Calculus in microeconomics

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Contents

1 Free market of one good

1.1 Basic concepts and theorems

Definition 1 (quantity)**.** *The **quantity** is a variable $Q \in [0, +\infty)$.*

The economic meaning of Q is to measure how much good there is involved in the market within a unit of time. Its unit is unit quantity per unit time.

Definition 2 (price)**.** *The **price** is a variable $P \in [0, +\infty)$.*

The economic meaning of P is to measure the value of good. Its unit is monetary unit per unit quantity.

Definition 3 (cost and benefit)**.** *The **cost** of the good is a function $C : [0, +\infty) \to [0, +\infty)$.*
   *The **benefit** of the good is a function $B : [0, +\infty) \to [0, +\infty)$.*

C maps Q to the total cost $C(Q)$ (whose unit is monetary unit per unit time) for producers to produce such amount of good within a unit of time. Similarly, B maps Q to the total benefit $B(Q)$ (whose unit is monetary unit per unit time) for consumers to consume such amount of good within a unit of time.

Here are some axioms concluded from principles of economics.

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Axiom 1. *C and B are increasing on $[0, +\infty)$.*

Axiom 1 is obvious when thinking about the economic meaning of C and B.

Axiom 2. *C and B are twice differentiable on $[0, +\infty)$.*

Axiom 2 is a mathematical requirement without which we will lack mathematical tools to conveniently prove some theorems.

Axiom 3.
$$C(0) = B(0) = 0.$$

Axiom 3 is as obvious as Axiom 1.

Axiom 4.
$$C'(0) < B'(0).$$

Axiom 4 is sometimes optional, but it is necessary if the market should exist.

Axiom 5.
$$\lim_{Q \to +\infty} C'(Q) = +\infty.$$
$$\lim_{Q \to +\infty} B'(Q) = 0.$$

Axiom 5 is as necessary as Axiom 4. Both of them restrict the boundary conditions of C' and B'.

Axiom 6 (law of supply and law of demand)**.** *C is strictly convex downward everywhere on $[0, +\infty)$.*
B is strictly concave downward everywhere on $[0, +\infty)$.

Axiom 6 is a important and basic law in economics. Note that law of demand is mathematically equivalent to law of diminishing marginal benefit.

From these axioms, we can easily deduce some corollaries using simple math.

Theorem 1. *The inverse function of C' and B' exists if their codomains are restricted to their ranges.*

Proof. According to Axiom 6, C' is strictly increasing on $[0, +\infty)$. According to Axiom 5, C' has range $[C'(0), +\infty)$. Thus, we can define its inverse function $C'^{-1} : [C'(0), +\infty) \to [0, +\infty)$.

The proof for existence of B'^{-1} is similar. $\qquad\qquad\qquad\square$

Definition 4 (supply and demand)**.** *Function C' is called **marginal cost (MC)** or **supply (S)**.*
*Function B' is called **marginal benefit (MB)** or **demand (D)**.*

Marginal cost and marginal benefit shares the unit with P (monetary unit per unit quantity).

It may be confusing why marginal cost is the same as supply. Here is a simple explanation: producers do not operate at a price below the shutdown point. It is similar to explain why marginal benefit is the same as demand.

By convention in the context of supply and demand graphs, the inverse functions C'^{-1} and B'^{-1} are used instead of C' and B'.

Definition 5 (surplus)**.** *The **producer surplus** $S_C : [0, +\infty)^2 \to \mathbb{R}$ is defined as*

$$S_C(Q, P) := PQ - C(Q).$$

*The **consumer surplus** $S_B : [0, +\infty)^2 \to \mathbb{R}$ is defined as*

$$S_B(Q, P) := B(Q) - PQ.$$

*The **social surplus** $S : [0, +\infty) \to \mathbb{R}$ is defined as*

$$S := B - C.$$

The unit of surplus is monetary unit per unit time. The economic meaning of surplus is the net gain within a unit of time. For producers, the net gain is part of income PQ that exceeds the cost $C(Q)$. For consumers, the net gain is part of benefit $B(Q)$ that exceeds the expenditure PQ.

