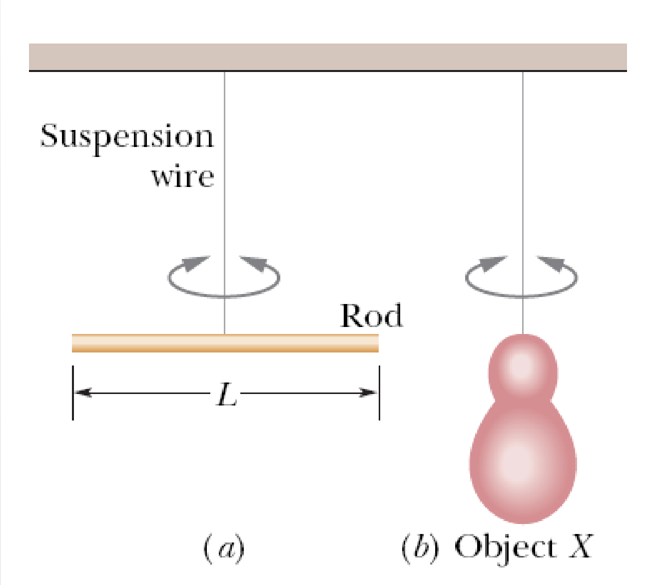
Oscillations

Example, angular SHM:

Figure *a shows a thin rod whose length L is 12.4 cm* and whose mass *m is 135 g, suspended at its midpoint* from a long wire. Its period T_a *of angular SHM is measured* to be 2.53 s. An irregularly shaped object, which we call object *X, is then hung from the same wire, as in Fig. b, and its period* T_b *is found to be 4.76 s. What is the* rotational inertia of object *X about its suspension axis?*



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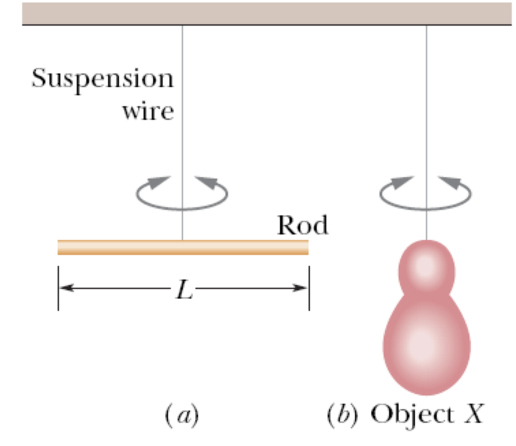


Answer: The rotational inertia of either the rod or object *X is related to the measured period.* The rotational inertia of a thin rod about a perpendicular axis through its midpoint is given as 1/12 mL^2*.Thus, we have, for the rod in Fig. a,*

$$I_a = \tfrac{1}{12} m L^2 = \left(\tfrac{1}{12}\right)(0.135\ \mathrm{kg})(0.124\ \mathrm{m})^2 = 1.73 \times 10^{-4}\ \mathrm{kg \cdot m^2}.$$

Now let us write the periods, once for the rod and once for object *X:*

$$T_a = 2\pi\sqrt{\frac{I_a}{\kappa}} \quad \text{and} \quad T_b = 2\pi\sqrt{\frac{I_b}{\kappa}}.$$

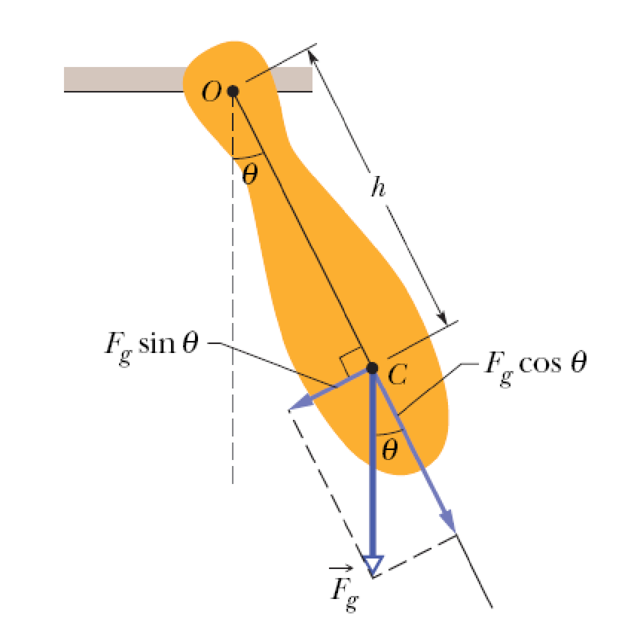
The constant κ, which is a property of the wire, is the same for both figures; only the periods and the rotational inertias differ.

