Constrained Optimization II

Nipun Batra

July 21, 2025

IIT Gandhinagar

Lagrangian and Duality

Nipun Batra

June 28, 2020

IIT Gandhinagar

Lectures heavily inspired by the Maths for Machine learning book

Minimax Inequality

- Minimax inequality states:$\max_{\boldsymbol{y}} \min_{\boldsymbol{x}} q(\boldsymbol{x}, \boldsymbol{y}) \leqslant \min_{\boldsymbol{x}} \max_{\boldsymbol{y}} q(\boldsymbol{x}, \boldsymbol{y})$

Minimax Inequality

- Minimax inequality states: $\max_{\boldsymbol{y}} \min_{\boldsymbol{x}} q(\boldsymbol{x}, \boldsymbol{y}) \leqslant \min_{\boldsymbol{x}} \max_{\boldsymbol{y}} q(\boldsymbol{x}, \boldsymbol{y})$
- We first prove For all $\boldsymbol{x}, \boldsymbol{y}$ $\quad \min_{\boldsymbol{x}} q(\boldsymbol{x}, \boldsymbol{y}) \leqslant \max_{\boldsymbol{y}} q(\boldsymbol{x}, \boldsymbol{y})$

Minimax Inequality

- Let us choose $q(x, y) = xy$

Minimax Inequality

- Let us choose $q(x, y) = xy$
- Let us first find $\max_{\boldsymbol{y}} q(\boldsymbol{x}, \boldsymbol{y})$

Minimax Inequality

- Let us choose $q(x, y) = xy$
- Let us first find $\max_{\boldsymbol{y}} q(\boldsymbol{x}, \boldsymbol{y})$

- Let us choose $q(x, y) = xy$
- Let us first find $\max_{\boldsymbol{y}} q(\boldsymbol{x}, \boldsymbol{y})$

x	1	1	2	3	4
$\Downarrow$	2	2	4	6	8
	3	3	6	9	12
	4	4	8	12	16
		1	2	3	4
		y	$\Rightarrow$		

Minimax Inequality

- For each value of x, we find y that maximizes $q(x, y)$

$$
\begin{array}{c|c|cccc}
x & 1 & 1 & 2 & 3 & 4 \\
\Downarrow & 2 & 2 & 4 & 6 & 8 \\
 & 3 & 3 & 6 & 9 & 12 \\
 & 4 & 4 & 8 & 12 & 16 \\
\hline
 & & 1 & 2 & 3 & 4 \\
 & & y & \Rightarrow & &
\end{array}
$$

Minimax Inequality

- For each value of x, we find y that maximizes $q(x, y)$
- $y = 4$ maximizes $q(x, y)\ \forall x$

x	1	1	2	3	4
$\Downarrow$	2	2	4	6	8
	3	3	6	9	12
	4	4	8	12	16
		1	2	3	4
		y	$\Rightarrow$		

Minimax Inequality

- For each value of y, we find x that minimizes $q(x, y)$

x	1	1	2	3	4
$\Downarrow$	2	2	4	6	8
	3	3	6	9	12
	4	4	8	12	16
		1	2	3	4
		y	$\Rightarrow$		

Minimax Inequality

- For each value of y, we find x that minimizes $q(x, y)$
- $x = 1$ minimizes $q(x, y)$ $\forall y$

x					
x	1	1	2	3	4
$\Downarrow$	2	2	4	6	8
	3	3	6	9	12
	4	4	8	12	16
		1	2	3	4
		y	$\Rightarrow$		

