

# KKT Conditions

Nipun Batra

IIT Gandhinagar

July 29, 2025

# KKT Conditions

Used for constrained optimization of the form

Minimize  $f(x)$ , where  $x \in \mathbb{R}^k$   
such that

$$\begin{aligned} h_i(x) &= 0, \forall i = 1, \dots, m \text{ (m equalities)} \\ g_j(x) &\leq 0, \forall j = 1, \dots, n \text{ (n inequalities)} \end{aligned}$$

## Step 2

- ▶ Minimize  $L(x, \lambda, \mu)$  w.r.t.  $x \implies \nabla_x L(x, \lambda, \mu) = 0$   
Gives  $k$  equations

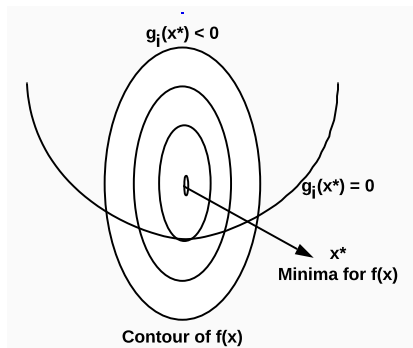
## Step 3

- ▶ Minimize  $L(x, \lambda, \mu)$  w.r.t.  $\lambda \implies \nabla_{\lambda} L(x, \lambda, \mu) = 0$   
Gives  $m$  equations

## Step 4

In both cases,  $\mu_i g_i(x^*) = 0$

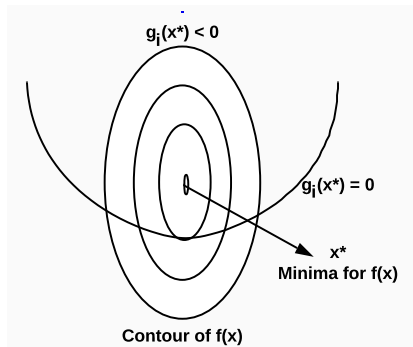
## Step 4



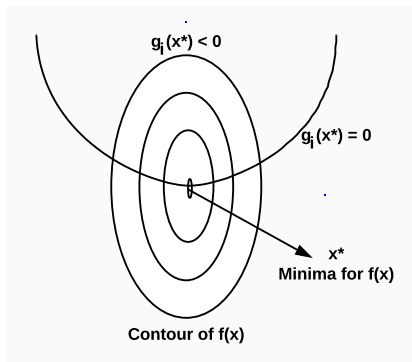
$$g_i(x^*) \leq 0$$
$$\mu_i = 0$$

In both cases,  $\mu_i g_i(x^*) = 0$

## Step 4



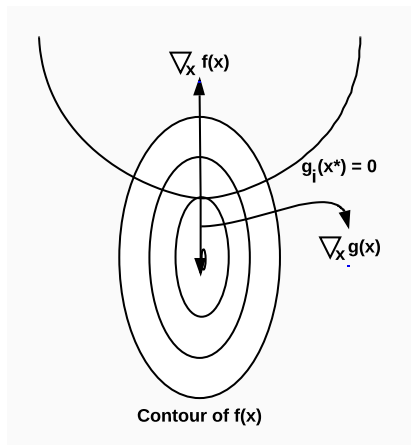
$$g_i(x^*) \leq 0$$
$$\mu_i = 0$$



$$g_i(x^*) = 0$$

In both cases,  $\mu_i g_i(x^*) = 0$

## Constraint on $\mu_i$ 's



$$\min_x L(x, \lambda, \mu) \implies \nabla_x f(x) + \nabla_x \mu_i g_i(x) = 0$$

$$\mu_i = \frac{\nabla_x f(x)}{\nabla_x \mu_i g_i(x)} = +ve$$



# KKT Conditions

## Stationarity (For minimization)

$$\nabla_x f(x) + \sum_{i=1}^m \nabla_x \lambda_i h_i(x) + \sum_{i=1}^n \nabla_x \mu_i g_i(x) = 0$$

# KKT Conditions

## Stationarity (For minimization)

$$\nabla_x f(x) + \sum_{i=1}^m \nabla_x \lambda_i h_i(x) + \sum_{i=1}^n \nabla_x \mu_i g_i(x) = 0$$

## Equality Constraints

$$\nabla_\lambda f(x) + \sum_{i=1}^m \nabla_\lambda \lambda_i h_i(x) + \sum_{i=1}^n \nabla_\lambda \mu_i g_i(x) = 0$$

$$\sum_{i=1}^m \nabla_\lambda \lambda_i h_i(x) = 0$$

# KKT Conditions

## Stationarity (For minimization)

$$\nabla_x f(x) + \sum_{i=1}^m \nabla_x \lambda_i h_i(x) + \sum_{i=1}^n \nabla_x \mu_i g_i(x) = 0$$

