Calculus in microeconomics

Ulysses

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1 Free market of one good

1.1 Basic concepts

Definition 1.1.1 (quantity)**.** *The **quantity** is a variable* $Q \in [0, +\infty)$.

The economic meaning of Q is to measure how much good there is involved in the market within a unit of time. Its unit is unit quantity per unit time.

Definition 1.1.2 (price)**.** *The **price** is a variable* $P \in [0, +\infty)$.

The economic meaning of P is to measure the value of good. Its unit is monetary unit per unit quantity.

Definition 1.1.3 (cost and benefit)**.** *The* ***cost*** *of the good is a function* $C : [0, +\infty) \to [0, +\infty)$.

The ***benefit*** *of the good is a function* $B : [0, +\infty) \to [0, +\infty)$.

They satisfies the axioms in Section 1.1.

C maps Q to the total cost $C(Q)$ (whose unit is monetary unit per unit time) for producers to produce such amount of good within a unit of time. Similarly, B maps Q to the total benefit $B(Q)$ (whose unit is monetary unit per unit time) for consumers to consume such amount of good within a unit of time.

Here are some axioms concluded from principles of economics.

Axiom 1.1.1. *C and B are strictly increasing on* $[0, +\infty)$.

Axiom 1.1.1 is obvious when thinking about the economic meaning of C and B.

Axiom 1.1.2. *C and B are twice differentiable on* $[0, +\infty)$.

Axiom 1.1.2 is a mathematical requirement without which we will lack mathematical tools to conveniently prove some theorems.

Axiom 1.1.3.

$$C(0) = B(0) = 0.$$

Axiom 1.1.3 is as obvious as Axiom 1.1.1.

Axiom 1.1.4.

$$C'(0) < B'(0)\,.$$

Axiom 1.1.4 is sometimes optional, but it is necessary if the market should exist.

Axiom 1.1.5.

$$\lim_{Q\to+\infty} C'(Q) = +\infty, \qquad \lim_{Q\to+\infty} B'(Q) = 0.$$

Axiom 1.1.5 is as necessary as Axiom 1.1.4. Both of them restrict the boundary conditions of C' and B'.

Axiom 1.1.6 (law of supply and law of demand)**.** *C is strictly convex downward everywhere on* $[0, +\infty)$.

B is strictly concave downward everywhere on $[0, +\infty)$.

Axiom 1.1.6 is a important and basic law in economics. Note that law of demand is mathematically equivalent to law of diminishing marginal benefit.

From these axioms, we can easily deduce some corollaries using simple math.

Theorem 1.1.1. *The inverse functions of C' and B' exist if their codomains are restricted to their ranges. In other words, C' and B' are injective.*

Proof. According to Axiom 1.1.6, C' is strictly increasing on $[0, +\infty)$. According to Axiom 1.1.5, C' has range $[C'(0), +\infty)$. Thus, we can define its inverse function $C'^{-1} : [C'(0), +\infty) \to [0, +\infty)$.

The proof for existence of B'^{-1} is similar. □

Definition 1.1.4 (supply and demand). *Function C' is called **marginal cost (MC)** or **supply (S)**.*

*Function B' is called **marginal benefit (MB)** or **demand (D)**.*

Marginal cost and marginal benefit shares the unit with P (monetary unit per unit quantity).

It may be confusing why marginal cost is the same as supply. Here is a simple explanation: producers do not operate at a price below the shutdown point. It is similar to explain why marginal benefit is the same as demand.

By convention in the context of supply and demand graphs, the inverse functions C'^{-1} and B'^{-1} are used instead of C' and B'.

1.2 Surplus

Definition 1.2.1 (surplus). *The **producer surplus** $S_C : [0, +\infty)^2 \to \mathbb{R}$ is defined as*

$$S_C(Q, P) := PQ - C(Q)\,.$$

*The **consumer surplus** $S_B : [0, +\infty)^2 \to \mathbb{R}$ is defined as*

$$S_B(Q, P) := B(Q) - PQ.$$

Definition 1.2.2 (social surplus). *The **social surplus** $S : [0, +\infty) \to \mathbb{R}$ is defined as*

$$S := B - C.$$

Definition 1.2.3 (formal surplus). *The **formal producer surplus** $S_{\mathrm{S}C} : [0, +\infty) \to \mathbb{R}$ is defined as*

$$S_{\mathrm{S}C}(Q) := S_C(Q, C'(Q))\,.$$

*The **formal consumer surplus** $S_{\mathrm{S}B} : [0, +\infty) \to \mathbb{R}$ is defined as*

$$S_{\mathrm{S}B}(Q) := S_B(Q, B'(Q))\,.$$

Definition 1.2.4 (actual social surplus). *The **actual social surplus** $S_{\mathrm{S}} : [0, +\infty) \to \mathbb{R}$ is defined as*

$$S_{\mathrm{S}} := S_{\mathrm{S}C} + S_{\mathrm{S}B}.$$

The unit of surplus is monetary unit per unit time. The economic meaning of surplus is the net gain within a unit of time. For producers, the net gain is part of income PQ that exceeds the cost $C(Q)$. For consumers, the net gain is part of benefit $B(Q)$ that exceeds the expenditure PQ.

Theorem 1.2.1. *For any $Q \geq 0$,*

$$S_{\mathrm{SC}}(Q) \geq 0, \qquad S_{\mathrm{SB}}(Q) \geq 0,$$

where the equality holds iff $Q = 0$.

Proof.

$$\begin{aligned}
S_{\mathrm{SC}}(Q) &= S_C(Q, C'(Q)) && \text{(Definition 1.2.3)}\\
&= C'(Q)\,Q - C(Q) && \text{(Definition 1.2.1)}\\
&= \int_0^Q C'(Q)\,\mathrm{d}x - \int_0^Q C'(x)\,\mathrm{d}x\\
&= \int_0^Q (C'(Q) - C'(x))\,\mathrm{d}x.
\end{aligned}$$

According to Axiom 1.1.6, C' is strictly increasing, so when $Q > 0$, for any $x \in [0, Q)$, $C'(Q) - C'(x) > 0$. Thus, the integral is an integral of an always-positive function, which means $S_{\mathrm{SC}}(Q) > 0$.

