Simple Harmonic Motion

Course Title : Applied Physics

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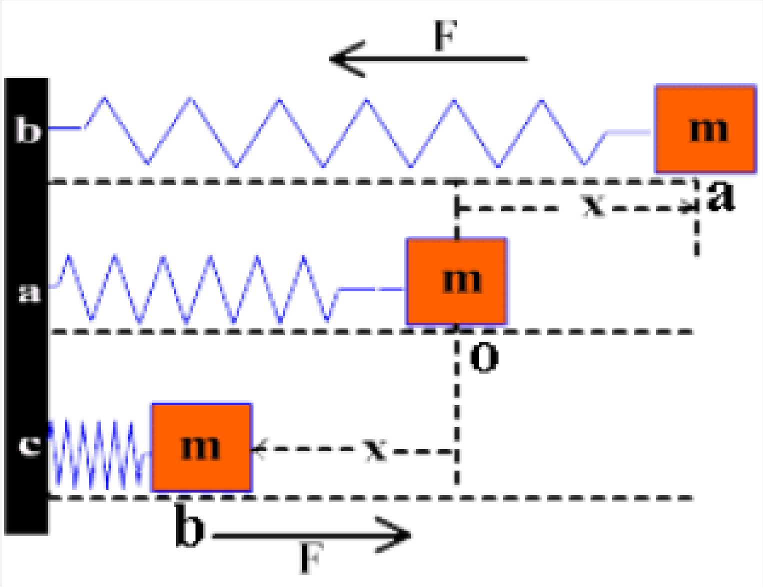
Definition:

- Such a motion in which acceleration is directly proportional to the displacement and is directed towards the mean position is called simple harmonic motion(SHM).



Condition FOR SHM:

- The system should haves restoring force.
- The system should have inertia.
- The system should be frictionless.

Hooke's Law

- force that is applied to spring is **directly proportional** to the displacement.
- If the spring is un stretched, there is no net force on the mass or the system is in equilibrium.
- if the mass is displaced from equilibrium, the spring will exert a restoring force, which is a force that tends to restore it to the equilibrium position.

$$F \propto x$$

where,
F → Elastic force
k → Spring constant
x → Displacement

$$\boxed{F = kx}$$

Expression for acceleration of the body executing SHM:

- Consider a mass 'm' attached to one end of elastic spring which can move freely on a frictionless horizontal surface.

- When the mass is released, it begins to vibrate about its mean or equilibrium position.

- But due to elasticity, spring opposes the applied force which produces the displacement. This opposing force is called restoring force.

Expression for acceleration of the body executing SHM:

The restoring force is written by;

$$F_r = -kx \text{........................(i)}$$

If ‘a’ is the acceleration produced by force ‘F’ in mass-spring system at any instant, then according to Newton’s law of motion.

$$F = ma$$

$$F = m\ddot{x} \text{ ....(ii)}$$

Comparing (i) and (ii)

$$m\ddot{x} = -kx \qquad \Rightarrow \ddot{x} = \left(\frac{-k}{m}\right)x$$

$$\because \frac{k}{m} = constant$$

$$a \; \alpha \; -x$$

From above equation we can write it as

$$\ddot{x} + \left(\frac{k}{m}\right)x = 0$$

Solution of the form

$$x = A\cos(\omega t + \phi).$$

$$\therefore \omega = \sqrt{\frac{k}{m}}$$

- Velocity can found by differentiating displacement

$$v = -A\omega \sin(\omega t + \emptyset)$$

- Acceleration can found by differentiating velocity

$$a = -A\omega^2 \cos(\omega t + \emptyset)$$

Simplifying acceleration in terms of displacement:

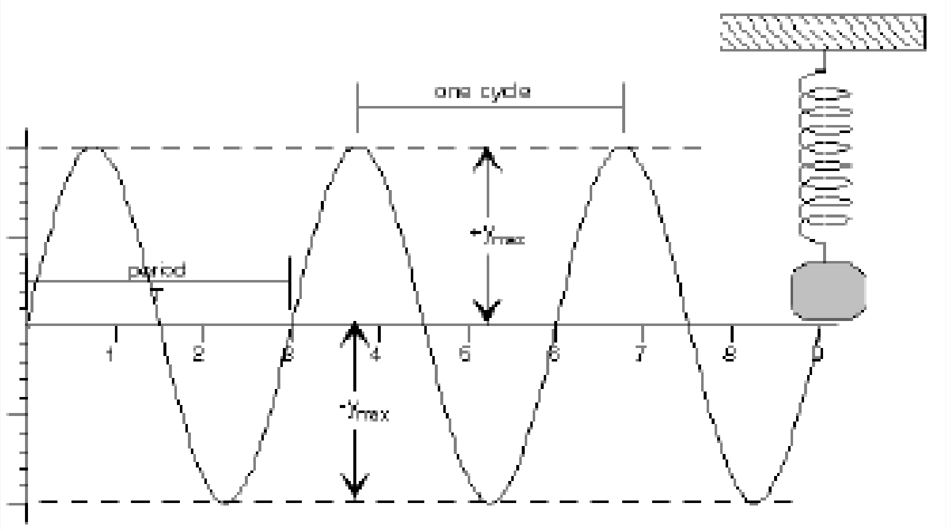
$$a = \frac{d^2x}{dt^2} = -\omega^2 x,$$

