TANGENT VECTORS

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1 Tangent Vectors

note that we need a way to define tangent for arbitary manifold, we know the notion of tangent vectors for shapes in euclidean plane as a subset of a bigger space, but when dealing with manifold we employ a more intristic approach where we dont use any ambient space in which manifold is embedded. first we will define tangent space for normal euclidean plane, we will do that as follows

Definition 1.1. For any $a \in \mathbb{R}^n$, The tangent space of $\mathbb{R}^n$ at a is defined as the set $(a, v) : v \in \mathbb{R}^n$ where v is a vector in $\mathbb{R}^n$.

Note that Tangent space of $\mathbb{R}^n$ is isomorphic to $\mathbb{R}^n$ hence we think of it like this in an intrinsic way instead of relying on any other embedding spaces but well note that this way of describing tangent space by vectors woudnt be fruitfull if we want to generalise it, we need an equivalent formalisation but something that can be abstracted into out manifolds hence we define same notion of direction but with directional derivative.

note that directional derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$ at a point a in the direction of a vector v is defined as

$$D_v f(a) = \lim_{t \to 0} \frac{f(a + tv) - f(a)}{t}$$

key motivation to do this way the fact that vectors as defined previously in tangent space and this set of directional derivatibe have one to one correspondence, which will see newcommand

now we define following objects

Definition 1.2. A map $\omega : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is called a **derivation** at $a \in \mathbb{R}^n$ if it satisfies the following properties:

- The map is linear over $\mathbb{R}$.
- For any $f, g \in C^\infty(\mathbb{R}^n)$, we have $\omega(fg) = f(a)\omega(g) + g(a)\omega(f)$.

We will later realise that our familiar directional derivatives are just derivations in disguise. we denote set of of all derivation at a as $T_a\mathbb{R}^n$,

note that this set is a vector space over $\mathbb{R}$, infact it is isomorphic to $\mathbb{R}^n$ Now we will study some properties of these derivations.

Theorem 1.1. *suppose that $a \in \mathbb{R}^n$ and $f, g \in C^\infty(\mathbb{R}^n)$, then the map $\omega : T_a\mathbb{R}^n \to \mathbb{R}$ defined by $\omega \in T_a\mathbb{R}^n$ is a derivation at a.*

- *if f is a constant function, then $\omega(f) = 0$.*
- *if $f(a) = g(a) = 0$, then $\omega(fg) = f(a)\omega(g) + g(a)\omega(f) = 0$.*

Proof. Left as an exercise for the reader. □

now think of these set of all derivation (i.e $T_a\mathbb{R}^n$) as a property of a point (coz thats kind of what it is), think of it as all the velocity vectors possible from that point are infintely compressed into a point. These intution would help in generalising the definition fro arbitary manifolds. now why should you think of them like this, and why do we exactly consider them to be same as direction out nect theorem will answer that question precisely.

Theorem 1.2. *let $a \in \mathbb{R}^n$*

- *For each geometric vector $v_a \in \mathbb{R}^n$, the map $D_v|_a : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ defined by $D_v|_a(f) = D_v f(a)$ is a derivation at a.(i.e Size of our derivation is "atleast" this Much, Now we will Claim that its exactly this Much)*
- *The map $v_a \to D_v|_a$ is an isomorphism from $\mathbb{R}^n_a$ onto $T_a\mathbb{R}^n$.*

Proof. 1) is trivial so will just prove ii) Now Linearity of the Map is Shown by writing the following element in range as a Gradient, Now we will Prove Injectivity, We Assume That $D_v|_a$ is a Zero derivation and suppose $v_a = v_i e_i$, where e_i is the standard basis of $\mathbb{R}^n$, then we have For $X^j : \mathbb{R}^n \to \mathbb{R}$ That $D_v|_a(X^j) = v_i \frac{\partial X^j}{\partial x_i}(a) = v_i$, which implies that $v_i = 0$ for all i, hence $v_a = 0$. Hence we have shown the injectivity now we will show Surjectivity Let $\omega \in T_a\mathbb{R}^n$ be a derivation at a, then we can define a vector v_a such that $D_v|_a(f) = \omega(f)$ for all $f \in C^\infty(\mathbb{R}^n)$, So first we need a candidate for such v_a , suppose that $v_i = \omega(X^i)$. Then we have $\omega = D_v|_a$, For this we will Use taylors formula

$$f(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a)(x^i - a^i) + \sum_{i,j=1}^{n} (x^i - a^i)(x^j - a^j) \int_0^1 (1-t) \frac{\partial^2 f}{\partial x^i \partial x^j}(a + t(x-a))dt$$

On operating this function on ω, we get that the integration terms vanish along with the constant term $f(a)$, and we get

$$\omega(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a) v_i$$

Hence we are done □

Note that from this we can derive a corollary that all the partial operators forms a basis for our $T_a\mathbb{R}^n$. for manifold it will be a bit complicated

Corollary 1.1. *The set of partial derivatives $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$ forms a basis for the tangent space $T_a\mathbb{R}^n$.*

Definition 1.3. A map $v : C^\infty(M) \to \mathbb{R}$ is called a **derivation** at a point $a \in M$ if it satisfies the following properties:

- The map is linear over $\mathbb{R}$.
- The map satisfies the Leibniz rule: $v(fg) = f(a)v(g) + g(a)v(f)$ for all $f, g \in C^\infty(M)$.

we had the nice correspondence in case of $\mathbb{R}^n$ between our tangent space and OG space but here since structure is locally euclidean we can have some complicacy and we will handle that with care so that we get everything out of this definition. Note that i have just realised that tangent vector are local object we always visualised them as arrows hence we thought it were result of abient space out original space was in but no we can define tangent vector like this in an intrinsic way too!!