When $Q = 0$, it can be easily proved that $S_{\mathrm{SC}}(Q) = 0$.

It is similar to show that $S_{\mathrm{SB}}(Q) \geq 0$. □

Theorem 1.2.2. *S_{SC} and S_{SB} are strictly increasing.*

Proof.

$$\begin{aligned}
S'_{\mathrm{SC}}(Q) &= \frac{\mathrm{d}}{\mathrm{d}Q} S_C(Q, C'(Q)) && \text{(Definition 1.2.3)}\\
&= \frac{\mathrm{d}}{\mathrm{d}Q}\left(C'(Q)\,Q - C(Q)\right) && \text{(Definition 1.2.1)}\\
&= C''(Q)\,Q\\
&> 0 && \text{(Axiom 1.1.6)},
\end{aligned}$$

so S_{SC} is strictly increasing.

It is similar to prove that S_{SB} is strictly increasing. □

Theorem 1.2.3. *For any $Q, P \in [0, +\infty)$,*

$$S(Q) = S_C(Q, P) + S_B(Q, P)\,,$$

which is independent of P.

Proof.

$$\begin{aligned}
S(Q) &= B(Q) - C(Q) && \text{(Definition 1.2.2)}\\
&= B(Q) - PQ + PQ - C(Q)\\
&= S_C(Q, P) + S_B(Q, P) && \text{(Definition 1.2.1)}.
\end{aligned}$$

□

1.3 Price elasticities

Definition 1.3.1 (price elasticity)**.** *The* ***price elasticity*** *of a function* $f : [0, +\infty) \to [0, +\infty)$ *with non-zero derivative everywhere on* $(0, +\infty)$ *is a function* $E_f : (0, +\infty) \to \mathbb{R}$ *defined as*

$$E_f(Q) := \frac{f(Q)}{f'(Q)\,Q}.$$

In Definition 1.3.1, the involved function f is a function mapping Q to P, usually C' or B'. In these two special cases, the price elasticity is respectively called **price elasticity of supply (PES)** and **price elasticity of demand (PED)**. The economic meaning of price elasticity is to show the responsiveness (elasticity) of Q to P. It is the ratio of percentage change in Q and that in P.

Theorem 1.3.1. *For any* $Q > 0$*,*

$$E_{C'}(Q) > 0, \qquad E_{B'}(Q) < 0.$$

Proof. According to Definition 1.3.1,

$$E_{C'}(Q) = \frac{C'(Q)}{C''(Q)\,Q}.$$

According to Axiom 1.1.1, $C'(Q) > 0$. According to Axiom 1.1.6, $C''(Q) > 0$. Thus, $E_{C'}(Q) > 0$.

Similarly, it can be proved that $E_{B'}(Q) < 0$. □

Theorem 1.3.2. *If* $\lim_{Q\to+\infty} E_{B'}(Q) \in (-1, 0]$*, then* B *has upper bound.*

Proof. According to Theorem 1.3.1, $\lim_{Q\to+\infty} E_{B'}(Q)$ cannot be positive.

Case 1: $\lim_{Q\to+\infty} E_{B'}(Q) = 0$.

According to definition of limit, for any $\varepsilon > 0$, there exists n such that for any $Q > n$, we have

$$|E_{B'}(Q)| < \varepsilon.$$

According to Theroem 1.3.1, $E_{B'}(Q) < 0$. Thus, the equation can be written as

$$E_{B'}(Q) > -\varepsilon.$$

Substitute Definition 1.3.1, and let $m := \frac{1}{\varepsilon}$, and then we can derive

$$\frac{B'(Q)}{B''(Q)\,Q} > -\frac{1}{m},$$

which means

$$\frac{B''(Q)}{B'(Q)} < -\frac{m}{Q}.$$

Note that it can be written as

$$\frac{\mathrm{d}}{\mathrm{d}Q}\ln(B'(Q)) < -\frac{\mathrm{d}}{\mathrm{d}Q}\left(m \ln Q\right),$$

or

$$\frac{\mathrm{d}}{\mathrm{d}Q}\ln(B'(Q)\,Q^m) < 0.$$

This means $B'(Q)\,Q^m$ is strictly decreasing on $(n, +\infty)$, so for any $Q > q > n$,

$$B'(Q)\,Q^m < B'(q)\,q^m,$$

which means that for any $Q > q$,

$$B'(Q) < aQ^{-m},$$

where $a := B'(q)\,q^m$.

Write $\lim_{Q\to+\infty} B(Q)$ in the integral form

$$\begin{aligned}\lim_{Q\to+\infty} B(Q) &= B(q) + \int_q^{+\infty} B'(Q)\,\mathrm{d}Q\\ &< B(q) + \int_q^{+\infty} aQ^{-m}\mathrm{d}Q.\end{aligned}$$

Since m is an arbitrary positive number, we can take $m > 1$. In this way, the integral above converges, and we can derive that

$$\lim_{Q\to+\infty} B(Q) < B(q) + \frac{C}{m-1}q^{1-m}.$$

Thus, according to comparison test, $\lim_{Q\to+\infty} B(Q)$ must exist.

It can be easily proved that $\lim_{Q\to+\infty} B(Q)$ is the supremum of B, so B has upper bound.

Case 2: $\lim_{Q\to+\infty} E_{B'}(Q) \in (-1, 0)$.

