Logistic Regression

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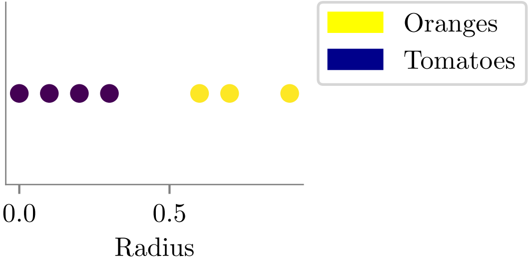
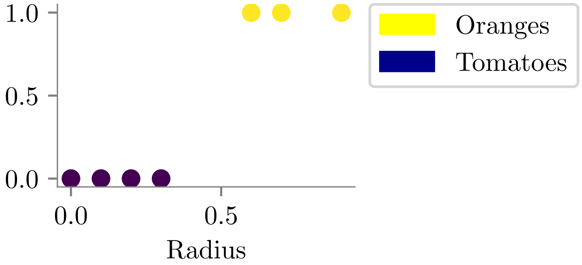
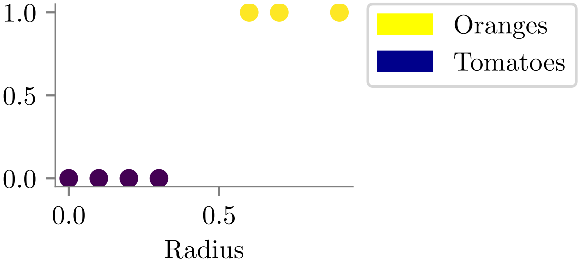
Classification Technique



Classification Technique

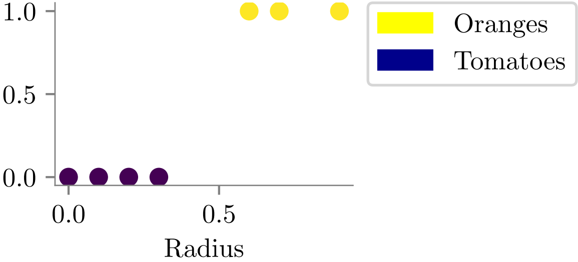


Classification Technique



Aim: Probability(Tomatoes | Radius) ? or

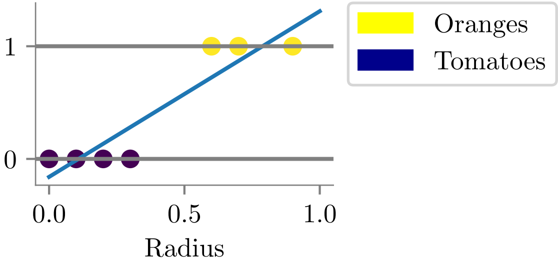
Classification Technique



Aim: Probability(Tomatoes | Radius) ? or

$\boxed{\text{More generally, } \mathrm{P}(y = 1 | \mathbf{X} = \mathbf{x})?}$

Idea: Use Linear Regression



$$P(X = Orange|Radius) = \theta_0 + \theta_1 \times Radius$$

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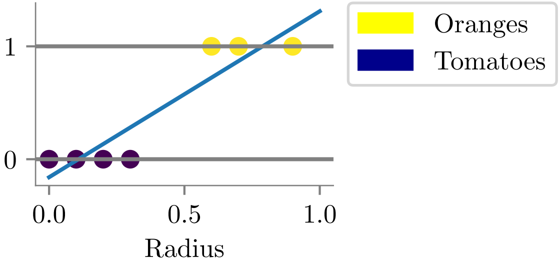


$$P(X = Orange|Radius) = \theta_0 + \theta_1 \times Radius$$

Generally,

$$P(y = 1|\mathbf{x}) = \mathbf{X}\boldsymbol{\theta}$$

Idea: Use Linear Regression

Prediction:

If $\theta_0 + \theta_1 \times \mathit{Radius} > 0.5 \rightarrow$ Orange

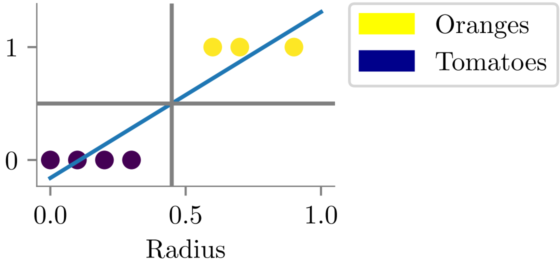
Else $\rightarrow$ Tomato

Problem:

Range of $\mathbf{X}\boldsymbol{\theta}$ is $(-\infty, \infty)$

But $P(y = 1|\ldots) \in [0, 1]$

Idea: Use Linear Regression



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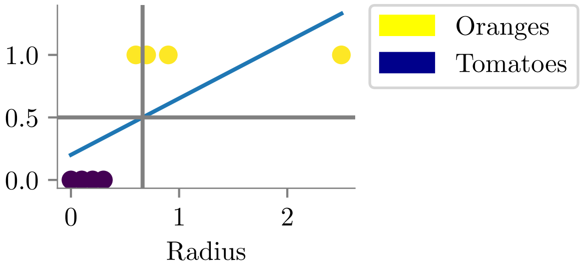
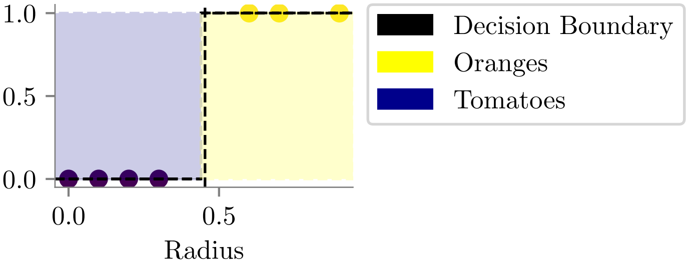