2 Differential of a Smooth map

Now note that if we have a smooth map then in case of vector space we could linearise the problem to find a derivative but here in case of manifold the structure is more complicated. We need to take into account the local charts and the transition maps between them. hence its not possible to always linearise the problem but instead we will generelise our definition of derivition as maps between tangent spaces rather then spaces itselfs.

Definition 2.1. Suppose $F : M \to N$ is a smooth map between manifolds M and N. Then the **differential of** F at a point $a \in M$ is a map

$$dF_a : T_aM \to T_{F(a)}N$$

defined by

$$(dF_a(v))(f) = v(f \circ F)$$

for all $v \in T_aM$ and $f \in C^\infty(N)$.

Now this is up to you to think why the element $v(f \circ F)$ in $T_{F(a)}N$ is a derivation :p.

Theorem 2.1. *Let M, N, P be smooth manifolds and $F : M \to N$, $G : N \to P$ be smooth maps. If F is a diffeomorphism, then $dF_p : T_p(M) \to T_{F(p)}(N)$ is an isomorphism, and $(dF_p)^{-1} = dF^{-1}_{F(p)}$.*

This Theorem Just says that if we have a same smooth structure, then Tangent spaces at all point in those two manifolds are indentical.

Proof. Let $p \in M$ and $q = F(p) \in N$. Since F is a diffeomorphism, there exists a smooth inverse $F^{-1} : N \to M$ such that $F^{-1}(q) = p$. The differential of F^{-1} at q is given by

$$dF_q^{-1} : T_qN \to T_pM$$

defined by

$$(dF_q^{-1}(w))(g) = w(g \circ F^{-1})$$

for all $w \in T_qN$ and $g \in C^\infty(M)$.

Now, we need to show that $(dF_p)^{-1} = dF_q^{-1}$. For any $v \in T_pM$, we have

$$(dF_p(v))(f) = v(f \circ F)$$

and

$$(dF_q^{-1}(w))(g) = w(g \circ F^{-1}).$$

By the chain rule, we have

$$(dF_p(v))(f) = v(f \circ F) = v(F^* f),$$

where $F^* f$ is the pullback of f under F. Similarly,

$$(dF_q^{-1}(w))(g) = w(g \circ F^{-1}) = w(F^* g).$$

Thus, we can conclude that $(dF_p)^{-1} = dF_q^{-1}$, completing the proof. □

Note that Derivation acts locally and following theorem solidifies it

Theorem 2.2. *Let M be a smooth manifold, $p \in M$, and $f, g \in C^\infty(M)$ such that these f,g agree on some neightbourhood of p, then $vf = vg$*

Proof. Let h = f - g, then h(p) = 0 , now we need to consider another function that would be zero at p to conclude this, we dont need to go far because such a fucntion is a smooth bump, consider ψ such that its 1 identically on support of h, now since h is zero in some neighbourhood og p there exist a circle around p for which ψ is 0, hence $v(\psi.h) = 0$ since both are zero at P □

Now Suppose we have Submanifold of An ambient Manifold, then how does the tangent space of the same point behave, in both cases?

Theorem 2.3. *Let M be a smooth manifold, and $U \subset M$ be open subset and $\tau : U \to M$ be inclusion map. for Every Point $P \in U$, The map $d\tau_p : T_pU \to T_pM$ is an isomorphism.*

Proof. TBD □

The following theorem just says that tangent space of submanifold is naturally identified with the tangent space of the ambient manifold at the same point. Now what is the dimension of this tangent for Rn it was N, is it same for manifold case too? lets see!

Theorem 2.4. *Suppose M is a N-dimensional smooth manifold, then for every point $p \in M$ The Tangent space T_pM is a n dimensional vector space.*

Proof. Suppose $p \in M$ then there exists a chart $\phi : U \to \tilde{U}$ where $\tilde{U} \subset \mathbb{R}^n$ is an open subset and U is an open subset of M. hence $d\phi_p : T_pU \to T_{\tilde{p}}\tilde{U}$ is a isomorphism, hence from dimension invariance the dimension of our vector space is n. □

Note that this is because since Tangent space are local objects, we can define , with a ambient vector space that being out coordinate chart. Though we cant do it in general.

