Cross Product

Muhammad Adeel

Vector Product or Cross Product

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Introduction

- When we calculate the vector product of two vectors the result, as the name suggests, is a **vector**.
- It is also named as **Cross Product**.
- It is a vector expressing the angular relationship between the vectors.

Definition

- Let's say, we have two vectors, a and b , if $|a|$ and $|b|$ represent the lengths of vectors a and b , respectively, and
- if θ is the angle between these vectors.
- Then, The cross product of vectors a and b will have the following relationship:

$$a \times b = |a||b|\sin\theta$$

Pictorial Presentation

Study the two vectors **a** and **b** drawn in Figure 1. Note that we have drawn the two vectors so that their tails are at the same point. The angle between the two vectors has been labelled θ.

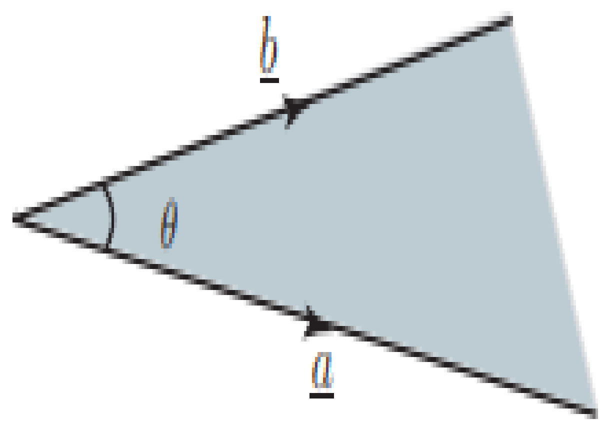


Figure 1. Two vectors **a** and **b** drawn so that the angle between them is θ.

Pictorial Presentation

As we stated before, when we find a **vector product** the result is a vector. We define the modulus, or magnitude, of this vector as

$$|\mathbf{a}|\,|\mathbf{b}|\,\sin\theta$$

so at this stage, a very similar definition to the scalar product, except now the sine of θ appears in the formula. However, this quantity is not a vector. To obtain a vector we need to specify a direction. By definition the direction of the vector product is such that it is at right angles to both $\mathbf{a}$ and $\mathbf{b}$. This means it is at right angles to the plane in which $\mathbf{a}$ and $\mathbf{b}$ lie. Figure 2 shows that we have two choices for such a direction.

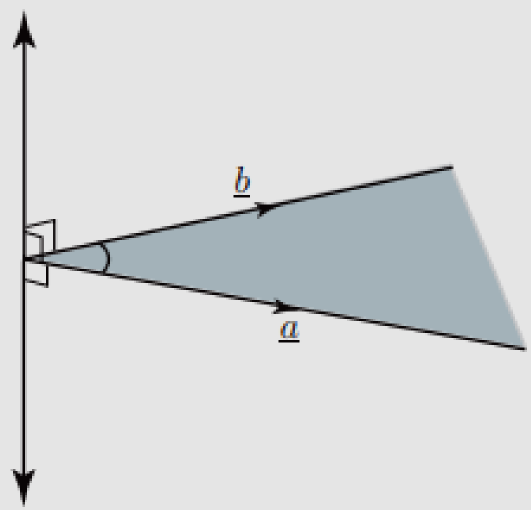


Figure 2. There are two directions which are perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

Pictorial Presentation

The convention is that we choose the direction specified by the right hand screw rule. This means that we imagine a screwdriver in the right hand. The direction of the vector product is the direction in which a screw would advance as the screwdriver handle is turned in the sense from $\mathbf{a}$ to $\mathbf{b}$. This is shown in Figure 3.

