Subgradient

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Outline

The Problem: Non-Differentiable Functions

Challenge: What happens when we can't compute gradients?

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Common Examples in ML

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Solution: Subgradients

Subgradients generalize gradients to convex but non-differentiable functions

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What is a Subgradient?

For convex function f at point x_0, vector g is a **subgradient** if:

$$f(x) \geq f(x_0) + g^T(x - x_0) \quad \forall x$$

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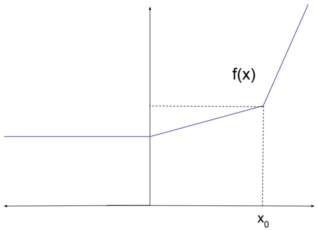
Intuition: g defines a linear lower bound that touches f at x_0

The Classic Example: $f(x) = |x|$

Challenge: Find "derivative" of $f(x) = |x|$ at $x = 0$



Non-differentiable function at $x = 0$

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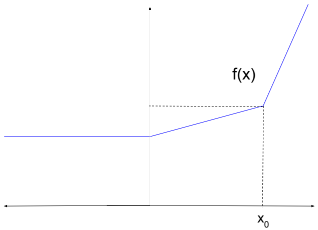

Non-differentiable function at $x = 0$

Classical calculus: Derivative doesn't exist at $x = 0$
Subgradient calculus: We can find a set of valid subgradients!

Pop Quiz: Subgradient Intuition

Quick Quiz 1

For $f(x) = |x|$ at $x = 0$, which values could be subgradients?

a) Only $g = 0$

Answer: b) Any value in $[-1, 1]$ satisfies the subgradient condition!

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Pop Quiz: Subgradient Intuition

Quick Quiz 1

For $f(x) = |x|$ at $x = 0$, which values could be subgradients?

a) Only $g = 0$
b) Any $g \in [-1, 1]$
c) Only $g = 1$ or $g = -1$

Answer: b) Any value in $[-1, 1]$ satisfies the subgradient condition!

Solution

- Construct a differentiable $g(x)$

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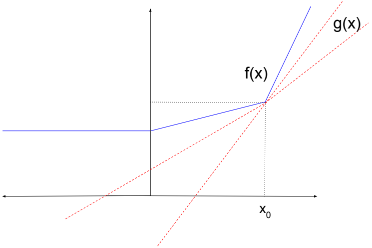
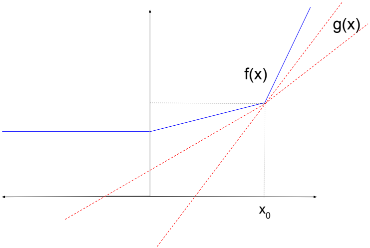
  - Below or on $f(x)$ for all x



Solution

- Compute slope of $g(x)$ at $x = x_0$



Another Example: $f(x) = |x|$

- Subgradient of $f(x)$ belongs to $[-1, 1]$