Applications of Gauss's Law (1)

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$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = \frac{q_{\text{in}}}{\epsilon_0} \qquad (24.6)$$

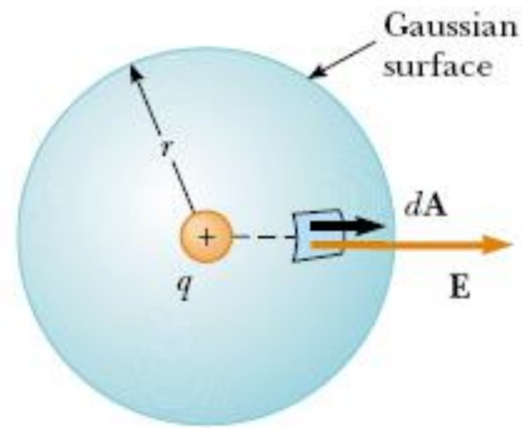
1. The value of the electric field can be argued by symmetry to be constant over the surface.
2. The dot product in Equation 24.6 can be expressed as a simple algebraic product $E\,dA$ because $\mathbf{E}$ and $d\mathbf{A}$ are parallel.
3. The dot product in Equation 24.6 is zero because $\mathbf{E}$ and $d\mathbf{A}$ are perpendicular.
4. The field can be argued to be zero over the surface.

Electric Field due to a Point Charge

Starting with Gauss's law, calculate the electric field due to an isolated point charge q.

Solution A single charge represents the simplest possible charge distribution, and we use this familiar case to show how to solve for the electric field with Gauss's law. Figure 24.10 and our discussion of the electric field due to a point charge in Chapter 23 help us to conceptualize the physical situation. Because the space around the single charge has spherical symmetry, we categorize this problem as one in which there is enough symmetry to apply Gauss's law. To analyze any Gauss's law problem, we consider the details of the electric field and choose a gaussian surface that satisfies some or all of the conditions that we have listed above. We choose a spherical gaussian surface of radius r centered on the point charge, as shown in Figure 24.10. The electric field due to a positive point charge is directed radially outward by symmetry and is therefore normal to the surface at every point. Thus, as in condition (2), $\mathbf{E}$ is parallel to $d\mathbf{A}$ at each point. Therefore, $\mathbf{E} \cdot d\mathbf{A} = E\,dA$ and Gauss's law gives

$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = \oint E\,dA = \frac{q}{\epsilon_0}$$

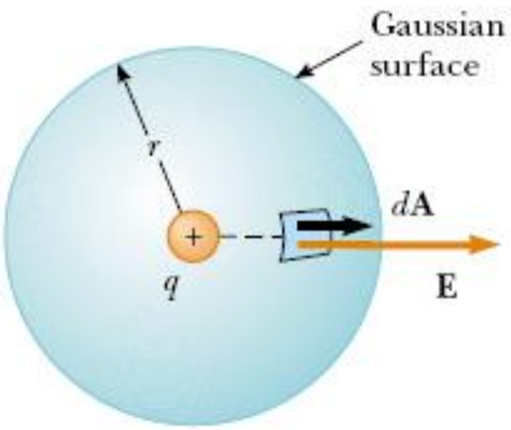


By symmetry, E is constant everywhere on the surface, which satisfies condition (1), so it can be removed from the integral. Therefore,

$$\oint E\,dA = E\oint dA = E(4\pi r^2) = \frac{q}{\epsilon_0}$$

where we have used the fact that the surface area of a sphere is $4\pi r^2$. Now, we solve for the electric field:

$$E = \frac{q}{4\pi\epsilon_0 r^2} = \boxed{k_e \frac{q}{r^2}}$$



What If? What if the charge in Figure 24.10 were not at the center of the spherical gaussian surface?

Answer In this case, while Gauss's law would still be valid, the situation would not possess enough symmetry to evaluate the electric field. Because the charge is not at the center, the magnitude of **E** would vary over the surface of the sphere and the vector **E** would not be everywhere perpendicular to the surface.

