

Convex Functions

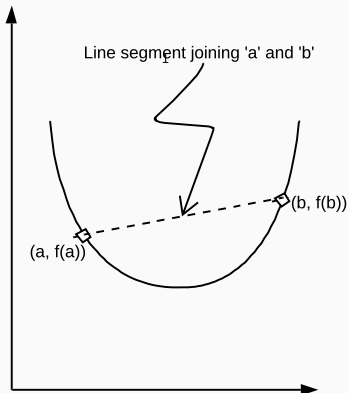
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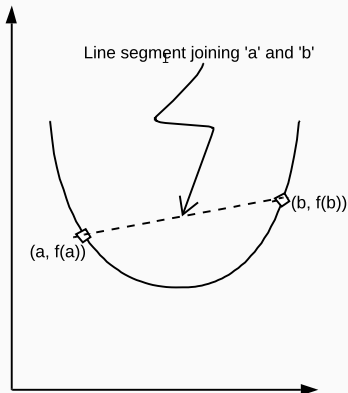
Definition

- Convexity is defined on an interval $[\alpha, \beta]$



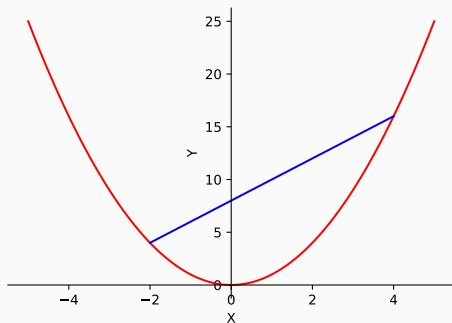
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- The line segment joining $(a, f(a))$ and $(b, f(b))$ should be *above or on* the function f for all points in interval $[\alpha, \beta]$.



