADM split is

$$\mathrm{d}s^2 = N^2\,\mathrm{d}t^2 - h_{rr}\,\mathrm{d}r^2 - h_{\theta\theta}\,\mathrm{d}\theta^2 - h_{\phi\phi}\left(\mathrm{d}\phi + N^\phi\,\mathrm{d}t\right)^2,$$

where

$$\begin{aligned}
N^2 &= \frac{\Delta - a^2 \sin^2\theta}{\Sigma} + \left(\frac{r^2 + a^2 - \Delta}{\Sigma R} a \sin^2\theta\right)^2, \\
N^\phi &= -\frac{r^2 + a^2 - \Delta}{\Sigma R^2} a \sin^2\theta, \\
h_{rr} &= \frac{\Sigma}{\Delta}, \\
h_{\theta\theta} &= \Sigma, \\
h_{\phi\phi} &= R^2 \coloneqq \frac{\left(r^2 + a^2\right)^2 - \Delta a^2 \sin^2\theta}{\Sigma} \sin^2\theta,
\end{aligned}$$

where the quantities Δ and Σ are defined as usual for the Kerr metric. Then, the tetrad can be chosen as (in matrix form in coordinate basis)

$$\begin{aligned}
&\left((e_t)^\mu, (e_r)^\mu, (e_\theta)^\mu, (e_\phi)^\mu\right) \\
&= \begin{pmatrix} N^{-1} & & & -N^{-1}N^\phi \\ & \sqrt{\Delta/\Sigma} & & \\ & & \sqrt{1/\Sigma} & \\ & & & R^{-1} \end{pmatrix}.
\end{aligned}$$

They give the gamma matrices

$$\gamma^t := N^{-1}\gamma^0, \quad \gamma^r := \sqrt{\frac{\Delta}{\Sigma}}\,\gamma^1, \quad \gamma^\theta := \sqrt{\frac{1}{\Sigma}}\,\gamma^2,$$
$$\gamma^\phi := N^{-1}N^\phi\gamma^0 + R^{-1}\gamma^3.$$

One important identity that we can prove is that $\chi_\mu\gamma^\mu = 0$ on H. To see this, substitute on the horizon $N = 0$ and $N^\phi = -\Omega$ into

$$\chi_\mu\gamma^\mu = N\gamma^0 - R\left(N^\phi + \Omega\right)\gamma^3.$$

Therefore, for a Dirac field ψ with charge current

$$j^a = \overline{\psi}\gamma^a\psi$$

and energy current

$$J_a = \frac{\mathrm{i}}{2}t^b\overline{\psi}\gamma_{(a}\left(\nabla_{b)} - \overline{\nabla}_{b)}\right)\psi,$$

we have

$$\chi_\mu j^\mu = 0, \quad \chi^\mu J_\mu = 0$$

on the horizon H by $\chi_\mu\gamma^\mu = 0$.

The difference between bosons (complex scalar field) and fermions (Dirac spinor field) in the black hole superradiance problem resembles the Klein paradox. The Klein paradox is the different ways in which bosons and fermions as they scatter off a potential barrier in 1D. A scalar (Klein–Gordon) field reflects off the potential with greater amplitude than the incident one in some conditions. For a Dirac spinor field, the Klein paradox happens: the reflected amplitude is smaller than the incident one, and the potential appears to be transparent as its height tends to infinity. In QFT, the interpretation of the Klein paradox is that particle–antiparticle pairs are spontaneously created in presence of strong potential, and stimulated emission happens in the boson case. The analogy of the Klein paradox to the superradiance suggests the presence of particle–antiparticle pair creation near the horizon of a black hole, which is the essence of the Hawking radiation.

IX. Conclusion

Spinor fields in curved spacetime require a refinement of the geometric framework of general relativity, involving spin structures and the spinor bundle formalism. Their proper definition hinges on the topological condition of vanishing second Stiefel–Whitney class, enabling the lift from the orthonormal frame bundle to a spin bundle. Once defined, spinors enrich the differential geometry of spacetime through spin connections and curvature decompositions not accessible through tensor methods alone. The formalism aligns naturally with the representation theory of the spin group, making it well-suited for analyzing fermionic field equations in gravitational backgrounds.