Let us square each of these equations, divide the second by the first, and solve the resulting equation for I_b*. The result is*

$$I_b = I_a \frac{T_b^2}{T_a^2} = (1.73 \times 10^{-4}\ \mathrm{kg \cdot m^2}) \frac{(4.76\ \mathrm{s})^2}{(2.53\ \mathrm{s})^2} = 6.12 \times 10^{-4}\ \mathrm{kg \cdot m^2}. \qquad \text{(Answer)}$$

Pendulums

A physical pendulum can have a complicated distribution of mass. If the center of mass, C, is at a distance of h from the pivot point (figure), then for *small angular amplitudes*, the motion is simple harmonic.

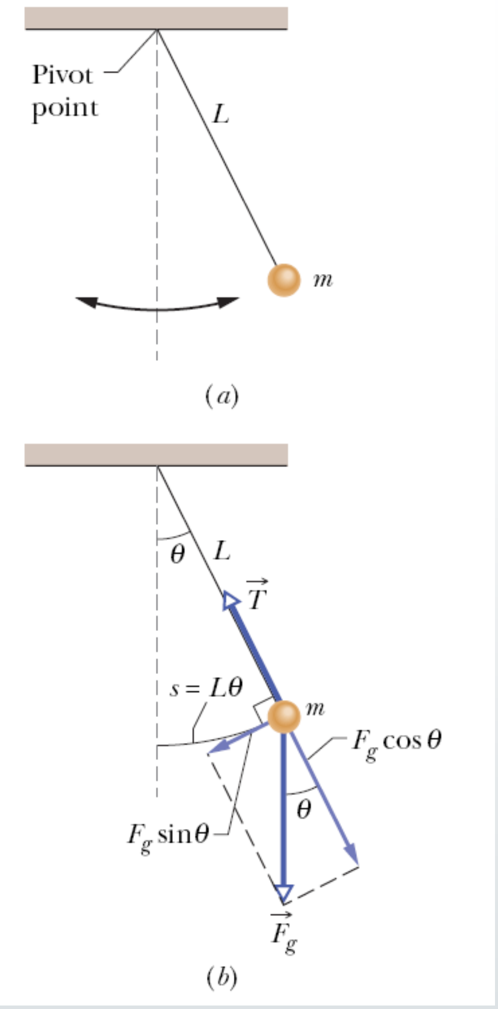


The period, T, is:

$$T = 2\pi\sqrt{\frac{I}{mgh}}$$

Here, I is the rotational inertia of the pendulum about O.

Pendulums



In a *simple pendulum*, a particle of mass m is suspended from one end of an unstretchable massless string of length L that is fixed at the other end.

The restoring torque acting on the mass when its angular displacement is θ, is:

$$\tau = -L(F_g \sin\theta) = I\alpha$$

α is the angular acceleration of the mass. Finally,

$$\alpha = -\frac{mgL}{I}\theta, and$$

$$T = 2\pi\sqrt{\frac{L}{g}}$$

This is true for *small angular displacements*, θ.

Pendulums

In the **small-angle approximation** we can assume that $\theta << 1$ and use the approximation $\sin\theta \cong \theta$. Let us investigate up to what angle θ is the approximation reasonably accurate?

