

Constrained Optimization II

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Lagrangian and Duality

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Lectures heavily inspired by the Maths for Machine learning book

Minimax Inequality

- Minimax inequality

$$\max_{\mathbf{y}} \min_{\mathbf{x}} q(\mathbf{x}, \mathbf{y}) \leq \min_{\mathbf{x}} \max_{\mathbf{y}} q(\mathbf{x}, \mathbf{y})$$

Minimax Inequality

- Minimax inequality
states: $\max_{\mathbf{y}} \min_{\mathbf{x}} q(\mathbf{x}, \mathbf{y}) \leq \min_{\mathbf{x}} \max_{\mathbf{y}} q(\mathbf{x}, \mathbf{y})$
- We first prove For all \mathbf{x}, \mathbf{y} $\min_{\mathbf{x}} q(\mathbf{x}, \mathbf{y}) \leq \max_{\mathbf{y}} q(\mathbf{x}, \mathbf{y})$

Minimax Inequality

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x	1	1	2	3	4
\Downarrow	2	2	4	6	8
	3	3	6	9	12
	4	4	8	12	16
		1	2	3	4
		$y \Rightarrow$			

Minimax Inequality

- For each value of x , we find y that maximizes $q(x, y)$

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Minimax Inequality

- For each value of x , we find y that maximizes $q(x, y)$
- $y = 4$ maximizes $q(x, y) \forall x$

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Minimax Inequality

- For each value of y , we find x that minimizes $q(x, y)$
- $x = 1$ minimizes $q(x, y) \forall y$

x \Downarrow	1	1	2	3	4
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	3	3	6	9	12
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Minimax Inequality

- We just showed For all \mathbf{x}, \mathbf{y} $\min_{\mathbf{x}} q(\mathbf{x}, \mathbf{y}) \leq \max_{\mathbf{y}} q(\mathbf{x}, \mathbf{y})$

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Minimax Inequality

- We just showed For all \mathbf{x}, \mathbf{y} $\min_{\mathbf{x}} q(\mathbf{x}, \mathbf{y}) \leq \max_{\mathbf{y}} q(\mathbf{x}, \mathbf{y})$
- The equality occurs at $x = 1, y = 4$

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Minimax Inequality

- Let us now find $\max_y \min_x q(x, y)$

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- Similarly, let us now find $\min_x \max_y q(x, y)$

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Minimax Inequality

- Similarly, let us now find $\min_x \max_y q(\mathbf{x}, \mathbf{y})$
- We can thus see our Minimax inequality
 $\max_y \min_x q(\mathbf{x}, \mathbf{y}) \leq \min_x \max_y q(\mathbf{x}, \mathbf{y})$

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Revisiting the Lagrange multipliers

Our problem is of the form

$$\begin{array}{ll} \min_{\mathbf{x}} & f(\mathbf{x}) \\ \text{subject to} & g_i(\mathbf{x}) \leq 0 \quad \text{for all } i = 1, \dots, m \end{array}$$

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Idea: Convert constrained problem to an unconstrained problem

$$J(\mathbf{x}) = f(\mathbf{x}) + \sum_{i=1}^m \mathbf{1}(g_i(\mathbf{x}))$$

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where $\mathbf{1}(z)$ is an infinite step function

$$\mathbf{1}(z) = \begin{cases} 0 & \text{if } z \leq 0 \\ \infty & \text{otherwise} \end{cases}$$

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This would give infinite penalty if constraint is not satisfied. But, this formulation is hard to solve too.

Revisiting the Lagrange multipliers

Idea: Introduce Lagrange multipliers ($\lambda_i \geq 0$) to “approximate” $J(\mathbf{x})$

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \sum_{i=1}^m \lambda_i g_i(\mathbf{x})$$

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What is the relationship between $\mathfrak{L}(\mathbf{x}, \boldsymbol{\lambda})$ and $J(\mathbf{x})$ given $\lambda_i \geq 0$?

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What is the relationship between $\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda})$ and $J(\mathbf{x})$ given $\lambda_i \geq 0$?

When $\lambda \geq 0$, the Lagrangian $\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda})$ is a lower bound of $J(\mathbf{x})$.

Hence, the maximum of $\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda})$ with respect to $\boldsymbol{\lambda}$ is

$$J(\mathbf{x}) = \max_{\boldsymbol{\lambda} \geq 0} \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda})$$

Revisiting the Lagrange multipliers

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$$\min_{\mathbf{x} \in \mathbb{R}^d} \max_{\lambda \geq 0} \mathcal{L}(\mathbf{x}, \lambda) \geq \max_{\lambda \geq 0} \min_{\mathbf{x} \in \mathbb{R}^d} \mathcal{L}(\mathbf{x}, \lambda)$$

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We can write the dual objective as a function of λ as

$$\mathcal{D}(\lambda) = \min_{\mathbf{x} \in \mathbb{R}^d} \mathcal{L}(\mathbf{x}, \lambda)$$

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- Primal objective:

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- Or, primal objective (in terms of \mathbf{x}) \geq dual objective (in terms of λ)
- For SVM like formulations, primal objective is the same as dual objective (strong duality)
- For some problems, there is a “duality-gap” between the two objectives