Let

$$l := -\frac{1}{\lim_{Q\to+\infty} E_{B'}(Q)} \in (1, +\infty)\,,$$

and then we have

$$\lim_{Q\to+\infty}\frac{1}{E_{B'}(Q)} = -l.$$

According to definition of limit, for any $m > 0$, there exists n such that for any $Q > n$, we have

$$\left|\frac{1}{E_{B'}(Q)} + l\right| < m,$$

so

$$\frac{1}{E_{B'}(Q)} < m - l.$$

Substitute Definition 1.3.1, and then we can derive

$$\frac{B''(Q)\,Q}{B'(Q)} < m - l.$$

Use similar tricks in Case 1, and then we can derive

$$\frac{\mathrm{d}}{\mathrm{d}Q}\ln\bigl(B'(Q)\,Q^{l-m}\bigr) < 0.$$

Use similar tricks in Case 1, and take $m \in (0, l-1)$, and we can prove that B has upper bound. ☐

Theorem 1.3.3. *If B has upper bound, then $\lim_{Q\to+\infty} E_{B'}(Q)$ can be any value in $[-1, 0]$, and can also non-exist, and cannot be less than -1.*

Proof. Example where $\lim_{Q\to+\infty} E_{B'}(Q)$ does not exist:

Let

$$B(Q) := \int_0^Q \frac{2+\sin\ln(1+x)}{(1+x)^2}\mathrm{d}x.$$

Then, B satisfies Axiom 1.1.1, Axiom 1.1.2, Axiom 1.1.3, Axiom 1.1.5, and Axiom 1.1.6 as can be justified.

B has upper bound because

$$\begin{aligned} B(Q) &< \int_0^Q \frac{3}{(1+x)^2}\mathrm{d}x \\ &< 3\int_1^{+\infty}\frac{\mathrm{d}x}{x^2} \\ &= 3. \end{aligned}$$

Calculate $E_{B'}(Q)$, and we have

$$\begin{aligned} E_{B'}(Q) &:= \frac{B'(Q)}{B''(Q)\,Q} && \text{(Definition 1.3.1)} \\ &= \frac{1+x}{x}\cdot\frac{2+\sin\ln(1+x)}{-2\,(2+\sin\ln(1+x))+\cos\ln(1+x)}. \end{aligned}$$

When $x = \mathrm{e}^{2k\pi}-1$,

$$E_{B'}(Q) = \frac{1+x}{x}\cdot\frac{-2}{3} \xrightarrow{x\to+\infty} -\frac{2}{3}.$$

When $x = \mathrm{e}^{\left(k+\frac{1}{4}\right)2\pi}-1$,

$$E_{B'}(Q) = \frac{1+x}{x}\cdot\frac{-1}{2} \xrightarrow{x\to+\infty} -\frac{1}{2}.$$

($k \in \mathbb{N}$.)

Thus, $\lim_{Q\to+\infty} E_{B'}(Q)$ does not exist.

Example where $\lim_{Q\to+\infty} E_{B'}(Q) = 0$:

Let

$$B(Q) := 1-\mathrm{e}^{-Q}.$$

Then, B satisfies Axiom 1.1.1, Axiom 1.1.2, Axiom 1.1.3, Axiom 1.1.5, and Axiom 1.1.6, and has upper bound 1 as can be justified.

Calculate $E_{B'}(Q)$, and we have

$$\begin{aligned} E_{B'}(Q) &:= \frac{B'(Q)}{B''(Q)\,Q} && \text{(Definition 1.3.1)} \\ &= -\frac{1}{Q} \\ &\xrightarrow{Q\to+\infty} 0. \end{aligned}$$

Example where $\lim_{Q\to+\infty} E_{B'}(Q) = -1$:

Let

$$B(Q) := 1 - \frac{1}{\ln(\mathrm{e}+Q)}.$$

Then, B satisfies Axiom 1.1.1, Axiom 1.1.2, Axiom 1.1.3, Axiom 1.1.5, and Axiom 1.1.6, and has upper bound 1 as can be justified.

Calculate $E_{B'}(Q)$, and we have

$$\begin{aligned} E_{B'}(Q) &:= \frac{B'(Q)}{B''(Q)\,Q} && \text{(Definition 1.3.1)} \\ &= -\frac{\mathrm{e}+Q}{Q} \cdot \frac{\ln(\mathrm{e}+Q)}{2+\ln(\mathrm{e}+Q)} \\ &\xrightarrow{Q\to+\infty} -1. \end{aligned}$$

Example where $\lim_{Q\to+\infty} E_{B'}(Q)$ can be any value in $(-1, 0)$:

Let

$$B(Q) := 1 - (1+x)^{1+\frac{1}{l}},$$

where $l \in (-1, 0)$ is an arbitrary parameter. Then, B satisfies Axiom 1.1.1, Axiom 1.1.2, Axiom 1.1.3, Axiom 1.1.5, and Axiom 1.1.6, and has upper bound 1 as can be justified.

Calculate $E_{B'}(Q)$, and we have

$$\begin{aligned} E_{B'}(Q) &:= \frac{B'(Q)}{B''(Q)\,Q} && \text{(Definition 1.3.1)} \\ &= l\left(1+\frac{1}{Q}\right) \\ &\xrightarrow{Q\to+\infty} l. \end{aligned}$$

Prove that $\lim_{Q\to+\infty} E_{B'}(Q) \geq -1$:

Use means of contradiction.

Suppose $\lim_{Q\to+\infty} E_{B'}(Q) < -1$, and use similar tricks in the proof of Theorem 1.3.2. We can derive that there exists $q > 0$ such that for any $Q > q$,

$$B'(Q)\,Q > B'(q)\,q,$$

which means

$$B'(Q) > \frac{a}{x},$$

where $a := B'(q)\, q$.