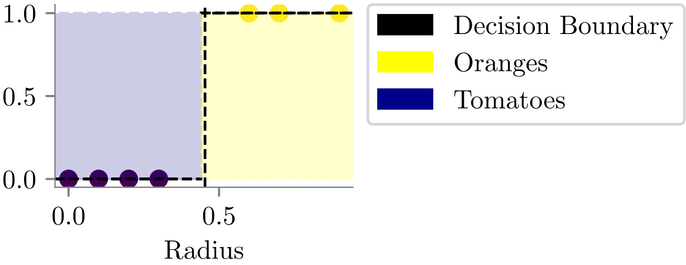
Linear regression for classification gives a poor prediction!

Ideal boundary



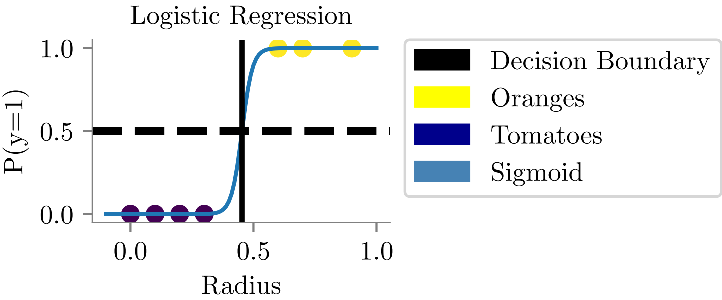
- Have a decision function similar to the above (but not so sharp and discontinuous)

Ideal boundary



- Have a decision function similar to the above (but not so sharp and discontinuous)
- Aim: use linear regression still!

Idea: Use Linear Regression



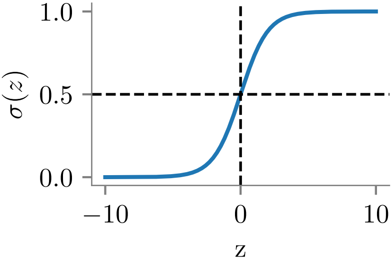
Question. Can we still use Linear Regression?

Answer. Yes! Transform $\hat{y} \to [0, 1]$

Logistic/Sigmoid function

$\hat{y} \in (-\infty, \infty)$
ϕ = Sigmoid / Logistic Function (σ)
$\phi(\hat{y}) \in [0, 1]$

$$\sigma(z) = \frac{1}{1 + e^{-z}}$$



Logistic / Sigmoid Function

$z \to \infty$

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$\sigma(z) \to 1$

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Logistic / Sigmoid Function

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$\sigma(z) \to 1$

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$\sigma(z) \to 0$

$z = 0$

$\sigma(z) = 0.5$

Logistic / Sigmoid Function

Question. Could you use some other transformation (ϕ) of $\hat{y}$ s.t.

$$\phi(\hat{y}) \in [0, 1]$$

Yes! But Logistic Regression works.

Logistic / Sigmoid Function

$$P(y = 1|\mathbf{X}) = \sigma(\mathbf{X}\boldsymbol{\theta}) = \frac{1}{1 + e^{-\mathbf{X}\boldsymbol{\theta}}}$$

Q. Write $\mathbf{X}\boldsymbol{\theta}$ in a more convenient form (as $P(y = 1|X)$, $P(y = 0|X)$)

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$$P(y = 0|X) = 1 - P(y = 1|X) = 1 - \frac{1}{1 + e^{-\mathbf{X}\boldsymbol{\theta}}} = \frac{e^{-\mathbf{X}\boldsymbol{\theta}}}{1 + e^{-\mathbf{X}\boldsymbol{\theta}}}$$

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$$\therefore \frac{P(y = 1|X)}{1 - P(y = 1|X)} = e^{\mathbf{X}\boldsymbol{\theta}} \implies \mathbf{X}\boldsymbol{\theta} = \log \frac{P(y = 1|X)}{1 - P(y = 1|X)}$$

Odds (Used in betting)

$$\frac{P(win)}{P(loss)}$$

Here,

$$Odds = \frac{P(y=1)}{P(y=0)}$$

$$\boxed{\text{log-odds} = \log \frac{P(y=1)}{P(y=0)} = \mathbf{X}\boldsymbol{\theta}}$$

Q. What is decision boundary for Logistic Regression?

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Decision Boundary: $P(y=1|X) = P(y=0|X)$

$$\text{or } \frac{1}{1+e^{-\mathbf{X}\boldsymbol{\theta}}} = \frac{e^{-\mathbf{X}\boldsymbol{\theta}}}{1+e^{-\mathbf{X}\boldsymbol{\theta}}}$$

$$\text{or } e^{\mathbf{X}\boldsymbol{\theta}} = 1$$

$$\text{or } \mathbf{X}\boldsymbol{\theta} = 0$$

Learning Parameters

Could we use cost function as:

$$J(\theta) = \sum (y_i - \hat{y}_i)^2$$

$$\hat{y}_i = \sigma(\mathbf{X}\boldsymbol{\theta})$$

Answer: **No (Non-Convex)**

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- Sigmoid $\sigma(z) = \frac{1}{1+e^{-z}}$ is non-linear

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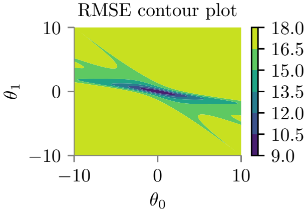
- No guarantee gradient descent finds global optimum
- This is why we need cross-entropy loss instead!