Applications range from constructing well-posed wave equations to classifying spacetime curvature via spinorial methods, offering both computational tools and conceptual clarity. The analysis of the Dirac field in the Kerr geometry underscores the physical consequences of this formalism, confirming the absence of fermion superradiance and providing insight into quantum field behavior near black holes. These developments highlight the essential role of spinors in any serious attempt to merge general relativity with quantum theory.

A. Complex tensor

For a finite-dimensional vector space W over $\mathbb{C}$, we denote W^* to be its dual space (the space of linear functionals $W \to \mathbb{C}$). There is a space $\overline{W}^*$ closely related to the dual space, which is called the **anti-dual space**, defined as the set of all antilinear functionals $W \to \mathbb{C}$ [18, p. 112].

For a vector space W, a map $f : W \to \mathbb{C}$ is said to be **antilinear** if [19, p. 20]

$$f(x+y) = f(x) + f(y)\,, \quad f(sx) = \overline{s}f(x)$$

holds for all $x, y \in W$ and every $s \in \mathbb{C}$, where $\overline{s}$ denotes the complex conjugate of s. Mathematicians would say that an antilinear map is a $\bar{\cdot}$-semilinear map [20, p. 223], where $\bar{\cdot} : \mathbb{C} \to \mathbb{C}$ is the field homomorphism given by complex conjugation.

There is a natural isomorphism between W^* and $\overline{W}^*$ called the **complex conjugation**, also denoted by $W^* \to \overline{W}^* \mu \mapsto \overline{\mu}$, defined as [7, p. 347]

$$\overline{\mu}(x) := \overline{\mu(x)}.$$

If we denote the dual space of $\overline{W}^*$ by $\overline{W}$, then there is also a natural isomorphism between W and $\overline{W}$, also called the **complex conjugation** and denoted by $W \to \overline{W} : \xi \mapsto \overline{\xi}$, defined as [7, p. 347]

$$\overline{\xi}(\psi) := \psi(\xi)\,.$$

A diagram that illustrates the construction of the four spaces W, W^*, $\overline{W}^*$, and $\overline{W}$ is shown in Figure 3.

$$\begin{tikzcd} W \arrow[r, "\text{dual space}"] & W^* \arrow[d, "\text{conjugate}"] \\ \overline{W} & \overline{W}^* \arrow[l, "\text{dual space}"'] \end{tikzcd}$$

Figure 3. The relation between the four spaces W, W^*, $\overline{W}^*$, and $\overline{W}$. The arrows indicate the construction path.

Just like how we construct tensors for real vector spaces using multilinear maps, we can construct a **tensor** of type (k,l,k',l') over W as a multilinear map [7, p. 347]

$$T: W^{*k}\times W^{l}\times\overline{W}^{*k'}\times\overline{W}^{l'}\to\mathbb{C}.$$

For any tensor of type (k,l,k',l'), we can define its **complex conjugate** as a tensor of type (k',l',k,l), denoted by $\overline{T}$, defined as

$$\overline{T}(\psi,\dots)\coloneqq\overline{T(\overline{\psi},\dots)},$$

where $\psi,\dots$ is any tuple of elements in $W^{*k'}\times W^{l'}\times\overline{W}^{*k}\times\overline{W}^{l}$, and $\overline{\psi},\dots$ is the corresponding tuple in $\overline{W}^{*k}\times\overline{W}^{l}\times W^{*k'}\times W^{l}$ derived by taking the complex conjugate of each component in the tuple.

We assign one type of abstract indices to each of the four spaces[11]: upper for W, lower for W^*, upper primed for $\overline{W}$, and lower primed for $\overline{W}^*$. All those indices use capitalized latin letters $A,B,\dots$. Therefore, we denote

$$\xi^A\in W,\quad \mu_A\in W^*,\quad \psi_{A'}\in\overline{W}^*,\quad \varphi^{A'}\in\overline{W}.$$

A tensor of type (k,l,k',l') has k upper indices, l lower indices, k' upper primed indices, and l' lower primed indices:

$$T^{\overbrace{A\cdots}^{k}}{}_{\underbrace{B\cdots}_{l}}{}^{\overbrace{A'\cdots}^{k'}}{}_{\underbrace{B'\cdots}_{l'}}.$$

In the abstract index notation for a tensor of type (k,l,k',l'), a pair of indices consisting of one upper index and one lower index can be contracted, resulting in a tensor of type $(k-1,l-1,k',l')$. Similarly, a pair of indices consisting of one upper primed index and one lower primed index can also be contracted, resulting in a tensor of type $(k,l,k'-1,l'-1)$.

