

# Multivariate Normal Distribution I

---

Nipun Batra

July 20, 2025

IIT Gandhinagar

# Univariate Normal Distribution

The probability density of univariate Gaussian is given as:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

also, given as

$$f(x) \sim \mathcal{N}(\mu, \sigma^2)$$

with mean  $\mu \in \mathbb{R}$  and variance  $\sigma^2 > 0$

# Univariate Normal Distribution

Pop Quiz: Why is the denominator the way it is? Let the normalizing constant be  $c$  and let  $g(x) = e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ .

$$1 = \int_{-\infty}^{\infty} c \cdot g(x) dx$$

# Univariate Normal Distribution

Pop Quiz: Why is the denominator the way it is? Let the normalizing constant be  $c$  and let  $g(x) = e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ .

$$1 = \int_{-\infty}^{\infty} c \cdot g(x) dx$$

$$1 = \int_{-\infty}^{\infty} c e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

# Univariate Normal Distribution

Pop Quiz: Why is the denominator the way it is? Let the normalizing constant be  $c$  and let  $g(x) = e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ .

$$1 = \int_{-\infty}^{\infty} c \cdot g(x) dx$$

$$1 = \int_{-\infty}^{\infty} c e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Let's substitute  $\frac{x-\mu}{\sqrt{2}\sigma}$  with  $t$ .

# Univariate Normal Distribution

Pop Quiz: Why is the denominator the way it is? Let the normalizing constant be  $c$  and let  $g(x) = e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ .

$$1 = \int_{-\infty}^{\infty} c \cdot g(x) dx$$

$$1 = \int_{-\infty}^{\infty} ce^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Let's substitute  $\frac{x-\mu}{\sqrt{2}\sigma}$  with  $t$ .

$$1 = \int_{-\infty}^{\infty} ce^{-t^2} dt \times \sqrt{2}\sigma$$

# Univariate Normal Distribution

Pop Quiz: Why is the denominator the way it is? Let the normalizing constant be  $c$  and let  $g(x) = e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ .

$$1 = \int_{-\infty}^{\infty} c \cdot g(x) dx$$

$$1 = \int_{-\infty}^{\infty} ce^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Let's substitute  $\frac{x-\mu}{\sqrt{2}\sigma}$  with  $t$ .

$$1 = \int_{-\infty}^{\infty} ce^{-t^2} dt \times \sqrt{2}\sigma$$

$$1 = \sqrt{2}\sigma c \times 2 \int_0^{\infty} e^{-t^2} dt$$

# Univariate Normal Distribution

$$\frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} dt$$

The above expression is called error function and its value is denoted by  $\text{erf}(t)$ . In our case, we want  $\text{erf}(\infty)$  which is equal to 1.



# Univariate Normal Distribution

$$\frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} dt$$

The above expression is called error function and its value is denoted by  $\text{erf}(t)$ . In our case, we want  $\text{erf}(\infty)$  which is equal to 1.

$$1 = \sqrt{2\pi}\sigma c \times \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} dt$$

# Univariate Normal Distribution

$$\frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} dt$$

The above expression is called error function and its value is denoted by  $\text{erf}(t)$ . In our case, we want  $\text{erf}(\infty)$  which is equal to 1.

$$1 = \sqrt{2\pi}\sigma c \times \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} dt$$

$$1 = \sqrt{2\pi}\sigma c \times 1$$

# Univariate Normal Distribution

$$\frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} dt$$

The above expression is called error function and its value is denoted by  $\text{erf}(t)$ . In our case, we want  $\text{erf}(\infty)$  which is equal to 1.

$$1 = \sqrt{2\pi\sigma}c \times \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} dt$$

$$1 = \sqrt{2\pi\sigma}c \times 1$$

$$\frac{1}{\sqrt{2\pi\sigma}} = c$$

# Univariate Normal Distribution

# Univariate Normal Distribution

# Univariate Normal Distribution

# Bivariate Normal Distribution

Bivariate normal distribution of two-dimensional random vector

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \mathcal{N}_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

where, mean vector  $\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} E[X_1] \\ E[X_2] \end{bmatrix}$

and, covariance matrix  $\boldsymbol{\Sigma}$

$$\Sigma_{i,j} := E[(X_i - \mu_i)(X_j - \mu_j)] = \text{Cov}[X_i, X_j]$$

# Bivariate Normal Distribution

Question: What is  $\text{Cov}(X, X)$ ?



# Bivariate Normal Distribution

Question: What is  $\text{Cov}(X, X)$ ?