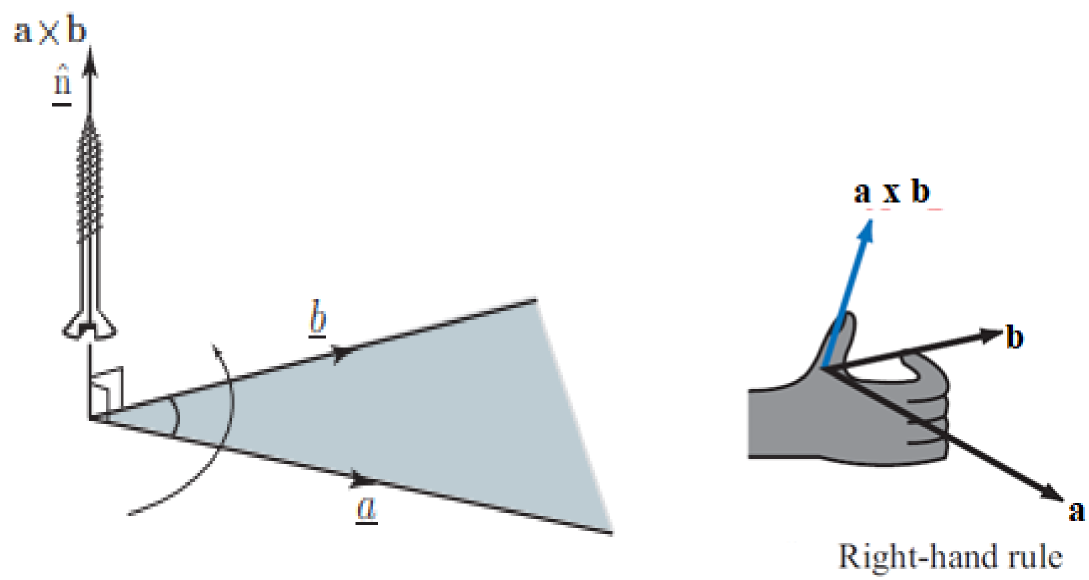
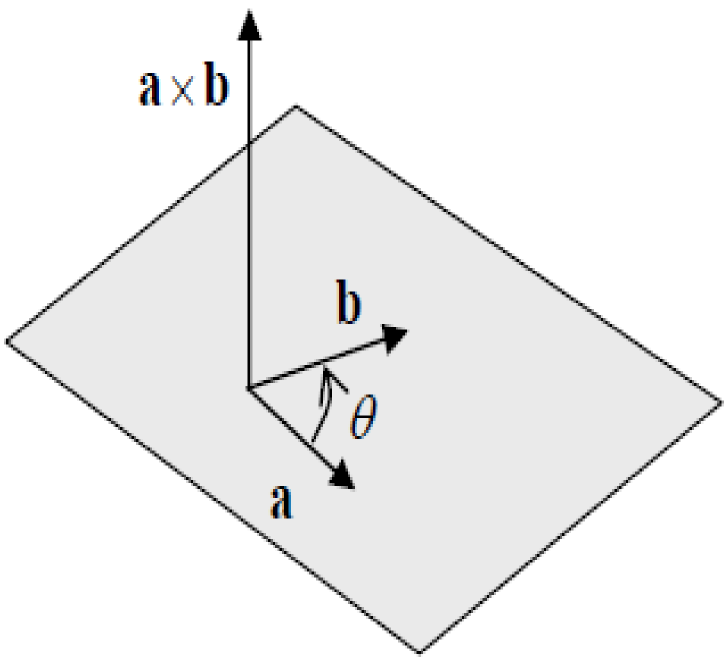
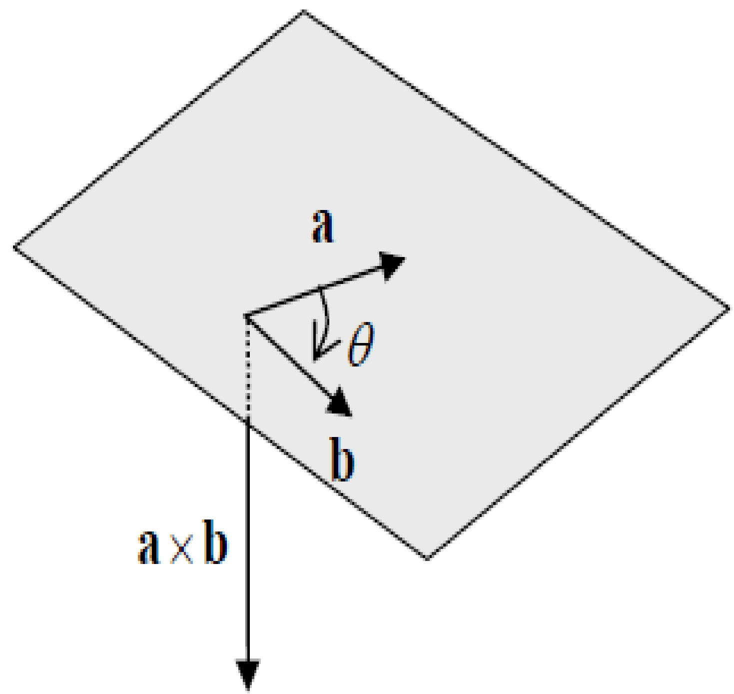


Figure 3. The direction of the vector product is determined by the right hand screw rule.