B. Symplectic group

A **symplectic form** is a non-degenerate 2-form. We can write it as an antisymmetric tensor ε_{AB} of type $(0,2)$ (or $(0,2,0,0)$ if the vector space is complex). A vector space W equipped with a symplectic form is called a **symplectic vector space** [22, p. 167]. When W is even-dimensional, by a procedure similar to the Gram–Schmidt process, it is always possible to choose a basis called the **symplectic basis** so that the matrix form of the symplectic form is [23, p. 7]

$$\begin{pmatrix} & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1\\ -1 & & & & & \\ & \ddots & & & & \\ & & -1 & & & \end{pmatrix}.$$

Because ε_{AB} is non-degenerate, we can define another tensor ε^{AB} of type $(2,0)$ (or $(2,0,0,0)$ if W is a complex vector space) such that

$$\varepsilon^{AB}\varepsilon_{AC}=\delta^{B}{}_{C},$$

where $\delta^{B}{}_{C}$ is the Kronecker delta. We define the raising and lowering of indices in terms of these two tensors:

$$\xi_B\coloneqq\varepsilon_{AB}\xi^A,\quad \mu^A\coloneqq\varepsilon^{AB}\mu_B.$$

If W is a complex vector space, we also define the raising and lowering of primed indices:

$$\varphi_{B'}\coloneqq\overline{\varepsilon}_{A'B'}\varphi^{A'},\quad \psi^{A'}\coloneqq\overline{\varepsilon}^{A'B'}\psi_{B'},$$

where $\overline{\varepsilon}_{A'B'}$ and $\overline{\varepsilon}^{A'B'}$ are the complex conjugates of ε_{AB} and ε^{AB}, respectively.

A linear map between two symplectic vector spaces is called a **symplectic map** if it preserves the symplectic form. All symplectic maps from W to itself form a group called the **symplectic group**, denoted as $\mathrm{Sp}(W)$. When W is finite-dimensional, one can use the matrix form of the symplectic form to easily show that the determinant $\det L$ of the matrix form of a symplectic map $L\in\mathrm{Sp}(W)$ must be 1, thus $\mathrm{Sp}(W)$ is a subgroup of the special linear group $\mathrm{SL}(W)$.

Specially, when W is 2-dimensional, we have $\mathrm{Sp}(W)=\mathrm{SL}(W)$. Because the proof is simple but important, I will sketch it here[12]. Any linear map $L:W\to W$ can be written as a tensor $L^{A}{}_{B}$ of type $(1,1)$, so the condition for L be a symplectic map is

$$\forall u,v\in W:\varepsilon_{AB}L^{A}{}_{C}u^{C}L^{B}{}_{D}v^{D}=\varepsilon_{CD}u^Cv^D.$$

Because ε_{AB} is non-degenerate, we have

$$\varepsilon_{AB}L^{A}{}_{C}L^{B}{}_{D}=\varepsilon_{CD}. \tag{B1}$$

When contracted with $\varepsilon^{CD}/2$, we get the desired result $\det L=1$.

---

[11] This notation aligns with [7] but differs from the more common notation where dotted and undotted lowercase alphabet letters are used for spinors (in e.g. [21]). The reason for using uppercase letters is to avoid confusion with the abstract indices on 4-vectors. The component indices, aligning with [7], are denoted by uppercase Greek letters $\Gamma,\Delta,\Sigma,\Lambda$, etc..

[12] This proof only works for the case when the field that W is over is not of characteristic 2, but the conclusion is still true in that case.

C. Spin group

For any finite-dimensional vector space V over $\mathbb{R}$ with an inner product $\langle\cdot,\cdot\rangle$ (may not be positive-definite[13]), a spin group may be defined. In this appendix, I will introduce the formal definition of the spin group and some notable representations.

The **Clifford algebra** $\mathrm{Cl}(V)$ is the quotient of the tensor algebra of V by the two-sided ideal generated by all elements of the form $v\otimes v-\langle v,v\rangle$ [24, p. 60][14]. From now on, a multiplication without any binary symbol between denotes the multiplication operation in $\mathrm{Cl}(V)$ as an algebra. The Clifford algebra is naturally graded, and the highest grade is $n\coloneqq\dim V$:

$$\mathrm{Cl}(V)=\bigoplus_{k=0}^{n}\mathrm{Cl}^k(V)\,.$$

For notational simplicity, we identify $\mathrm{Cl}^0(V)=\mathbb{R}$ and $\mathrm{Cl}^1(V)=V$.

