Mathematical Conventions Tutorial
Essential Notation for Machine Learning

ES335 - Machine Learning
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Abstract

This tutorial introduces the mathematical notation and conventions used throughout the Machine Learning course. Understanding these conventions is crucial for following lectures, reading research papers, and implementing algorithms. We cover scalar, vector, and matrix notation, common spaces like $\mathbb{R}$ and $\mathbb{R}^n$, and provide both theoretical exercises and coding problems to reinforce understanding.

Contents

1   Scalar Notation

1.1   Basic Scalars

A **scalar** is a single number. We typically use lowercase letters to denote scalars:

- a, b, c - generic scalars
- α, β, γ - Greek letters for parameters
- n, m, d - dimensions and counts
- ϵ - small positive number (tolerance)
- λ - regularization parameter

Example

If we have a learning rate $\alpha = 0.01$ and a regularization parameter $\lambda = 0.1$, both are scalars.

1.2   Special Scalars

- y_i - the i-th target value (scalar output)
- $\hat{y}_i$ - predicted value for the i-th sample
- $\ell(\hat{y}_i, y_i)$ - loss function (returns a scalar)

2   Vector Notation

2.1   Vector Representation

A **vector** is an ordered list of numbers. We use lowercase bold letters or arrows:

- $\mathbf{x}, \mathbf{y}, \mathbf{z}$ - generic vectors
- $\mathbf{w}$ - weight vector (parameters)
- $\boldsymbol{\theta}$ - parameter vector
- $\boldsymbol{\alpha}$ - vector of dual variables

Vector Examples

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix} \in \mathbb{R}^d \tag{1}$$

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \in \mathbb{R}^3 \tag{2}$$

2.2 Vector Operations

- $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{d} x_i y_i$ - dot product (inner product)

- $\|\mathbf{x}\|$ - norm of vector $\mathbf{x}$

- $\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^{d} x_i^2}$ - L2 norm (Euclidean norm)

- $\|\mathbf{x}\|_1 = \sum_{i=1}^{d} |x_i|$ - L1 norm (Manhattan norm)

- $\mathbf{x}^T$ - transpose of vector (row vector)

3 Matrix Notation

3.1 Matrix Representation

A **matrix** is a rectangular array of numbers. We use uppercase bold letters:

- $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ - generic matrices

- $\mathbf{A}, \mathbf{B}$ - coefficient matrices

- $\mathbf{I}$ - identity matrix

- $\boldsymbol{\Sigma}$ - covariance matrix

Matrix Example

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1d} \\ x_{21} & x_{22} & \cdots & x_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nd} \end{bmatrix} \in \mathbb{R}^{n \times d} \tag{3}$$

where n is the number of samples and d is the number of features.

3.2 Matrix Operations

- $\mathbf{A}^T$ - transpose of matrix $\mathbf{A}$

- $\mathbf{A}^{-1}$ - inverse of matrix $\mathbf{A}$ (if it exists)

- $\mathbf{AB}$ - matrix multiplication

- $\text{tr}(\mathbf{A})$ - trace of matrix $\mathbf{A}$

- $\det(\mathbf{A})$ - determinant of matrix $\mathbf{A}$

4 Common Mathematical Spaces

4.1 Real Number Spaces

- $\mathbb{R}$ - the set of all real numbers

- $\mathbb{R}^d$ - d-dimensional real vector space

- $\mathbb{R}^{n \times d}$ - space of $n \times d$ real matrices

- $\mathbb{R}_+$ - positive real numbers

- $\mathbb{R}_{++}$ - strictly positive real numbers

> **Space Examples**
>
> - A house price: $p \in \mathbb{R}_+$ (positive real number)
>
> - Feature vector: $\mathbf{x} \in \mathbb{R}^d$ (d-dimensional vector)
>
> - Dataset: $\mathbf{X} \in \mathbb{R}^{n \times d}$ (n samples, d features)

4.2 Other Important Spaces

- $\{0, 1\}$ - binary values

- $\{1, 2, \ldots, k\}$ - discrete classes (for k-class classification)

- $[0, 1]$ - unit interval (for probabilities)