Minimax Inequality

- We just showed For all $\boldsymbol{x}, \boldsymbol{y}$ $\min_{\boldsymbol{x}} q(\boldsymbol{x}, \boldsymbol{y}) \leqslant \max_{\boldsymbol{y}} q(\boldsymbol{x}, \boldsymbol{y})$

x $\Downarrow$					
	1	1	2	3	4
	2	2	4	6	8
	3	3	6	9	12
	4	4	8	12	16
		1	2	3	4

$y \Rightarrow$

Minimax Inequality

- We just showed For all $\boldsymbol{x}, \boldsymbol{y}$ $\quad \min_{\boldsymbol{x}} q(\boldsymbol{x}, \boldsymbol{y}) \leqslant \max_{\boldsymbol{y}} q(\boldsymbol{x}, \boldsymbol{y})$
- The equality occurs at $x = 1, y = 4$

x	1	1	2	3	4
$\Downarrow$	2	2	4	6	8
	3	3	6	9	12
	4	4	8	12	16
		1	2	3	4
		y	$\Rightarrow$		

Minimax Inequality

- Let us now find $\max_{\boldsymbol{y}} \min_{\boldsymbol{x}} q(\boldsymbol{x}, \boldsymbol{y})$

x					
	1	1	2	3	**4**
$\Downarrow$	2	2	4	6	8
	3	3	6	9	12
	4	4	8	12	16
		1	2	3	4
		y	$\Rightarrow$		

Minimax Inequality

- Similarly, let us now find $\min_{\boldsymbol{x}} \max_{\boldsymbol{y}} q(\boldsymbol{x}, \boldsymbol{y})$

x	1	1	2	3	**4**
$\Downarrow$	2	2	4	6	8
	3	3	6	9	12
	4	4	8	12	16
		1	2	3	4
		y	$\Rightarrow$		

Minimax Inequality

- Similarly, let us now find $\min_{\boldsymbol{x}} \max_{\boldsymbol{y}} q(\boldsymbol{x}, \boldsymbol{y})$
- We can thus see our Minimax inequality $\max_{\boldsymbol{y}} \min_{\boldsymbol{x}} q(\boldsymbol{x}, \boldsymbol{y}) \leqslant \min_{\boldsymbol{x}} \max_{\boldsymbol{y}} q(\boldsymbol{x}, \boldsymbol{y})$

x	1	1	2	3	**4**
$\Downarrow$	2	2	4	6	8
	3	3	6	9	12
	4	4	8	12	16
		1	2	3	4
		y	$\Rightarrow$		

Revisiting the Lagrange multipliers

Our problem is of the form

$$\begin{array}{ll} \min_{\boldsymbol{x}} & f(\boldsymbol{x}) \\ \text{subject to} & g_i(\boldsymbol{x}) \leqslant 0 \quad \text{for all} \quad i = 1, \ldots, m \end{array}$$

Revisiting the Lagrange multipliers

Our problem is of the form

$$\begin{array}{lll} \min_{\boldsymbol{x}} & f(\boldsymbol{x}) & \\ \text{subject to} & g_i(\boldsymbol{x}) \leqslant 0 \quad \text{for all} & i = 1, \ldots, m \end{array}$$

Idea: Convert constrained problem to an unconstrained problem

$$J(\boldsymbol{x}) = f(\boldsymbol{x}) + \sum_{i=1}^{m} \mathbf{1}\left(g_i(\boldsymbol{x})\right)$$

Revisiting the Lagrange multipliers

Our problem is of the form

$$\begin{array}{lll} \min_{\boldsymbol{x}} & f(\boldsymbol{x}) & \\ \text{subject to} & g_i(\boldsymbol{x}) \leqslant 0 \quad \text{for all} & i = 1, \ldots, m \end{array}$$

Idea: Convert constrained problem to an unconstrained problem

$$J(\boldsymbol{x}) = f(\boldsymbol{x}) + \sum_{i=1}^{m} \mathbf{1}\left(g_i(\boldsymbol{x})\right)$$

where $1(z)$ is an infinite step function

$$\mathbf{1}(z) = \begin{cases} 0 & \text{if } z \leqslant 0 \\ \infty & \text{otherwise} \end{cases}$$

Revisiting the Lagrange multipliers

Our problem is of the form

$$\begin{array}{ll} \min_{\boldsymbol{x}} & f(\boldsymbol{x}) \\ \text{subject to} & g_i(\boldsymbol{x}) \leqslant 0 \quad \text{for all} \quad i = 1, \ldots, m \end{array}$$

Idea: Convert constrained problem to an unconstrained problem

$$J(\boldsymbol{x}) = f(\boldsymbol{x}) + \sum_{i=1}^{m} \mathbf{1}\left(g_i(\boldsymbol{x})\right)$$

where $1(z)$ is an infinite step function

$$\mathbf{1}(z) = \begin{cases} 0 & \text{if } z \leqslant 0 \\ \infty & \text{otherwise} \end{cases}$$

This would give infinte penalty if constraint is not satisfied. But, this formulation is hard to solve too.

