Matrix Factorization for Movie Recommendation Systems

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Outline

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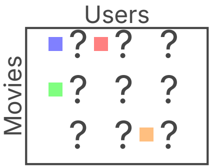

Sparse Rating Matrix

!

The Magic Formula:

$$\text{Alice's rating} = \text{Alice's preferences} \cdot \text{Sholay's features}$$

b Sholay's features: $\mathbf{h}_1 = [0.95, 0.10, 0.85]^T$

$$\hat{a}_{11} = \mathbf{w}_1^T \mathbf{h}_1 \tag{1}$$
$$= w_{11} \cdot 0.95 + w_{12} \cdot 0.10 + w_{13} \cdot 0.85 \tag{2}$$

Goal: Find w_{11}, w_{12}, w_{13} such that $\hat{a}_{11} \approx 5$ (Alice's actual rating)
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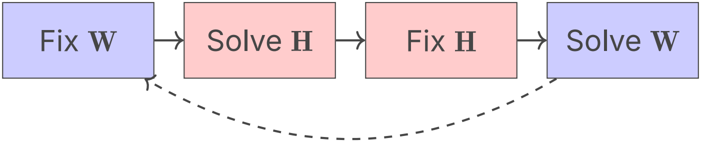
1. $\mathbf{A} \in \mathbb{R}^{N \times M}$
2. $\mathbf{W} \in \mathbb{R}^{N \times r}$
3. $\mathbf{H} \in \mathbb{R}^{r \times M}$
4. Total parameters: $Nr + rM = r(N + M)$

bInitialize: $\mathbf{W}^{(0)}$ and $\mathbf{H}^{(0)}$ randomly



$$\text{minimize}_{\mathbf{w}_i} \sum_{j:(i,j)\in\Omega} (a_{ij} - \mathbf{w}_i^T \mathbf{h}_j)^2$$

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Matrix Form for User *i*: Let $\Omega_i = \{j : (i,j) \in \Omega\}$ (movies rated by user *i*)

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$$\mathbf{y}_i = [a_{i,j_1}, a_{i,j_2}, \ldots, a_{i,j_{|\Omega_i|}}]^T \tag{3}$$

$$\mathbf{X}_i = [\mathbf{h}_{j_1}, \mathbf{h}_{j_2}, \ldots, \mathbf{h}_{j_{|\Omega_i|}}]^T \tag{4}$$

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Least Squares Solution:

$$\boxed{\mathbf{w}_i^* = (\mathbf{X}_i^T \mathbf{X}_i)^{-1} \mathbf{X}_i^T \mathbf{y}_i}$$

$$\mathbf{y}_1 = \begin{bmatrix} 5 \\ 4 \\ 2 \\ 3 \\ 2 \end{bmatrix} \tag{5}$$

$$\mathbf{X}_1 = \begin{bmatrix} 0.95 & 0.10 & 0.85 \\ 1.00 & 0.20 & 0.90 \\ 0.05 & 0.80 & 0.30 \\ 0.05 & 0.95 & 0.70 \\ 0.05 & 0.15 & 0.95 \end{bmatrix} \tag{6}$$

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Solution: $\mathbf{w}_1^* = (\mathbf{X}_1^T \mathbf{X}_1)^{-1} \mathbf{X}_1^T \mathbf{y}_1$
This gives us Alice's feature preferences!

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$$\mathbf{y}_j = [a_{i_1,j}, a_{i_2,j}, \ldots, a_{i_{|\Omega_j|},j}]^T \quad (7)$$

$$\mathbf{X}_j = [\mathbf{w}_{i_1}, \mathbf{w}_{i_2}, \ldots, \mathbf{w}_{i_{|\Omega_j|}}]^T \quad (8)$$

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Objective Function:

$$L(\mathbf{W}, \mathbf{H}) = \sum_{(i,j) \in \Omega} (a_{ij} - \mathbf{w}_i^T \mathbf{h}_j)^2$$

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Gradients:

$$\frac{\partial L}{\partial \mathbf{w}_i} = -2 \sum_{j:(i,j)\in\Omega} (a_{ij} - \mathbf{w}_i^T \mathbf{h}_j)\mathbf{h}_j \tag{9}$$

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SGD does exactly this!

- One rating at a time
- Small adjustments
- Gradually improves

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3. **Update:**

$$\mathbf{w}_i \leftarrow \mathbf{w}_i + \alpha \cdot e_{ij} \cdot \mathbf{h}_j \tag{11}$$

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- If $e_{ij} > 0$: Predicted rating too low → Increase similarity

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Intuition:

- If $e_{ij} > 0$: Predicted rating too low → Increase similarity
- If $e_{ij} < 0$: Predicted rating too high → Decrease similarity

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Intuition:

- If $e_{ij} > 0$: Predicted rating too low → Increase similarity
- If $e_{ij} < 0$: Predicted rating too high → Decrease similarity
- Learning rate α controls step size

$$\text{Current: } \mathbf{w}_1 = [0.4, 0.2, 0.3], \quad \mathbf{h}_1 = [0.95, 0.10, 0.85] \tag{13}$$

$$\text{Prediction: } \hat{a}_{11} = 0.4 \times 0.95 + 0.2 \times 0.10 + 0.3 \times 0.85 = 0.655 \tag{14}$$

$$\text{Error: } e_{11} = 5 - 0.655 = 4.345 \tag{15}$$

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Updates with $\alpha = 0.01$:

$$\mathbf{w}_1 \leftarrow [0.4, 0.2, 0.3] + 0.01 \times 4.345 \times [0.95, 0.10, 0.85] \tag{16}$$

$$= [0.4413, 0.2043, 0.3369] \tag{17}$$

$$\mathbf{h}_1 \leftarrow [0.95, 0.10, 0.85] + 0.01 \times 4.345 \times [0.4, 0.2, 0.3] \tag{18}$$

$$= [0.9674, 0.1087, 0.8631] \tag{19}$$

!

Answers:

b

$X_1 = \begin{bmatrix} 1.0 & 0.3 \\ 0.2 & 0.8 \end{bmatrix}$ (columns 1,3 of $\mathbf{H}^{(0)T}$)

Solve: $\mathbf{w}_1^{(1)} = (\mathbf{X}_1^T\mathbf{X}_1)^{-1}\mathbf{X}_1^T\mathbf{y}_1$

Continue for all users and movies...

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- Distributed computing, approximate algorithms, caching batrix factorization can only work with explicit ratings

Answers:

stability requirements