Pictorial Presentation

RIGHT HAND SCREW RULE





Geometrical Interpretation

Given the characteristics of the cross product of two vectors by the relation

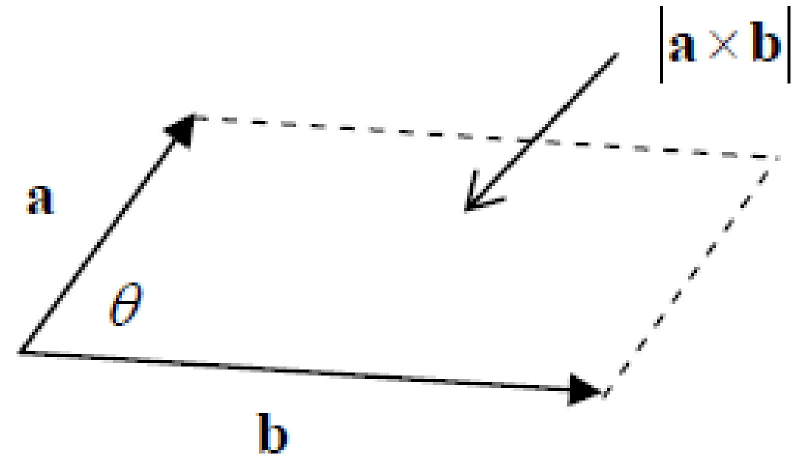
$$a \times b = |a|\,|b|\sin\theta$$

Now, we can interpret three possible conditions:

- $a \times b$ is perpendicular to both the vectors a and b .
- $a \times b$ represents the area of parallelogram determined by these vectors as adjacent sides.
- If a and b are parallel vectors then $a \times b = 0$

Area of a Parallelogram

$a \times b$ represents the area of parallelogram determined by these vectors as adjacent sides.



the magnitude of the cross product

Key Points

We let a unit vector in this direction be labelled $\hat{\mathbf{n}}$. We then define the vector product of $\mathbf{a}$ and $\mathbf{b}$ as follows:

The **vector product** of $\mathbf{a}$ and $\mathbf{b}$ is defined to be

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\sin\theta\,\hat{\mathbf{n}}$$

where

$|\mathbf{a}|$ is the modulus, or magnitude of $\mathbf{a}$,

$|\mathbf{b}|$ is the modulus of $\mathbf{b}$,

θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, and $\hat{\mathbf{n}}$ is a unit vector, perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ in a sense defined by the right hand screw rule.

Illustration of Cross Product

Q:Why Cross Product?

- For accumulation of interactions between different dimensions.

Method to Find Cross Product

If ***a*** *and* ***b*** *are two vectors of form*

- $\boldsymbol{a = a_1 i + a_2 j + a_3 k}$
- $\boldsymbol{b = b_1 i + b_2 j + b_3 k}$

Then the cross Product of ***a and b*** *can be calculated as,*

$$\boldsymbol{a \times b} = \begin{vmatrix} i & j & k \\ a1 & a2 & a3 \\ b1 & b2 & b3 \end{vmatrix}$$

Method to Find Cross Product

- The angular relationship of two vectors **a** and **b** is ,

$$\vec{a} \times \vec{b} = \left| |a| \, |b| \sin\theta \right|$$

Solved Example

- Calculate the cross product between a=(3,−3,1) and b=(4,9,2)
- **Solution:** The cross product is

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & -3 & 1\\ 4 & 9 & 2\end{vmatrix}\\ &= \mathbf{i}(-3\cdot 2-1\cdot 9)-\mathbf{j}(3\cdot 2-1\cdot 4)+\mathbf{k}(3\cdot 9+3\cdot 4)\\ &= -15\mathbf{i}-2\mathbf{j}+39\mathbf{k}\end{aligned}$$

Solved Example

- Calculate the area of the parallelogram spanned by the vectors

$$a=(3,-3,1) \text{ and } b=(4,9,2)$$

Solution: The area is $\|\mathbf{a} \times \mathbf{b}\|$. Using the above expression for the cross product, we find that the area is $\sqrt{15^2+2^2+39^2}=5\sqrt{70}$.

Practice Problem

TASK:

- Calculate the area of the parallelogram spanned by the vectors ,

  $a=(3,-3,1)$ and $c=(-12,12,-4)$.