Integrate both sides, and we can derive

$$B(Q) > B(q) + a \ln \frac{Q}{q} \xrightarrow{Q \to +\infty} +\infty,$$

which contradicts with it that B has upper bound. $\square$

1.4 Revenue–expenditure

Definition 1.4.1 (revenue–expenditure)**.** *The **revenue–expenditure** of a function $f : [0, +\infty) \to [0, +\infty)$ is a function $R_f : [0, +\infty) \to [0, +\infty)$ defined as*

$$R_f(Q) := f(Q)\, Q.$$

In Definition 1.4.1, the involved function f is a function mapping Q to P, usually C' or B'. One may usually find that the surplus can often be represented with difference of revenue–expenditure and cost/benefit.

Theorem 1.4.1. *If $f : [0, +\infty) \to [0, +\infty)$ is a function with non-zero derivative everywhere on $(0, +\infty)$, and $Q_0 \neq 0$ is an extremal of R_f such that $f(Q_0) \neq 0$, then $E_f(Q_0) = -1$.*

Proof. Because R_f is a differentiable function, its extremal is a solution to equation $R_f' = 0$ or the boundary of its domain. Since $Q_0 \neq 0$, Q_0 is not the boundary of the domain of R_f. Thus $R_f'(Q_0) = 0$.

$$\begin{aligned}
R_f'(Q) &= \frac{\mathrm{d}}{\mathrm{d}Q} \left(f(Q)\, Q\right) && \text{(Definition 1.4.1)} \\
&= f'(Q)\, Q + f(Q) \\
&= \frac{f(Q)}{E_f(Q)} + f(Q) && \text{(Definition 1.3.1)} \\
&= \left(1 + \frac{1}{E_f(Q)}\right) f(Q)\, .
\end{aligned}$$

From the calculations above, since $R_f'(Q_0) = 0$ and $f(Q_0) \neq 0$, we have $E_f(Q) = -1$. $\square$

Theorem 1.4.2. *For any $Q \geq 0$,*

$$C'(Q) \leq R_{C'}'(Q)\, , \qquad B'(Q) \geq R_{B'}'(Q)\, ,$$

where the equality holds iff $Q = 0$.

Proof. Consider $Q > 0$, and then we have

$$\begin{aligned} R'_{C'}(Q) &= \left(1 + \frac{1}{E_{C'}(Q)}\right) C'(Q) && \text{(Theorem 1.4.1)} \\ &> C'(Q) && \text{(Theorem 1.3.1).} \end{aligned}$$

As for $Q = 0$, it can be easily proved that $R'_{C'}(0) = C'(0)$.

It is similar to prove $B'(Q) \geq R'_{B'}(Q)$. $\square$

Theorem 1.4.3. *If B has upper bound, then $\lim_{Q\to+\infty} R_{B'}(Q) = 0$.*

Proof. Because B has upper bound and is strictly increasing, $\lim_{Q\to+\infty} B(Q)$ exists. According to Cauchy's convergence test, for any $\frac{\varepsilon}{2} > 0$, there exists n such that when $Q > \frac{Q}{2} > n$,

$$B(Q) - B\left(\frac{Q}{2}\right) < \frac{\varepsilon}{2}$$

(the absolute value bracket is omitted due to Axiom 1.1.1).

On the other hand,

$$\begin{aligned} B(Q) - B\left(\frac{Q}{2}\right) &= \int_{\frac{Q}{2}}^{Q} B'(x)\,\mathrm{d}x \\ &> \int_{\frac{Q}{2}}^{Q} B'(Q)\,\mathrm{d}x && \text{(Axiom 1.1.6)} \\ &= \frac{1}{2}B'(Q)\,Q \\ &= \frac{1}{2}R_{B'}(Q) && \text{(Definition 1.4.1).} \end{aligned}$$

Thus, when $Q > n$, $0 < R_{B'}(Q) < \varepsilon$, which means $\lim_{Q\to+\infty} R_{B'}(Q) = 0$. $\square$

1.5 Perfect competition

Theorem 1.5.1. *S has a unique maximal on $(0, +\infty)$, which is the unique solution of equation $C' = B'$.*

Proof. It can be just proved by showing that the equation $C' = B'$ has a unique solution on $(0, +\infty)$. In other words, if we construct a function

$$f := S' = C' - B',$$

then f has a unique zero on $(0, +\infty)$.

According to Axiom 1.1.4, we have

$$f(0) = C'(0) - B'(0) < 0.$$

According to Axiom 1.1.6, C' is strictly increasing, and B' is strictly decreasing, so f is strictly increasing.

According to Axiom 1.1.5, we have

$$\lim_{x \to \infty} f(x) = +\infty,$$

which means that f has range $[f(0), +\infty)$, so f has its inverse function f^{-1} : $[f(0), +\infty) \to [0, +\infty)$.

Because $f(0) < 0$, we have $0 \in [f(0), +\infty)$. Thus, $f^{-1}(0)$ is the unique solution of equation $f = 0$.

After that, it can be easily proved that $f^{-1}(0) \in (0, +\infty)$ is the unique maximal of S. □

Definition 1.5.1 (equilibrium of a perfect competition market)**.** *The* ***equilibrium of a perfect competition market*** Q^* *is defined as the unique maximal of S, whose existence is ensured by Theorem 1.5.1.*

Theorem 1.5.2. *For any $Q \in (0, Q^*)$, $S_{\mathrm{S}}(Q) < S(Q)$. For any $Q \in (Q^*, +\infty)$, $S_{\mathrm{S}}(Q) > S(Q)$.*

Proof.

$$\begin{aligned}
S_{\mathrm{S}}(Q) &= S_{\mathrm{S}C}(Q) + S_{\mathrm{S}B}(Q) && \text{(Definition 1.2.4)}\\
&= S_C(Q, C'(Q)) + S_B(Q, B'(Q)) && \text{(Definition 1.2.3)}\\
&= C'(Q)\,Q - C(Q) + B(Q) - B'(Q)\,Q && \text{(Definition 1.2.1)}\\
&= S(Q) - S'(Q)\,Q && \text{(Definition 1.2.2).}
\end{aligned}$$