Acceleration can also be expressed as:

$$a(t) = -(2\pi f)^2 x(t)$$

Characteristic of Mass-spring system executing SHM:

When a body is vibrating, its displacement from the mean position changes with time. The value of its distance from the mean position at any time is known as its instantaneous displacement.



- The maximum value of displacement is known as its amplitude.

- A vibration means one complete round trip of the body in motion.

- The time required to complete one vibration is called time period

$$T = 2\pi\sqrt{\frac{m}{k}}$$

- The number of cycles per second. A cycle is a complete round trip is called Frequency

$$f = \frac{1}{T}$$

$$f = \frac{1}{2\pi\sqrt{\frac{m}{k}}}$$

$$f = 2\pi\sqrt{\frac{k}{m}}$$

- If T is time period of a body executing SHM, its angular frequency can be written as;

$$\omega = \frac{2\pi}{T} = 2\pi f$$

Phase angle:

The angle $\theta = \omega t$ which specifies the displacement as well as the direction of the point executing SHM is known as phase angle.

General Equation

$$x(t) = A\cos\left(2\pi f t + \phi\right)$$

where,

x → Displacement

A → Amplitude of the oscillation

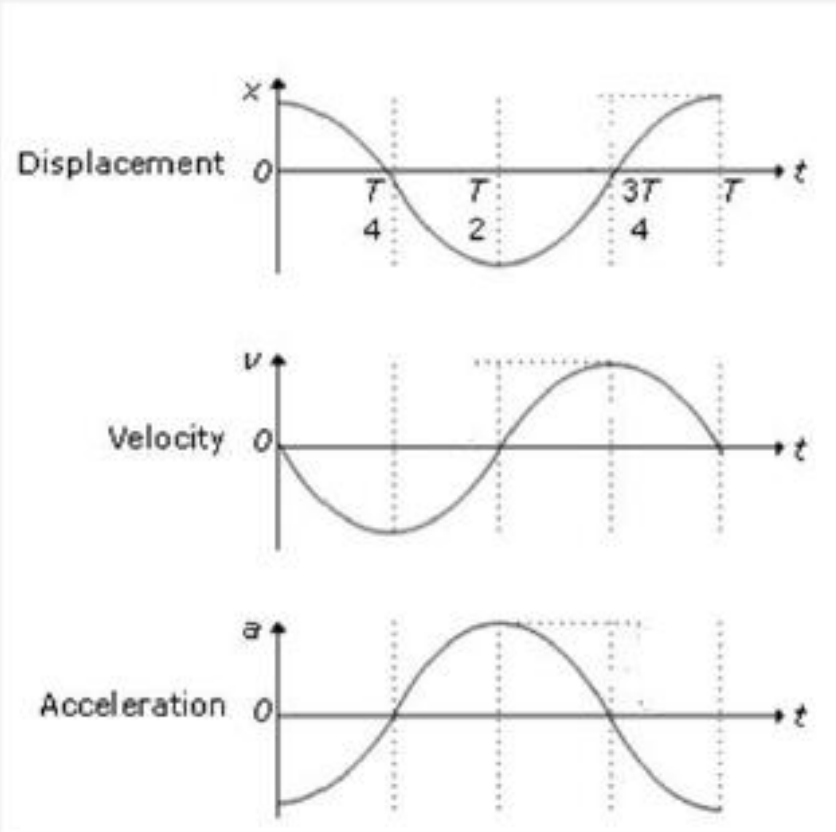
f → Frequency

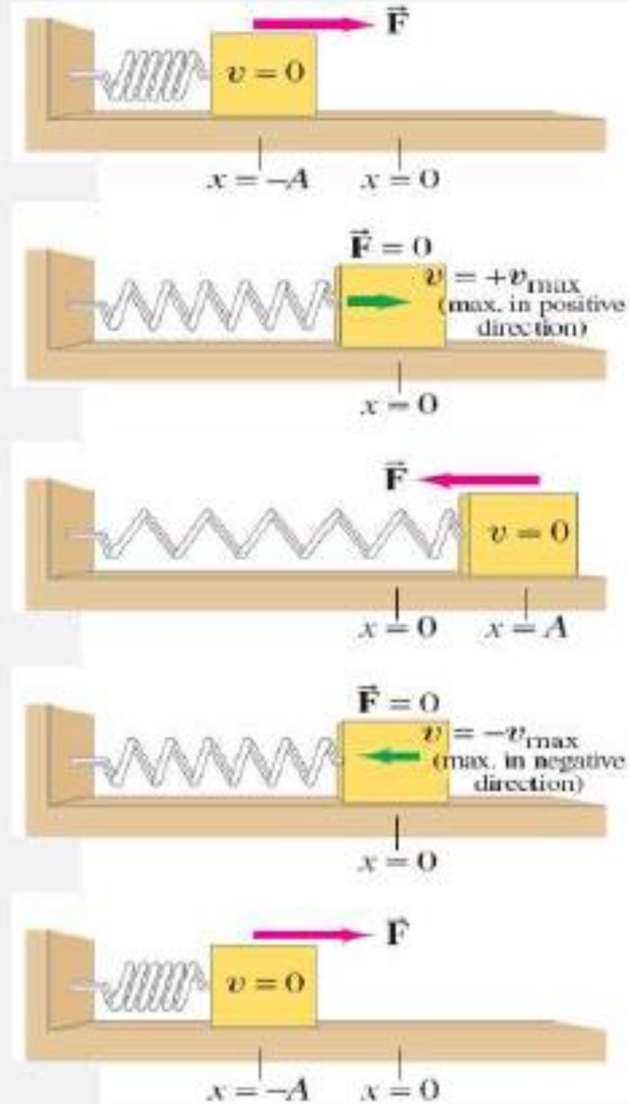
t → time

ϕ → Phase of oscillation

If there is no displacement at time t = 0, the phase is $\phi = \pi/2$

Graphical representation





A block whose mass m is 680 g is fastened to a spring whose spring constant k is 65 N/m. The block is pulled a distance $x = 11$ cm from its equilibrium position at $x = 0$ on a frictionless surface and released from rest at $t = 0$.

(a) What are the angular frequency, the frequency, and the period of the resulting motion?

KEY IDEA The block–spring system forms a linear simple harmonic oscillator, with the block undergoing SHM.