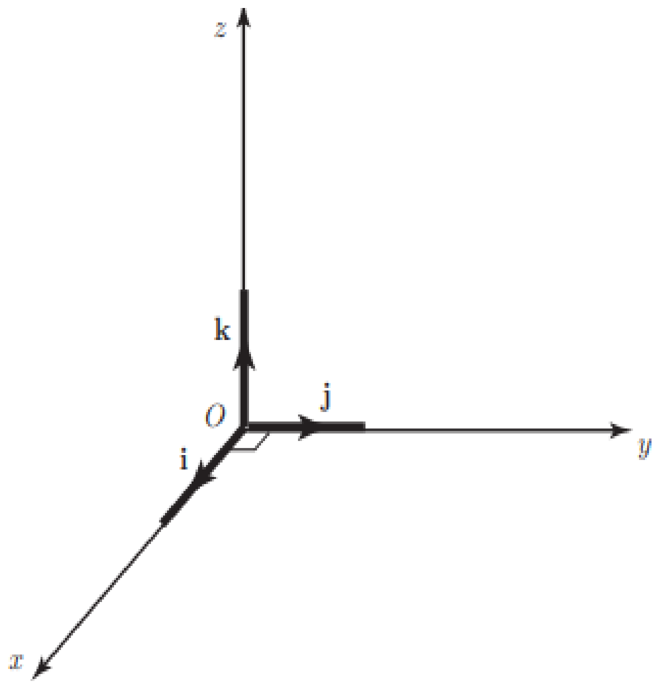
Vector Product in Cartesian Form

Suppose we want to find $\mathbf{i} \times \mathbf{j}$. The vectors $\mathbf{i}$ and $\mathbf{j}$ are shown in Figure 4 Note that because these vectors lie along the x and y axes they must be perpendicular.

Figure 4 The unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. Note that $\mathbf{k}$ is a unit vector perpendicular to $\mathbf{i}$ and $\mathbf{j}$.

The angle between $\mathbf{i}$ and $\mathbf{j}$ is $90°$, and $\sin 90° = 1$. Further, if we apply the right hand screw rule, a vector perpendicular to both $\mathbf{i}$ and $\mathbf{j}$ is $\mathbf{k}$. Therefore

$$\begin{aligned} \mathbf{i} \times \mathbf{j} &= |\mathbf{i}|\,|\mathbf{j}| \sin 90° \, \mathbf{k} \\ &= (1)(1)(1)\,\mathbf{k} \\ &= \mathbf{k} \end{aligned}$$



Key Points

$$\mathrm{i} \times \mathrm{i} = 0 \qquad \mathrm{j} \times \mathrm{j} = 0 \qquad \mathrm{k} \times \mathrm{k} = 0$$

$$\mathrm{i} \times \mathrm{j} = \mathrm{k} \qquad \mathrm{j} \times \mathrm{k} = \mathrm{i} \qquad \mathrm{k} \times \mathrm{i} = \mathrm{j}$$

$$\mathrm{j} \times \mathrm{i} = -\mathrm{k} \qquad \mathrm{k} \times \mathrm{j} = -\mathrm{i} \qquad \mathrm{i} \times \mathrm{k} = -\mathrm{j}$$

Prove that:

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

Properties of Vector Product

- The vector product is not commutative.