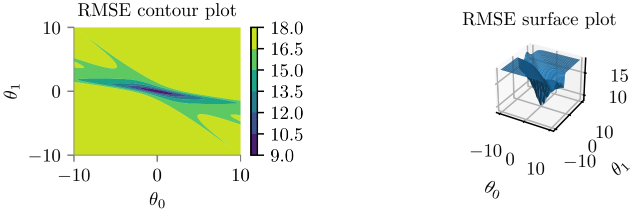
Deriving Cost Function via Maximum Likelihood Estimation

Cost function convexity

Cost function convexity



Cost function convexity



Learning Parameters

Likelihood $= P(D|\theta)$

$P(y|X,\theta) = \prod_{i=1}^{n} P(y_i|x_i,\theta)$
where y = 0 or 1

Learning Parameters

$$\text{Likelihood} = P(D|\theta)$$

$$P(y|X,\theta) = \prod_{i=1}^{n} P(y_i|x_i,\theta) = \prod_{i=1}^{n} \left\{\frac{1}{1+e^{-x_i^T\theta}}\right\}^{y_i} \left\{1 - \frac{1}{1+e^{-x_i^T\theta}}\right\}^{1-y_i}$$

[Above: Similar to $P(D|\theta)$ for Linear Regression;

Difference Bernoulli instead of Gaussian]

$-\log P(y|\mathbf{X},\boldsymbol{\theta})$ = Negative Log Likelihood = Cost function will be minim

Aside on Bernoulli Likelihood

- Assume you have a coin and flip it ten times and get (H, H, T, T, T, H, H, T, T, T).

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- We might think it to be: $4/10 = 0.4$. But why?

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- Answer 2: What is likelihood of seeing the above sequence when the p(Head)$=\theta$?

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- Answer 2: What is likelihood of seeing the above sequence when the p(Head)$=\theta$?
- Idea find MLE estimate for θ

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- Log-likelihood $= \mathcal{LL}(\theta) = n_h \log(\theta) + n_t \log(1-\theta)$
- $\frac{\partial \mathcal{LL}(\theta)}{\partial \theta} = 0 \implies \frac{n_h}{\theta} + \frac{n_t}{1-\theta} = 0 \implies \theta_{MLE} = \frac{n_h}{n_h + n_t}$

Cross Entropy Cost Function

$$J(\theta) = -\log\left\{\prod_{i=1}^{n}\left\{\frac{1}{1+e^{-x_i^T\theta}}\right\}^{y_i}\left\{1-\frac{1}{1+e^{-x_i^T\theta}}\right\}^{1-y_i}\right\}$$

$$J(\theta) = -\left\{\sum_{i=1}^{N} y_i\log(\sigma_\theta(x_i)) + (1-y_i)\log(1-\sigma_\theta(x_i))\right\}$$

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This cost function is called cross-entropy.

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Why?

Interpretation of Cross-Entropy Cost Function

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What is the interpretation of the cost function?

Interpretation of Cross-Entropy Cost Function

What is the interpretation of the cost function?

Let us try to write the cost function for a single example:

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$$J(\theta) = -y_i \log \hat{y}_i - (1 - y_i) \log(1 - \hat{y}_i)$$

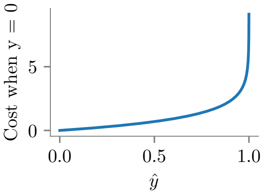
Interpretation of Cross-Entropy Cost Function

What is the interpretation of the cost function?

Let us try to write the cost function for a single example:

$$J(\theta) = -y_i \log \hat{y}_i - (1 - y_i) \log(1 - \hat{y}_i)$$

First, assume y_i is 0, then if $\hat{y}_i$ is 0, the loss is 0; but, if $\hat{y}_i$ is 1, the loss tends towards infinity!



Notebook: logits-usage

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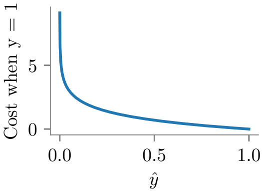
What is the interpretation of the cost function?

$$J(\theta) = -y_i \log \hat{y}_i - (1 - y_i) \log(1 - \hat{y}_i)$$

Now, assume y_i is 1, then if $\hat{y}_i$ is 0, the loss is huge; but, if $\hat{y}_i$ is 1, the loss is zero!



Cost function convexity

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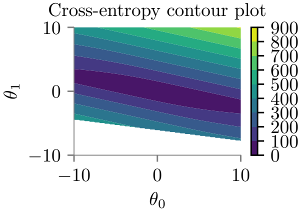

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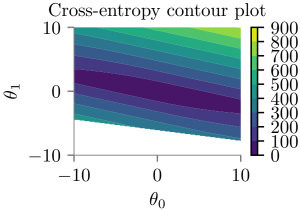

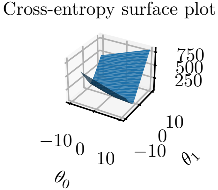

Learning Parameters

$$\frac{\partial J(\theta)}{\partial \theta_j} = -\frac{\partial}{\partial \theta_j}\left\{ \sum_{i=1}^{N} y_i log(\sigma_\theta(x_i)) + (1 - y_i) log(1 - \sigma_\theta(x_i))\right\}$$

$$= -\sum_{i=1}^{N} \left[y_i \frac{\partial}{\partial \theta_j} \log(\sigma_\theta(x_i)) + (1 - y_i) \frac{\partial}{\partial \theta_j} log(1 - \sigma_\theta(x_i))\right]$$