Calculations: The angular frequency is given by Eq. 15-12:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{65\ \text{N/m}}{0.68\ \text{kg}}} = 9.78\ \text{rad/s}$$

$$\approx 9.8\ \text{rad/s.} \qquad \text{(Answer)}$$

The frequency follows from Eq. 15-5, which yields

$$f = \frac{\omega}{2\pi} = \frac{9.78\ \text{rad/s}}{2\pi\ \text{rad}} = 1.56\ \text{Hz} \approx 1.6\ \text{Hz.} \qquad \text{(Answer)}$$

The period follows from Eq. 15-2, which yields

$$T = \frac{1}{f} = \frac{1}{1.56\ \text{Hz}} = 0.64\ \text{s} = 640\ \text{ms.} \qquad \text{(Answer)}$$

A block whose mass m is 680 g is fastened to a spring whose spring constant k is 65 N/m. The block is pulled a distance $x = 11$ cm from its equilibrium position at $x = 0$ on a frictionless surface and released from rest at $t = 0$.

(b) What is the amplitude of the oscillation?

KEY IDEA With no friction involved, the mechanical energy of the spring–block system is conserved.

Reasoning: The block is released from rest 11 cm from its equilibrium position, with zero kinetic energy and the elastic potential energy of the system at a maximum. Thus, the block will have zero kinetic energy whenever it is again 11 cm from its equilibrium position, which means it will never be farther than 11 cm from that position. Its maximum displacement is 11 cm:

$$x_m = 11 \text{ cm.} \qquad \text{(Answer)}$$

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(c) What is the maximum speed v_m of the oscillating block, and where is the block when it has this speed?

KEY IDEA The maximum speed v_m is the velocity amplitude ωx_m in Eq. 15-6.

Calculation: Thus, we have

$$v_m = \omega x_m = (9.78 \text{ rad/s})(0.11 \text{ m})$$
$$= 1.1 \text{ m/s.} \qquad \text{(Answer)}$$

This maximum speed occurs when the oscillating block is rushing through the origin; compare Figs. 15-4*a* and 15-4*b*, where you can see that the speed is a maximum whenever $x = 0$.