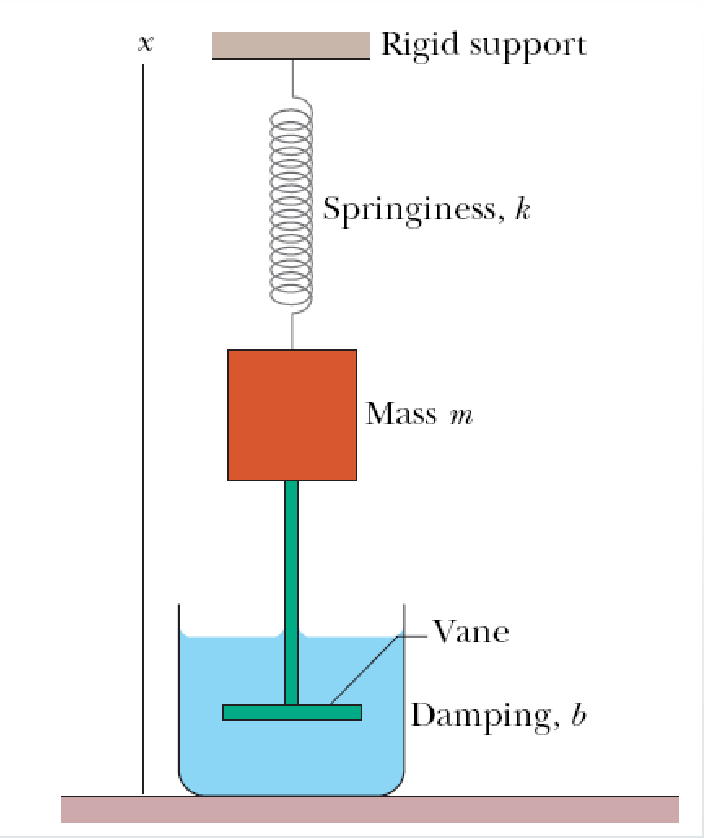
θ **(degrees)**	θ **(radians)**	**sin** θ
5	0.087	0.087
10	0.174	0.174
15	0.262	0.259 (1% off)
20	0.349	0.342 (2% off)

Conclusion: If we keep $\theta < 10\,°$ we make less than 1 % error.

Damped Oscillations

In a damped oscillation, the motion of the oscillator is reduced by an external force.

Example: A block of mass *m* oscillates vertically on a spring with spring constant *k*.
From the block a rod extends to a vane which is submerged in a liquid.
The liquid provides the external damping force, F_d.

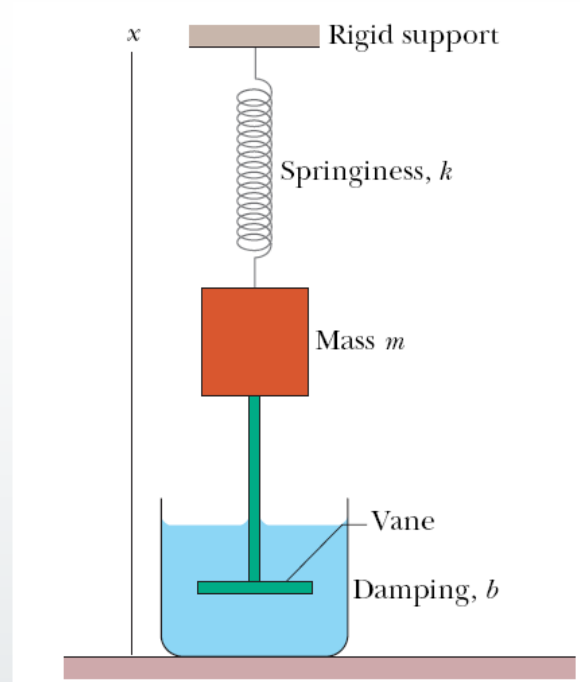


Damped Oscillations

Often the damping force, F_d, is proportional to the 1st power of the velocity v. That is,

$$F_d = -\gamma v$$

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = 0$$



The solution is:

$$x(t) = x_0 e^{\frac{-\gamma t}{2m}} \cos(\omega' t + \phi)$$

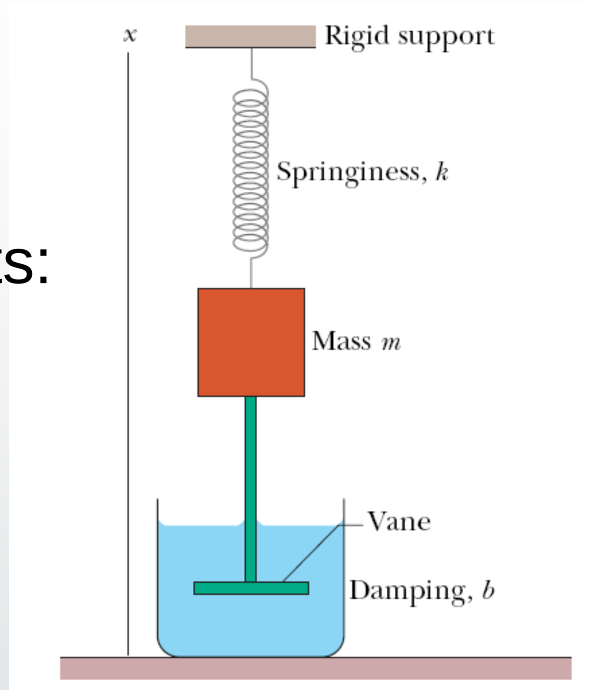
$$\omega' = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2m}\right)^2}$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

Damped SHM

Often the damping force, F_d, is proportional to the 1st power of the velocity v. That is,

$$F_d = -bv$$



From Newton's 2nd law, the following DE results:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

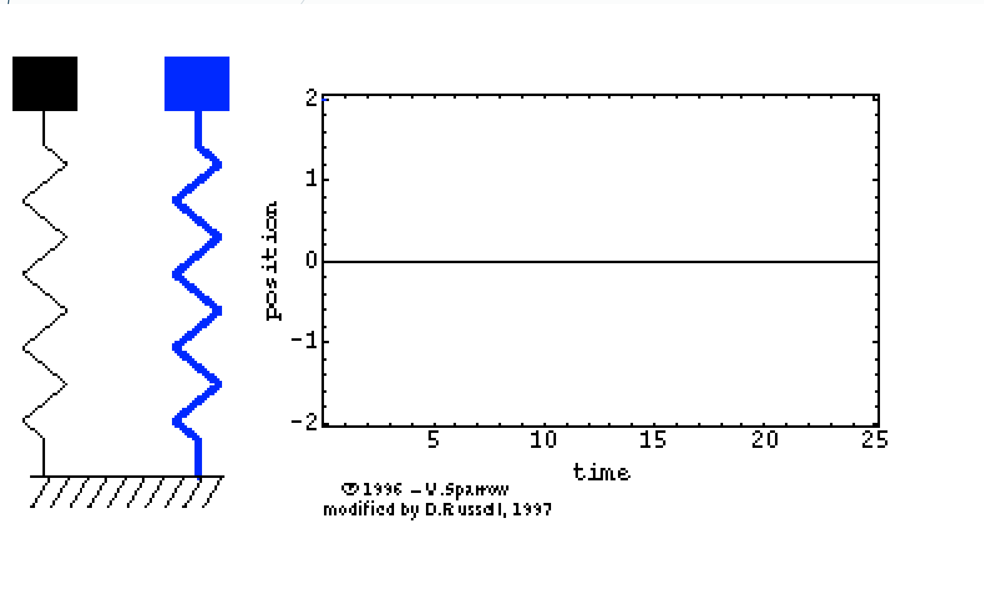
The solution is:

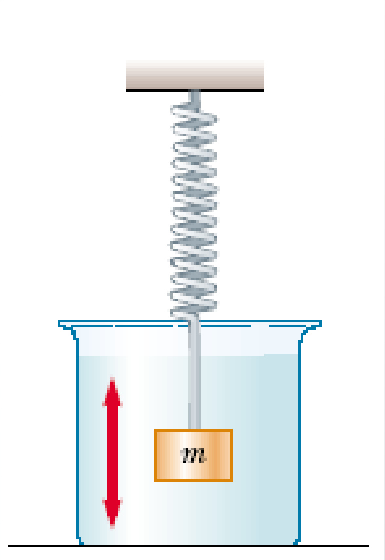
$$x(t) = x_m e^{\frac{-bt}{2m}} \cos(\omega' t + \phi)$$

Here ω' is the angular frequency, and is given by:

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

DAMPED OSCILLATIONS

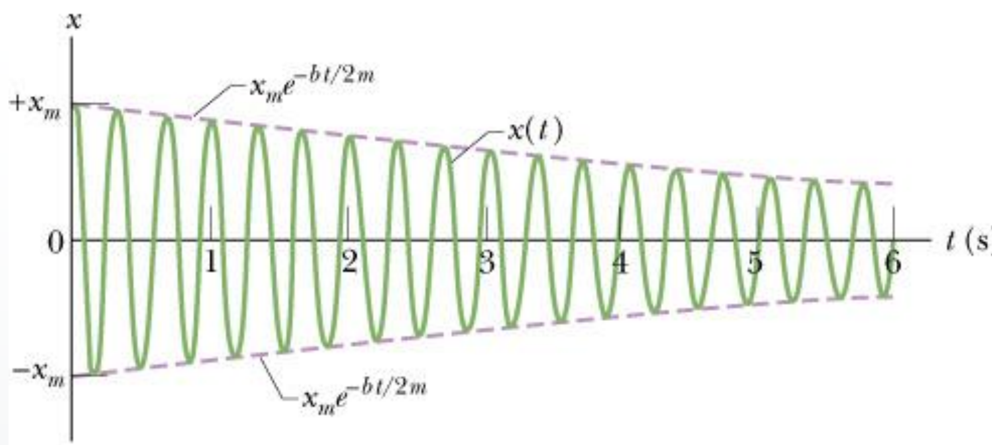




http://www.lon-capa.org/~mmp/applist/damped/d.htm

15.5 Damped Oscillations

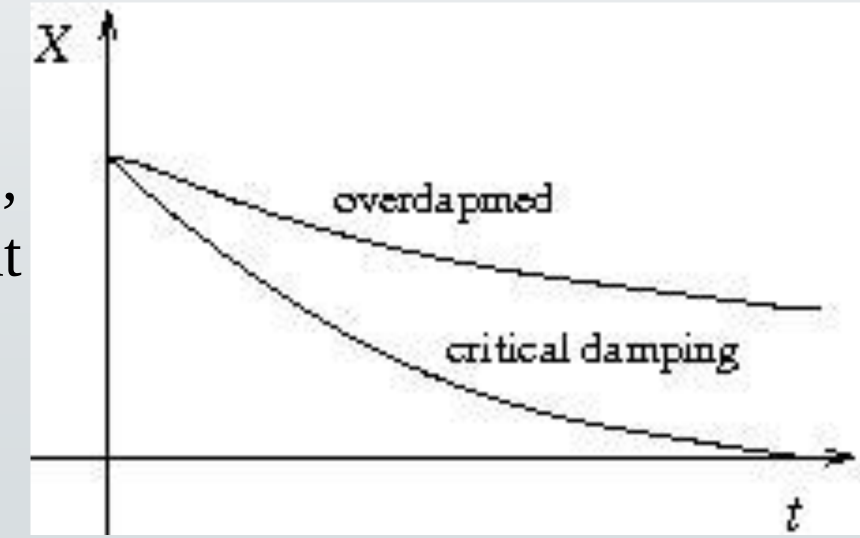
$$x(t) = x_0 e^{\frac{-\gamma t}{2m}} \cos(\omega' t + \varphi)$$



The above figure shows the displacement function *x(t)* for the damped oscillator described before.
The amplitude decreases as x_0 exp ($-\gamma t$ / $2m$) with time.