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$$= -\sum_{i=1}^{N} \left[\frac{y_i}{\sigma_\theta(x_i)} \frac{\partial}{\partial \theta_j} \sigma_\theta(x_i) + \frac{1 - y_i}{1 - \sigma_\theta(x_i)} \frac{\partial}{\partial \theta_j} (1 - \sigma_\theta(x_i)) \right]$$

Aside:

$$\frac{\partial}{\partial z} \sigma(z) = \frac{\partial}{\partial z} \frac{1}{1 + e^{-z}} = -(1 + e^{-z})^{-2} \frac{\partial}{\partial z} (1 + e^{-z})$$

$$= \frac{e^{-z}}{(1 + e^{-z})^2} = \left(\frac{1}{1 + e^{-z}} \right) \left(\frac{e^{-z}}{1 + e^{-z}} \right) = \sigma(z) \left\{ \frac{1 + e^{-z}}{1 + e^{-z}} - \frac{1}{1 + e^{-z}} \right\}$$

$$= \sigma(z)(1 - \sigma(z))$$

Learning Parameters

Resuming from (1)

$$\frac{\partial J(\theta)}{\partial \theta_j} = -\sum_{i=1}^{N} \left[\frac{y_i}{\sigma_\theta(x_i)} \frac{\partial}{\partial \theta_j} \sigma_\theta(x_i) + \frac{1 - y_i}{1 - \sigma_\theta(x_i)} \frac{\partial}{\partial \theta_j} (1 - \sigma_\theta(x_i)) \right]$$

$$= -\sum_{i=1}^{N} \left[\frac{y_i \sigma_\theta(x_i)}{\sigma_\theta(x_i)} (1 - \sigma_\theta(x_i)) \frac{\partial}{\partial \theta_j} (x_i \theta) + \frac{1 - y_i}{1 - \sigma_\theta(x_i)} (1 - \sigma_\theta(x_i)) \frac{\partial}{\partial \theta_j} (1 - \sigma_\theta(x_i)) \right]$$

$$= -\sum_{i=1}^{N} \left[y_i (1 - \sigma_\theta(x_i)) x_i^j - (1 - y_i) \sigma_\theta(x_i) x_i^j \right]$$

$$= -\sum_{i=1}^{N} \left[(y_i - y_i \sigma_\theta(x_i) - \sigma_\theta(x_i) + y_i \sigma_\theta(x_i)) x_i^j \right]$$

$$= \sum_{i=1}^{N} \left[\sigma_\theta(x_i) - y_i \right] x_i^j$$

Learning Parameters

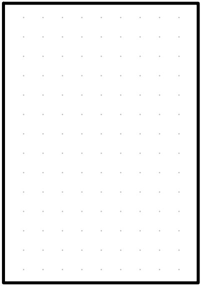
$$\boxed{\frac{\partial J(\theta)}{\theta_j} = \sum_{i=1}^{N} \left[\sigma_\theta(x_i) - y_i\right] x_i^j}$$

Now, just use Gradient Descent!

$$\frac{\partial J(\theta)}{\partial \theta_j} = \sum_{i=1}^{N} (\hat{y}_i - y_i) x_i^j$$

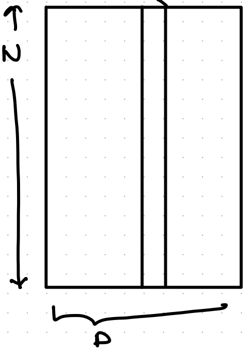
$$\frac{\partial J(\theta)}{\partial \theta_j} = \sum_{i=1}^{N} (\hat{y}_i - y_i) x_i^j$$

MATRIX X



$$\frac{\partial J(\theta)}{\partial \theta_j} = \sum_{i=1}^{N} (\hat{y}_i - y_i)\, x_i^j$$

MATRIX X



j^{th} column of X

$= X^j$

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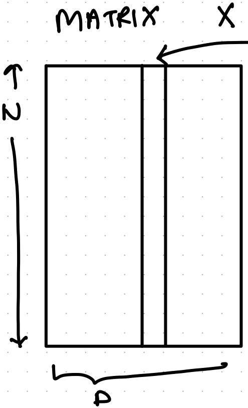

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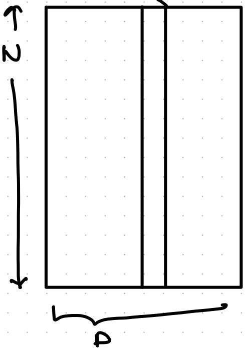
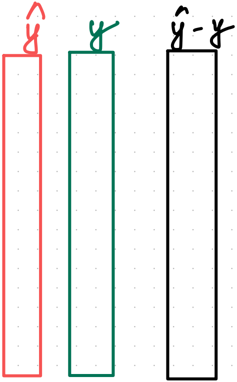
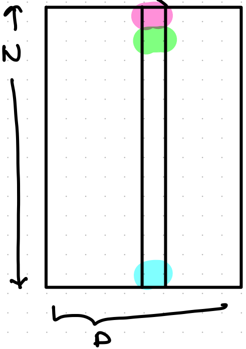




$$\frac{\partial J(\theta)}{\partial \theta_j} = \sum_{i=1}^{N} (\hat{y}_i - y_i)\, x_i^j$$