Theorem 2. *For any $Q \geq 0$,*

$$S_C(Q, C'(Q)) \geq 0,$$

$$S_B(Q, B'(Q)) \geq 0,$$

where the equality holds only when $Q = 0$.

Proof.

$$\begin{aligned} S_C(Q, C'(Q)) &= C'(Q)Q - C(Q) \qquad \text{(Definition 5)} \\ &= \int_0^Q C'(Q)\,\mathrm{d}x - \int_0^Q C'(x)\,\mathrm{d}x \\ &= \int_0^Q (C'(Q) - C'(x))\,\mathrm{d}x. \end{aligned}$$

According to Axiom 6, C' is strictly increasing, so when $Q > 0$, for any $x \in [0, Q)$, $C'(Q) - C'(x) > 0$. Thus, the integral is an integral of an always-positive function, which means

$$S_C(Q, C'(Q)) > 0.$$

When $Q = 0$, it can be easily proved that $S_C(Q, C'(Q)) = 0$.

It is similar to show that

$$S_B(Q, B'(Q)) > 0.$$

$\square$

Theorem 3. $Q \mapsto S_C(Q, C'(Q))$ *is strictly increasing.*

$Q \mapsto S_B(Q, B'(Q))$ *is strictly increasing.*

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}Q} S_C\left(Q, C'\left(Q\right)\right) = \frac{\mathrm{d}}{\mathrm{d}Q}\left(C'\left(Q\right)Q - C\left(Q\right)\right) \qquad \text{(Definition 5)}$$
$$= C''\left(Q\right)Q$$
$$> 0 \qquad \text{(Axiom 6)},$$

so $Q \mapsto S_C\left(Q, C'\left(Q\right)\right)$ is strictly increasing.

It is similar to prove that $Q \mapsto S_B\left(Q, B'\left(Q\right)\right)$ is strictly increasing. $\qquad \square$

Theorem 4. *For any $Q, P \in [0, +\infty)$,*

$$S\left(Q\right) = S_C\left(Q, P\right) + S_B\left(Q, P\right),$$

which is independent of P.

Proof.

$$S\left(Q\right) = B\left(Q\right) - C\left(Q\right) \qquad \text{(Definition 5)}$$
$$= B\left(Q\right) - PQ + PQ - C\left(Q\right)$$
$$= S_C\left(Q, P\right) + S_B\left(Q, P\right) \qquad \text{(Definition 5)}.$$

$$\square$$

Definition 6 (price elasticity)**.** *The **price elasticity** of a function $f : [0, +\infty) \to [0, +\infty)$ with non-zero derivative everywhere on $(0, +\infty)$ is a function $E_f : (0, +\infty) \to \mathbb{R}$ defined as*

$$E_f\left(Q\right) := \frac{f\left(Q\right)}{f'\left(Q\right)Q}.$$

In Definition 6, the involved function f is a function mapping Q to P, usually C' or B'. In these two special cases, the price elasticity is called **price elasticity of supply (PES)** or **price elasticity of demand (PED)**. The economic meaning of price elasticity is to show the responsiveness (elasticity) of Q to P. It is the ratio of percentage change in Q and that in P.

Theorem 5. *For any $Q > 0$,*

$$E_{B'}\left(Q\right) < 0,$$

$$E_{C'}\left(Q\right) > 0.$$

Proof. According to Definition 6,

$$E_{B'}\left(Q\right) = \frac{B'\left(Q\right)}{B''\left(Q\right)Q}.$$

According to Axiom 1, $B'\left(Q\right) > 0$. According to Axiom 6, $B''\left(Q\right) < 0$. Thus, $E_{B'}\left(Q\right) < 0$. $\qquad \square$

Theorem 6. *If* $\lim_{Q \to +\infty} E_{B'}(Q) \in (-1, 0]$, *then B has upper bound.*

Proof. According to Theorem 5, $\lim_{Q \to +\infty} E_{B'}(Q)$ cannot be positive.