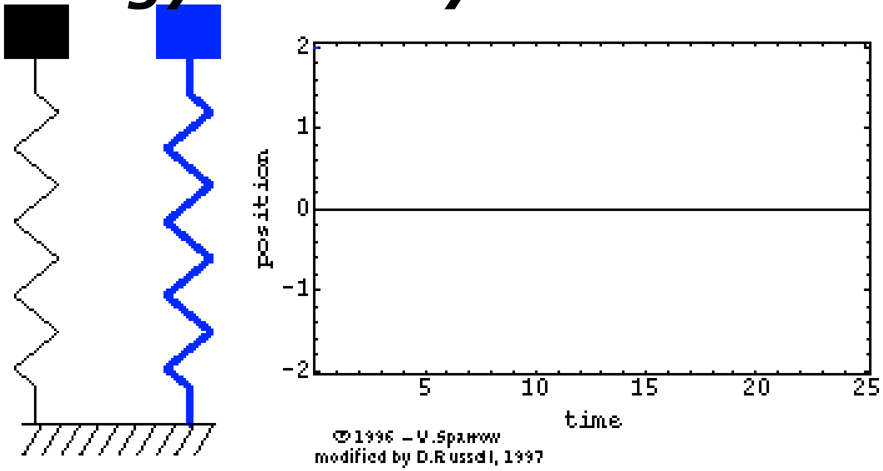
\# The above is for $\gamma < 2m\omega_0$ (underdapmed).

\# For $\gamma > 2m\omega_0$ (overdapmed) and $\gamma = 2m\omega_0$ (critical damping), the oscillation goes like the right figure.



DAMPED OSCILLATIONS

In many real systems, dissipative forces, such as friction, retard the motion.
Consequently, the mechanical energy of the system diminishes in time, and the motion is said to be Damped.
Retarding force $\mathbf{R} = -b\mathbf{v}$



, we can write Newton's second law as

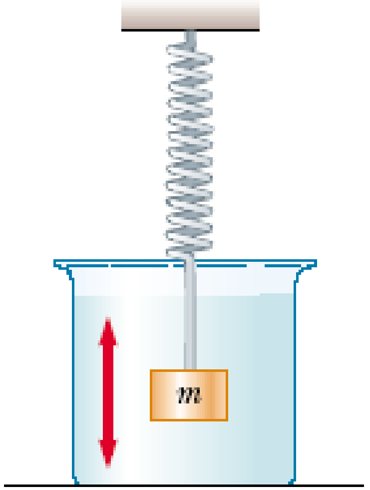
$$\sum F_x = -kx - bv = ma_x$$

$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

The solution of this equation $x = Ae^{-\frac{b}{2m}t}\cos(\omega t + \phi)$

where the angular frequency of oscillation is

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

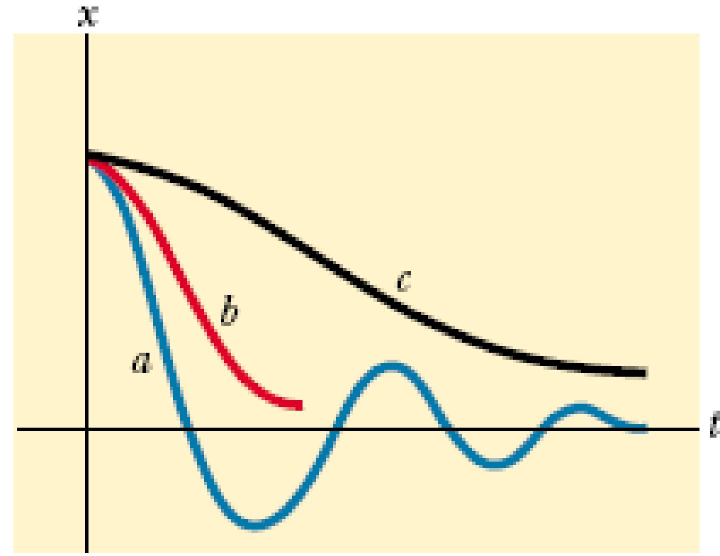


http://www.lon-capa.org/~mmp/applist/damped/d.htm

DAMPED OSCILLATIONS

$$\omega' = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2m}\right)^2}$$

where $\omega_0 = \sqrt{k/m}$ represents the angular frequency in the absence of a retarding force (the undamped oscillator) and is called the **natural frequency**