MATRIX X



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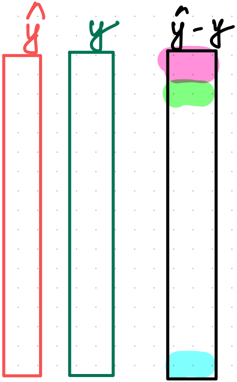
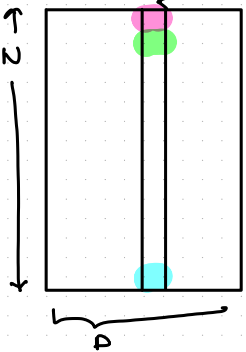


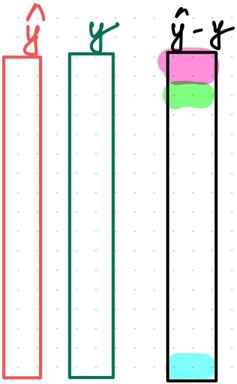
$$\frac{\partial J(\theta)}{\partial \theta_j} = \sum_{i=1}^{N} (\hat{y}_i - y_i)\, x_i^j = \underset{1\times N}{X^{j^T}} (\hat{y} - y)$$

MATRIX X



j^{th} column of X

$= X^j$



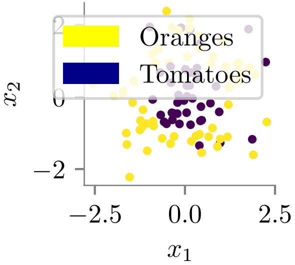
$$\frac{\partial J(\theta)}{\partial \theta_j} = \sum_{i=1}^{N} (\hat{y}_i - y_i) x_i^j = \underset{1\times N}{x^{j^T}} (\hat{y} - y)$$

$$\begin{bmatrix} \frac{\partial J(\theta)}{\partial \theta_1} \\ \frac{\partial J(\theta)}{\partial \theta_2} \\ \vdots \\ \\ \frac{\partial J(\theta)}{\partial \theta_D} \end{bmatrix} = \begin{pmatrix} x^{1^T} (\hat{y} - y) \\ x^{2^T} (\hat{y} - y) \\ \vdots \\ x^{D^T} (\hat{y} - y) \end{pmatrix}$$

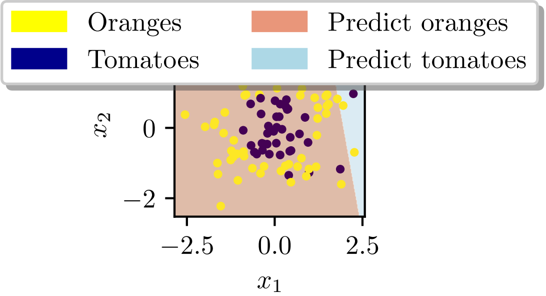
$$\frac{\partial J(\theta)}{\partial \theta_j} = \sum_{i=1}^{N} (\hat{y}_i - y_i) x_i^j = \underset{1\times N}{{x^j}^T} (\hat{y} - y)$$

$$\begin{bmatrix} \frac{\partial J(\theta)}{\partial \theta_1} \\ \frac{\partial J(\theta)}{\partial \theta_2} \\ \vdots \\ \\ \frac{\partial J(\theta)}{\partial \theta_D} \end{bmatrix} = \begin{pmatrix} {x^1}^T (\hat{y} - y) \\ {x^2}^T (\hat{y} - y) \\ \vdots \\ {x^D}^T (\hat{y} - y) \end{pmatrix} = X^T (\hat{y} - y)$$

Logistic Regression with feature transformation

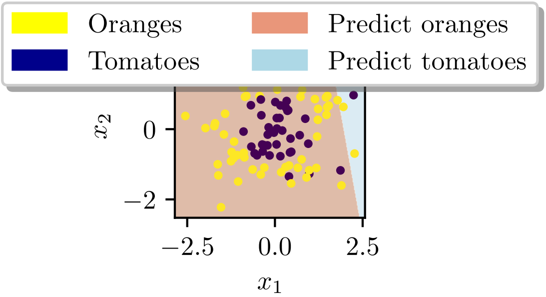


What happens if you apply logistic regression on the above data?



Linear boundary will not be accurate here. What is the technical name of the problem?

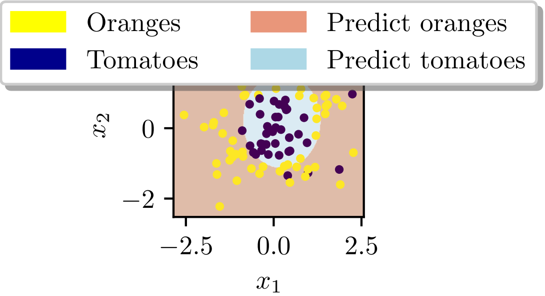
Logistic Regression with feature transformation



Linear boundary will not be accurate here. What is the technical name of the problem? Bias!