A spherically symmetric Charge Distribution

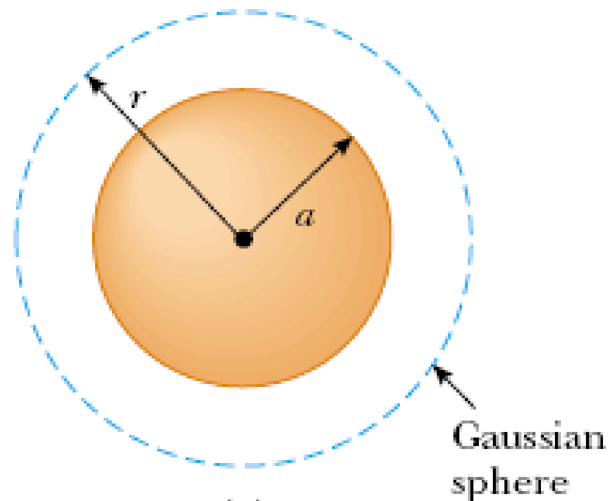
An insulating solid sphere of radius a has a uniform volume charge density ρ and carries a total positive charge Q (Fig. 24.11).

(A) Calculate the magnitude of the electric field at a point outside the sphere.

Solution Because the charge distribution is spherically symmetric, we again select a spherical gaussian surface of radius r, concentric with the sphere, as shown in Figure 24.11a. For this choice, conditions (1) and (2) are satisfied, as they were for the point charge in Example 24.4. Following the line of reasoning given in Example 24.4, we find that

$$(1) \qquad E = \boxed{k_e \frac{Q}{r^2}} \qquad (\text{for } r > a)$$

Note that this result is identical to the one we obtained for a point charge. Therefore, we conclude that, **for a uniformly charged sphere, the field in the region external to the sphere is *equivalent* to that of a point charge located at the center of the sphere.**



(a)

(B) Find the magnitude of the electric field at a point inside the sphere.

Solution In this case we select a spherical gaussian surface having radius $r < a$, concentric with the insulating sphere (Fig. 24.11b). Let us denote the volume of this smaller sphere by V'. To apply Gauss's law in this situation, it is important to recognize that the charge q_{in} within the gaussian surface of volume V' is less than Q. To calculate q_{in}, we use the fact that $q_{\text{in}} = \rho V'$:

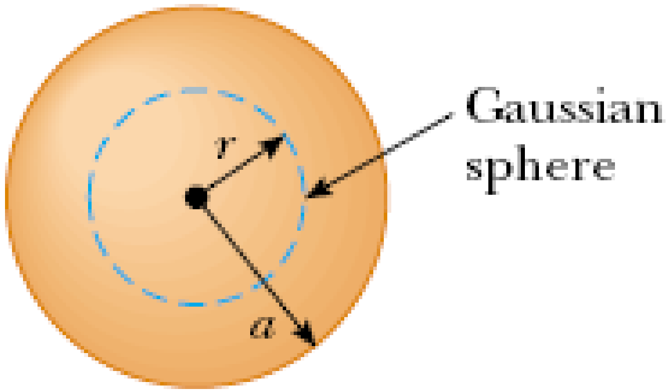
$$q_{\text{in}} = \rho V' = \rho \left(\tfrac{4}{3}\pi r^3\right)$$

By symmetry, the magnitude of the electric field is constant everywhere on the spherical gaussian surface and is normal to the surface at each point—both conditions (1) and (2) are satisfied. Therefore, Gauss's law in the region $r < a$ gives

$$\oint E\, dA = E \oint dA = E\,(4\pi r^2) = \frac{q_{\text{in}}}{\epsilon_0}$$

Solving for E gives

$$E = \frac{q_{\text{in}}}{4\pi\epsilon_0 r^2} = \frac{\rho\left(\frac{4}{3}\pi r^3\right)}{4\pi\epsilon_0 r^2} = \frac{\rho}{3\epsilon_0}\, r$$

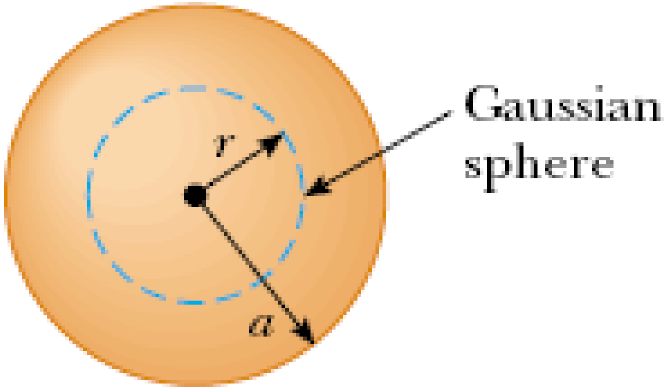


(b)