3 Computation in coordinates

Now Note That After all This abstract Bullshit, How exactly do we USe them to do the real dirty work? Now note that if $\phi : U \to \mathbb{R}^n$ is a chart, Then $\phi(U) = \tilde{U}$ and $d\phi_p : T_pM \to T_{\tilde{p}}\mathbb{R}^n$ is an isomorphism, hence we can use the standard basis of $\mathbb{R}^n$ to compute the tangent space at p. Note that Standard basis for tangent space Rn at p, would be set of operators, $\{\frac{\partial}{\partial x^i}\big|_{\phi(p)}\}_{i=1}^n$.

Note that since our tangent space are isomorphic , preimage of these operators would give us basis for T_pM. Hence

$$\frac{\partial}{\partial x^i}\big|_p = (d\phi_p)^{-1}\left(\frac{\partial}{\partial x^i}\big|_{\phi(p)}\right)$$

$$\frac{\partial}{\partial x^i}\big|_p = d(\phi_p^{-1})\left(\frac{\partial}{\partial x^i}\big|_{\phi(p)}\right)$$

suppose if f is an arbitary $C^\infty(M)$ function on M, then we can write its coordinate representation in the chart ϕ as $\tilde{f} = f \circ \phi^{-1}$, then we can compute the directional derivative of f at p in the direction of $\frac{\partial}{\partial x^i}\big|_p$ as follows:

$$\frac{\partial}{\partial x^i}\big|_p(f) = \frac{\partial}{\partial x^i}\big|_{\phi(p)}(f \circ \phi^{-1})$$

$$\frac{\partial}{\partial x^i}\big|_p(f) = \frac{\partial}{\partial x^i}\big|_{\phi(p)}(\tilde{f})$$

where $\tilde{f}$ is the coordinate representation of f in the chart ϕ.

Hence note that it means even though we dont have our friend (Euclidean structure), still we can do differentiation which would just be differentiation of coordinate representation.

Corollary 3.1. *Suppose M is a smooth n-manifold, let $p \in M$ Then Basis for $T_p(M)$ can be written as*

$$\{\frac{\partial}{\partial x^i}\big|_p\}_{i=1}^n$$

4 Differential in coordinate

now what does the diffrential look like in coordinates??
note that in case of a function $F : \mathbb{R}^n \to \mathbb{R}^m$ we had the derivative of the fucntion as the jacobian matrix, note that the matrix itself didnt depend on the coordinate we used, infact there were properties in linear algebra that we studied that were done purely without ever mentioning the basis of matrix, but when we describe this matrix or when we have to compute example this comes in handy, and would help us do calculation from cooridnates.
we woould want the defination of diffrential map of function $F : U \to V$ where U, V are open subsets of $\mathbb{R}^n and \mathbb{R}$ to be the same as the diffrential map of function $F : \mathbb{R}^n \to \mathbb{R}^m$ repectively to co-incide with out notion of Jacobian
Hence Suppse $F : U \to V$ is a function where U,V are as above, then the differential is $dF_p : T_pU \to T_{F(p)}V$ Now let $(x^1, x^2, \ldots, x^n)$ and $(y^1, y^2, \ldots, y^m)$ be basis for $\mathbb{R}^n$ and $\mathbb{R}^m$
Here Note that these coordinates give us a way to define direction derivatives which are the language of diffrential map. hence we would start with

$$dF_p(\frac{\partial}{\partial x^i}\big|_p)f = \frac{\partial}{\partial x^i}\big|_{F(p)}(f \circ F) = \frac{\partial}{\partial y^j}\big|_{F(p)}(f)\frac{\partial F^j}{\partial x^i}(p)$$

We now know That $\frac{\partial}{\partial y^j}$ is a basis for $T_{F(p)}V$, and $\frac{\partial F^j}{\partial x^i}(p)$ is a basis for T_pU, hence we can write and since diffrential is a linear map, we can write that by a finite matrix that will be

$$\begin{pmatrix} \frac{\partial F^1}{\partial x^1}(p) & \frac{\partial F^1}{\partial x^2}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\ \frac{\partial F^2}{\partial x^1}(p) & \frac{\partial F^2}{\partial x^2}(p) & \cdots & \frac{\partial F^2}{\partial x^n}(p) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial x^1}(p) & \frac{\partial F^m}{\partial x^2}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p) \end{pmatrix}$$

This Exactly is out Jacobian Matrix. hence the diffrential map works fine for R^n case Hence we need to generalise this for M manifold. Suppose we have a map $F : M \to N$ where M,N are smooth manifolds then let $p \in M$ let $(U, \phi), (V, \psi)$ are two charts with usual properties, then

$$F \circ \phi^{-1} = \psi^{-1} \circ F$$

now we know that $\frac{\partial}{\partial x^i}$ is a basis for $T_{(p)}U$, and $\frac{\partial}{\partial y^j}$ is a basis for $T_(F(p))V$, hence we can write

$$dF_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right) = \sum_{j=1}^{m} \frac{\partial F^j}{\partial x^i}(p)\, \frac{\partial}{\partial y^j}\bigg|_{F(p)}$$