Constrained Optimization II

Nipun Batra

IIT Gandhinagar

August 2, 2025

Lagrangian and Duality

Nipun Batra

June 28, 2020

IIT Gandhinagar

Lectures heavily inspired by the Maths for Machine learning book

Minimax inequality
 states:max_y min_x q(x, y) ≤ min_x max_y q(x, y)

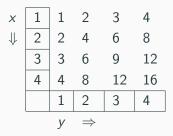
- Minimax inequality
 states: max_y min_x q(x, y) ≤ min_x max_y q(x, y)
- We first prove For all $x, y = \min_{x} q(x, y) \leqslant \max_{y} q(x, y)$

• Let us choose q(x, y) = xy

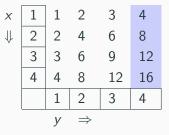
- Let us choose q(x, y) = xy
- ullet Let us first find $\max_{oldsymbol{y}} q(oldsymbol{x}, oldsymbol{y})$

- Let us choose q(x, y) = xy
- ullet Let us first find $\max_{oldsymbol{y}} q(oldsymbol{x}, oldsymbol{y})$

- Let us choose q(x, y) = xy
- Let us first find $\max_{y} q(x, y)$

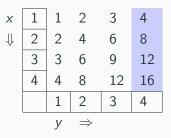


• For each value of x, we find y that maximizes q(x,y)



3

- For each value of x, we find y that maximizes q(x, y)
- y = 4 maximizes $q(x, y) \forall x$



• For each value of y, we find x that minimizes q(x, y)

X	1	1	2	3	4
\Downarrow	2	2	4	6	8
	3	3	6	9	12
	4	4	8	12	16
		1	2	3	4
		У	\Rightarrow		

Л

- For each value of y, we find x that minimizes q(x, y)
- x = 1 minimizes $q(x, y) \forall y$

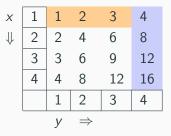
X	1	1	2	3	4
\Downarrow	2	2	4	6	8
	3	3	6	9	12
	4	4	8	12	16
		1	2	3	4
		У	\Rightarrow		

• We just showed For all $x, y = \min_{x} q(x, y) \leqslant \max_{y} q(x, y)$

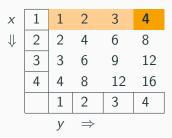


5

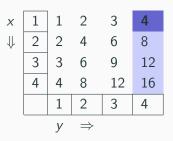
- We just showed For all $x, y = \min_{x} q(x, y) \leqslant \max_{y} q(x, y)$
- The equality occurs at x = 1, y = 4



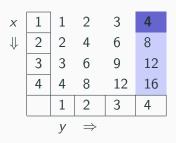
• Let us now find $\max_{y} \min_{x} q(x, y)$



• Similarly, let us now find $\min_{x} \max_{y} q(x, y)$



- Similarly, let us now find $\min_{x} \max_{y} q(x, y)$
- We can thus see our Minimax inequality $\max_{\mathbf{y}} \min_{\mathbf{x}} q(\mathbf{x}, \mathbf{y}) \leqslant \min_{\mathbf{x}} \max_{\mathbf{y}} q(\mathbf{x}, \mathbf{y})$



Our problem is of the form

$$\min_{m{x}} f(m{x})$$
 subject to $g_i(m{x}) \leqslant 0$ for all $i=1,\ldots,m$

Our problem is of the form

$$\min_{\mathbf{x}} f(\mathbf{x})$$

subject to $g_i(\mathbf{x}) \leqslant 0$ for all $i = 1, \dots, m$

Idea: Convert constrained problem to an unconstrained problem

$$J(\mathbf{x}) = f(\mathbf{x}) + \sum_{i=1}^{m} \mathbf{1} \left(g_i(\mathbf{x}) \right)$$

Our problem is of the form

$$\min_{m{x}} f(m{x})$$
 subject to $g_i(m{x}) \leqslant 0$ for all $i=1,\ldots,m$

Idea: Convert constrained problem to an unconstrained problem

$$J(\mathbf{x}) = f(\mathbf{x}) + \sum_{i=1}^{m} \mathbf{1} \left(g_i(\mathbf{x}) \right)$$

where 1(z) is an infinite step function

$$\mathbf{1}(z) = \begin{cases} 0 & \text{if } z \leqslant 0 \\ \infty & \text{otherwise} \end{cases}$$

Our problem is of the form

$$\min_{\boldsymbol{x}} f(\boldsymbol{x})$$
 subject to $g_i(\boldsymbol{x}) \leqslant 0$ for all $i=1,\ldots,m$

Idea: Convert constrained problem to an unconstrained problem

$$J(\mathbf{x}) = f(\mathbf{x}) + \sum_{i=1}^{m} \mathbf{1}(g_i(\mathbf{x}))$$

where 1(z) is an infinite step function

$$\mathbf{1}(z) = \begin{cases} 0 & \text{if } z \leqslant 0 \\ \infty & \text{otherwise} \end{cases}$$

This would give infinte penalty if constraint is not satisfied. But, this formulation is hard to solve too.

Idea: Introduce Lagrange multipliers $(\lambda_i \ge 0)$ to "approximate" J(x)

$$\mathfrak{L}(\mathbf{x}, \lambda) = f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x})$$

Idea: Introduce Lagrange multipliers $(\lambda_i \ge 0)$ to "approximate" J(x)

$$\mathfrak{L}(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x})$$

What is the relationship between $\mathfrak{L}(\mathbf{x}, \lambda)$ and $J(\mathbf{x})$ given $\lambda_i \geq 0$?