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(d) What is the magnitude a_m of the maximum acceleration of the block?

KEY IDEA The magnitude a_m of the maximum acceleration is the acceleration amplitude $\omega^2 x_m$ in Eq. 15-7.

Calculation: So, we have

$$a_m = \omega^2 x_m = (9.78 \text{ rad/s})^2(0.11 \text{ m})$$
$$= 11 \text{ m/s}^2. \qquad \text{(Answer)}$$

This maximum acceleration occurs when the block is at the ends of its path. At those points, the force acting on the block has its maximum magnitude; compare Figs. 15-4a and 15-4c, where you can see that the magnitudes of the displacement and acceleration are maximum at the same times.

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(e) What is the phase constant ϕ for the motion?

Calculations: Equation 15-3 gives the displacement of the block as a function of time. We know that at time $t = 0$, the block is located at $x = x_m$. Substituting these *initial conditions*, as they are called, into Eq. 15-3 and canceling x_m give us

$$1 = \cos\phi. \tag{15-14}$$

Taking the inverse cosine then yields

$$\phi = 0 \text{ rad.} \qquad \text{(Answer)}$$

(Any angle that is an integer multiple of 2π rad also satisfies Eq. 15-14; we chose the smallest angle.)

A block whose mass m is 680 g is fastened to a spring whose spring constant k is 65 N/m. The block is pulled a distance $x = 11$ cm from its equilibrium position at $x = 0$ on a frictionless surface and released from rest at $t = 0$.

(f) What is the displacement function $x(t)$ for the spring–block system?

Calculation: The function $x(t)$ is given in general form by Eq. 15-3. Substituting known quantities into that equation gives us

$$\begin{aligned} x(t) &= x_m \cos(\omega t + \phi) \\ &= (0.11\text{ m}) \cos[(9.8\text{ rad/s})t + 0] \\ &= 0.11 \cos(9.8t), \qquad\qquad \text{(Answer)} \end{aligned}$$

where x is in meters and t is in seconds.

Loudspeaker.

The cone of a loudspeaker oscillates in SHM at a frequency of 262 Hz. The amplitude at the center of the cone is $A = 1.5 \times 10^{-4}$ m, and at $t = 0$, $x = A$. (a) What equation describes the motion of the center of the cone? (b) What are the velocity and acceleration as a function of time? (c) What is the position of the cone at $t = 1.00$ min $(= 1.00 \times 10^{-3}$ s)?

Energy conservation in Simple harmonic motion:

- if the friction effect are neglected, total mechanical energy of vibrating mass spring system remains constant
- The velocity and position of the vibrating body are continually changing
- The kinetic and potential energies also change, but their sum must have the same values at any instant.
- By hook's law
- F=-kx
- $W = \int f\, dx$
- U=-W
- U=$-\int f\, dx$
- $U = -\int -kx\, dx$
- $U = k\int x\, dx$
- $U = k\frac{x^2}{2}$

- Putting value of displacement

$$U(t) = \tfrac{1}{2}k\left(x_m \cos(\omega t + \phi)\right)^2$$
$$U(t) = \tfrac{1}{2}k\, x_m^2 \cos^2(\omega t + \phi)$$

- Kinetic energy is given by $\mathrm{K.E} = \frac{1}{2}\mathrm{m}v^2$

- Kinetic energy is maximum if x=0 when the mass is at equilibrium position.

$$K(t) = \tfrac{1}{2}m\left(-\omega\, x_m \sin(\omega t + \phi)\right)^2$$

$$k = m\omega^2 \qquad K(t) = \tfrac{1}{2}k\, x_m^2 \sin^2(\omega t + \phi)$$

- The energy is partly P.E and partly K.E.

$$\mathrm{E = P.E + K.E}$$

$$\therefore E = \tfrac{1}{2}k\, x_m^2 = constant$$

Many tall building have mass dampers, which are anti-sway devices to prevent them from oscillating in a wind. The device might be a block oscillating at the end of a spring and on a lubricated track. If the building sways, say eastward, the block also moves eastward but delayed enough so that when it finally moves, the building is then moving back westward. Thus, the motion of the oscillator is out of step with the motion of the building.

Suppose that the block has mass $m = 2.72 \times 10^5$ kg and is designed to oscillate at frequency $f = 10.0$ Hz and with amplitude $x_m = 20.0$ cm.

(a) What is the total mechanical energy E of the spring-block system?

KEY IDEA The mechanical energy E (the sum of the kinetic energy $K = \frac{1}{2}mv^2$ of the block and the potential energy $U = \frac{1}{2}kx^2$ of the spring) is constant throughout the motion of the oscillator. Thus, we can evaluate E at any point during the motion.