One may then prove that $\mathfrak{spin}(V)\coloneqq\mathrm{Cl}^2(V)$ is a Lie algebra under the commutator operation $[\cdot,\cdot]$ and that it is isomorphic to $\mathfrak{so}V$, where the isomorphism is given by [24, p. 61]

$$\mathrm{Cl}^2(V)\to\mathrm{End}(V):a\mapsto(v\mapsto[a,v])\,.$$

One can then exponentiate $\mathfrak{spin}(V)$ to obtain a Lie group $\mathrm{Spin}(V)$ called the **spin group**. It turns out that

$$\mathrm{Spin}(V)=\{v_1\cdots v_{2k}\mid v_i\in V,\ \langle v_i,v_i\rangle=1,\ k\in\mathbb{N}\}\,.$$

One can prove that

$$\mathrm{Spin}(V)\to\mathrm{SO}(V):a\mapsto\left(v\mapsto ava^{\mathrm{t}}\right)$$

is a surjective homomorphism and thus a double cover of $\mathrm{SO}(V)$, where the superscript "t" is defined as

$$(v_1\cdots v_k)^{\mathrm{t}}\coloneqq v_k\cdots v_1,\quad v_i\in V.$$

Consider the complexification $V\otimes\mathbb{C}$ of V, with the inner product extended to be $\langle\cdot,\cdot\rangle_{\mathbb{C}}$, a $\mathbb{C}$-linear in both arguments (so it is no longer an inner product but a bilinear map). We then try to find two maximal isotropic subspaces W and W^* with only trivial intersection. When $n=2m$ is even (which we are going to assume throughout the rest of this section), one construction is given as follows. Given an orthonormal basis $\{\gamma_i\}$ on V, define a bunch of vectors

$$\eta_j^\pm\coloneqq\frac{1}{\sqrt{2}}\left(\nu_{2j-1}^+\gamma_{2j-1}\pm\nu_{2j}^-\gamma_{2j}\right),\quad j=1,\ldots,m,\tag{C1}$$

where $\nu_i^\pm\coloneqq\sqrt{\pm\gamma_i\gamma_i}$ is either 1 or i. Then, the two complex vector spaces $W\coloneqq\mathrm{Span}_{\mathbb{C}}\left\{\eta_j^-\right\}$ and $W^*\coloneqq\mathrm{Span}_{\mathbb{C}}\left\{\eta_j^+\right\}$ are both isotropic subspaces of $V\otimes\mathbb{C}$ w.r.t. $\langle\cdot,\cdot\rangle_{\mathbb{C}}$. They complement each other:

$$V\otimes\mathbb{C}=W\oplus W^*.$$

One can identify W^* to be the dual space of W by defining [24, p. 72]

$$\mu(\xi)\coloneqq\langle\xi,\mu\rangle_{\mathbb{C}}\,,\quad\mu\in W^*,\quad\xi\in W.$$

Call the exterior algebra $S\coloneqq\bigwedge W$ the **spinor space**. Then, there is an algebra isomorphism $\rho:\mathrm{Cl}^{\mathbb{C}}(V)\to\mathrm{End}(S)$, where $\mathrm{Cl}^{\mathbb{C}}(V)=\mathrm{Cl}(V)\otimes_{\mathbb{R}}\mathbb{C}$ is the complexification of $\mathrm{Cl}(V)$, and $\mathrm{End}(S)$ is regarded as an algebra by composition. The isomorphism is given by

$$\rho(\xi\in W)\coloneqq\left(s\mapsto\sqrt{2}\,\xi\wedge s\right),\quad\rho(\mu\in W^*)\coloneqq-\sqrt{2}\,\iota_\mu,$$

where $\iota_\mu:S\to S$ is the interior product. This then gives a representation, called the **spin representation**, of $\mathrm{Spin}(V)$ (as a subgroup of $\mathrm{Cl}^{\mathbb{C}}(V)$) on S.