According to Theorem 1.5.1, for $Q < Q^*$, $S'(Q) > 0$, and for $Q > Q^*$, $S'(Q) < 0$. In this way, the conclusion can be proved. □

1.6 Monopoly and monopsony

Theorem 1.6.1. $Q \mapsto S_C(Q, B'(Q))$ *has at least one maximal on* $(0, +\infty)$*, which is a solution of equation* $R'_{B'} = C'$*.*

$Q \mapsto S_B(Q, C'(Q))$ *has at least one maximal on* $(0, +\infty)$*, which is a solution of equation* $R'_{C'} = B'$*.*

Proof. According to Definition 1.2.1 and Definition 1.4.1,

$$f(Q) := S_C(Q, B'(Q)) = R_{B'}(Q) - C(Q)\,.$$

According to Axiom 1.1.4 and Theorem 1.4.2,

$$f'(0) = B'(0) - C'(0) > 0,$$

so 0 cannot be a maximal of f.

According to Axiom 1.1.5 and Theorem 1.4.2,

$$\lim_{Q \to +\infty} f'(Q) < \lim_{Q \to +\infty} (B'(Q) - C'(Q)) = -\infty.$$

Thus, using definition of limit, it can be easily proved that there exists $Q_0 > 0$ such that $f'(Q_0) < 0$, so Q_0 cannot be a maximal of f.

Since f is continuous, its range on closed interval $[0, Q_0]$ is a closed interval. Suppose the supremum of the range is $f(z)$, where $z \in (0, Q_0)$. Then $z \in (0, +\infty)$ is a maximal of f.

Because f is differentiable everywhere, its maximal must be a solution of equation $f' = 0$, which is the same equation as $R'_{B'} = C'$.

It is similar to prove that $Q \mapsto S_B(Q, C'(Q))$ has at least one maximal on $(0, +\infty)$, which is a solution of equation $R'_{C'} = B'$. □

According to principles of economics, the maximal of $Q \mapsto S_C(Q, B'(Q))$ mentioned in Theorem 1.6.1 is the **equilibrium of a monopoly market**. Similarly, the maximal of $Q \mapsto S_B(Q, C'(Q))$ mentioned in 1.6.1 is the **equilibrium of a monopsony market**.

2 Market of one good with gov intervention

Gov is something able to operate the market directly.

2.1 Taxes and subsidies

Definition 2.1.1 (tax–subsidy). *A* ***tax–subsidy*** *is two functions $T_C : [0, +\infty)^2 \to [0, +\infty)$ and $T_B : [0, +\infty)^2 \to [0, +\infty)$ such that functions $C_@(Q) := T_C(Q, C(Q))$ and $B_@(Q) := T_B(Q, B(Q))$ satisfies the axioms in Section 1.1 as if they were C and B.*

In this article, we use $X_@$ to denote the X after applying a tax–subsidy.

Definition 2.1.2 (tax and subsidy). *A* ***tax*** *is a tax–subsidy such that for any $Q \geq 0$,*

$$C'_@(Q) \geq C'(Q)\,, \qquad B'_@(Q) \leq B'(Q)\,.$$

A ***subsidy*** *is a tax–subsidy such that for any $Q \geq 0$,*

$$C'_@(Q) \leq C'(Q)\,, \qquad B'_@(Q) \geq B'(Q)\,.$$

The two functions mentioned above maps a pair of Q and cost/benefit to a new cost/benefit.

Note that normally taxes and subsidies are not called by a joint name, so here the word "tax–subsidy" is an unconventional name. What is more, "tax–subsidy" is not only a joint name because there can be some tax–subsidy that is neither a tax nor a subsidy.

Theorem 2.1.1. *$Q^*_@ \leq Q^*$ if a tax is applied, and $Q^*_@ \geq Q^*$ if a subsidy is applied.*

Proof. Consider the case of applying a tax.

Use means of contradiction. Suppose $Q^*_@ > Q^*$, and then

$$\begin{aligned}
C'_@(Q^*_@) &> C'(Q^*_@) && \text{(Definition 2.1.2)}\\
&\geq C'(Q^*) && \text{(Axiom 1.1.1)}\\
&= B'(Q^*) && \text{(Theorem 1.5.1)}\\
&> B'(Q^*_@) && \text{(Axiom 1.1.1)}\\
&\geq B'_@(Q^*_@) && \text{(Definition 2.1.2)},
\end{aligned}$$

which contradicts with $C'_@(Q^*_@) = B'_@(Q^*_@)$ from Theorem 1.5.1. The proof for the case of applying a subsidy is similar. □

Theorem 2.1.1 shows how a tax–subsidy can affect a perfect competition market.

Theorem 2.1.2. *$S_{\mathrm{S}}(Q^*_@) \leq S(Q^*_@)$ if a tax is applied, and $S_{\mathrm{S}}(Q^*_@) \geq S(Q^*_@)$ if a subsidy is applied.*

Proof. Use Theorem 1.5.2 and Theorem 2.1.1, it can be easily proved. □

Definition 2.1.3 (tax–subsidy revenue–expenditure)**.** *The **tax–subsidy revenue–expenditure** is the difference $S(Q^*_@) - S_{\mathrm{S}}(Q^*_@)$.*

As can be seen, the tax–subsidy revenue–expenditure is positive if a tax is applied, which is called the **tax revenue**. The tax–subsidy revenue–expenditure is negative if a subsidy is applied, which is called the **subsidy expenditure**.

2.2 Price controls

This is so mathematically trivial that I do not want to write anything here.

3 Free market of multiple goods

3.1 Basic concepts

We need to use multi-variable calculus instead of single-variable calculus. We are going to rewrite Section 1.1 in a multi-variable form.