Answer:  $\text{Var}(X) = \text{Cov}(X, X) = E[(X - E[X])]^2$

In the case of univariate normal,  $\text{Var}(X)$  is written as  $\sigma^2$

# Bivariate Normal Distribution

Question: What is  $\text{Cov}(X, X)$ ?

Answer:  $\text{Var}(X) = \text{Cov}(X, X) = E[(X - E[X])]^2$

In the case of univariate normal,  $\text{Var}(X)$  is written as  $\sigma^2$

Question: What is the relation between  $\Sigma_{i,j}$  and  $\Sigma_{j,i}$ ?

# Bivariate Normal Distribution

Question: What is  $\text{Cov}(X, X)$ ?

Answer:  $\text{Var}(X) = \text{Cov}(X, X) = E[(X - E[X])]^2$

In the case of univariate normal,  $\text{Var}(X)$  is written as  $\sigma^2$

Question: What is the relation between  $\Sigma_{i,j}$  and  $\Sigma_{j,i}$ ?

Answer: They are the same!

# Bivariate Normal Distribution

Question: What is  $\text{Cov}(X, X)$ ?

Answer:  $\text{Var}(X) = \text{Cov}(X, X) = E[(X - E[X])]^2$

In the case of univariate normal,  $\text{Var}(X)$  is written as  $\sigma^2$

Question: What is the relation between  $\Sigma_{i,j}$  and  $\Sigma_{j,i}$ ?

Answer: They are the same!

Question: What can we say about the covariance matrix  $\Sigma$ ?

# Bivariate Normal Distribution

Question: What is  $\text{Cov}(X, X)$ ?

Answer:  $\text{Var}(X) = \text{Cov}(X, X) = E[(X - E[X])]^2$

In the case of univariate normal,  $\text{Var}(X)$  is written as  $\sigma^2$

Question: What is the relation between  $\Sigma_{i,j}$  and  $\Sigma_{j,i}$ ?

Answer: They are the same!

Question: What can we say about the covariance matrix  $\Sigma$ ?

Answer: It is symmetric. Thus  $\Sigma = \Sigma^T$

## Correlation and Covariance

If  $X$  and  $Y$  are two random variables, with means (expected values)  $\mu_X$  and  $\mu_Y$  and standard deviations  $\sigma_X$  and  $\sigma_Y$ , respectively, then their covariance and correlation are as follows:

$$\text{cov}_{XY} = \sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)]$$

## Correlation and Covariance

If  $X$  and  $Y$  are two random variables, with means (expected values)  $\mu_X$  and  $\mu_Y$  and standard deviations  $\sigma_X$  and  $\sigma_Y$ , respectively, then their covariance and correlation are as follows:

$$\text{cov}_{XY} = \sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)]$$

$$\text{corr}_{XY} = \rho_{XY} = E[(X - \mu_X)(Y - \mu_Y)]/(\sigma_X\sigma_Y)$$

## Correlation and Covariance

If  $X$  and  $Y$  are two random variables, with means (expected values)  $\mu_X$  and  $\mu_Y$  and standard deviations  $\sigma_X$  and  $\sigma_Y$ , respectively, then their covariance and correlation are as follows:

$$\text{cov}_{XY} = \sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)]$$

$$\text{corr}_{XY} = \rho_{XY} = E[(X - \mu_X)(Y - \mu_Y)]/(\sigma_X\sigma_Y)$$

so that

$$\rho_{XY} = \sigma_{XY}/(\sigma_X\sigma_Y)$$

where  $E$  is the expected value operator.



We might have seen that

$$f_X(X_1, X_2) = \frac{\exp(\frac{-1}{2}(X - \mu)^T \Sigma^{-1}(X - \mu))}{2\pi |\Sigma|^{\frac{1}{2}}}$$

How do we get such a weird looking formula?!

## PDF of bivariate normal with no cross-correlation

Let us assume no correlation between  $X_1$  and  $X_2$ .

$$\text{We have } \Sigma = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$$

$$\text{We have } f_X(X_1, X_2) = f_X(X_1)f_X(X_2)$$

$$= \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X_1-\mu_1}{\sigma_1}\right)^2} \times \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X_2-\mu_2}{\sigma_2}\right)^2}$$

$$= \frac{1}{\sigma_1\sigma_2 2\pi} e^{-\frac{1}{2}\left\{\left(\frac{X_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{X_2-\mu_2}{\sigma_2}\right)^2\right\}}$$