The representation is reducible because the even-grade subspace $S^+\coloneqq\bigwedge^{\mathrm{even}}W$ or the odd-grade subspace $S^-\coloneqq\bigwedge^{\mathrm{odd}}W$ of S are respectively invariant. The two subspaces are the projection image of the projectors

$$P_\pm\coloneqq\frac{1}{2}\left(1\pm\gamma_5\right)\in\mathrm{Cl}^{\mathbb{C}}(V)\,,$$

where

$$\gamma_5\coloneqq\mathrm{i}^{(q-p)/2}\prod_\mu\gamma_\mu\in\mathrm{Cl}^{\mathbb{C}}(V)\,,$$

where p and q are the number of positive and negative directions of the inner product on V respectively, and $\{\gamma_\mu\}$ is an orthonormal basis of V (but γ_5 is independent of the choice). When restricted to one of $S^\pm$, the spin representation is called the **half spin representation**, and it is irreducible.

For $v\in V$, define the analog of the Pauli matrices as

$$\sigma_v^\pm\coloneqq P_\mp vP_\pm.\tag{C2}$$

Then, for $v\neq0$, one can prove that $\rho(\sigma_v^\pm)$ is an isomorphism from $S^\pm$ to $S^\mp$ (in the sense that it connects the two representations of $\mathrm{Cl}^{\mathbb{C}}(V)$). As such maps, they form an algebraic relation similar to the Clifford algebra:

$$\sigma_u^\mp\sigma_v^\pm+\sigma_v^\mp\sigma_u^\pm=2\langle u,v\rangle\,\mathrm{id}_{S^\pm}\,.$$

Choose a specific basis on S so that for any $v\in V$, the matrix form of $\rho(v)$ takes form of **Weyl basis**

$$\rho(v)=\begin{pmatrix}&\sigma_v^-\\\sigma_v^+&\end{pmatrix}.\tag{C3}$$

[13] The main reference [24] for this appendix only talked about positive-definite inner products, but all the results mentioned in this particular appendix are either applicable for the more general case or modified to suit the more general case.

[14] The convention in [24] uses $v\otimes v+\langle v,v\rangle$ instead probably because of the $-,+,+,+$ convention of the Minkowski metric. The current article uses the other convention, so the minus sign is used.

Suppose there exists an invertible matrix $C \in \rho\Big(\mathrm{Cl}^{\mathbb{C}}(V)\Big)$ such that

$$\forall v \in V : C\rho(v)\, C^{-1} = \rho(v)^{\mathrm{T}} , \tag{C4}$$

where the superscript "T" denotes the usual transpose of a matrix, and the superscript "-1" denotes the usual inverse of a matrix. For any matrix $A \in \rho\Big(\mathrm{Cl}^{\mathbb{C}}(V)\Big)$, define

$$A^{\ddagger} \coloneqq C^{-1} A^{\dagger} C,$$

where the superscript "†" is the usual Hermitian conjugate of a matrix. Then, for any $L \in \mathrm{Spin}(V)$ and $v \in V$,

$$\rho(L)\, \rho\big(\sigma_v^-\big)\, \rho(L)^{\ddagger} = \rho\big(\sigma_{Lv}^-\big) , \tag{C5}$$

where Lv denotes the $\mathrm{SO}(V)$ action of the double covering image of L on v. Thus

$$\varepsilon : S^- \times S^+ \to \mathbb{C} : (\xi, \mu) \mapsto \langle C\xi, \mu\rangle_{\mathbb{C}} \tag{C6}$$

is an invariant bilinear form of $\mathrm{Spin}(V)$.

D. Fiber bundle

A **principal bundle** [25, p. 42] (manifold) is a manifold P (called the total space or bundle manifold) together with another manifold M (called the base manifold) and a Lie group G (called the structure group), satisfying: G has a free right action $\phi : P \times G \to P$ on P; there is a surjective smooth projection $\pi : P \to M$ such that

$$\forall p \in P : \pi^{-1}(\pi(p)) = \{pg \mid g \in G\} ;$$

and for any $x \in M$, there is a neighborhood $U \subseteq M$ and a diffeomorphism $\tau : \pi^{-1}(U) \to U \times G$ such that it takes the form

$$\forall p \in \pi^{-1}(U) : \tau(p) = (\pi(p)\, , s(p))\, ,$$

where $s : \pi^{-1}(U) \to G$ satisfies

$$\forall g \in G : s(pg) = s(p)\, g.$$

The right action $\phi : P \times G \to P$ naturally induces a map $\phi_p : G \to P$ for each $p \in P$, which one can prove to be an embedding, and also a diffeomorphism from G to the fiber $\pi^{-1}\pi(p)$; it also naturally induces a map $\phi_g : P \to P$ for each $g \in G$, which is a diffeomorphism from P to itself. The relation between different spaces in terms of ϕ and π is illustrated in Figure 4.