Definition 3.1.1 (number of goods)**.** *The **number of goods** is $N \in \mathbb{Z}^+$.*

Definition 3.1.2 (prices)**.** *The **prices** are a vector $\mathbf{P} \in [0, +\infty)^N$.*

Definition 3.1.3 (quantities)**.** *The **quantities** are a vector $\mathbf{Q} \in [0, +\infty)^N$.*

Definition 3.1.4 (costs and benefits)**.** *The **costs** and **benefits** are scalar fields $C : [0, +\infty)^N \to [0, +\infty)$ and $B : [0, +\infty)^N \to [0, +\infty)$.*

Axiom 3.1.1. *C and B are twice differentiable on $[0, +\infty)^N$.*

Definition 3.1.5 (supplies and demands)**.** *The* ***supplies*** *or* ***marginal costs*** *are a vector field* $\mathbf{C}' : [0,+\infty)^N \to [0,+\infty)^N$*, and the* ***demands*** *or* ***marginal benefits*** *are a vecotr field* $\mathbf{B}' : [0,+\infty)^N \to [0,+\infty)^N$*. They are respectively the gradient of* C *and* B*.*

If $\mathbf{X}$ is a function defined on $\mathbb{R}^N$, then $\mathbf{X}'$ denotes its Jacobi matrix. Specially, if X is a scalar field, $\mathbf{X}'$ is its gradient, and $\mathbf{X}''$ is its Hessian matrix.

Axiom 3.1.2. *For any* $\mathbf{Q} \in [0,+\infty)^N$*,*

$$\mathbf{C}'(\mathbf{Q}) > 0, \qquad \mathbf{B}'(\mathbf{Q}) > 0.$$

A matrix/vector $\mathbf{X} > 0$ means each component of $\mathbf{X}$ is positive.

Axiom 3.1.3 (law of supply and law of demand)**.** *For any* $\mathbf{Q} \in [0,+\infty)^N$*,*

$$\mathbf{C}''(\mathbf{Q}) \succ 0, \qquad \mathbf{B}''(\mathbf{Q}) \prec 0.$$

A real symmetric matrix $\mathbf{X} \succ 0$ means $\mathbf{X}$ is positive-definite.

Axiom 3.1.4.

$$\mathbf{C}'(\mathbf{0}) - \mathbf{B}'(\mathbf{0}) < 0.$$

Axiom 3.1.5.

$$C(\mathbf{0}) = B(\mathbf{0}) = 0.$$

Axiom 3.1.6. *For each component index* j*,*

$$\lim_{Q_j \to +\infty} C_j'(\mathbf{Q}) = +\infty, \qquad \lim_{Q_j \to +\infty} B_j'(\mathbf{Q}) = 0.$$

Theorem 3.1.1. *The inverse functions of* $\mathbf{C}'$ *and* $\mathbf{B}'$ *exist if their codomains are restricted to their ranges. In other words,* $\mathbf{C}'$ *and* $\mathbf{B}'$ *are injective.*

Proof. Suppose $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are two distinct points in $[0,+\infty)^N$. Then, we can depict a line segment connecting the two points by using the function

$$\mathbf{c}(t) := \mathbf{Q}_1 + t\left(\mathbf{Q}_2 - \mathbf{Q}_1\right).$$

Then, we can write the difference in $\mathbf{C}'$ at the two different points as a line integral

$$\begin{aligned}\mathbf{C}'(\mathbf{Q}_2) - \mathbf{C}'(\mathbf{Q}_1) &= \int_0^1 \mathbf{C}''(\mathbf{c}(t))\,\mathbf{c}'(t)\,\mathrm{d}t \\ &= \int_0^1 \mathbf{C}''(\mathbf{c}(t))\left(\mathbf{Q}_2 - \mathbf{Q}_1\right)\mathrm{d}t.\end{aligned}$$

Multiply both sides with $\mathbf{Q}_2 - \mathbf{Q}_1$, and we can derive

$$\left(\mathbf{Q}_2 - \mathbf{Q}_1\right) \cdot \left(\mathbf{C}'(\mathbf{Q}_2) - \mathbf{C}'(\mathbf{Q}_1)\right) = \int_0^1 \left(\mathbf{Q}_2 - \mathbf{Q}_1\right)^{\mathrm{T}} \mathbf{C}''(\mathbf{c}(t))\left(\mathbf{Q}_2 - \mathbf{Q}_1\right)\mathrm{d}t.$$

Because $\mathbf{C}''(\mathbf{c}'(t))$ is positive definite for all $t \in [0,1]$ according to Axiom 3.1.3, the right-hand side of the equation is positive. Therefore, $\mathbf{C}'(\mathbf{Q}_2) - \mathbf{C}'(\mathbf{Q}_1)$ cannot be $\mathbf{0}$.

Therefore, $\mathbf{C}'$ is an injection.

Similarly, we can prove that $\mathbf{B}'$ is an injection. ∎

3.2 Inter-good effects

Definition 3.2.1 (complements and substitutes)**.** *Two goods j and k are* ***complements*** *at* $\mathbf{Q}$ *if* $B''_{j,k}(\mathbf{Q}) < 0$.

Two goods j and k are ***substitutes*** *at* $\mathbf{Q}$ *if* $B''_{j,k}(\mathbf{Q}) > 0$.