Graphs of displacement versus time for (a) an underdamped oscillator, (b) a critically damped oscillator, and (c) an overdamped oscillator.

Forced Oscillations and Resonance

When the oscillator is subjected to an external force that is periodic, the oscillator will exhibit forced/driven oscillations.

There are two frequencies involved in a forced oscillator:

I. w_0, the natural angular frequency of the oscillator, without the presence of any external force, and
II. w_e, the angular frequency of the applied external force.

The equation of motion is like the following:

$$m\frac{d^2x}{dt^2} + g\frac{dx}{dt} + kx = F_0\cos(w_e t)$$

Forced Oscillations and Resonance

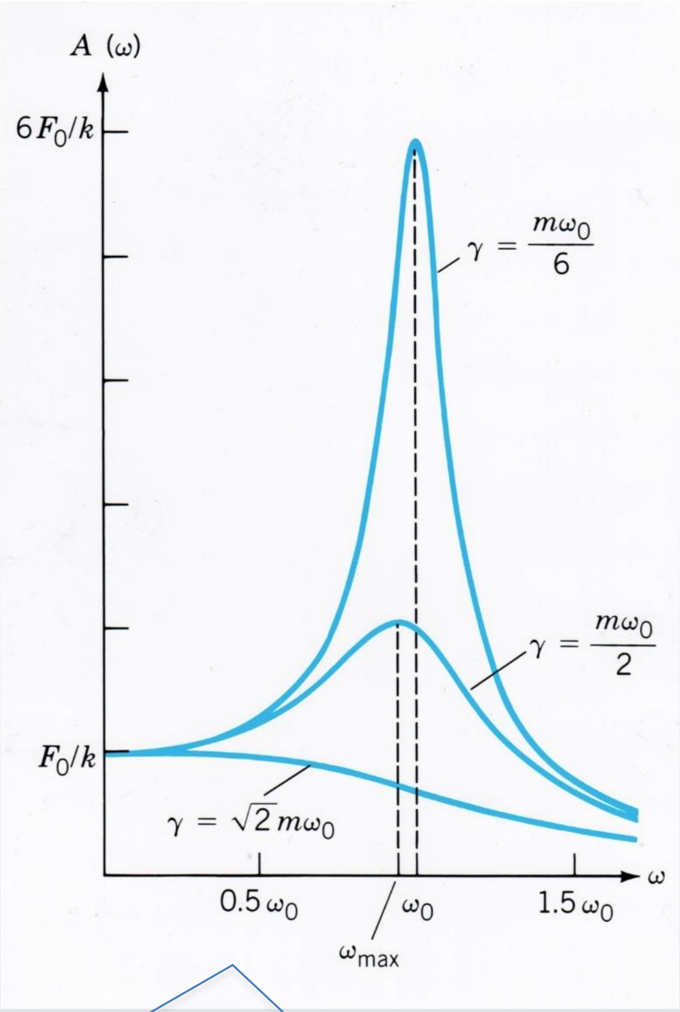
$$m\frac{d^2x}{dt^2}+\gamma\frac{dx}{dt}+kx=F_0\cos(\omega_e t)$$

The *steady state* solution is

$$x(t)=A\cos(\omega_e t+\delta)$$

$$A=\frac{F_0/m}{\sqrt{(\omega_0^2-\omega_e^2)^2+\left(\frac{\gamma}{m}\omega_e\right)^2}}$$

$$\tan\delta=\frac{\gamma}{m}\frac{\omega_e}{\omega_0^2-\omega_e^2}\qquad \omega_0=\sqrt{\frac{k}{m}}$$