Sometimes U can be as large as the whole M, in which case P is diffeomorphic to $M \times G$. Such a principal bundle is called a **trivial principal bundle**. For this case, generally when U is a neighborhood of $x \in M$, the diffeomorphism τ is called a **local trivialization**. The relation between different spaces in terms of τ and s is illustrated

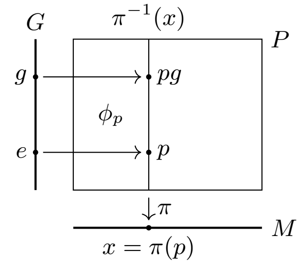

Figure 4. The relation between the total space P, the base manifold M, and the structure group G in a principal bundle. The arrows indicate the projection π and the right action ϕ_p.

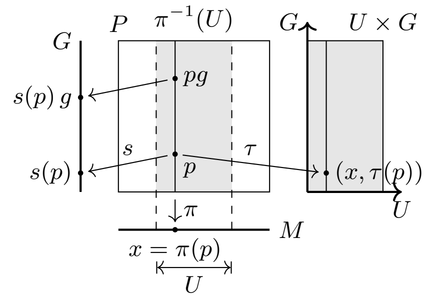

Figure 5. The relation between the total space P, the base manifold M, and the structure group G in a principal bundle with a local trivialization τ.

in Figure 5. A function $\sigma : U \to P$ such that $\pi \circ \sigma = \mathrm{id}_U$ is called a (local) **section**.

The definition of a principal bundle defines a naturally distinguished subspace in the tangent space of P at each point $p \in P$, called the **vertical subspace** V_p, defined as

$$\mathrm{V}_p \coloneqq \{X \in \mathrm{T}_p P \mid \pi_*(X) = 0\}\, ,$$

where $\pi_* : \mathrm{T}_p P \to \mathrm{T}_{\pi(p)} M$ is the pushforward by π. A smooth choice of a **horizontal subspace** H_p complement to V_p at each point $p \in P$ such that $\phi_{g*}(\mathrm{H}_p) = \mathrm{H}_{pg}$ (where $\phi_{g*} : \mathrm{T}_p P \to \mathrm{T}_{pg} P$ is the pushforward by ϕ_g) for all $g \in G$ is called a **connection** on the principal bundle [10, p. 29]. To give such a choice, one can use a $\mathfrak{g}$-valued 1-form field $\tilde{\omega}$ called the **connection 1-form** (or commonly also called the connection) on P (where $\mathfrak{g}$ is the Lie algebra of G), which intuitively defines the normal direction of H_p. It needs to satisfy

$$\forall A \in \mathfrak{g}, p \in P : \tilde{\omega}_p(\phi_{p*} A) = A,$$
$$\forall p \in P, g \in G, X \in \mathrm{T}_p P : \tilde{\omega}_{pg}(\phi_{g*} X) = \mathrm{Ad}_{g^{-1}} \tilde{\omega}_p(X)\, ,$$

where $\phi_{p*} : \mathfrak{g} \to \mathfrak{g}$ is the pushforward by ϕ_p (tangent spaces at different points of G are identified to be $\mathfrak{g}$ by the Maurer–Cartan form), and Ad denotes the adjoint action of G on $\mathfrak{g}$. The horizontal subspace H_p is then given by [10, p. 31]

$$\mathrm{H}_p \coloneqq \{X \in \mathrm{T}_p P \mid \tilde{\omega}_p(X) = 0\}\, .$$

Given a connection $\tilde{\omega}$ on a principal bundle P, we can define the **exterior covariant derivative** of any k-form α on P as

$$\mathrm{D}\alpha(X,\dots) \coloneqq \mathrm{d}\alpha\big(X^{\mathrm{H}},\dots\big)\,,$$

where $\mathrm{d}\alpha$ is the usual exterior derivative of α, and $X^{\mathrm{H}} \in \mathrm{H}_p$ is the horizontal component of $X \in \mathrm{T}_pP$. The exterior covariant derivative $\tilde{\Omega} \coloneqq \mathrm{D}\tilde{\omega}$ of $\tilde{\omega}$ itself is called the **curvature** of the connection. The second Cartan structural equation states that [26, p. 77]

$$\tilde{\Omega} = \mathrm{d}\tilde{\omega} + \frac{1}{2}\left[\tilde{\omega},\tilde{\omega}\right],$$

where $[\cdot,\cdot]$ is the bracket operation of the graded Lie superalgebra of $\mathfrak{g}$-valued forms.