Logistic Regression with feature transformation

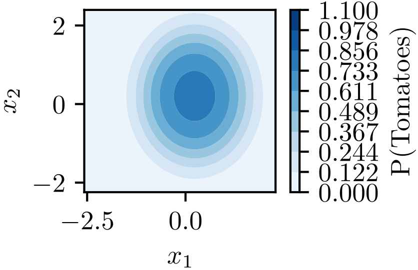
$$\phi(x) = \begin{bmatrix} \phi_0(x) \\ \phi_1(x) \\ \vdots \\ \phi_{K-1}(x) \end{bmatrix} = \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ \vdots \\ x^{K-1} \end{bmatrix} \in \mathbb{R}^K$$



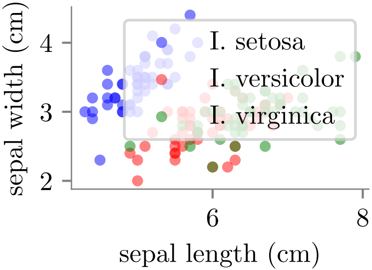
Using x_1^2, x_2^2 as additional features, we are able to learn a more accurate classifier.

How would you expect the probability contours look like?

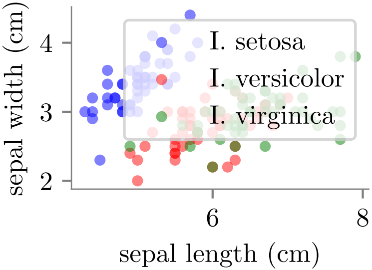
How would you expect the probability contours look like?



Multi-Class Prediction

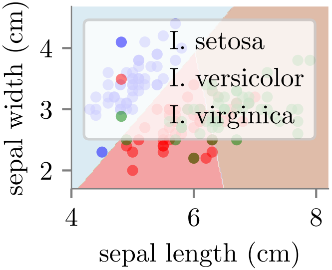


Multi-Class Prediction

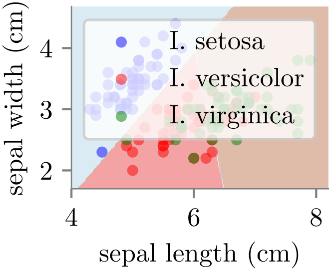


How would you learn a classifier? Or, how would you expect the classifier to learn decision boundaries?

Multi-Class Prediction

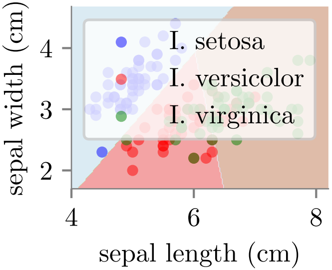


Multi-Class Prediction

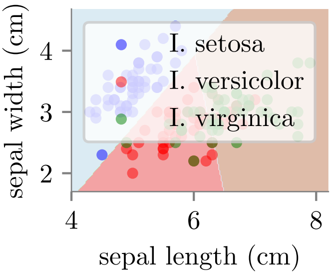


1. Use one-vs.-all on Binary Logistic Regression

Multi-Class Prediction

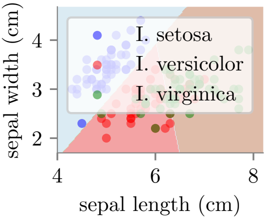


1. Use one-vs.-all on Binary Logistic Regression
2. Use one-vs.-one on Binary Logistic Regression

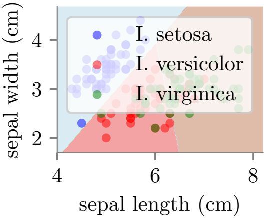


1. Use one-vs.-all on Binary Logistic Regression
2. Use one-vs.-one on Binary Logistic Regression
3. Extend Binary Logistic Regression to Multi-Class Logistic Regression

Multi-Class Prediction

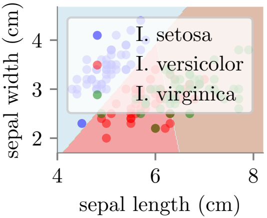


Multi-Class Prediction



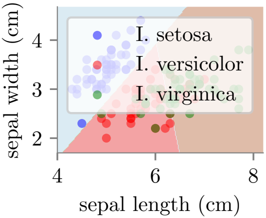
1. Learn P(setosa (class 1)) $= \mathcal{F}(\boldsymbol{X}\boldsymbol{\theta}_1)$

Multi-Class Prediction



1. Learn P(setosa (class 1)) $= \mathcal{F}(\mathbf{X}\boldsymbol{\theta}_1)$
2. P(versicolor (class 2)) $= \mathcal{F}(\mathbf{X}\boldsymbol{\theta}_2)$

Multi-Class Prediction



1. Learn P(setosa (class 1)) $= \mathcal{F}(\mathbf{X}\boldsymbol{\theta}_1)$
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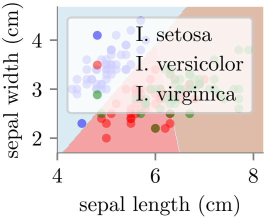
3. P(virginica (class 3)) $= \mathcal{F}(\mathbf{X}\boldsymbol{\theta}_3)$

Multi-Class Prediction



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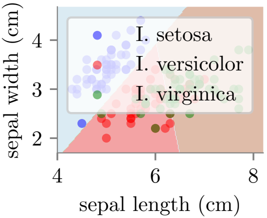
4. Goal: Learn $\theta_i \forall i \in \{1, 2, 3\}$

Multi-Class Prediction



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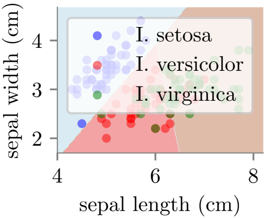
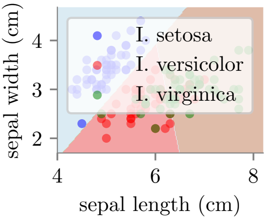
4. Goal: Learn $\theta_i \forall i \in \{1, 2, 3\}$
5. Question: What could be an $\mathcal{F}$?