3.3 Surplus

Definition 3.3.1 (surplus)**.** *The* ***producer surplus*** $S_C : [0, +\infty)^{2N} \to \mathbb{R}$ *is defined as*

$$S_C(\mathbf{Q}, \mathbf{P}) := \mathbf{P} \cdot \mathbf{Q} - C(\mathbf{Q})\,.$$

The ***consumer surplus*** $S_B : [0, +\infty)^{2N} \to \mathbb{R}$ *is defined as*

$$S_C(\mathbf{Q}, \mathbf{P}) := B(\mathbf{Q}) - \mathbf{P} \cdot \mathbf{Q}.$$

Definition 3.3.2 (social surplus)**.** *The* ***social surplus*** $S : [0, +\infty)^{N} \to \mathbb{R}$ *is defined as*

$$S := B - C.$$

Definition 3.3.3 (formal surplus)**.** *The* ***formal producer surplus*** $S_{\mathrm{S}C} : [0, +\infty)^{N} \to \mathbb{R}$ *is defined as*

$$S_{\mathrm{S}C}(\mathbf{Q}) := S_C(\mathbf{Q}, \mathbf{C}'(\mathbf{Q}))\,.$$

The ***formal consumer surplus*** $S_{\mathrm{S}B} : [0, +\infty)^{N} \to \mathbb{R}$ *is defined as*

$$S_{\mathrm{S}B}(\mathbf{Q}) := S_B(\mathbf{Q}, \mathbf{B}'(\mathbf{Q}))\,.$$

Definition 3.3.4 (actual social surplus)**.** *The* ***actual social surlus*** $S_{\mathrm{S}} : [0, +\infty)^{N} \to \mathbb{R}$ *is defined as*

$$S_{\mathrm{S}} := S_{\mathrm{S}C} + S_{\mathrm{S}B}.$$

Theorem 3.3.1. *For any* $\mathbf{Q} \in [0, +\infty)^N$,

$$S_{\mathrm{S}C}(\mathbf{Q}) \geq 0, \qquad S_{\mathrm{S}B}(\mathbf{Q}) \geq 0,$$

where the equality holds iff $\mathbf{Q} = \mathbf{0}$.

Proof.

$$\begin{aligned}
S_{\mathrm{S}C}(\mathbf{Q}) &= S_C(\mathbf{Q}, \mathbf{C}'(\mathbf{Q})) && \text{(Definition 3.3.3)} \\
&= \mathbf{C}'(\mathbf{Q}) \cdot \mathbf{Q} - C(\mathbf{Q}) && \text{(Definition 3.3.1)} \\
&= \int_{\mathbf{0}}^{\mathbf{Q}} \mathbf{C}'(\mathbf{Q}) \cdot \mathrm{d}\mathbf{x} - \int_{\mathbf{0}}^{\mathbf{Q}} \mathbf{C}'(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} \\
&= \int_{\mathbf{0}}^{\mathbf{Q}} (\mathbf{C}'(\mathbf{Q}) - \mathbf{C}'(\mathbf{x})) \cdot \mathrm{d}\mathbf{x}.
\end{aligned}$$

The integrated expression is obviously a conservative field, which means the integral path is arbitrary. Choose such a path that any $\mathbf{x}$ within the path satisfies $\mathbf{Q} - \mathbf{x} > 0$ and $\mathrm{d}\mathbf{x} > 0$ (such a path obviously exists). Thus, the integral can be written as an integral of an always-positive function, which means $S_{\mathrm{S}C}(\mathbf{Q}) > 0$.

When $\mathbf{Q} = \mathbf{0}$, it can be easily proved that $S_{\mathrm{S}C}(\mathbf{Q}) = 0$.

It is similar to show that $S_{\mathrm{S}B}(\mathbf{Q}) \geq 0$. $\square$

Theorem 3.3.2. *For any* $\mathbf{Q} \in [0, +\infty)$*,* $\mathbf{S}'_{\mathrm{S}C}(\mathbf{Q})$ *and* $\mathbf{S}'_{\mathrm{S}B}(\mathbf{Q})$ *have respectively at least one positive component.*

Proof.

$$\begin{aligned}
\mathbf{S}'_{\mathrm{S}C}(\mathbf{Q}) &= \frac{\partial}{\partial \mathbf{Q}} S_C(\mathbf{Q}, \mathbf{C}'(\mathbf{Q})) && \text{(Definition 3.3.3)} \\
&= \frac{\partial}{\partial \mathbf{Q}} \left(\mathbf{C}'(\mathbf{Q}) \cdot \mathbf{Q} - C(\mathbf{Q})\right) && \text{(Definition 3.3.1)} \\
&= \mathbf{C}''(\mathbf{Q})\, \mathbf{Q}.
\end{aligned}$$

Use means of contradiction. Suppose $\mathbf{S}'_{\mathrm{S}C}(\mathbf{Q}) \leq 0$, then

$$\mathbf{Q} \cdot \mathbf{S}'_{\mathrm{S}C}(\mathbf{Q}) \leq 0.$$

However,

$$\mathbf{Q} \cdot \mathbf{S}'_{\mathrm{S}C}(\mathbf{Q}) = \mathbf{Q}^{\mathrm{T}} \mathbf{C}''(\mathbf{Q})\, \mathbf{Q},$$

which, by Axiom 3.1.3, is positive, contradicting.

Thus, we can prove that $\mathbf{S}'_{\mathrm{S}C}(\mathbf{Q})$ has at least one positive component.

It is similar to prove that $\mathbf{S}'_{\mathrm{S}B}(\mathbf{Q})$ has at least one positive component. $\square$

Not necessarily all components of the vector is positive, because the product of a positive-definite matrix and a positive vector is not necessarily a positive vector.

Theorem 3.3.3. *For any* $\mathbf{Q}, \mathbf{P} \in [0, +\infty)^N$*,*

$$S(\mathbf{Q}) = S_C(\mathbf{Q}, \mathbf{P}) + S_B(\mathbf{Q}, \mathbf{P}),$$

which is independent of $\mathbf{P}$*.*

Proof.

$$\begin{aligned}
S(\mathbf{Q}) &= B(\mathbf{Q}) - C(\mathbf{Q}) && \text{(Definition 3.3.1)} \\
&= B(\mathbf{Q}) - \mathbf{P} \cdot \mathbf{Q} + \mathbf{P} \cdot \mathbf{Q} - C(\mathbf{Q}) \\
&= S_C(\mathbf{Q}, \mathbf{P}) + S_B(\mathbf{Q}, \mathbf{P}) && \text{(Definition 3.3.1).}
\end{aligned}$$