Case 1: $\lim_{Q \to +\infty} E_{B'}(Q) = 0$.

According to definition of limit, for any $\varepsilon > 0$, there exists n such that for any $Q > n$, we have

$$|E_{B'}(Q)| < \varepsilon.$$

According to Theroem 5, $E_{B'}(Q) < 0$. Thus, the equation can be written as

$$E_{B'}(Q) > -\varepsilon.$$

Substitute Definition 6, and let $m := \frac{1}{\varepsilon}$, and then we can derive

$$\frac{B'(Q)}{B''(Q)Q} > -\frac{1}{m},$$

which means

$$\frac{B''(Q)}{B'(Q)} < -\frac{m}{Q}.$$

Note that it can be written as

$$\frac{\mathrm{d}}{\mathrm{d}Q} \ln (B'(Q)) < -\frac{\mathrm{d}}{\mathrm{d}Q} (m \ln Q),$$

or

$$\frac{\mathrm{d}}{\mathrm{d}Q} \ln (B'(Q)Q^m) < 0.$$

This means $B'(Q)Q^m$ is strictly decreasing on $(n, +\infty)$, so for any $Q > q > n$,

$$B'(Q)Q^m < B'(q)q^m,$$

which means that for any $Q > q$,

$$B'(Q) < CQ^{-m},$$

where $C := B'(q)q^m$.

Write $\lim_{Q \to +\infty} B(Q)$ in the integral form

$$\lim_{Q \to +\infty} B(Q) = B(q) + \int_q^{+\infty} B'(Q)\,\mathrm{d}Q$$

$$< B(q) + \int_q^{+\infty} CQ^{-m}\mathrm{d}Q.$$

Since m is an arbitrary positive number, we can take $m > 1$. In this way, the integral above converges, and we can derive that

$$\lim_{Q \to +\infty} B(Q) < B(q) + \frac{C}{m-1}q^{1-m}.$$

Thus, according to comparison test, $\lim_{Q \to +\infty} B(Q)$ must exist.

It can be easily proved that $\lim_{Q \to +\infty} B(Q)$ is the supremum of B, so B has upper bound.

Case 2: $\lim_{Q \to +\infty} E_{B'}(Q) \in (-1, 0)$.

Let

$$l := -\frac{1}{\lim_{Q \to +\infty} E_{B'}(Q)} \in (1, +\infty),$$

and then we have

$$\lim_{Q \to +\infty} \frac{1}{E_{B'}(Q)} = -l.$$

According to definition of limit, for any $m > 0$, there exists n such that for any $Q > n$, we have

$$\left| \frac{1}{E_{B'}(Q)} + l \right| < m,$$

so

$$\frac{1}{E_{B'}(Q)} < m - l.$$

Substitute Definition 6, and then we can derive

$$\frac{B''(Q)Q}{B'(Q)} < m - l.$$

Use similar tricks in Case 1, and then we can derive

$$\frac{\mathrm{d}}{\mathrm{d}Q} \ln \left(B'(Q) Q^{l-m} \right) < 0.$$

Use similar tricks in Case 1, and take $m \in (0, l - 1)$, and we can prove that B has upper bound. □

Theorem 7. *If B has upper bound, then $\lim_{Q \to +\infty} E_{B'}(Q)$ can be any value in $[-1, 0]$, and can also non-exist, and cannot be less than -1.*

Proof. Example where $\lim_{Q \to +\infty} E_{B'}(Q)$ does not exist:

Let

$$B(Q) := \int_0^Q \frac{2 + \sin \ln (1 + x)}{(1 + x)^2} \mathrm{d}x.$$

Then, B satisfies Axiom 1, Axiom 2, Axiom 3, Axiom 5, and Axiom 6 as can be justified.