5 Dataset and ML-Specific Notation

5.1 Dataset Representation

- $\mathcal{D} = \{(\mathbf{x}_i, y_i)\}_{i=1}^n$ - dataset with n samples

- $\mathbf{x}_i \in \mathbb{R}^d$ - i-th feature vector (input)

- $y_i \in \mathbb{R}$ - i-th target value (for regression)

- $y_i \in \{1, 2, \ldots, k\}$ - i-th class label (for classification)

- $\hat{y}_i$ - predicted output for sample i

5.2 Model Parameters

- $\boldsymbol{\theta} \in \mathbb{R}^p$ - parameter vector with p parameters

- $\mathbf{w} \in \mathbb{R}^d$ - weight vector

- $b \in \mathbb{R}$ - bias term (intercept)

- $f(\mathbf{x}; \boldsymbol{\theta})$ - model function parameterized by $\boldsymbol{\theta}$

6 Exercises

6.1 Theoretical Exercises

Exercise #1: Vector Operations

Given vectors $\mathbf{x} = [2, -1, 3]^T$ and $\mathbf{y} = [1, 4, -2]^T$:

(a) Calculate the dot product $\mathbf{x} \cdot \mathbf{y}$

(b) Find the L2 norm $\|\mathbf{x}\|_2$

(c) Compute the L1 norm $\|\mathbf{y}\|_1$

Solutions:

(a) $\mathbf{x} \cdot \mathbf{y} = 2(1) + (-1)(4) + 3(-2) = 2 - 4 - 6 = -8$

(b) $\|\mathbf{x}\|_2 = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{4 + 1 + 9} = \sqrt{14}$

(c) $\|\mathbf{y}\|_1 = |1| + |4| + |-2| = 1 + 4 + 2 = 7$

Exercise #2: Matrix Dimensions

For the following expressions, determine if they are valid and find the dimensions:

(a) $\mathbf{A} \in \mathbb{R}^{3 \times 4}$, $\mathbf{B} \in \mathbb{R}^{4 \times 2}$. What is the dimension of $\mathbf{AB}$?

(b) $\mathbf{x} \in \mathbb{R}^5$, $\mathbf{W} \in \mathbb{R}^{3 \times 5}$. What is the dimension of $\mathbf{Wx}$?

(c) $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{y} \in \mathbb{R}^d$. What is the dimension of $\mathbf{x}^T \mathbf{y}$?

Solutions:

(a) Valid. $\mathbf{AB} \in \mathbb{R}^{3 \times 2}$

(b) Valid. $\mathbf{Wx} \in \mathbb{R}^3$

(c) Valid. $\mathbf{x}^T \mathbf{y} \in \mathbb{R}$ (scalar)

Exercise #3: ML Notation

Given a dataset with 100 samples and 5 features:

(a) Write the dimensions of the feature matrix $\mathbf{X}$

(b) If this is a regression problem, what are the dimensions of the target vector $\mathbf{y}$?

(c) If the weight vector is $\mathbf{w} \in \mathbb{R}^5$ and bias is $b \in \mathbb{R}$, write the prediction for sample i

Solutions:

(a) $\mathbf{X} \in \mathbb{R}^{100 \times 5}$

(b) $\mathbf{y} \in \mathbb{R}^{100}$

(c) $\hat{y}_i = \mathbf{w}^T \mathbf{x}_i + b$ or $\hat{y}_i = \mathbf{x}_i^T \mathbf{w} + b$

6.2   Coding Exercises

Coding Problem #1: Vector Operations in Python

Implement the following functions in Python using NumPy:

```python
import numpy as np

def compute_dot_product(x, y):
    """Compute dot product of two vectors"""
    # TODO: Implement
    pass

def compute_l2_norm(x):
    """Compute L2 norm of a vector"""
    # TODO: Implement
    pass

def compute_l1_norm(x):
    """Compute L1 norm of a vector"""
    # TODO: Implement
    pass

# Test your functions
x = np.array([2, -1, 3])
y = np.array([1, 4, -2])

print(f"Dot product: {compute_dot_product(x, y)}")
print(f"L2 norm of x: {compute_l2_norm(x)}")
print(f"L1 norm of y: {compute_l1_norm(y)}")
```

Solution:

```python
def compute_dot_product(x, y):
    return np.dot(x, y)  # or x @ y

def compute_l2_norm(x):
    return np.linalg.norm(x, ord=2)  # or np.sqrt(np.sum(x**2))

def compute_l1_norm(x):
    return np.linalg.norm(x, ord=1)  # or np.sum(np.abs(x))
```