$$b \times a = -a \times b$$

- The vector product is distributive over addition. This means

  $a \times (b + c) = (a \times b) + (a \times c)$

Equivalently,

  $(b + c) \times a = (b \times a) + (c \times a)$

- For two parallel vectors ., $a \times b = 0$

Properties of Vector Product

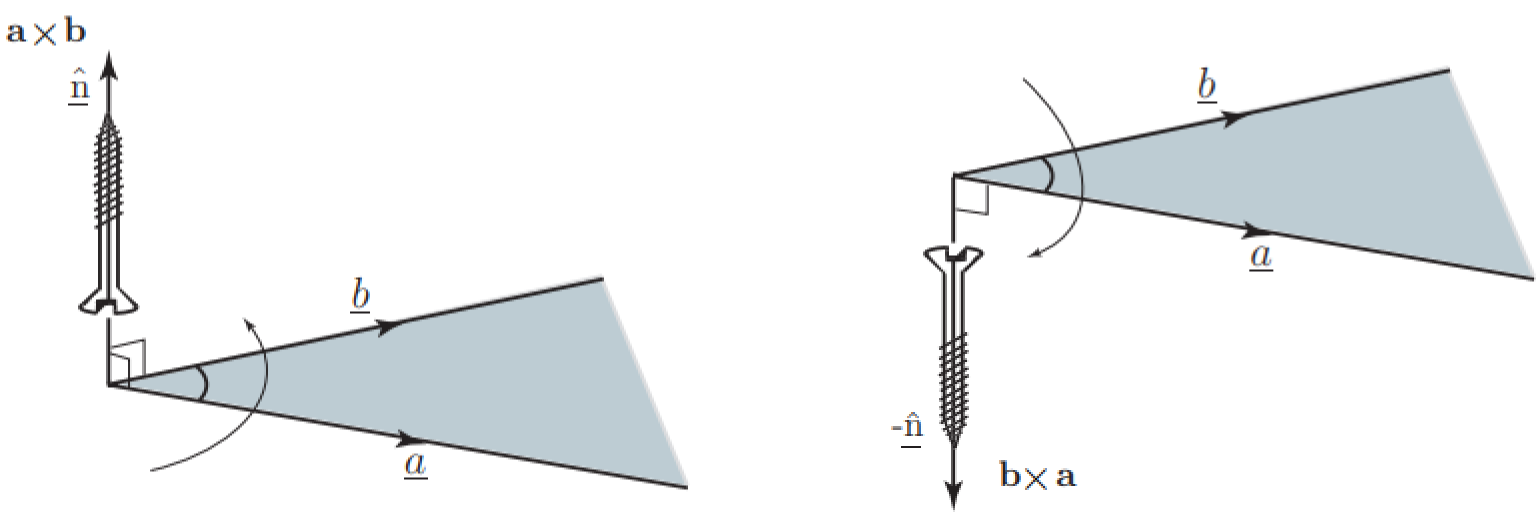
Suppose, for the two vectors $\mathbf{a}$ and $\mathbf{b}$ we calculate the product in a different order. That is, suppose we want to find $\mathbf{b} \times \mathbf{a}$. Using the definition of $\mathbf{b} \times \mathbf{a}$ and using the right-hand screw rule to obtain the required direction we find

$$\mathbf{b} \times \mathbf{a} = |\mathbf{b}|\,|\mathbf{a}|\,\sin\theta\,(-\hat{\mathbf{n}})$$

We see that the direction of $\mathbf{b} \times \mathbf{a}$ is opposite to that of $\mathbf{a} \times \mathbf{b}$ as shown in Figure 5. So

$$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$$

So the vector product is **not commutative**. In practice, this means that the order in which we do the calculation does matter. $\mathbf{b} \times \mathbf{a}$ is in the opposite direction to $\mathbf{a} \times \mathbf{b}$.



Comparison b/w Dot & Cross Product

Dot product	Cross product
Result of a dot product is a scalar quantity.	Result of a cross product is a vector quantity.
It follows commutative law.	It doesn't follow commutative law.
Dot product of vectors in the same direction is maximum.	Cross product of vectors in same direction is zero.
Dot product of orthogonal vectors is zero.	Cross product of orthogonal vectors is maximum.
It doesn't follow right hand system.	It follows right hand system.
It is used to find projection of vectors.	It is used to find a third vector.
It is represented by a dot (.)	It is represented by a cross (x)

Exercise 1

1. Use the formula $\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$ to find the vector product $\mathbf{a} \times \mathbf{b}$ in each of the following cases.

(a) $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = -2\mathbf{i} + 9\mathbf{j}$.
(b) $\mathbf{a} = 4\mathbf{i} - 2\mathbf{j}$, $\mathbf{b} = 5\mathbf{i} - 7\mathbf{j}$.

Comment upon your solutions.
2. Use the formula in Q1 to find the vector product $\mathbf{a} \times \mathbf{b}$ in each of the following cases.

(a) $\mathbf{a} = 5\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 8\mathbf{j} + 9\mathbf{k}$.
(b) $\mathbf{a} = \mathbf{i} + \mathbf{j} - 12\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$.

3. Use determinants to find the vector product $\mathbf{p} \times \mathbf{q}$ in each of the following cases.

(a) $\mathbf{p} = \mathbf{i} + 4\mathbf{j} + 9\mathbf{k}$, $\mathbf{q} = 2\mathbf{i} - \mathbf{k}$.
(b) $\mathbf{p} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{q} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$.