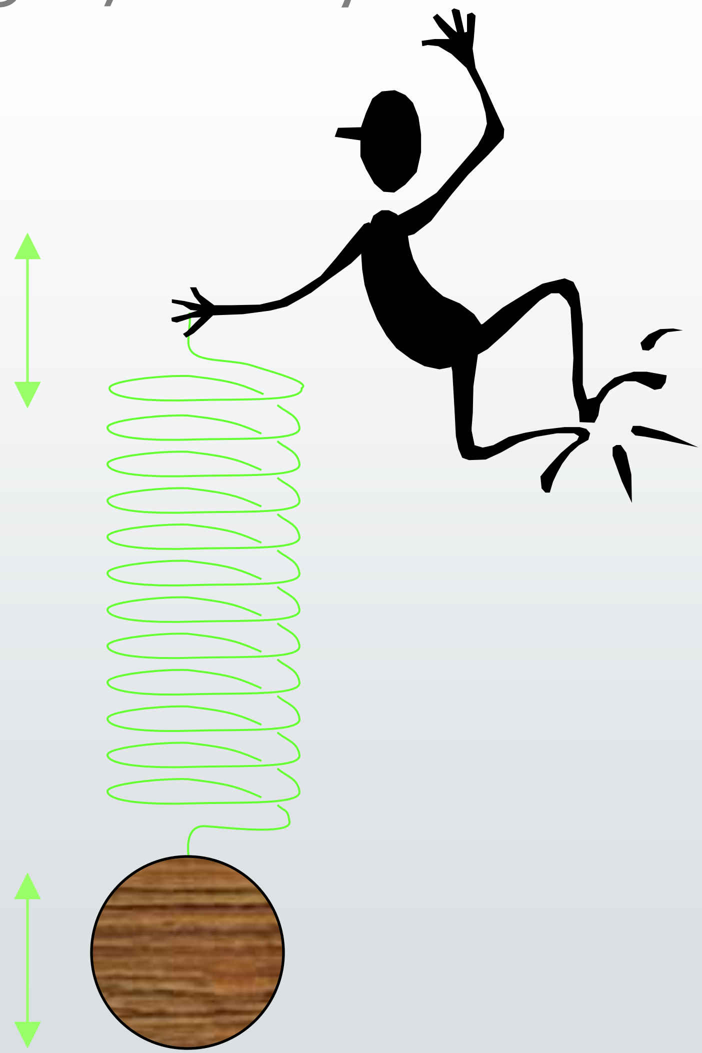


Resonance occurs at $\omega_e \sim \omega_{max} < \omega_0$, for $\gamma<\sqrt{2}m\omega_0$

Example (Mass-Spring System)

Periodic driving force of freq. f

Oscillating with natural freq. f_o



Resonance

When a system is disturbed by a periodic driving force which frequency is ***equal to the natural frequency (fo)*** of the system the system will oscillate with ***LARGE amplitude***.

Resonance is said to occur.

http://www.acoustics.salford.ac.uk/feschools/waves/shm3.htm

Example 1

Breaking Glass

System : ***glass***

Driving Force : ***sound wave***

Example 2

Collapse of the Tacoma Narrows suspension bridge in America in 1940

System : ***bridge***

Driving Force : ***strong wind***

FORCED OSCILLATIONS

When a system is disturbed by a ***periodic***
driving force and then oscillate, this is called ***forced oscillation***.
The system will oscillate with ***<u>its</u> natural frequency (f_o)*** which is ***independent of*** the frequency of the driving force

$$x = A\cos(\omega t + \phi)$$

$$F_{\text{ext}}\cos\omega t - kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

Where,

$$A = \frac{F_{\text{ext}}/m}{\sqrt{(\omega^2 - {\omega_0}^2)^2 + \left(\frac{b\omega}{m}\right)^2}}$$