Because $\rho = Q/\frac{4}{3}\pi a^3$ by definition and because $k_e = 1/4\pi\epsilon_0$, this expression for E can be written as

$$(2) \qquad E = \frac{Qr}{4\pi\epsilon_0 a^3} = \boxed{k_e \frac{Q}{a^3} r} \qquad (\text{for } r < a)$$

Note that this result for E differs from the one we obtained in part (A). It shows that $E \to 0$ as $r \to 0$. Therefore, the result eliminates the problem that would exist at $r = 0$ if E varied as $1/r^2$ inside the sphere as it does outside the sphere. That is, if $E \propto 1/r^2$ for $r < a$, the field would be infinite at $r = 0$, which is physically impossible.



(b)

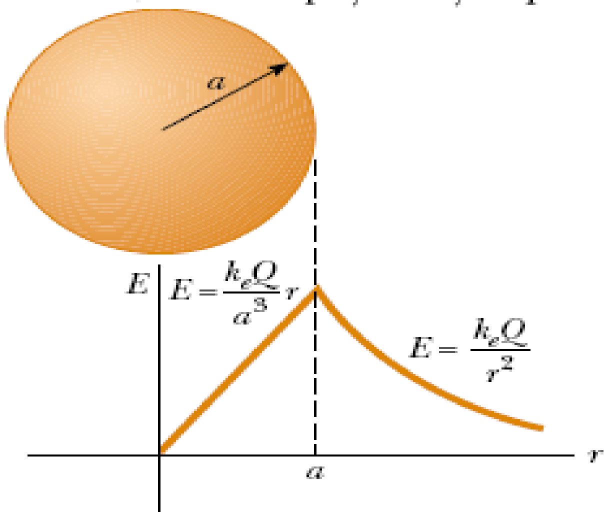


Figure 24.12 (Example 24.5) A plot of E versus r for a uniformly charged insulating sphere. The electric field inside the sphere ($r < a$) varies linearly with r. The field outside the sphere ($r > a$) is the same as that of a point charge Q located at $r = 0$.

What If? Suppose we approach the radial position $r = a$ from inside the sphere and from outside. Do we measure the same value of the electric field from both directions?

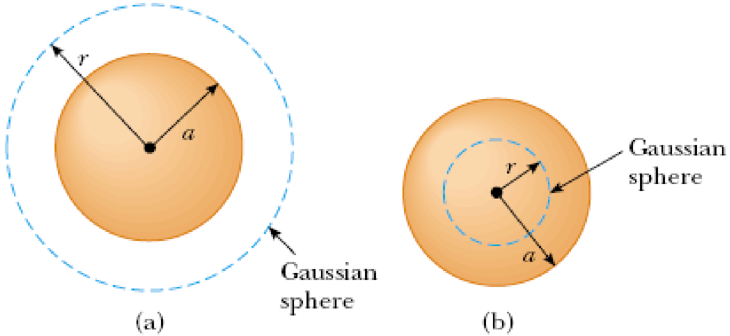
Answer From Equation (1), we see that the field approaches a value from the outside given by

$$E = \lim_{r \to a} \left(k_e \frac{Q}{r^2} \right) = k_e \frac{Q}{a^2}$$

From the inside, Equation (2) gives us

$$E = \lim_{r \to a} \left(k_e \frac{Q}{a^3} r \right) = k_e \frac{Q}{a^3} a = k_e \frac{Q}{a^2}$$