Calculations: Because we are given amplitude x_m of the oscillations, let's evaluate E when the block is at position $x = x_m$, where it has velocity $v = 0$. However, to evaluate U at that point, we first need to find the spring constant k. From Eq. 15-12 ($\omega = \sqrt{k/m}$) and Eq. 15-5 ($\omega = 2\pi f$), we find

$$\begin{aligned} k &= m\omega^2 = m(2\pi f)^2 \\ &= (2.72 \times 10^5 \text{ kg})(2\pi)^2(10.0 \text{ Hz})^2 \\ &= 1.073 \times 10^9 \text{ N/m}. \end{aligned}$$

We can now evaluate E as

$$\begin{aligned} E &= K + U = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 \\ &= 0 + \tfrac{1}{2}(1.073 \times 10^9 \text{ N/m})(0.20 \text{ m})^2 \\ &= 2.147 \times 10^7 \text{ J} \approx 2.1 \times 10^7 \text{ J}. \qquad \text{(Answer)} \end{aligned}$$

(b) What is the block's speed as it passes through the equilibrium point?

Calculations: We want the speed at $x = 0$, where the potential energy is $U = \frac{1}{2}kx^2 = 0$ and the mechanical energy is entirely kinetic energy. So, we can write

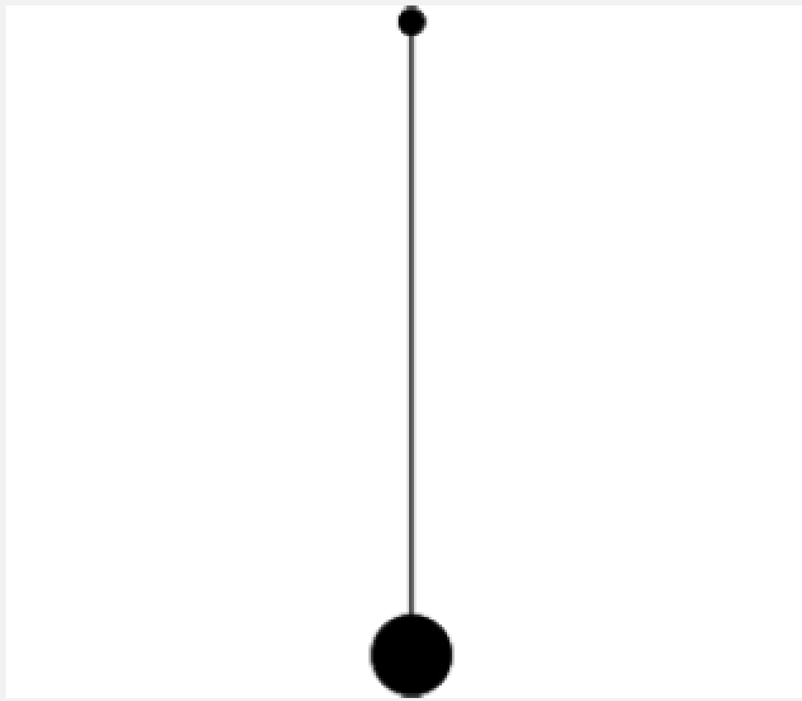
$$E = K + U = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2$$

$$2.147 \times 10^7 \text{ J} = \tfrac{1}{2}(2.72 \times 10^5 \text{ kg})v^2 + 0,$$

or $$v = 12.6 \text{ m/s.} \qquad \text{(Answer)}$$

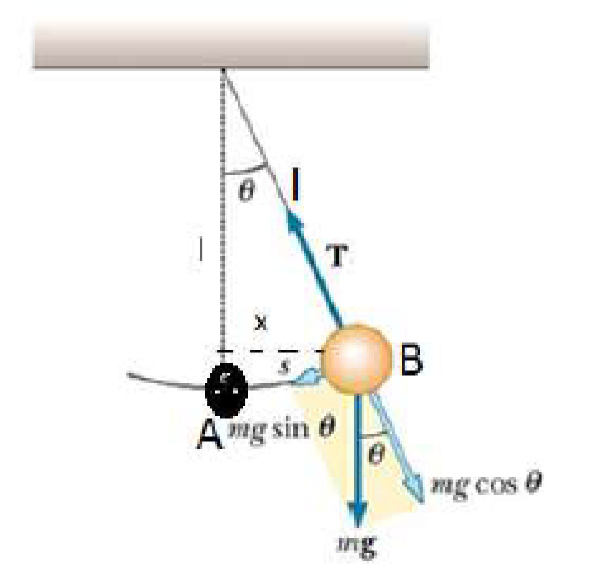
Because E is entirely kinetic energy, this is the maximum speed v_m.

