Constrained Optimization II

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July 21, 2025

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Lagrangian and Duality

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June 28, 2020

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Lectures heavily inspired by the Maths for Machine learning book

• Minimax inequality states: $\max_{\mathbf{y}} \min_{\mathbf{x}} q(\mathbf{x}, \mathbf{y}) \leqslant \min_{\mathbf{x}} \max_{\mathbf{y}} q(\mathbf{x}, \mathbf{y})$

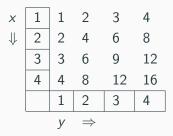
- Minimax inequality
 states:max_y min_x q(x, y) ≤ min_x max_y q(x, y)
- We first prove For all $x, y = \min_{x} q(x, y) \leqslant \max_{y} q(x, y)$

• Let us choose q(x, y) = xy

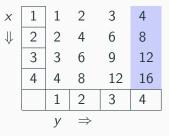
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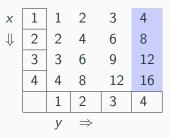


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- y = 4 maximizes $q(x, y) \forall x$



• For each value of y, we find x that minimizes q(x, y)

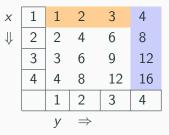
X	1	1	2	3	4
\Downarrow	2	2	4	6	8
	3	3	6	9	12
	4	4	8	12	16
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		У	\Rightarrow		

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- For each value of y, we find x that minimizes q(x, y)
- x = 1 minimizes $q(x, y) \forall y$

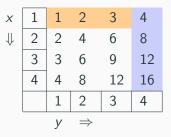
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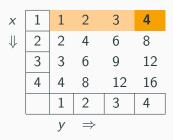


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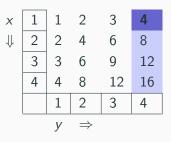
- We just showed For all $x, y = \min_{x} q(x, y) \leqslant \max_{y} q(x, y)$
- The equality occurs at x = 1, y = 4



• Let us now find $\max_{y} \min_{x} q(x, y)$

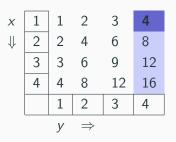


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- Similarly, let us now find $\min_{x} \max_{y} q(x, y)$
- We can thus see our Minimax inequality $\max_{\mathbf{y}} \min_{\mathbf{x}} q(\mathbf{x}, \mathbf{y}) \leqslant \min_{\mathbf{x}} \max_{\mathbf{y}} q(\mathbf{x}, \mathbf{y})$



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This would give infinte penalty if constraint is not satisfied. But, this formulation is hard to solve too.

Idea: Introduce Lagrange multipliers ($\lambda_i \geq 0$) to "approximate" J(x)

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What is the relationship between $\mathfrak{L}(\mathbf{x}, \lambda)$ and $J(\mathbf{x})$ given $\lambda_i \geq 0$?

When $\lambda \geqslant 0$, the Lagrangian $\mathcal{L}(x,\lambda)$ is a lower bound of J(x). Hence, the maximum of $\mathfrak{L}(x,\lambda)$ with respect to λ is

$$J(\mathbf{x}) = \max_{\mathbf{\lambda} \geqslant 0} \mathfrak{L}(\mathbf{x}, \mathbf{\lambda})$$

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But, our original problem was minimizing J(x), which is equivalent to:

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We can write the dual objective as a function of λ as

$$\mathfrak{D}(\boldsymbol{\lambda}) = \min_{\boldsymbol{x} \in \mathbb{R}^d} \mathfrak{L}(\boldsymbol{x}, \boldsymbol{\lambda})$$

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- For SVM like formulations, primal objective is the same as dual objective (strong duality)
- For some problems, there is a "daulity-gap" between the two objectives