Submersions, Immersions, Embbedings

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1 Introduction

Note that The Maps That are best described by their Linear approximation (The Differential) are the ones that have Constant Rank, such as submersions, immersions, and embeddings. we will discuss them in Differential Topology.

2 Map of Constant Rank

Rank is a well-known Concept from Linear Algebra, Infact it is a the only Thing that distinguishes Linear Transformations from one another.
In Context of smooth manifolds Suppose we have $F : M \to N$ as a smooth map between two smooth manifolds M, N, then we can define the rank of F at a point $p \in M$ as the rank of the differential of F at p, i.e. $\text{rank}(F_p) = \text{rank}(dF_p)$. A Map F is said of be of Rank r if rank of map is r for all point p in its domain.
We know That rank of a Linear Transformation is bounded by $\text{Min}\{\dim(M), \dim(N)\}$.
Suppose it is equal to one of This upper Bound then The Map is called A map of Full Rank.
There are Two Intresting Example of Maps of Full rank

- Submersion: If dF_p is surjective (i.e Rank(F) = dim(N)), then we call F a submersion.
- Immersion: If dF_p is injective(i.e Rank(F) = dim(M)), we call it an immersion.

What if A map has Rank r just at a point p can we say something about it?? the answer is freaking yes!!

Proposition 2.1. *Suppose $F : M \to N$ is a smooth map where M , N are smooth Manifolds and $p \in M$ is such that dF_p is surjective. Then There exist a neighbourhood $U \subset M$ of p such that $F|_U : U \to N$ is a submersion.*

Proof. Suppose We have Jacobian of F at p Then Since Set set of all matrices of full rank are open subset of $M(m \times n, \mathbb{R})$ From Continuity We can find analogous Open Subset in M too we will name that U. □

Now we will look into some examples

- Suppose that $M_1, M_2, \ldots, M_k$ are smooth manifolds then, Each projection map $\pi_i : M_1 \times M_2 \times \ldots \times M_k \to M_i$ is a smooth submersion.
- if $\gamma : J \to M$ then γ is a smooth immersion if and only if $\gamma'(t) \neq 0$ for all $t \in J$.
- The projection map From tangent bundle to a manifold is also a smooth submersion.
- Suppose $X : \mathbb{R}^2 \to \mathbb{R}^3$ is a smooth map defined by $X(u, v) = [(2 + cos2\pi u)cos2\pi v, (2 + cos2\pi u)sin2\pi u, sin2\pi u]$ then X is a smooth immersion.

3 Local Diffeomorphisms

Definition 3.1. A Map $F : M \to N$ is called a local diffeomorphism if for each point $p \in M$, there exist an open neighbourhood U of p such that $F(U) \subset N$ is open and $F|_U : U \to F(U)$ is a diffeomorphism.

Now we will look into the GOAT Theorem that is Inverse Function Theorem.

Theorem 3.1. *Let M, N be smooth manifolds and $F : M \to N$ be a Smooth map, If $p \in M$ is a point such that dF_p is invertible, Then there exist connected neighbourhoods of p , F(p) say $U_0 and V_0$ respectively Such that $F|_{U_0} : U_0 \to V_0$ is a diffeomorphism.*

Proof. Now Since dF_p is invertible it is surjective too hence The point P has neighbourhood (say U_1) where dF_p is surjective. Same there exists a neighbourhood U_2 such that dF_P is a immersion. Hence taking intersection of these two sets which we know is non-empty we get an open neighbourhood where differential is isomorphism, Hence from previous Theorem F restricted to this set U is a diffeomorphism. □

Now we will just state more results with the same proof strategy.

Proposition 3.1. *Suppose M, N are smooth manifolds and $F : M \to N$ is a map.*

- *F is a local diffeomorphism if and only if it is both a smooth immersion and a smooth submersion.*
- *If $dim(M) = dim(N)$ then F is a local diffeomorphism if and only if it is either a smooth immersion or a smooth submersion.*

4 Constant Rank Theorem

Now we have the main Theorem of this chapter without any further delay we will jump right into it.

Theorem 4.1. *Let M, N be Smooth manifolds of dimension m and n respectively, And $F : M \to N$ be a smooth map of Constant Rank r, Then for every point $P \in M$ there Exists Chart (U, ψ) centered at p, and chart (V, ϕ) centered at $F(p)$ such that We can write the coordinate representation of F in following form*

$$\hat{F}(x^1, x^2, \dots, x^m) = (x^1, x^2, \dots, x^r, 0, 0, \dots, 0)$$

Proof. □