B has upper bound because

$$B(Q) < \int_0^Q \frac{3}{(1 + x)^2} \mathrm{d}x$$

$$< 3 \int_1^{+\infty} \frac{\mathrm{d}x}{x^2}$$

$$= 3.$$

Calculate $E_{B'}(Q)$, and we have

$$E_{B'}(Q) := \frac{B'(Q)}{B''(Q)Q} \qquad \text{(Definition 6)}$$

$$= \frac{1+x}{x} \cdot \frac{2 + \sin\ln(1+x)}{-2(2 + \sin\ln(1+x)) + \cos\ln(1+x)}.$$

When $x = \mathrm{e}^{2k\pi} - 1$,

$$E_{B'}(Q) = \frac{1+x}{x} \cdot \frac{-2}{3} \xrightarrow{x \to +\infty} -\frac{2}{3}.$$

When $x = \mathrm{e}^{\left(k+\frac{1}{4}\right)2\pi} - 1$,

$$E_{B'}(Q) = \frac{1+x}{x} \cdot \frac{-1}{2} \xrightarrow{x \to +\infty} -\frac{1}{2}.$$

$(k \in \mathbb{N}.)$

Thus, $\lim_{Q \to +\infty} E_{B'}(Q)$ does not exist.

Example where $\lim_{Q \to +\infty} E_{B'}(Q) = 0$:

Let

$$B(Q) := 1 - \mathrm{e}^{-Q}.$$

Then, B satisfies Axiom 1, Axiom 2, Axiom 3, Axiom 5, and Axiom 6, and has upper bound 1 as can be justified.

Calculate $E_{B'}(Q)$, and we have

$$E_{B'}(Q) := \frac{B'(Q)}{B''(Q)Q} \qquad \text{(Definition 6)}$$

$$= -\frac{1}{Q}$$

$$\xrightarrow{Q \to +\infty} 0.$$

Example where $\lim_{Q \to +\infty} E_{B'}(Q) = -1$:

Let

$$B(Q) := 1 - \frac{1}{\ln(\mathrm{e} + Q)}.$$

Then, B satisfies Axiom 1, Axiom 2, Axiom 3, Axiom 5, and Axiom 6, and has upper bound 1 as can be justified.

Calculate $E_{B'}(Q)$, and we have

$$E_{B'}(Q) := \frac{B'(Q)}{B''(Q)Q} \qquad \text{(Definition 6)}$$

$$= -\frac{\mathrm{e} + Q}{Q} \cdot \frac{\ln(\mathrm{e} + Q)}{2 + \ln(\mathrm{e} + Q)}$$

$$\xrightarrow{Q \to +\infty} -1.$$

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Example where $\lim_{Q \to +\infty} E_{B'}(Q)$ can be any value in $(-1, 0)$:

Let
$$B(Q) := 1 - (1 + x)^{1 + \frac{1}{l}},$$

where $l \in (-1, 0)$ is an arbitrary parameter. Then, B satisfies Axiom 1, Axiom 2, Axiom 3, Axiom 5, and Axiom 6, and has upper bound 1 as can be justified.

Calculate $E_{B'}(Q)$, and we have

$$E_{B'}(Q) := \frac{B'(Q)}{B''(Q) Q} \qquad \text{(Definition 6)}$$
$$= l \left(1 + \frac{1}{Q} \right)$$
$$\xrightarrow{Q \to +\infty} l.$$

Prove that $\lim_{Q \to +\infty} E_{B'}(Q) \geq -1$:

Use means of contradiction.

Suppose $\lim_{Q \to +\infty} E_{B'}(Q) < -1$, and use similar tricks in the proof of Theorem 6. We can derive that there exists $q > 0$ such that for any $Q > q$,

$$B'(Q) Q > B'(q) q,$$

which means

$$B'(Q) > \frac{C}{x},$$

where $C := B'(q) q$.