## Equality Constraints

$$\nabla_\lambda f(x) + \sum_{i=1}^m \nabla_\lambda \lambda_i h_i(x) + \sum_{i=1}^n \nabla_\lambda \mu_i g_i(x) = 0$$

$$\sum_{i=1}^m \nabla_\lambda \lambda_i h_i(x) = 0$$

## Inequality Constraints (Complementary Slackness)

$$\mu_i g_i(x) = 0 \forall i = 1, \dots, n$$

$$\mu_i \geq 0$$

## Example

Minimize  $x^2 + y^2$  such that,

$$x^2 + y^2 \leq 5$$

$$x + 2y = 4$$

$$x, y \geq 0$$

## Example

$$f(x, y) = x^2 + y^2$$

## Example

$$f(x, y) = x^2 + y^2$$

$$h(x, y) = x + 2y - 4$$

## Example

$$f(x, y) = x^2 + y^2$$

$$h(x, y) = x + 2y - 4$$

$$g_1(x, y) = x^2 + y^2 - 5$$

## Example

$$f(x, y) = x^2 + y^2$$

$$h(x, y) = x + 2y - 4$$

$$g_1(x, y) = x^2 + y^2 - 5$$

$$g_2(x, y) = -x$$



## Example

$$f(x, y) = x^2 + y^2$$

$$h(x, y) = x + 2y - 4$$

$$g_1(x, y) = x^2 + y^2 - 5$$

$$g_2(x, y) = -x$$

$$g_3(x, y) = -y$$

## Example

$$f(x, y) = x^2 + y^2$$

$$h(x, y) = x + 2y - 4$$

$$g_1(x, y) = x^2 + y^2 - 5$$

$$g_2(x, y) = -x$$

$$g_3(x, y) = -y$$

$$L(x, y, \lambda, \mu_1, \mu_2, \mu_3) = \\ x^2 + y^2 + \lambda(x + 2y - 4) + \mu_1(x^2 + y^2 - 5) + \mu_2(-x) + \mu_3(-y)$$

# Example

## Stationarity

$$\nabla_x L(x, y, \lambda, \mu_1, \mu_2, \mu_3) = 0$$

$$\implies 2x + \lambda + 2\mu_1 x - \mu_2 = 0 \dots\dots\dots (1)$$

$$\nabla_y L(x, y, \lambda, \mu_1, \mu_2, \mu_3) = 0$$

$$\implies 2y + 2\lambda + 2\mu_1 y - \mu_3 = 0 \dots\dots\dots (2)$$

# Example

## Stationarity

$$\begin{aligned}\nabla_x L(x, y, \lambda, \mu_1, \mu_2, \mu_3) &= 0 \\ \implies 2x + \lambda + 2\mu_1 x - \mu_2 &= 0 \dots\dots\dots (1)\end{aligned}$$

$$\begin{aligned}\nabla_y L(x, y, \lambda, \mu_1, \mu_2, \mu_3) &= 0 \\ \implies 2y + 2\lambda + 2\mu_1 y - \mu_3 &= 0 \dots\dots\dots (2)\end{aligned}$$

## Equality Constraint

$$x + 2y = 4 \dots\dots\dots (3)$$

# Example

## Stationarity

$$\begin{aligned}\nabla_x L(x, y, \lambda, \mu_1, \mu_2, \mu_3) &= 0 \\ \implies 2x + \lambda + 2\mu_1 x - \mu_2 &= 0 \dots\dots\dots (1)\end{aligned}$$

$$\begin{aligned}\nabla_y L(x, y, \lambda, \mu_1, \mu_2, \mu_3) &= 0 \\ \implies 2y + 2\lambda + 2\mu_1 y - \mu_3 &= 0 \dots\dots\dots (2)\end{aligned}$$

## Equality Constraint

$$x + 2y = 4 \dots\dots\dots (3)$$

## Slackness

$$\mu_1(x^2 + y^2 - 5) = 0 \dots\dots\dots (4)$$

$$\mu_2 x = 0 \dots\dots\dots (5)$$

$$\mu_3 y = 0 \dots\dots\dots (6)$$

## Example

From (6),  $\mu_3 = 0$  or  $y = 0$

But if,  $y = 0$ , then  $x = 4$  according to (3) . This violates (1).

Hence,  $y \neq 0$  and  $\mu_3 = 0$

## Example

From (6),  $\mu_3 = 0$  or  $y = 0$

But if,  $y = 0$ , then  $x = 4$  according to (3) . This violates (1).

Hence,  $y \neq 0$  and  $\mu_3 = 0$

From (5),  $\mu_1 = 0$  or  $x = 0$

If  $x = 0$ ,  $y = 2$ , which implies  $x^2 + y^2 = 4 (\leq 5)$

Since  $(x,y) = (0,2)$  gives smaller  $x^2 + y^2$  terms than 5,

Using (4),  $\mu_1 = 0$

## Example

From (6),  $\mu_3 = 0$  or  $y = 0$

But if,  $y = 0$ , then  $x = 4$  according to (3) . This violates (1).

Hence,  $y \neq 0$  and  $\mu_3 = 0$

From (5),  $\mu_1 = 0$  or  $x = 0$

If  $x = 0$ ,  $y = 2$ , which implies  $x^2 + y^2 = 4 (\leq 5)$

Since  $(x,y) = (0,2)$  gives smaller  $x^2 + y^2$  terms than 5,

Using (4),  $\mu_1 = 0$

On further solving we get,

$$x = 0.8$$

$$y = 1.6$$