Simple pendulum:



Simple pendulum:

- A simple pendulum is idealized model consist of a point mass suspended by an inextensible string of length *l* fixed
- When pulled to one side of its equilibrium position A to the position B through a small angle θ and released, it starts oscillating to and fro over the small.



Simple pendulum performs SHM:

- Condition for SHM is that restoring force F should be directly proportional to the displacement oppositely directed. The path of the bob is not straight line, but the arc of the circle of radius *l*

- Let T is the tension in the spring. When the particle is at point B two forces are acting on it:

1. mg, the weight of the point bob acting vertically downward.
2. T, the tension along the string.

- The weight mg can be resolved into two rectangular components.

1. Component of weight mg along the spring=$mg\cos\theta$
2. Component of weight mg perpendicular to the string=$mg\sin\theta$

- Since there is no motion of bob along the string, so the component mg cosθ must be equal to tension in the string T

$$T = mg\cos\theta$$

- Component mg sinθ is responsible for the motion of the bob towards the mean position. Thus, the restoring force F is;

$$F = -mg\sin\theta \qquad ....(iv)$$

$$\text{So } \sin\theta = \frac{x}{l}$$

$$F = -mg\frac{x}{l}$$

- By 2nd law of motion

F=ma ....(v)

- Comparing (iv) and (v)

$$ma = -\,mg\frac{x}{l}$$

$$a = -\,\frac{gx}{l}$$

$$a = -(\text{constant})x$$

$$a \propto -x$$

Expression for time period:

We know that

$a = -x\omega^2 \quad \Rightarrow \quad a = -\frac{gx}{l}$

$-x\omega^2 = -\frac{gx}{l} \quad \Rightarrow \omega = \sqrt{g/l}$

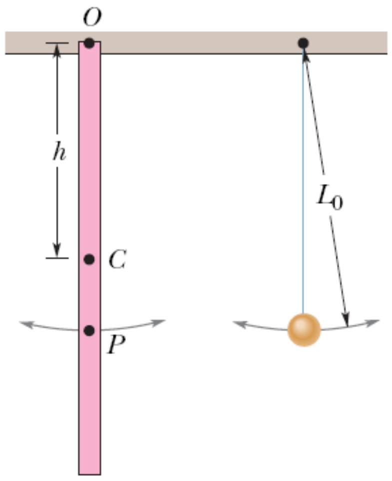
$T = \frac{2\pi}{\omega} \quad \Rightarrow T = 2\pi\sqrt{\frac{l}{g}}$

Frequency:

$F = {}^{1}/_{T}$

$F = {}^{1}/_{2\pi} \times \sqrt{g/l}$

In Fig. *a, a meter stick swings about a pivot point* at one end, at distance *h from the stick's center of mass.*
(a)What is the period of oscillation *T*?



(*a*) (*b*)

KEY IDEA: The stick is not a simple pendulum because its mass is not concentrated in a bob at the end opposite the pivot point—so the stick is a physical pendulum.

Calculations: The period for a physical pendulum depends on the rotational inertia, I, of the stick about the pivot point. We can treat the stick as a uniform rod of length L and mass m. Then I =1/3 mL^2, where the distance h is L.
Therefore,

$$T = 2\pi\sqrt{\frac{I}{mgh}} = 2\pi\sqrt{\frac{\frac{1}{3}mL^2}{mg(\frac{1}{2}L)}} = 2\pi\sqrt{\frac{2L}{3g}}$$

$$= 2\pi\sqrt{\frac{(2)(1.00\text{ m})}{(3)(9.8\text{ m/s}^2)}} = 1.64\text{ s.} \qquad \text{(Answer)}$$

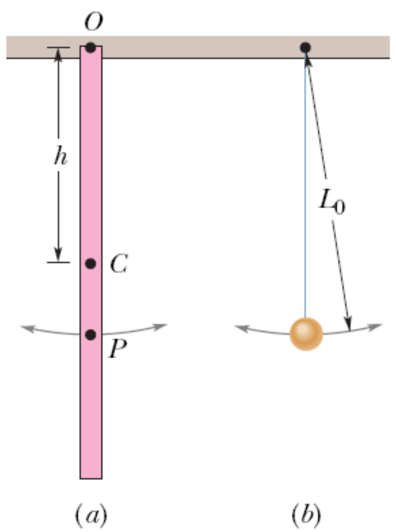
Note the result is independent of the pendulum's mass *m*.