## PDF of bivariate normal with no cross-correlation

Let us consider only the exponential part for now

$$Q = \left( \frac{X_1 - \mu_1}{\sigma_1} \right)^2 + \left( \frac{X_2 - \mu_2}{\sigma_2} \right)^2$$

Question: Can you write Q in the form of vectors X and  $\mu$ ?

$$= \begin{bmatrix} X_1 - \mu_1 & X_2 - \mu_2 \end{bmatrix}_{1 \times 2} g(\Sigma)_{2 \times 2} \begin{bmatrix} X_1 - \mu_1 \\ X_2 - \mu_2 \end{bmatrix}_{2 \times 1}$$

Here  $g(\Sigma)$  is a matrix function of  $\Sigma$  that will result in  $\sigma_1^2$  like terms in the denominator; also there is no cross-terms indicating zeros in right diagonal!

$$g(\Sigma) = \begin{bmatrix} \frac{1}{\sigma_1^2} & 0 \\ 0 & \frac{1}{\sigma_2^2} \end{bmatrix}_{2 \times 2} = \frac{1}{\sigma_1^2 \sigma_2^2} \begin{bmatrix} \sigma_2^2 & 0 \\ 0 & \sigma_1^2 \end{bmatrix}_{2 \times 2} = \frac{1}{|\Sigma|} \text{adj}(\Sigma) = \Sigma^{-1}$$

## PDF of bivariate normal with no cross-correlation

Let us consider the normalizing constant part now.  $M = \frac{1}{\sigma_1 \sigma_2 2\pi}$   
 $= \frac{1}{2\pi \times |\Sigma|^{\frac{1}{2}}}$

## Bivariate Gaussian samples with cross-correlation $\neq 0$

## Bivariate Gaussian samples with cross-correlation = 0

## Intuition for Multivariate Gaussian

Let us assume no correlation between the elements of  $\mathbf{X}$ . This means  $\Sigma$  is a diagonal matrix.

$$\text{We have } \Sigma = \begin{bmatrix} \sigma_1^2 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \sigma_n^2 \end{bmatrix}$$

And,

$$p(\mathbf{X}; \mu, \Sigma) = \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}}} \exp \left( -\frac{1}{2} (\mathbf{X} - \mu)^T \Sigma^{-1} (\mathbf{X} - \mu) \right)$$

As seen in the case for univariate Gaussians, we can write the following for the multivariate case,

We have  $f_X(X_1, \dots, X_n) = f_X(X_1) \times \dots \times f_X(X_n)$

# Intuition for Multivariate Gaussian

Now,

$$\begin{aligned} &= \frac{1}{\sigma_1 \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x_1 - \mu_1}{\sigma_1} \right)^2} \times \dots \times \frac{1}{\sigma_n \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x_n - \mu_n}{\sigma_n} \right)^2} \\ &= \frac{1}{\sigma_1 \cdots \sigma_n (2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2} \left\{ \left( \frac{x_1 - \mu_1}{\sigma_1} \right)^2 + \dots + \left( \frac{x_n - \mu_n}{\sigma_n} \right)^2 \right\}} \end{aligned}$$

Taking all  $\sqrt{2\pi}$  together, we get  $(2\pi)^{\frac{n}{2}}$ .

Similarly, taking all  $\sigma$  together, we get  $\sigma_1 \cdots \sigma_n$ . Which can be written as  $|\Sigma|^{\frac{1}{2}}$ , given the determinant of a diagonal matrix is the multiplication of its diagonal elements.



Now, let us remove the assumption of no covariance among the elements of  $\mathbf{X}$

Main idea: A correlated Gaussian is a rotated independent Gaussian<sup>1</sup>

Rotate input space using rotation matrix  $R$ .

$$p(\mathbf{X}; \mu, \Sigma) = \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}}} \exp \left( -\frac{1}{2} (\mathbf{R}^T \mathbf{X} - R^T \mu)^T \Sigma^{-1} (\mathbf{R}^T \mathbf{X} - R^T \mu) \right)$$

$$p(\mathbf{X}; \mu, \Sigma) = \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}}} \exp \left( -\frac{1}{2} (\mathbf{X} - \mu)^T R \Sigma^{-1} R^T (\mathbf{X} - \mu) \right)$$

---

<sup>1</sup>Neil Lawrence GPSS 2016

$$C = R\Sigma^{-1}R^T$$

$$p(\mathbf{X}; \mu, \Sigma) = \frac{1}{(2\pi)^{\frac{n}{2}} |C|^{\frac{1}{2}}} \exp \left( -\frac{1}{2} (\mathbf{X} - \mu)^T C^{-1} (\mathbf{X} - \mu) \right)$$