Given a principal bundle P and a left action $\chi : G \times F \to F$ of G on another manifold F, we can construct an **associated bundle** $Q \coloneqq (P \times F)/{\sim}$, where the equivalence relation $\sim$ is defined such that

$$\forall g \in G : (p,f) \sim \big(pg, g^{-1}f\big)\,.$$

From now on, we will denote the equivalence class represented by (p,f) as $[p,f]$. A natural differentiable structure can be defined on Q so that it is a manifold [26, p. 55]. The bundle projection $\hat{\pi} : Q \to M$ can be defined as

$$\hat{\pi}([p,f]) \coloneqq \pi(p)\,,$$

which can be easily proven to be independent of the choice of p.

The space F is called the **typical fiber** of Q, and people often take F to be a vector space and take χ to be a representation of G on F, in which case the associated bundle is a **vector bundle**. In this case, each fiber $\hat{\pi}^{-1}(x)$ can also be made into a vector space by defining that

$$[p,f_1] + [p,f_2] \coloneqq [p, f_1+f_2]\,,\quad c\,[p,f] \coloneqq [p,cf]\,,$$

where $c \in \mathbb{R}$ (or $\mathbb{C}$, depending on the field that F is over); this definition is independent of the choice of p.

One defines the vertical subspace at any point $q \in Q$ as

$$V_q \coloneqq \{X \in \mathrm{T}_qQ \mid \hat{\pi}_*(X) = 0\}\,.$$

If $q \in \hat{\pi}^{-1}(x)$, then there is a natural linear isomorphism $\mathrm{V}_q \to \hat{\pi}^{-1}(x)$. The construction needs a basis $\{e_\mu\}$ on $\hat{\pi}^{-1}(x)$, then construct $\partial_\mu|_q \mapsto e_\mu$, where $\{\partial_\mu\}$ is the partial derivative w.r.t. $\{v^\mu\}$, defined as the components of vectors in $\hat{\pi}^{-1}(x)$ for the basis $\{e_\mu\}$. We will identify those two vector spaces for each q from now on.

Similar to a principal bundle, one defines a section on Q to be a function $\hat{\sigma} : U \to Q$ for some open set $U \subseteq M$ such that $\hat{\pi} \circ \hat{\sigma} = \mathrm{id}_U$; and one defines a connection on Q to be a smooth choice of a horizontal subspace $H_q \subseteq \mathrm{T}_qQ$ that is complementary to V_q at each point $q \in Q$ and satisfies that

$$c_*(\mathrm{H}_q) = \mathrm{H}_{cq}\,,\quad c \neq 0,$$

where c_* is the pushforward by $q \mapsto cq$. With a connection on Q, the **covariant derivative** of a section $\hat{\sigma}$ equals another section[15] [26, p. 115]

$$\nabla_T\hat{\sigma} \coloneqq \left(\hat{\sigma}_*T\right)^{\mathrm{V}},$$

where T is a tangent vector field on U, and the superscript V means taking the vertical component. This expression is linear in T, so one may use physicists' notation to write as

$$\nabla_T\hat{\sigma} = T^a\nabla_a\hat{\sigma}, \tag{D1}$$

where a is the abstract index for the tangent spaces of M.

Now, we can apply this machinary to reproduce some familiar results. Denote $\mathrm{F}M$ to be the **frame bundle** of an n-dimensional smooth manifold M, defined as a principal bundle with structure group $\mathrm{GL}(n)$. Its total space is a $\big(n+n^2\big)$-dimensional manifold defined as [26, p. 55]

$$\mathrm{F}M \coloneqq \bigsqcup_{x\in M}\{\text{ordered bases of } \mathrm{T}_xM\}\,,$$

where T_xM is the tangent space at x. The free right action of $g \in \mathrm{GL}(n)$ on $\mathrm{F}M$ is defined by

$$g\left(x,\{e_\mu\}\right) \coloneqq \big(x, e_\nu g^\nu{}_\mu\big)\,.$$

The bundle projection and the local trivializations are easy to construct. The **tangent bundle** $\mathrm{T}M$, then, is defined as the associated vector bundle of $\mathrm{F}M$ with the defining representation of $\mathrm{GL}(n)$ on $\mathbb{R}^n$.