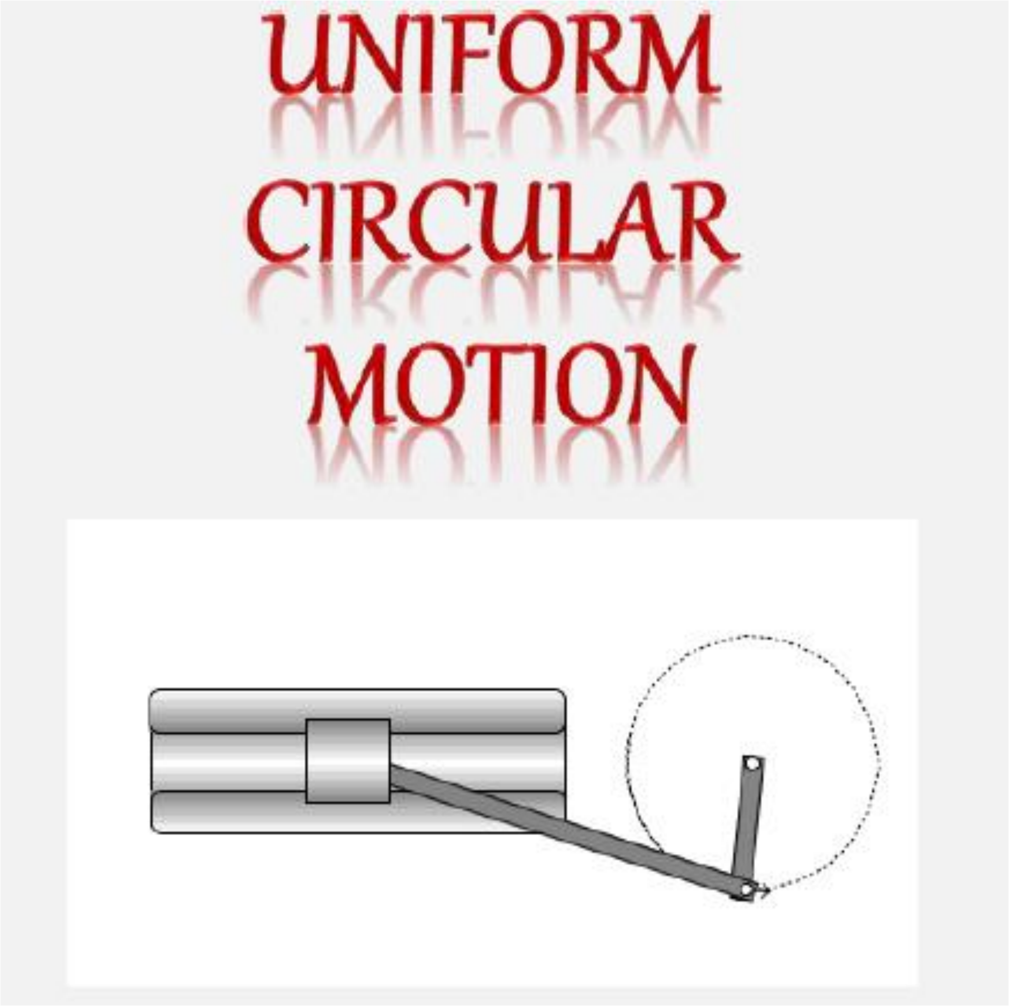
(b) What is the distance L_0 *between the pivot point O* of the stick and the center of oscillation of the stick?

Calculations: We want the length L_0 of the simple pendulum (drawn in Fig. *b)* that has the same period as the physical pendulum (the stick) of Fig. *a*.

$$T = 2\pi\sqrt{\frac{L_0}{g}} = 2\pi\sqrt{\frac{2L}{3g}}.$$

$$L_0 = \tfrac{2}{3}L = (\tfrac{2}{3})(100 \text{ cm}) = 66.7 \text{ cm.} \quad \text{(Answer)}$$




