KKT Conditions

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Outline

KKT Conditions: The Foundation of Constrained Optimization

Constrained Optimization Problem

Minimize: $f(x)$ where $x \in \mathbb{R}^k$
Subject to:

$$h_i(x) = 0, \quad i = 1, \ldots, m \quad \text{(equality constraints)} \tag{1}$$
$$g_j(x) \leq 0, \quad j = 1, \ldots, n \quad \text{(inequality constraints)} \tag{2}$$

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Why KKT Conditions Matter

They tell us when we've found the optimal solution!
Essential for SVM, regularized regression, and many ML algorithms.

Step 2

- Minimize $L(x, \lambda, \mu)$ w.rt. $x \implies \nabla_x L(x, \lambda, \mu) = 0$
  Gives k equations

Step 3

- Minimize $L(x, \lambda, \mu)$ w.rt. $\lambda \implies \nabla_\lambda L(x, \lambda, \mu) = 0$
  Gives m equations

Step 4

In both cases, $\mu_i g_i(x^*) = 0$

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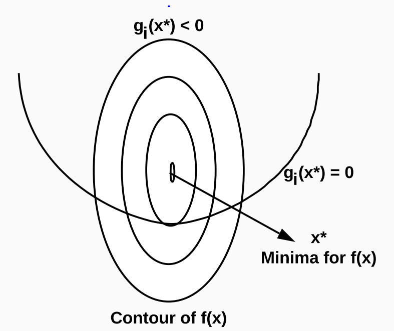

$$g_i(x^*) \leq 0$$
$$\mu_i = 0$$

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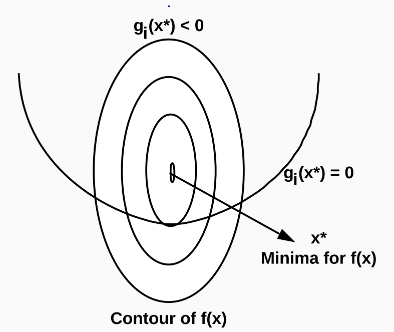

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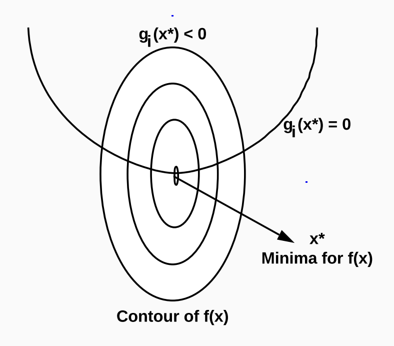

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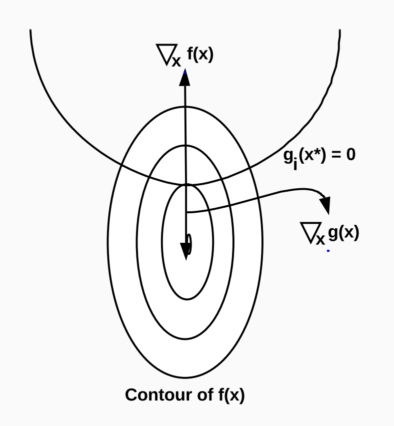
Constraint on μ_i's



$$min_x L(x, \lambda, \mu) \implies \nabla_x f(x) + \nabla_x \mu_i g_i(x) = 0$$

$$\mu_i = \frac{\nabla_x f(x)}{\nabla_x \mu_i g_i(x)} = +ve$$

KKT Conditions

Stationarity (For minimization)

$$\nabla_x f(x) + \sum_{i=1}^{m} \nabla_x \lambda_i h_i(x) + \sum_{i=1}^{n} \nabla_x \mu_i g_i(x) = 0$$

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Equality Constraints

$$\nabla_\lambda f(x) + \sum_{i=1}^{m} \nabla_\lambda \lambda_i h_i(x) + \sum_{i=1}^{n} \nabla_\lambda \mu_i g_i(x) = 0$$

$$\sum_{i=1}^{m} \nabla_\lambda \lambda_i h_i(x) = 0$$

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Stationarity (For minimization)

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Inequality Constraints (Complementary Slackness)

$$\mu_i g_i(x) = 0 \forall i = 1, \ldots, n$$

$$\mu_i \geq 0$$

Example

Minimize $x^2 + y^2$ such that,

$$x^2 + y^2 \leq 5$$
$$x + 2y = 4$$
$$x, y \geq 0$$

Example

$$f(x, y) = x^2 + y^2$$

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$$h(x, y) = x + 2y - 4$$

Example

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\begin{aligned}
&f(x,y) = x^2 + y^2 \\
&h(x,y) = x + 2y - 4 \\
&g_1(x,y) = x^2 + y^2 - 5
\end{aligned}
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Example

$$f(x, y) = x^2 + y^2$$
$$h(x, y) = x + 2y - 4$$
$$g_1(x, y) = x^2 + y^2 - 5$$
$$g_2(x, y) = -x$$
$$g_3(x, y) = -y$$

$$L(x, y, \lambda, \mu_1, \mu_2, \mu_3) =$$
$$x^2 + y^2 + \lambda(x + 2y - 4) + \mu_1(x^2 + y^2 - 5) + \mu_2(-x) + \mu_3(-y)$$

Example

Stationarity

$\nabla_x L(x, y, \lambda, \mu_1, \mu_2, \mu_3) = 0$

$\implies 2x + \lambda + 2\mu_1 x - \mu_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$

$\nabla_y L(x, y, \lambda, \mu_1, \mu_2, \mu_3) = 0$

$\implies 2y + 2\lambda + 2\mu_1 y - \mu_3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

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Equality Constraint

$x + 2y = 4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$

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Equality Constraint

$x + 2y = 4 \dots\dots\dots\dots (3)$

Slackness

$\mu_1(x^2 + y^2 - 5) = 0 \dots\dots\dots\dots (4)$

$\mu_2 x = 0 \dots\dots\dots\dots (5)$

$\mu_3 y = 0 \dots\dots\dots\dots (6)$

Example

From (6), $\mu_3 = 0$ or $y = 0$
But if, $y = 0$, then $x = 4$ according to (3) . This violates (1).
Hence, $y \neq 0$ and $\mu_3 = 0$

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If $x = 0$, $y = 2$, which implies $x^2 + y^2 = 4 (\leq 5)$
Since (x,y) = (0,2) gives smaller $x^2 + y^2$ terms than 5,
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On further solving we get,
$x = 0.8$
$y = 1.6$