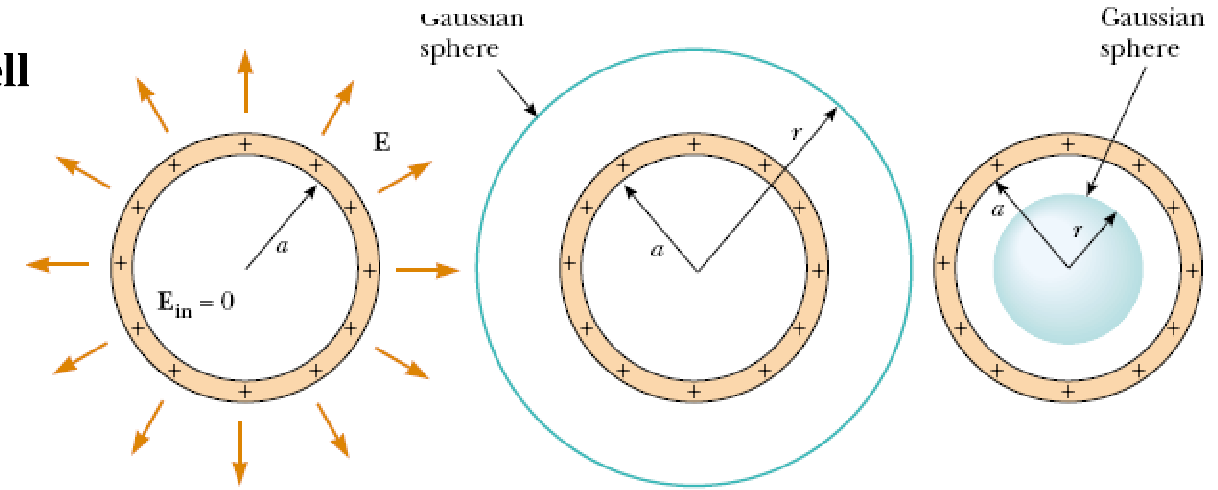
Thus, the value of the field is the same as we approach the surface from both directions. A plot of E versus r is shown in Figure 24.12. Note that the magnitude of the field is continuous, but the derivative of the field magnitude is not.



Electric Field due to Thin Spherical Shell

A thin spherical shell of radius a has a total charge Q distributed uniformly over its surface (Fig. 24.13a). Find the electric field at points

(A) Outside the Shell



(a) (b) (c)

Solution

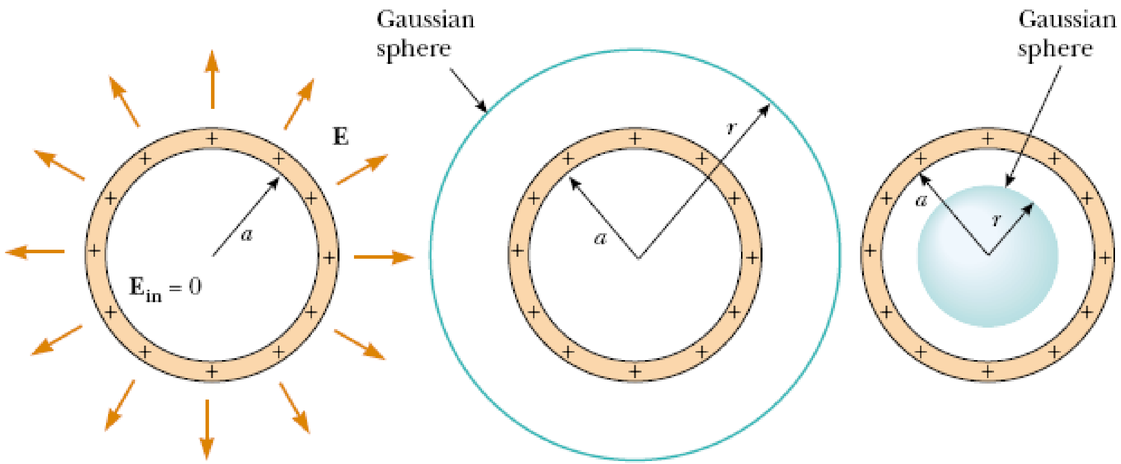
(A) The calculation for the field outside the shell is identical to that for the solid sphere shown in Example 24.5a. If we construct a spherical gaussian surface of radius $r > a$ concentric with the shell (Fig. 24.13b), the charge inside this surface is Q. Therefore, the field at a point outside the shell is equivalent to that due to a point charge Q located at the center:

$$E = \boxed{k_e \frac{Q}{r^2}} \qquad (\text{for } r > a)$$

(B) inside the shell.

(B) The electric field inside the spherical shell is zero. This follows from Gauss's law applied to a spherical surface of radius $r < a$ concentric with the shell (Fig. 24.13c). Because of the spherical symmetry of the charge distribution and because the net charge inside the surface is zero—satisfaction of conditions (1) and (2) again—application of Gauss's law shows that $E = 0$ in the region $r < a$. We obtain the same results using Equation 23.11 and integrating over the charge distribution. This calculation is rather complicated. Gauss's law allows us to determine these results in a much simpler way.

$$\mathbf{E} = k_e \lim_{\Delta q_i \to 0} \sum_i \frac{\Delta q_i}{r_i^2} \hat{\mathbf{r}}_i = k_e \int \frac{dq}{r^2} \hat{\mathbf{r}} \qquad (23.11)$$



(a) (b) (c)