UNIFORM
CIRCULAR
MOTION

Relation between uniform circular motion and SHM

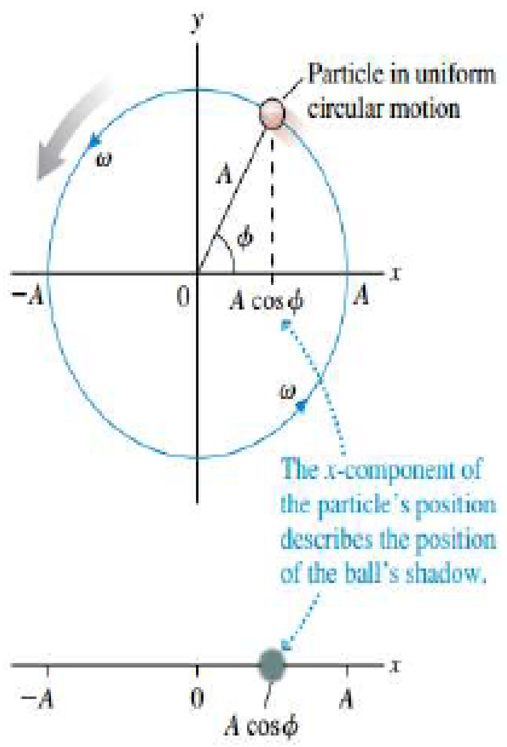
An object in simple harmonic motion has the same motion as of an object in uniform circular motion:

- Consider the particle in uniform circular motion with radius A and angle ϕ
- x= A cos ϕ
- Particle's angular velocity, in rad/s, is
- $\frac{d\varphi}{dt}$=ω
- This is the rate at which the angle ϕ is increasing.

If the particle starts from $\emptyset_0$= 0 at t = 0, its angle at a later time t is simply

- φ =ωt

A particle in uniform circular motion with radius A and angular velocity ω.



- As φ increases, the particle's x-component is

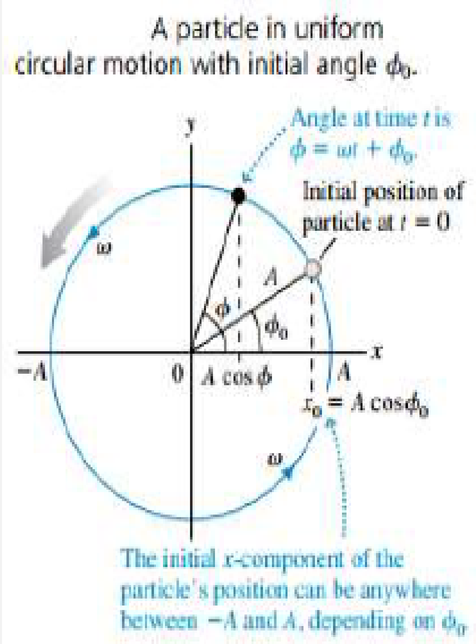
$x(t) = A \cos \omega t$

- The particle in started at $\varnothing_0$ = 0. fig shows a more general situation in which the initial angle $\varnothing_0$ can have any value. The angle at a later time t is then

$$\varphi = \omega t + \varnothing_0$$

- $v(t) = -\omega A \sin(\omega t + \varnothing_0) =$

$v(t) = -v_{max} \sin(\omega t + \varnothing_0)$



Example

A particle rotates counterclockwise in a circle of radius 3.00 m with a constant angular speed of 8.00 rad/s. As the particle has an *x* coordinate of 2.00 m and is moving to the right. (a) Determine the *x* coordinate as a function of time.

Solution Because the amplitude of the particle's motion equals the radius of the circle and ω = 8.00 rad/s, we have

$$x = A\cos(\omega t + \phi) = (3.00\text{ m})\cos(8.00t + \phi)$$

We can evaluate ϕ by using the initial condition that x = 2.00 m at $t = 0$:

$$2.00\text{ m} = (3.00\text{ m})\cos(0 + \phi)$$

$$\phi = \cos^{-1}\left(\frac{2.00\text{ m}}{3.00\text{ m}}\right)$$

If we were to take our answer as $\phi = 48.2°$, then the coordinate $x = (3.00\text{ m})\cos(8.00t + 48.2°)$ would be decreasing at time $t = 0$ (that is, moving to the left). Because our particle is first moving to the right, we must choose $\phi = -48.2° = -0.841$ rad. The x coordinate as a function of time is then

$$x = \boxed{(3.00\text{ m})\cos(8.00t - 0.841)}$$

Note that ϕ in the cosine function must be in radians.

(b) Find the x components of the particle's velocity and acceleration at any time t.

Solution

$$v_x = \frac{dx}{dt} = (-3.00\text{ m})(8.00\text{ rad/s})\sin(8.00t - 0.841)$$

$$= \boxed{-(24.0\text{ m/s})\sin(8.00t - 0.841)}$$

$$a_x = \frac{dv_x}{dt} = (-24.0\text{ m/s})(8.00\text{ rad/s})\cos(8.00t - 0.841)$$

$$= \boxed{-(192\text{ m/s}^2)\cos(8.00t - 0.841)}$$

From these results, we conclude that v_{max} = 24.0 m/s and that a_{max} = 192 m/s^2. Note that these values also equal the tangential speed ωA and the centripetal acceleration $\omega^2 A$.