4. For the vectors $\mathbf{p} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{q} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$ show that, in this special case, $\mathbf{p} \times \mathbf{q} = \mathbf{q} \times \mathbf{p}$.
5. For the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $\mathbf{c} = 7\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, show that

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$$

Applications of DOT & CROSS Product

Application of Dot Product

In order to find PROJECTION

From the figure,

$$\cos\theta = \frac{ON}{B}$$

$$ON = B\cos\theta$$

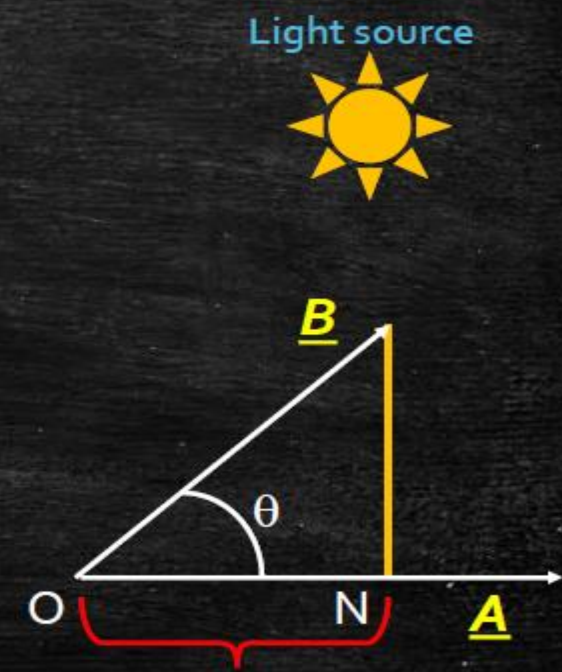
As we know,

$$\cos\theta = \frac{\boldsymbol{A}.\boldsymbol{B}}{AB}$$

$$B\cos\theta = \frac{\boldsymbol{A}.\boldsymbol{B}}{A}$$

So we reach to,

$$\boxed{ON = \frac{\boldsymbol{A}.\boldsymbol{B}}{A}}$$



Real Life Application of Dot Product

- Calculating the total cost.
- Electromagnetism, from which we get light, electricity, computers etc.
- Gives the combined effect of the coordinates in different dimensions on each other.

Application of Cross Product

- To find the area of a parallelogram

Consider the parallelogram shown in Figure. which has sides given by vectors $\mathbf{b}$ and $\mathbf{c}$.

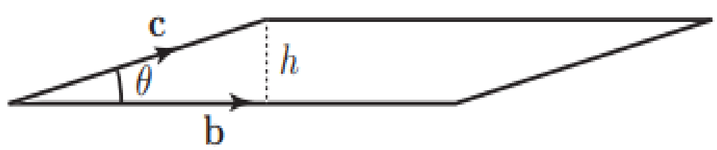


Figure. A parallelogram with two sides given by $\mathbf{b}$ and $\mathbf{c}$.

The area of the parallelogram is the length of the base multiplied by the perpendicular height, h. Now $\sin\theta = \dfrac{h}{|\mathbf{c}|}$ and so $h = |\mathbf{c}|\sin\theta$. Therefore

$$\text{area} = |\mathbf{b}|\,|\mathbf{c}|\sin\theta$$

which is simply the modulus of the vector product of $\mathbf{b}$ and $\mathbf{c}$. We deduce that the area of the parallelogram is given by

$$\text{area} = |\mathbf{b}\times\mathbf{c}|$$

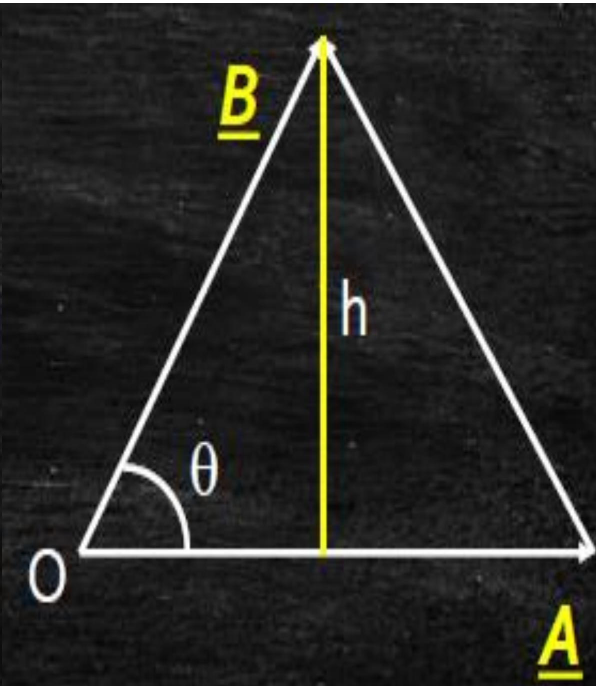
Application of Cross Product

To find the area of a triangle:

$$\text{Area of triangle} = \frac{1}{2} h \, |\underline{B}|$$

$$= \frac{1}{2} |\underline{A}| \sin\theta \, |\underline{B}|$$

$$= \frac{1}{2} |\underline{A} \times \underline{B}|$$



Real Life Application

- Finding moment
- Finding torque
- Rowing a boat
- Finding the most effective path

