Support Vector Machines: Maximum Margin Classifiers

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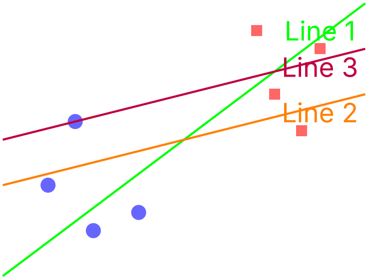
Outline

The Problem with Many Decision Boundaries

Question: For linearly separable data, which line should we choose?



All lines separate the data perfectly! Which is best for new, unseen data?

Pop Quiz: Decision Boundary Selection

Quick Quiz 1

Among all possible separating lines, which criterion should we use to choose the best one?

a) The line closest to the data points

Answer: b) Maximum margin! The line farthest from both classes generalizes better.

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a) The line closest to the data points
b) The line that maximizes distance to closest points
c) Any line works equally well

Answer: b) Maximum margin! The line farthest from both classes generalizes better.

SVM's Big Idea: Maximum Margin

SVM Principle

Among all separating hyperplanes, choose the one that **maximizes the margin** to the nearest data points.

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Intuition

Larger margin = More confident predictions = Better generalization to new data

Mathematical Formulation: Distance Between Hyperplanes

Given two parallel hyperplanes:

- $\mathbf{w} \cdot \mathbf{x} + b_1 = 0$

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- Move perpendicular distance t in direction $\mathbf{w}$: $\mathbf{x}_2 = \mathbf{x}_1 + t\frac{\mathbf{w}}{||\mathbf{w}||}$

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- Substitute into second hyperplane equation
- Solve for t to get distance

Pop Quiz: Hyperplane Distance

Quick Quiz 2

What is the distance between hyperplanes $2x + y - 3 = 0$ and $2x + y + 1 = 0$?

a) 1 unit

Answer: c) $D = \frac{|(-3)-1|}{\sqrt{2^2+1^2}} = \frac{4}{\sqrt{5}}$ units

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The SVM Optimization Problem

Goal: Find hyperplane that maximizes margin

Hard-Margin SVM Formulation

$$\text{Minimize: } \frac{1}{2}\|\mathbf{w}\|^2 \tag{1}$$

$$\text{Subject to: } y_i(\mathbf{w} \cdot \mathbf{x}_i + b) \geq 1 \quad \forall i \tag{2}$$

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Why this formulation?

- Minimizing $\|\mathbf{w}\|^2 \Rightarrow$ Maximizing margin $\frac{2}{\|\mathbf{w}\|}$
- Constraint ensures all points are correctly classified
- Support vectors lie exactly on margin boundary: $y_i(\mathbf{w} \cdot \mathbf{x}_i + b) = 1$

Pop Quiz: SVM Constraints

Quick Quiz 3

In the SVM constraint $y_i(\mathbf{w} \cdot \mathbf{x}_i + b) \geq 1$, what does $y_i = +1$ and $\mathbf{w} \cdot \mathbf{x}_i + b = 0.5$ mean?

a) Point is correctly classified and satisfies constraint

Answer: b) Since $1 \times 0.5 = 0.5 < 1$, the constraint is violated!

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b) Point violates the constraint

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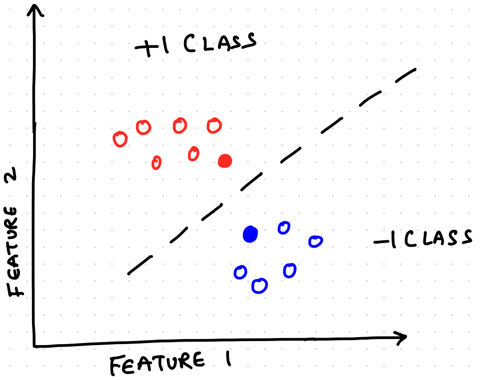
a) Point is correctly classified and satisfies constraint
b) Point violates the constraint
c) Point is a support vector

Answer: b) Since $1 \times 0.5 = 0.5 < 1$, the constraint is violated!

FORMULATION



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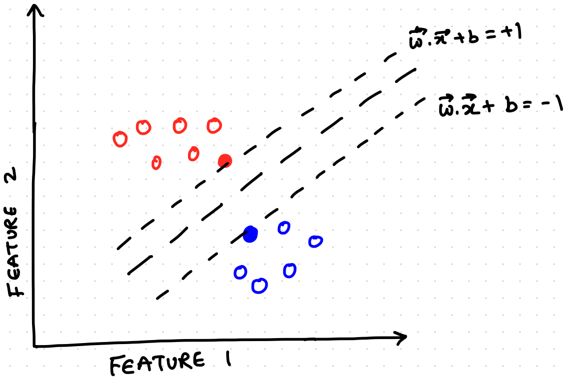

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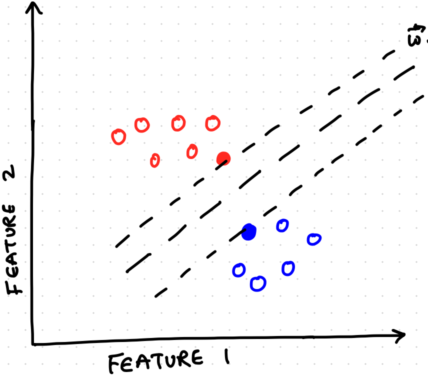

$$\text{MARGIN} = \frac{(b+1)-(b-1)}{\|\vec{w}\|}$$

$$= \frac{2}{\|\vec{w}\|}$$

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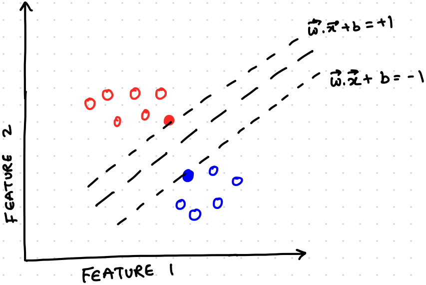

GOAL: MAXIMIZE MARGIN

$\Rightarrow$ MAXIMIZE $\dfrac{2}{||\vec{w}||}$

$\Rightarrow$ MINIMIZE $||\vec{w}||$

S.T. Correctly label points

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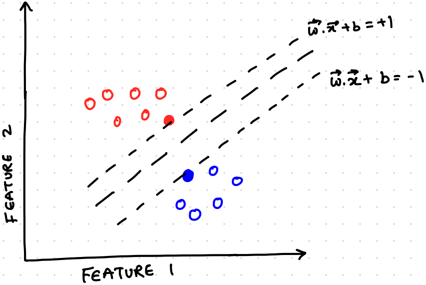

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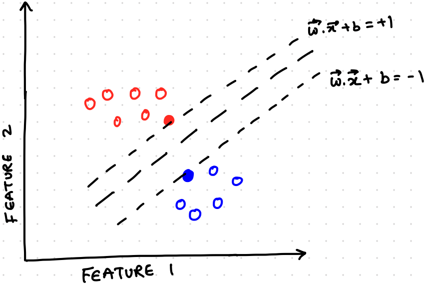

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$$\boxed{y_i(\vec{w}.\vec{x}+b) \geq 1}$$

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$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

$$\begin{aligned} \|\mathbf{w}\| &= \sqrt{\mathbf{w}^\top \mathbf{w}} \\ &= \sqrt{\begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}} \end{aligned}$$

EXAMPLE (IN 1D)