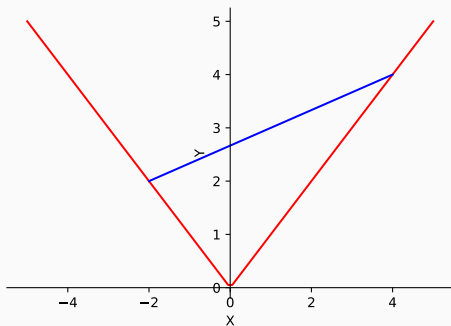
Example: $y = x^2$

Convex on the entire real line i.e. $(-\infty, \infty)$



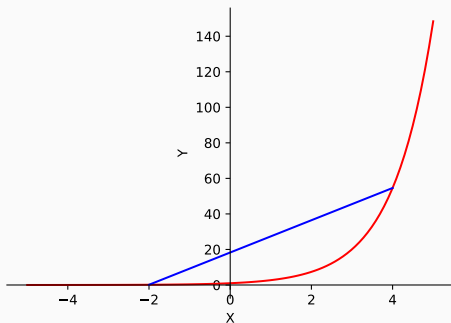
Example: $y = |x|$

Convex on the entire real line i.e. $(-\infty, \infty)$



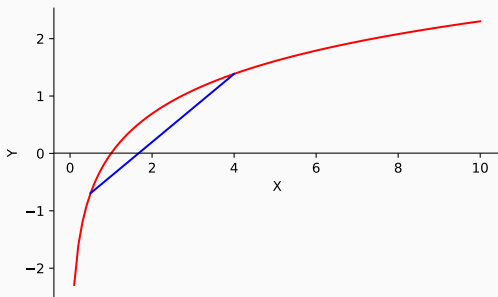
Example: $y = e^x$

Convex on the entire real line i.e. $(-\infty, \infty)$



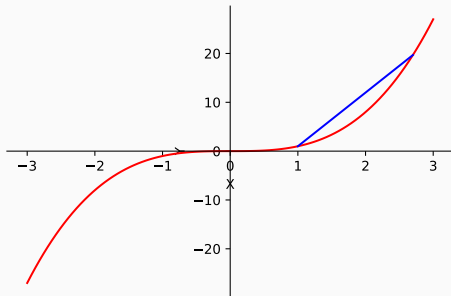
Example: $y = \ln x$

Not convex on the entire real line i.e. $(-\infty, \infty)$



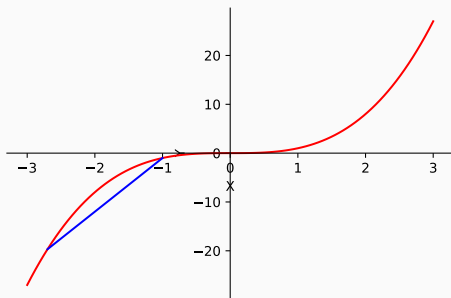
Example: $y = x^3$

It is convex for the interval $[0, \infty)$



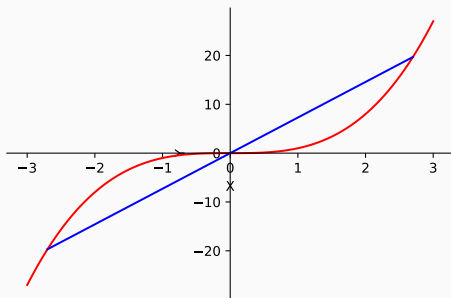
Example: $y = x^3$

It is concave for the interval $(-\infty, 0]$



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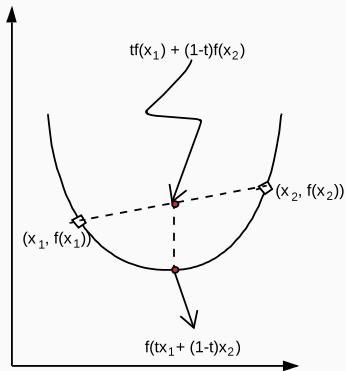
But, it is not convex for the interval $(-\infty, \infty)$



Mathematical Formulation

Function f is convex on set X , if $\forall x_1, x_2 \in X$ and $\forall t \in [0, 1]$

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$



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$$\text{LHS} = f(tx_1 + (1-t)x_2) = t^2x_1^2 + (1-t)^2x_2^2 + 2t(1-t)x_1x_2$$

$$\text{RHS} = tf(x_1) + (1-t)f(x_2) = tx_1^2 + (1-t)x_2^2$$

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Here,

$$\begin{aligned}\text{LHS} - \text{RHS} &= (t^2 - t)x_1^2 + [(1-t)^2 - (1-t)]x_2^2 + 2t(1-t)x_1x_2 \\ &= (t^2 - t)x_1^2 + (t^2 - t)x_2^2 - 2(t^2 - t)x_1x_2 \\ &= (t^2 - t)(x_1 - x_2)^2\end{aligned}$$

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Here, $(t^2 - t) \leq 0$ since $t \in [0, 1]$ and $(x_1 - x_2)^2 \geq 0$

Hence, $\text{LHS} - \text{RHS} \leq 0$

Hence $\text{LHS} \leq \text{RHS}$

Hence proved.

Alternative ways to prove convexity

The Double-Derivative Test

If $f''(x) > 0$, the function is convex.

For example,

$$\frac{\partial^2(x^2)}{\partial x^2} = 2 > 0 \Rightarrow x^2 \text{ is a convex function.}$$

Alternative ways to prove convexity

The double derivative test for multi-parameter function is equal to using the Hessian Matrix

A function $f(x_1, x_2, \dots, x_n)$ is convex iff its $n \times n$ Hessian Matrix is positive semidefinite for all possible values of (x_1, x_2, \dots, x_n)

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

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Eigenvalues of \mathbf{H} are 2 and $2 > 0 \Rightarrow \mathbf{H}$ is positive semidefinite.

Hence, $f(x_1, x_2) = x_1^2 + x_2^2$ is convex.

Convexity of linear least squares

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$$\frac{df}{d\boldsymbol{\theta}} = \frac{d(\|\mathbf{y}\|^2 - 2\mathbf{y}^T \mathbf{X}\boldsymbol{\theta} + \|\mathbf{X}\boldsymbol{\theta}\|^2)}{d\boldsymbol{\theta}} = -2\mathbf{y}^T \mathbf{X} + 2(\mathbf{X}\boldsymbol{\theta})^T \mathbf{X}$$

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$\mathbf{X}^T\mathbf{X}$ is positive semidefinite for any $\mathbf{X} \in \mathbb{R}^{m \times n}$.

Hence, linear least squares function is convex.

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Using this we can say that:

- $(\mathbf{y} - \mathbf{X}\boldsymbol{\theta})^T(\mathbf{y} - \mathbf{X}\boldsymbol{\theta}) + \boldsymbol{\theta}^T\boldsymbol{\theta}$ is convex
- $(\mathbf{y} - \mathbf{X}\boldsymbol{\theta})^T(\mathbf{y} - \mathbf{X}\boldsymbol{\theta}) + \|\boldsymbol{\theta}\|_1$ is convex