9

Idea: Introduce Lagrange multipliers $(\lambda_i \ge 0)$ to "approximate" J(x)

$$\mathfrak{L}(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x})$$

What is the relationship between $\mathfrak{L}(\mathbf{x}, \lambda)$ and $J(\mathbf{x})$ given $\lambda_i \geq 0$?

When $\lambda \geqslant 0$, the Lagrangian $\mathcal{L}(x,\lambda)$ is a lower bound of J(x). Hence, the maximum of $\mathfrak{L}(x,\lambda)$ with respect to λ is

$$J(\mathbf{x}) = \max_{\mathbf{\lambda} \geqslant 0} \mathfrak{L}(\mathbf{x}, \mathbf{\lambda})$$

9

$$J(\mathbf{x}) = \max_{\mathbf{\lambda} \geqslant 0} \mathfrak{L}(\mathbf{x}, \mathbf{\lambda})$$

$$J(\mathbf{x}) = \max_{\mathbf{\lambda} \geqslant 0} \mathfrak{L}(\mathbf{x}, \mathbf{\lambda})$$

But, our original problem was minimizing J(x), which is equivalent to:

$$\min_{\pmb{x} \in \mathbb{R}^d} \max_{\pmb{\lambda} \geqslant \pmb{0}} \mathfrak{L}(\pmb{x}, \pmb{\lambda})$$

$$J(\mathbf{x}) = \max_{\mathbf{\lambda} \geqslant 0} \mathfrak{L}(\mathbf{x}, \mathbf{\lambda})$$

But, our original problem was minimizing J(x), which is equivalent to:

$$\min_{oldsymbol{x} \in \mathbb{R}^d} \max_{oldsymbol{\lambda} \geqslant \mathbf{0}} \mathfrak{L}(oldsymbol{x}, oldsymbol{\lambda})$$

Using the Minimax inequality, we can write:

$$J(\mathbf{x}) = \max_{\mathbf{\lambda} \geqslant 0} \mathfrak{L}(\mathbf{x}, \mathbf{\lambda})$$

But, our original problem was minimizing J(x), which is equivalent to:

$$\min_{oldsymbol{x} \in \mathbb{R}^d} \max_{oldsymbol{\lambda} \geqslant \mathbf{0}} \mathfrak{L}(oldsymbol{x}, oldsymbol{\lambda})$$

Using the Minimax inequality, we can write:

$$\min_{\mathbf{x} \in \mathbb{R}^d} \max_{\lambda \geqslant 0} \mathfrak{L}(\mathbf{x}, \lambda) \geqslant \max_{\mathbf{\lambda} \geqslant 0} \min_{\mathbf{x} \in \mathbb{R}^d} \mathfrak{L}(\mathbf{x}, \lambda)$$

$$J(\mathbf{x}) = \max_{\mathbf{\lambda} \geqslant 0} \mathfrak{L}(\mathbf{x}, \mathbf{\lambda})$$

But, our original problem was minimizing J(x), which is equivalent to:

$$\min_{oldsymbol{x} \in \mathbb{R}^d} \max_{oldsymbol{\lambda} \geqslant \mathbf{0}} \mathfrak{L}(oldsymbol{x}, oldsymbol{\lambda})$$

Using the Minimax inequality, we can write:

$$\min_{oldsymbol{x} \in \mathbb{R}^d} \max_{oldsymbol{\lambda} \geqslant \mathbf{0}} \mathfrak{L}(oldsymbol{x}, oldsymbol{\lambda}) \geqslant \max_{oldsymbol{\lambda} \geqslant \mathbf{0}} \min_{oldsymbol{x} \in \mathbb{R}^d} \mathfrak{L}(oldsymbol{x}, oldsymbol{\lambda})$$

We can write the dual objective as a function of λ as

$$\mathfrak{D}(\lambda) = \min_{\mathbf{x} \in \mathbb{R}^d} \mathfrak{L}(\mathbf{x}, \lambda)$$

• Primal objective:

$$\min_{m{x}} f(m{x})$$
 subject to $g_i(m{x}) \leqslant 0$ for all $i=1,\ldots,m$

• Primal objective:

$$\min_{\mathbf{x}} f(\mathbf{x})$$

subject to $g_i(\mathbf{x}) \leqslant 0$ for all $i = 1, \dots, m$

• Or, primal objective $= J(x) \ge \max_{\lambda} \mathfrak{D}(\lambda)$

Primal objective:

$$\min_{\mathbf{x}} f(\mathbf{x})$$

subject to $g_i(\mathbf{x}) \leqslant 0$ for all $i = 1, \dots, m$

- Or, primal objective = $J(x) \ge \max_{\lambda} \mathfrak{D}(\lambda)$
- Or, primal objective (in terms of x) ≥ dual objective (in terms of λ)

• Primal objective:

$$\min_{\mathbf{x}} f(\mathbf{x})$$

subject to $g_i(\mathbf{x}) \leqslant 0$ for all $i = 1, \dots, m$

- Or, primal objective = $J(x) \ge \max_{\lambda} \mathfrak{D}(\lambda)$
- Or, primal objective (in terms of x) ≥ dual objective (in terms of λ)
- For SVM like formulations, primal objective is the same as dual objective (strong duality)

• Primal objective:

$$\min_{\mathbf{x}} f(\mathbf{x})$$

subject to $g_i(\mathbf{x}) \leq 0$ for all $i = 1, ..., m$

- Or, primal objective = $J(x) \ge \max_{\lambda} \mathfrak{D}(\lambda)$
- Or, primal objective (in terms of x) ≥ dual objective (in terms of λ)
- For SVM like formulations, primal objective is the same as dual objective (strong duality)
- For some problems, there is a "daulity-gap" between the two objectives