Multi-Class Prediction

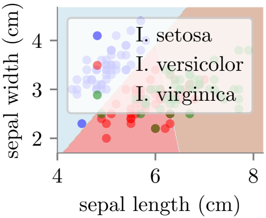


Multi-Class Prediction



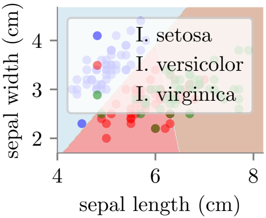
1. Question: What could be an $\mathcal{F}$?

Multi-Class Prediction



1. Question: What could be an $\mathcal{F}$?
2. Property: $\sum_{i=1}^{3} \mathcal{F}(\mathbf{X}\boldsymbol{\theta}_i) = 1$

Multi-Class Prediction



1. Question: What could be an $\mathcal{F}$?
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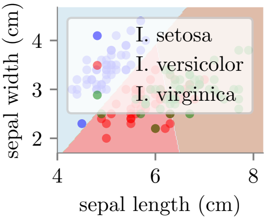
3. Also $\mathcal{F}(z) \in [0, 1]$

Multi-Class Prediction



1. Question: What could be an $\mathcal{F}$?
2. Property: $\sum_{i=1}^{3} \mathcal{F}(\mathbf{X}\boldsymbol{\theta}_i) = 1$
3. Also $\mathcal{F}(z) \in [0, 1]$
4. Also, $\mathcal{F}(z)$ has squashing proprties: $R \mapsto [0, 1]$

Softmax

$$Z \in \mathbb{R}^d$$

$$\mathcal{F}(z_i) = \frac{e^{z_i}}{\sum_{i=1}^{d} e^{z_i}}$$

$$\therefore \sum \mathcal{F}(z_i) = 1$$

$\mathcal{F}(z_i)$ refers to <u>probability</u> of class <u>i</u>

Softmax for Multi-Class Logistic Regression

$$k = \{1, \ldots, k\}\text{classes}$$

$$\theta = \begin{bmatrix} \vdots\vdots\vdots\vdots \\ \theta_1 \theta_2 \cdots \theta_k \\ \vdots\vdots\vdots\vdots \end{bmatrix}$$

$$\boxed{P(y = k | X, \theta) = \frac{e^{\mathbf{X}\boldsymbol{\theta}_k}}{\sum_{k=1}^{K} e^{\mathbf{X}\boldsymbol{\theta}_k}}}$$

Softmax for Multi-Class Logistic Regression

For K $=2$ classes,

$$P(y=k|X,\theta) = \frac{e^{\mathbf{X}\boldsymbol{\theta}_k}}{\sum_{k=1}^{K} e^{\mathbf{X}\boldsymbol{\theta}_k}}$$

$$P(y=0|X,\theta) = \frac{e^{\mathbf{X}\boldsymbol{\theta}_0}}{e^{\mathbf{X}\boldsymbol{\theta}_0} + e^{\mathbf{X}\boldsymbol{\theta}_1}}$$

$$P(y=1|X,\theta) = \frac{e^{\mathbf{X}\boldsymbol{\theta}_1}}{e^{\mathbf{X}\boldsymbol{\theta}_0} + e^{\mathbf{X}\boldsymbol{\theta}_1}} = \frac{e^{\mathbf{X}\boldsymbol{\theta}_1}}{e^{\mathbf{X}\boldsymbol{\theta}_1}\{1 + e^{X(\theta_0 - \theta_1)}\}}$$

$$= \frac{1}{1 + e^{-\mathbf{X}\boldsymbol{\theta}'}}$$

$$= \text{Sigmoid!}$$

Multi-Class Logistic Regression Cost

Assume our prediction and ground truth for the three classes for i^{th} point is:

$$\hat{y}_i = \begin{bmatrix} 0.1 \\ 0.8 \\ 0.1 \end{bmatrix} = \begin{bmatrix} \hat{y}_i^1 \\ \hat{y}_i^2 \\ \hat{y}_i^3 \end{bmatrix}$$

$$y_i = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} y_i^1 \\ y_i^2 \\ y_i^3 \end{bmatrix}$$

meaning the true class is Class $\#2$

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Let us calculate $-\sum_{k=1}^{3} y_i^k \log \hat{y}_i^k$

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$= -(0 \times \log(0.1) + 1 \times \log(0.8) + 0 \times \log(0.1))$

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$= -(0 \times \log(0.1) + 1 \times \log(0.8) + 0 \times \log(0.1))$

Tends to zero

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Let us calculate $-\sum_{k=1}^{3} y_i^k \log \hat{y}_i^k$

$= -(0 \times \log(0.1) + 1 \times \log(0.4) + 0 \times \log(0.1))$

High number! Huge penalty for misclassification!