Coding Problem #2: Dataset Creation

Create a synthetic dataset and verify its dimensions:

```python
import numpy as np

def create_regression_dataset(n_samples, n_features, noise_std=0.1):
    """Create a synthetic regression dataset"""
    # TODO: Create feature matrix X
    # TODO: Create true weight vector w
    # TODO: Create target vector y = X @ w + noise
    # Return X, y, w
    pass

# Test with 50 samples, 3 features
X, y, w_true = create_regression_dataset(50, 3)

print(f"X shape: {X.shape}")
print(f"y shape: {y.shape}")
print(f"w_true shape: {w_true.shape}")

# Verify dimensions are correct for linear model
print(f"X @ w_true shape: {(X @ w_true).shape}")
```

Solution:

```python
def create_regression_dataset(n_samples, n_features, noise_std=0.1):
    np.random.seed(42)  # For reproducibility
    X = np.random.randn(n_samples, n_features)
    w = np.random.randn(n_features)
    noise = np.random.normal(0, noise_std, n_samples)
    y = X @ w + noise
    return X, y, w
```

Coding Problem #3: Linear Model Implementation

Implement a simple linear regression model:

```python
class LinearRegression:
    def __init__(self):
        self.w = None
        self.b = None

    def fit(self, X, y):
        """Fit linear regression using normal equation"""
        # Add bias column to X
        # TODO: Implement normal equation solution
        pass

    def predict(self, X):
        """Make predictions"""
        # TODO: Implement prediction
        pass

    def mse(self, X, y):
        """Compute mean squared error"""
        # TODO: Implement MSE calculation
        pass

# Test your implementation
X, y, _ = create_regression_dataset(100, 2)
model = LinearRegression()
model.fit(X, y)
predictions = model.predict(X)
error = model.mse(X, y)
print(f"MSE: {error}")
```

7 Common Mistakes and Tips

7.1 Notation Mistakes to Avoid

1. **Scalar vs Vector confusion:** Always check if you're dealing with scalars (lowercase) or vectors (bold lowercase)

2. **Matrix dimension mismatch:** Always verify matrix multiplication dimensions: $(m \times n) \times (n \times p) = (m \times p)$

3. **Transpose confusion:** Remember $\mathbf{x}^T \in \mathbb{R}^{1 \times d}$ is a row vector, $\mathbf{x} \in \mathbb{R}^d$ is a column vector

4. **Index notation:** x_i is the i-th element, $\mathbf{x}_i$ is the i-th vector

7.2 Best Practices

1. Always write down matrix/vector dimensions when working through problems

2. Use consistent notation throughout your work

3. When coding, use meaningful variable names that reflect the mathematical notation

4. Verify your implementations with simple test cases where you know the answer

8   Reference Quick Sheet

Notation	Type	Example/Description
a, α, λ	Scalar	Single number
$\mathbf{x}, \mathbf{w}, \boldsymbol{\theta}$	Vector	Column vector in $\mathbb{R}^d$
$\mathbf{X}, \mathbf{A}, \boldsymbol{\Sigma}$	Matrix	2D array in $\mathbb{R}^{n \times d}$
$\mathbb{R}$	Space	Set of real numbers
$\mathbb{R}^d$	Space	d-dimensional real vectors
$\mathbb{R}^{n \times d}$	Space	n×d real matrices
$\mathbf{x} \cdot \mathbf{y}$	Operation	Dot product (inner product)
$\|\mathbf{x}\|$	Operation	Vector norm
$\mathbf{A}^T$	Operation	Matrix transpose
$\mathcal{D}$	Set	Dataset
$y_i, \hat{y}_i$	Scalar	True/predicted output

Table 1: Mathematical Notation Reference

9   Further Reading

- **Linear Algebra:** Gilbert Strang, "Introduction to Linear Algebra"

- **ML Math:** Deisenroth, Faisal, Ong, "Mathematics for Machine Learning"

- **Online Resources:** Khan Academy Linear Algebra, 3Blue1Brown Essence of Linear Algebra

- **NumPy Documentation:** `https://numpy.org/doc/stable/`