Triple Product

- Dot and cross products of three vectors ***A , B and C may*** produce meaningful products of the form (***A.B)xC, A.(BxC)*** and ***Ax(BxC) then phenomenon is called triple product.***
- ***Two types of triple product***

1. ***Scalar Triple Product***
2. ***Vector Triple Product***

Types of Triple Product

Scalar Triple Product

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}).$$

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Vector Triple Product

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}),$$

which is known as the "bac-cab" rule.

Application of Triple Product

Consider a parallelepiped. This is a six sided solid, the sides of which are parallelograms. Opposite parallelograms are identical. The volume, V, of a parallelepiped with edges $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is given by

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

This formula can be obtained by understanding that the volume is the product of the area of the base and the perpendicular height. Because the base is a parallelogram its area is $|\mathbf{b} \times \mathbf{c}|$. The perpendicular height is the component of $\mathbf{a}$ in the direction perpendicular to the plane containing $\mathbf{b}$ and $\mathbf{a}$, and this is $h = \mathbf{a} \cdot \widehat{\mathbf{b} \times \mathbf{c}}$. So the volume is given by

$$\begin{aligned} V &= (\text{height})(\text{ area of base}) \\ &= \mathbf{a} \cdot \widehat{\mathbf{b} \times \mathbf{c}}\, |\mathbf{b} \times \mathbf{c}| \\ &= \mathbf{a} \cdot \frac{\mathbf{b} \times \mathbf{c}}{|\mathbf{b} \times \mathbf{c}|}\, |\mathbf{b} \times \mathbf{c}| \\ &= \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \end{aligned}$$

Application of Triple Product

This could turn out to be negative, so in fact, for the volume we take its modulus: $V = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$.

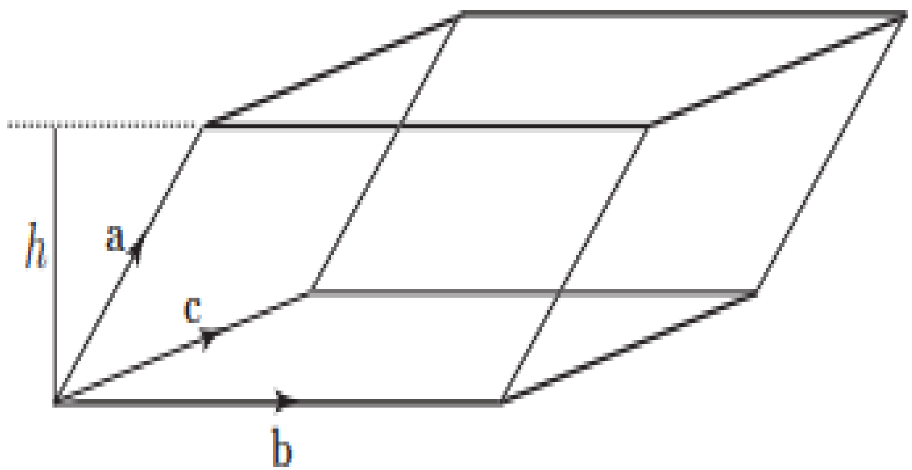


Figure . A parallelepiped with edges given by **a**, **b** and **c**.

Solved Example:(Triple Product)

Example

Suppose we wish to find the volume of the parallelepiped with edges $\mathbf{a} = 3\mathbf{i}+2\mathbf{j}+\mathbf{k}$, $\mathbf{b} = 2\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{c} = \mathbf{i}+2\mathbf{j}+4\mathbf{k}$.

We first evaluate the vector product $\mathbf{b}\times\mathbf{c}$.

$$\begin{aligned}
\mathbf{b}\times\mathbf{c} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix} \\
&= (1\times 4 - 1\times 2)\mathbf{i} - (2\times 4 - 1\times 1)\mathbf{j} + (2\times 2 - 1\times 1)\mathbf{k} \\
&= 2\mathbf{i} - 7\mathbf{j} + 3\mathbf{k}
\end{aligned}$$

Then we need to find the scalar product of $\mathbf{a}$ with $\mathbf{b}\times\mathbf{c}$.