$\square$

3.4 Price elasticities

Definition 3.4.1 (price elasticity)**.** *The **price elasticity** of a function* $\mathbf{f} : [0, +\infty)^N \to [0, +\infty)^N$ *whose Jacobi matrix has no zero component everywhere on* $(0, +\infty)^N$ *is a function* $\mathbf{E_f} : (0, +\infty)^N \to \mathbb{R}^{N\times N}$ *defined as*

$$E_{\mathbf{f},j,k}(\mathbf{Q}) := \frac{f_j(\mathbf{Q})}{f'_{j,k}(\mathbf{Q})\, Q_k}.$$

In Definition 3.4.1, the involved function $\mathbf{f}$ is a function mapping $\mathbf{Q}$ to $\mathbf{P}$, usually $\mathbf{C}'$ or $\mathbf{B}'$. In these two special cases, the price elasticity is respectively called **cross price elasticity of supply (XES)** and **cross price elasticity of demand (XED)**.

Theorem 3.4.1. *For any* $\mathbf{Q} > 0$*,*

$$\mathbf{diag}\,\mathbf{E_{C'}}(\mathbf{Q}) > 0, \qquad \mathbf{diag}\,\mathbf{E_{B'}}(\mathbf{Q}) < 0.$$

Proof. According to Definition 3.4.1,

$$E_{\mathbf{C}',j,j}(\mathbf{Q}) = \frac{C'_j(\mathbf{Q})}{C''_{j,j}(\mathbf{Q})\, Q_j}.$$

According to Axiom 3.1.2, $C'_j(\mathbf{Q}) > 0$. According to Axiom 3.1.3, since the diagonal elements of a positive-definite matrix are all positive, $C''_{j,j}(\mathbf{Q}) > 0$. Thus, $E_{C',j,j}(\mathbf{Q}) > 0$. The conclusion is true for any j, so $\mathbf{diag}\,\mathbf{E_{C'}}(\mathbf{Q}) > 0$.

Similarly, it can be proved that $\mathbf{diag}\,\mathbf{E_{B'}}(\mathbf{Q}) < 0$ □

3.5 revenue–expenditure

Definition 3.5.1 (revenue–expenditure)**.** *The **revenue–expenditure** of a function* $\mathbf{f} : [0, +\infty)^N \to [0, +\infty)^N$ *is a function* $R_\mathbf{f} : [0, +\infty)^N \to [0, +\infty)$ *defined as*

$$R_\mathbf{f}(\mathbf{Q}) := \mathbf{f}(\mathbf{Q}) \cdot \mathbf{Q}.$$

Theorem 3.5.1. *If* $\mathbf{f} : [0, +\infty)^N \to [0, +\infty)^N$ *is a function whose Jacobi matrix has no zero component anywhere on* $(0, +\infty)^N$*, and* $\mathbf{Q}_0$*, which has no zero component, is an extremal of* $R_\mathbf{f}$ *such that* $\mathbf{f}(\mathbf{Q}_0)$ *has no zero component, then for any good* j*,*

$$\sum_k \frac{1}{E_{\mathbf{f},j,k}(\mathbf{Q}_0)} = -1.$$

Proof. Because $R_\mathbf{f}$ is a differentiable function, its extremal is a solution to equation $\mathbf{R}'_\mathbf{f} = \mathbf{0}$ or the boundary of its domain. Since $\mathbf{Q}_0 \neq \mathbf{0}$, $\mathbf{Q}_0$ is not the boundary of the domain of $R_\mathbf{f}$. Thus $\mathbf{R}'_\mathbf{f}(\mathbf{Q}_0) = \mathbf{0}$.

$$\begin{aligned}\mathbf{R}'_\mathbf{f} &= \frac{\partial}{\partial \mathbf{Q}} \left(\mathbf{f}(\mathbf{Q}) \cdot \mathbf{Q}\right) && \text{(Definition 3.5.1)} \\ &= \mathbf{f}'(\mathbf{Q})\, \mathbf{Q} + \mathbf{f}(\mathbf{Q})\,,\end{aligned}$$

so we have the equation

$$\mathbf{f}'(\mathbf{Q})\,\mathbf{Q} + \mathbf{f}(\mathbf{Q}) = \mathbf{0}.$$

Since the jth component of $\mathbf{f}'(\mathbf{Q})\,\mathbf{Q}$ is $\sum_k f'_{j,k}(\mathbf{Q}_0)\,Q_k$, which is $\sum_k \frac{f_j(\mathbf{Q}_0)}{E_{\mathbf{f},j,k}(\mathbf{Q}_0)}$ according to Definition 3.4.1, for each j, we have the equation

$$\sum_k \frac{f_j(\mathbf{Q}_0)}{E_{\mathbf{f},j,k}(\mathbf{Q}_0)} + f_j(\mathbf{Q}_0) = 0.$$

Because $\mathbf{f}(\mathbf{Q}_0)$ has no zero component, we must have

$$\sum_k \frac{1}{E_{\mathbf{f},j,k}(\mathbf{Q}_0)} = -1.$$

□

Theorem 3.5.2. *If B has upper bound, then $R_{\mathbf{B}'}(\mathbf{Q}) \to 0$ as each component of $\mathbf{Q}$ approaches $+\infty$.*

Proof. Write the limit as the composition of N limits, each of which lets one component of $\mathbf{Q}$ approach $+\infty$, and then use Axiom 3.1.2 and similar tricks in the proof of Theorem 1.4.3. □

3.6 Perfect competition

Theorem 3.6.1. *S has a unique maximal on $(0, +\infty)^N$, which is the unique solution of equation $\mathbf{C}' = \mathbf{B}'$.*

Proof. The gradient of a scalar field must be zero at its maximal if the maximal is an interior point of the domain. At the same time, the sufficient condition for a stationary point to be a maximal is that the field is strictly concave at that point.

Let $\mathbf{f} : [0, +\infty)^N \to \mathbb{R}^N$ □