Integrate both sides, and we can derive

$$B(Q) > B(q) + C \ln \frac{Q}{q} \xrightarrow{Q \to +\infty} +\infty,$$

which contradicts with it that B has upper bound. $\qquad \square$

Definition 7 (revenue). *The **revenue** of a function $f : [0, +\infty) \to [0, +\infty)$ is a function $R_f : [0, +\infty) \to [0, +\infty)$ defined as*

$$R_f(Q) := f(Q) Q.$$

In Definition 7, the involved function f is a function mapping Q to P, usually C' or B'. One may usually find that the surplus can often be represented with difference of revenue and cost/benefit.

Theorem 8. *If $f : [0, +\infty) \to [0, +\infty)$ is a function with non-zero derivative everywhere on $(0, +\infty)$, and $Q_0 \neq 0$ is an extremal of R_f such that $f(Q_0) \neq 0$, then $E_f(Q_0) = -1$.*

Proof. Because R_f is a differentiable function, its extremal is a solution of equation $R'_f = 0$ or the boundary of its domain. Since $Q_0 \neq 0$, Q_0 is not the boundary of the domain of R_f. Thus $R'_f(Q_0) = 0$.

$$
\begin{aligned}
R'_f(Q) &= \frac{\mathrm{d}}{\mathrm{d}Q}\left(f(Q)\,Q\right) && \text{(Definition 7)} \\
&= f'(Q)\,Q + f(Q) \\
&= \frac{f(Q)}{E_f(Q)} + f(Q) && \text{(Definition 6)} \\
&= \left(1 + \frac{1}{E_f(Q)}\right) f(Q).
\end{aligned}
$$

From the calculations above, since $R'_f(Q_0) = 0$ and $f(Q_0) \neq 0$, we have $E_f(Q) = -1$. $\qquad\square$

Theorem 9. *For any $Q \geq 0$,*

$$
C'(Q) \leq R'_{C'}(Q),
$$

$$
B'(Q) \geq R'_{B'}(Q),
$$

where the equality holds only when $Q = 0$.

Proof. Consider $Q > 0$, and then we have

$$
\begin{aligned}
R'_{C'}(Q) &= \left(1 + \frac{1}{E_{C'}(Q)}\right) C'(Q) && \text{(Theorem 8)} \\
&> C'(Q) && \text{(Theorem 5)}.
\end{aligned}
$$

As for $Q = 0$, it can be easily proved that $R'_{C'}(0) = C'(0)$.

It is similar to prove $B'(Q) \geq R'_{B'}(Q)$. $\qquad\square$

Theorem 10. *If B has upper bound, then $\lim_{Q \to +\infty} R_{B'}(Q) = 0$.*

Proof. Because B has upper bound and is strictly increasing, $\lim_{Q \to +\infty} B(Q)$ exists. According to Cauchy's convergence test, for any $\frac{\varepsilon}{2} > 0$, there exists n such that when $Q > \frac{Q}{2} > n$,

$$
B(Q) - B\left(\frac{Q}{2}\right) < \frac{\varepsilon}{2}
$$

(the absolute value bracket is omitted due to Axiom 1).

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On the other hand,

$$
\begin{aligned}
B\left(Q\right) - B\left(\frac{Q}{2}\right) &= \int_{\frac{Q}{2}}^{Q} B'\left(x\right) \mathrm{d}x \\
&> \int_{\frac{Q}{2}}^{Q} B'\left(Q\right) \mathrm{d}x \qquad \text{(Axiom 6)} \\
&= \frac{1}{2} B'\left(Q\right) Q \\
&= \frac{1}{2} R_{B'}\left(Q\right) \qquad \text{(Definition 7)}.
\end{aligned}
$$

Thus, when $Q > n$, $0 < R_{B'}\left(Q\right) < \varepsilon$, which means $\lim_{Q \to +\infty} R_{B'}\left(Q\right) = 0$. $\qquad\square$

1.2 Perfect competition

Theorem 11. *S has a unique maximal on $(0, +\infty)$, which is the unique solution of equation $C' = B'$.*

Proof. It can be just proved by showing that the equation $C' = B'$ has a unique solution on $(0, +\infty)$. In other words, if we construct a function

$$
f := S' = C' - B',
$$

then f has a unique zero on $(0, +\infty)$.