Multi-Class Logistic Regression Cost

For 2 class we had:

$$J(\theta) = -\left\{ \sum_{i=1}^{N} y_i \log(\sigma_\theta(x_i)) + (1 - y_i) \log(1 - \sigma_\theta(x_i)) \right\}$$

Multi-Class Logistic Regression Cost

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More generally,

Multi-Class Logistic Regression Cost

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More generally,

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Extend to K-class:

$$J(\theta) = -\left\{ \sum_{i=1}^{N} \sum_{k=1}^{K} y_i^k \log(\hat{y}_i^k) \right\}$$

Multi-Class Logistic Regression Cost

Now:

$$\frac{\partial J(\theta)}{\partial \theta_k} = \sum_{i=1}^{N} \left[x_i \left\{ I(y_i = k) - P(y_i = k | x_i, \theta) \right\} \right]$$

Hessian Matrix

The Hessian matrix of f(.) with respect to θ, written $\nabla^2_\theta f(\theta)$ or simply as $\mathbb{H}$, is the $d \times d$ matrix of partial derivatives,

$$\nabla^2_\theta f(\theta) = \begin{bmatrix} \frac{\partial^2 f(\theta)}{\partial \theta_1^2} & \frac{\partial^2 f(\theta)}{\partial \theta_1 \partial \theta_2} & \cdots & \frac{\partial^2 f(\theta)}{\partial \theta_1 \partial \theta_n} \\ \frac{\partial^2 f(\theta)}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 f(\theta)}{\partial \theta_2^2} & \cdots & \frac{\partial^2 f(\theta)}{\partial \theta_2 \partial \theta_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(\theta)}{\partial \theta_n \partial \theta_1} & \frac{\partial^2 f(\theta)}{\partial \theta_n \partial \theta_2} & \cdots & \frac{\partial^2 f(\theta)}{\partial \theta_n^2} \end{bmatrix}$$

Newton's Algorithm

The most basic second-order optimization algorithm is Newton's algorithm, which consists of updates of the form,

$$\theta_{k+1} = \theta_k - \mathbb{H}_k^1 g_k$$

where g_k is the gradient at step k. This algorithm is derived by making a second-order Taylor series approximation of $f(\theta)$ around θ_k:

$$f_{quad}(\theta) = f(\theta_k) + g_k^T(\theta - \theta_k) + \frac{1}{2}(\theta - \theta_k)^T \mathbb{H}_k(\theta - \theta_k)$$

differentiating and equating to zero to solve for θ_{k+1}.

Learning Parameters

Now assume:

$$g(\theta) = \sum_{i=1}^{N} \Big[\sigma_\theta(x_i) - y_i\Big] x_i^j = \mathbf{X}^\top(\sigma_\theta(X) - y)$$

$$\pi_i = \sigma_\theta(x_i)$$

Let $\mathbb{H}$ represent the Hessian of $J(\theta)$

$$\mathbb{H} = \frac{\partial}{\partial\theta} g(\theta) = \frac{\partial}{\partial\theta} \sum_{i=1}^{N} \Big[\sigma_\theta(x_i) - y_i\Big] x_i^j$$

$$= \sum_{i=1}^{N} \left[\frac{\partial}{\partial\theta}\sigma_\theta(x_i) x_i^j - \frac{\partial}{\partial\theta} y_i x_i^j\right] = \sum_{i=1}^{N} \sigma_\theta(x_i)(1 - \sigma_\theta(x_i)) x_i x_i^T$$

$$= \mathbf{X}^\top \mathrm{diag}(\sigma_\theta(x_i)(1 - \sigma_\theta(x_i)))\mathbf{X}$$

Iterately reweighted least squares (IRLS)

For binary logistic regression, recall that the gradient and Hessian of the negative log-likelihood are given by:

$$g(\theta)_k = \mathbf{X}^\top(\pi_k - y)$$

$$\mathbf{H}_k = \mathbf{X}^\top S_k \mathbf{X}$$

$$\mathbf{S}_k = diag(\pi_{1k}(1-\pi_{1k}), \ldots, \pi_{nk}(1-\pi_{nk}))$$

$$\pi_{ik} = sigm(\mathbf{x_i}\theta_\mathbf{k})$$

The Newton update at iteraion $k+1$ for this model is as follows:

$$\theta_{k+1} = \theta_k - \mathbb{H}^{-1}g_k = \theta_k + (X^T S_k X)^{-1} X^T (y - \pi_k)$$

$$= (X^T S_k X)^{-1}[(X^T S_k X)\theta_k + X^T(y-\pi_k)] = (X^T S_k X)^{-1} X^T [S_k \mathbf{X}\boldsymbol{\theta}_k + y - \pi$$

Regularized Logistic Regression

Unregularised:

$$J_1(\theta) = -\left\{ \sum_{i=1}^{N} y_i \log(\sigma_\theta(x_i)) + (1 - y_i) \log(1 - \sigma_\theta(x_i)) \right\}$$

L2 Regularization:

$$J(\theta) = J_1(\theta) + \lambda \theta^T \theta$$

L1 Regularization:

$$J(\theta) = J_1(\theta) + \lambda |\theta|$$

Class Imbalance Handling

The Problem: Imbalanced Data

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- **Naive approach fails**: Predicting all samples as majority class

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With 99% class 0, 1% class 1:

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- **Need**: Better evaluation metrics and techniques

Solution 1: Weighted Loss Function

Modify the cost function to penalize minority class errors more:

$$J(\boldsymbol{\theta}) = -\sum_{i=1}^{N} w_i \left[y_i \log(\sigma(\boldsymbol{\theta}^\top \mathbf{x}_i)) + (1 - y_i) \log(1 - \sigma(\boldsymbol{\theta}^\top \mathbf{x}_i)) \right]$$

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- **Implementation**: Available in most ML libraries (sklearn: `class_weight='balanced'`)

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- **Trade-off**: Lower threshold $\rightarrow$ higher recall, lower precision

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  - More sophisticated than simple duplication

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Practice and Review

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- **Regularization**: L1/L2 help prevent overfitting