The interesting point of this construction is that, a section of $\mathrm{T}M$ is just a vector field on M. To see this, explicitly write the action of $\mathrm{GL}(n)$ on an element of $\mathrm{F}M$:

$$g\left(x,\{e_m u\}\,;f^\rho\right) = \Big(x, e_\nu g^\nu{}_\mu; \left(g^{-1}\right)^\rho{}_\sigma f^\sigma\Big)\,.$$

Naturally, for each $x \in M$, one can then construct

$$v \coloneqq e_\mu f^\mu \in \mathrm{T}_xM.$$

It is then easy to check that v is invariant under $\mathrm{GL}(n)$.

Another interesting point is that, given a connection 1-form $\tilde{\omega}$ on $\mathrm{F}M$, the induced covariant derivative for

[15] This is not the actual definition of the covariant derivative. Conventionally the definition is in terms of the derivative on the horizontal lift of an arbitrarily chosen curve. However, the expression shown here is more manifestly independent of any arbitrary choices, so I think it would be more beneficial to use this expression.

sections of $\mathrm{T}M$ then recovers a covariant derivative for vector fields on M. Assume that we have a vector field v given by a section $\hat{\sigma} : M \to \mathrm{T}M$. To express the covariant derivative $\nabla_T v$ (where T is a vector field on M) in terms of numbers, we need to first choose a section $\sigma : M \to \mathrm{F}M$ (equivalent to choose the basis vectors on which we will express the vector components). Let us say it is the coordinate basis given by a set of coordinates $\{x^\mu\}$:

$$\sigma(x) := \left(x, \left\{ (\partial_\mu)^a\big|_x \right\}\right) .$$

Then, after some calculations, we can show that [26, p. 144]

$$(\nabla_T v)^a = T^\sigma \left((\partial_\mu)^a \left(\frac{\partial v^\mu}{\partial x^\sigma} + \omega^\mu{}_{\nu\sigma} v^\nu \right) \right),$$

where $\omega^\mu{}_{\nu\sigma}$ are the components of the pullback $\omega := \sigma^* \tilde{\omega}$ of the connection 1-form $\tilde{\omega}$ under the section σ. One can see that $\omega^\mu{}_{\nu\sigma}$ are exactly the **Christoffel symbols** of the covariant derivative on this coordinate basis [26, p. 141].

For an orientable pseudo-Riemannian manifold M, it is often also beneficial to consider the **oriented orthonormal frame bundle** $\mathrm{F_{SO}}M$, defined very similarly to $\mathrm{F}M$, but consists of orthonormal bases instead of general bases, and its structure group is the special orthogonal group of the metric on M. Using this principal bundle, one can define the tangent bundle $\mathrm{T}M$ as the associated bundle with the defining representation of the special orthogonal group, and it is exactly isomorphic (and thus identified) to the tangent bundle defined using the frame bundle. The only difference is that, when calculating the covariant derivative of a vector field, the **Ricci rotation coefficients** (see e.g. [7, p. 50]) instead of the Christoffel symbols appears. This corresponds to the so-called tetrad formalism in general relativity, and a **tetrad** is just a section of $\mathrm{F_{SO}}M$. Specially, when the induced covariant derivative is torsion-free, the connection is called the **Levi-Civita connection**.

The Riemann curvature tensor $R_{abc}{}^d$ can be interpreted as the pullback $\Omega = \sigma^* \tilde{\Omega}$ of the curvature 2-form $\tilde{\Omega}$, which can be directly related by using the definition of $R_{abc}{}^d$ and the second Cartan structural equation. Explicitly,

$$-\Omega(e^\rho, e^\sigma)^\nu{}_\mu = R_{\rho\sigma\mu}{}^\nu,$$

where the left-hand side identified $\Omega(e^\rho, e^\sigma)$ in the Lie algebra of the special orthogonal group as an element in $\mathrm{End}(\mathbb{R}^n)$ by the Lie algebra representation.

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