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (3\mathbf{i}+2\mathbf{j}+\mathbf{k})\cdot(2\mathbf{i}-7\mathbf{j}+3\mathbf{k}) = 6-14+3 = -5$$

Finally, we want the modulus, or absolute value, of this result. We conclude the parallelepiped has volume 5 (units cubed).

Exercise 2

1. Find a unit vector which is perpendicular to both $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

2. Find the area of the parallelogram with edges represented by the vectors $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $7\mathbf{i} + \mathbf{j} + \mathbf{k}$.

3. Find the volume of the parallelepiped with edges represented by the vectors $\mathbf{i}+\mathbf{j}+\mathbf{k}$, $2\mathbf{i}+3\mathbf{j}+4\mathbf{k}$ and $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

4. Calculate the **triple scalar product** $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ when $\mathbf{a} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{c} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

Table Summary of vector relations.

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation $\mathbf{A}$ =	$A_x\hat{\imath}+A_y\hat{\jmath}+A_z\hat{k}$	$\hat{\mathbf{r}}A_r+\hat{\boldsymbol{\phi}}A_\phi+\hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R+\hat{\boldsymbol{\theta}}A_\theta+\hat{\boldsymbol{\phi}}A_\phi$
Magnitude of A $\lvert\mathbf{A}\rvert =$	$\sqrt[+]{A_x^2+A_y^2+A_z^2}$	$\sqrt[+]{A_r^2+A_\phi^2+A_z^2}$	$\sqrt[+]{A_R^2+A_\theta^2+A_\phi^2}$
Position vector $\overrightarrow{OP_1} =$	$x_1\hat{\imath}+y_1\hat{\jmath}+z_1\hat{k}$ for $P=(x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1+\hat{\mathbf{z}}z_1$, for $P=(r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$, for $P=(R_1, \theta_1, \phi_1)$
Base vectors properties $\hat{\imath}\times\hat{\jmath}=\hat{k}$ $\hat{\jmath}\times\hat{k}=\hat{\imath}$ $\hat{k}\times\hat{\imath}=\hat{\jmath}$	$\hat{\imath}.\hat{\imath}=\hat{\jmath}.\hat{\jmath}=\hat{k}.\hat{k}=1$ $\hat{\imath}.\hat{\jmath}=\hat{\jmath}.\hat{k}=\hat{k}.\hat{\imath}=0$ $\hat{\imath}\times\hat{\imath}=\hat{\jmath}\times\hat{\jmath}=\hat{k}\times\hat{k}=0$	$\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}=\hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}}=\hat{\mathbf{z}}\cdot\hat{\mathbf{z}}=1$ $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\phi}}=\hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{z}}=\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}=0$ $\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}}=\hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}}=\hat{\mathbf{r}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{r}}=\hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}}\cdot\hat{\mathbf{R}}=\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}}=\hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}}=1$ $\hat{\mathbf{R}}\cdot\hat{\boldsymbol{\theta}}=\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}}=\hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{R}}=0$ $\hat{\mathbf{R}}\times\hat{\boldsymbol{\theta}}=\hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}}=\hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{R}}=\hat{\boldsymbol{\theta}}$
Dot product $\mathbf{A}\cdot\mathbf{B} =$	$A_xB_x+A_yB_y+A_zB_z$	$A_rB_r+A_\phi B_\phi+A_zB_z$	$A_RB_R+A_\theta B_\theta+A_\phi B_\phi$
Cross product $\mathbf{A}\times\mathbf{B} =$	$\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}$	$\begin{vmatrix}\hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}}\\ A_r & A_\phi & A_z\\ B_r & B_\phi & B_z\end{vmatrix}$	$\begin{vmatrix}\hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}}\\ A_R & A_\theta & A_\phi\\ B_R & B_\theta & B_\phi\end{vmatrix}$
Differential length $d\mathbf{l} =$	$\hat{\imath}\ dx+\hat{\jmath}\ dy+\hat{k}\ dz$	$\hat{\mathbf{r}}\,dr+\hat{\boldsymbol{\phi}}r\,d\phi+\hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR+\hat{\boldsymbol{\theta}}R\,d\theta+\hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$ds_x=\hat{\imath}\,dy\,dz$ $ds_y=\hat{\jmath}\,dx\,dz$ $ds_z=\hat{k}\,dx\,dy$	$ds_r=\hat{\mathbf{r}}r\,d\phi\,dz$ $ds_\phi=\hat{\boldsymbol{\phi}}\,dr\,dz$ $ds_z=\hat{\mathbf{z}}r\,dr\,d\phi$	$ds_R=\hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $ds_\theta=\hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $ds_\phi=\hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume $dV =$	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$