Mathematical Foundations for ML: Contour Plots and Gradients

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Table of Contents

Understanding Contour Plots

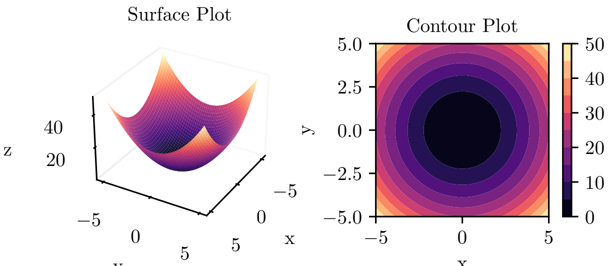
Introduction to Contour Plots

Definition: What is a Contour Plot?

Concept: A contour plot shows curves where a function $f(x, y) = K$ for different constant values K

Example: Example Function: Circular Contours

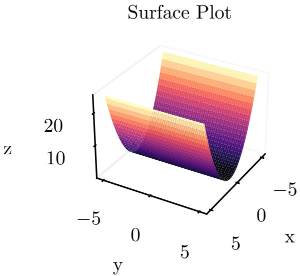
$$z = f(x, y) = x^2 + y^2$$

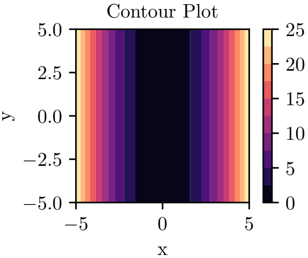


Contour Example: Parabolic Function

Example: Function: $z = f(x, y) = x^2$

Note: This function depends only on x, not on y!





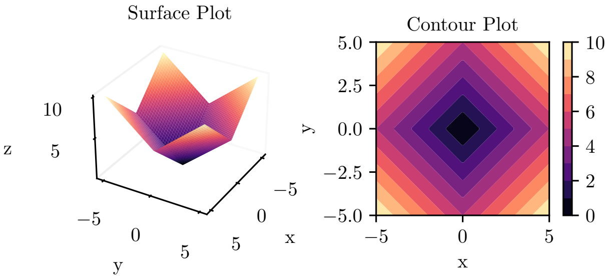
Key Points

Observation: Contour lines are vertical because $f(x, y) = x^2$ is constant for all y values when x is fixed

Contour Example: Manhattan Distance

Example: Function: $z = f(x, y) = |x| + |y|$

Also known as: Manhattan distance or L1 norm



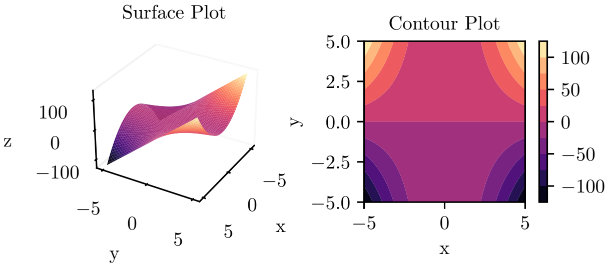
Key Points

Shape: Diamond-shaped contours due to absolute value functions

Contour Example: Polynomial Function

Example: Function: $z = f(x, y) = x^2 \cdot y$

Type: Mixed polynomial (quadratic in x, linear in y)



Key Points

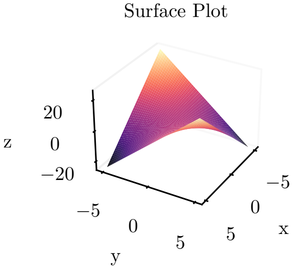
Key Features:

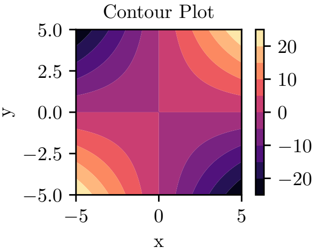
- Asymmetric contours

Contour Example: Hyperbolic Function

Example: Function: $z = f(x, y) = xy$

Type: Bilinear function (linear in each variable separately)





Key Points

Shape: Hyperbolic contours with saddle point at origin

Gradients and Contour Plots

Understanding Gradients

Definition: What is a Gradient?

Gradient ∇f: Vector pointing in the direction of steepest increase of function f

Key Points

Key Properties

- **Direction:** Points toward steepest ascent
- **Magnitude:** Rate of steepest change
- **Contour relationship:** Always perpendicular to contour lines

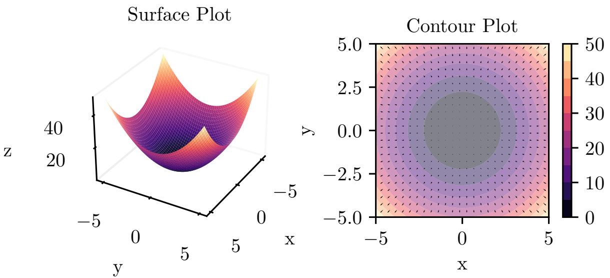
Example: Fundamental Insight

All points on the same contour have identical $f(x, y)$ values
Moving along a contour: No change in function value

Gradients Visualized: Circular Contours

Example: Function: $z = f(x, y) = x^2 + y^2$

Gradient: $\nabla f = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$



Key Points

Observations:

Gradient Properties: Key Insights

Important: Direction Interpretation

Steepest Ascent: Gradient ∇f points toward maximum increase in $f(x, y)$
Steepest Descent: $-\nabla f$ points toward maximum decrease in $f(x, y)$

Key Points

Contour Relationship

- **Same contour:** All points have identical $f(x, y)$ values
- **Gradient direction:** Always perpendicular to contour lines
- **Zero gradient:** Occurs at critical points (minima, maxima, saddle points)

Definition: Machine Learning Connection

Summary: Contours and Gradients in ML

Key Points

What We Learned

- **Contour plots:** Visualize function behavior in 2D
- **Different shapes:** Circular, diamond, hyperbolic, asymmetric
- **Gradients:** Point toward steepest function increase
- **Perpendicular relationship:** Gradients contours

Important: ML Applications

- **Loss landscapes:** Understanding optimization challenges
- **Gradient descent:** Following steepest descent direction
- **Regularization:** L1/L2 penalties create different contour shapes
- **Saddle points:** Common in deep learning optimization