According to Axiom 4, we have

$$
f\left(0\right) = C'\left(0\right) - B'\left(0\right) < 0.
$$

According to Axiom 6, C' is strictly increasing, and B' is strictly decreasing, so f is strictly increasing.

According to Axiom 5, we have

$$
\lim_{x \to \infty} f\left(x\right) = +\infty,
$$

which means that f has range $[f\left(0\right), +\infty)$, so f has its inverse function $f^{-1}: [f\left(0\right), +\infty) \to [0, +\infty)$.

Because $f\left(0\right) < 0$, we have $0 \in [f\left(0\right), +\infty)$. Thus, $f^{-1}\left(0\right)$ is the unique solution of equation $f = 0$.

After that, it can be easily proved that $f^{-1}\left(0\right) \in (0, +\infty)$ is the unique maximal of S. $\qquad\square$

According to principles of economics, the maximal of S mentioned in Theorem 11 is the **equilibrium of a perfect competition market**.

1.3 Monopoly and monopsony

Theorem 12. $Q \mapsto S_C\left(Q, B'\left(Q\right)\right)$ *has at least one maximal on* $(0, +\infty)$, *which is a solution of equation* $R'_{B'} = C'$.

$Q \mapsto S_B\left(Q, C'\left(Q\right)\right)$ *has at least one maximal on* $(0, +\infty)$, *which is a solution of equation* $R'_{C'} = B'$.

Proof. According to Definition 5 and Definition 7,

$$f\left(Q\right) := S_C\left(Q, B'\left(Q\right)\right) = R_{B'}\left(Q\right) - C\left(Q\right).$$

According to Axiom 4 and Theorem 9,

$$f'\left(0\right) = B'\left(0\right) - C'\left(0\right) > 0,$$

so 0 cannot be a maximal of f.

According to Axiom 5 and Theorem 9,

$$\lim_{Q \to +\infty} f'\left(Q\right) < \lim_{Q \to +\infty} \left(B'\left(Q\right) - C'\left(Q\right)\right) = -\infty.$$

Thus, using definition of limit, it can be easily proved that there exists $Q_0 > 0$ such that $f'\left(Q_0\right) < 0$, so Q_0 cannot be a maximal of f.

Since f is continuous, its range on closed interval $[0, Q_0]$ is a closed interval. Suppose the supremum of the range is $f\left(z\right)$, where $z \in (0, Q_0)$. Then $z \in (0, +\infty)$ is a maximal of f.

Because f is differentiable everywhere, its maximal must be a solution of equation $f' = 0$, which is the same equation as $R'_{B'} = C'$.

It is similar to prove that $Q \mapsto S_B\left(Q, C'\left(Q\right)\right)$ has at least one maximal on $(0, +\infty)$, which is a solution of equation $R'_{C'} = B'$. □

According to principles of economics, the maximal of $Q \mapsto S_C\left(Q, B'\left(Q\right)\right)$ mentioned in Theorem 12 is the **equilibrium of a monopoly market**. Similarly, the maximal of $Q \mapsto S_B\left(Q, C'\left(Q\right)\right)$ mentioned in 12 is the **equilibrium of a monopsony market**.

2 Market of one good with gov intervention

2.1 Taxes and subsidies

2.2 Price controls

3 Free market of multiple goods

3.1 Basic concepts