Revisiting the Lagrange multipliers

Idea: Introduce Lagrange multipliers ($\lambda_i \geq 0$) to "approximate" $J(x)$

$$\mathfrak{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = f(\boldsymbol{x}) + \sum_{i=1}^{m} \lambda_i g_i(\boldsymbol{x})$$

Revisiting the Lagrange multipliers

Idea: Introduce Lagrange multipliers ($\lambda_i \geq 0$) to "approximate" $J(x)$

$$\mathfrak{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = f(\boldsymbol{x}) + \sum_{i=1}^{m} \lambda_i g_i(\boldsymbol{x})$$

What is the relationship between $\mathfrak{L}(\boldsymbol{x}, \boldsymbol{\lambda})$ and $J(\boldsymbol{x})$ given $\lambda_i \geq 0$?

Revisiting the Lagrange multipliers

Idea: Introduce Lagrange multipliers ($\lambda_i \geq 0$) to “approximate” $J(x)$

$$\mathfrak{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = f(\boldsymbol{x}) + \sum_{i=1}^{m} \lambda_i g_i(\boldsymbol{x})$$

What is the relationship between $\mathfrak{L}(\boldsymbol{x}, \boldsymbol{\lambda})$ and $J(\boldsymbol{x})$ given $\lambda_i \geq 0$?

When $\lambda \geqslant 0$, the Lagrangian $\mathcal{L}(x, \lambda)$ is a lower bound of $J(\boldsymbol{x})$. Hence, the maximum of $\mathfrak{L}(\boldsymbol{x}, \boldsymbol{\lambda})$ with respect to $\boldsymbol{\lambda}$ is

$$J(\boldsymbol{x}) = \max_{\boldsymbol{\lambda} \geqslant 0} \mathfrak{L}(\boldsymbol{x}, \boldsymbol{\lambda})$$

Revisiting the Lagrange multipliers

$$J(\boldsymbol{x}) = \max_{\boldsymbol{\lambda} \geqslant 0} \mathfrak{L}(\boldsymbol{x}, \boldsymbol{\lambda})$$

Revisiting the Lagrange multipliers

$$J(\boldsymbol{x}) = \max_{\boldsymbol{\lambda} \geqslant 0} \mathfrak{L}(\boldsymbol{x}, \boldsymbol{\lambda})$$

But, our original problem was minimizing $J(\boldsymbol{x})$, which is equivalent to:

$$\min_{\boldsymbol{x} \in \mathbb{R}^d} \max_{\boldsymbol{\lambda} \geqslant \mathbf{0}} \mathfrak{L}(\boldsymbol{x}, \boldsymbol{\lambda})$$

Revisiting the Lagrange multipliers

$$J(\boldsymbol{x}) = \max_{\boldsymbol{\lambda} \geqslant 0} \mathfrak{L}(\boldsymbol{x}, \boldsymbol{\lambda})$$

But, our original problem was minimizing $J(\boldsymbol{x})$, which is equivalent to:

$$\min_{\boldsymbol{x} \in \mathbb{R}^d} \max_{\boldsymbol{\lambda} \geqslant \mathbf{0}} \mathfrak{L}(\boldsymbol{x}, \boldsymbol{\lambda})$$

Using the Minimax inequality, we can write:

Revisiting the Lagrange multipliers

$$J(\boldsymbol{x}) = \max_{\boldsymbol{\lambda} \geqslant 0} \mathfrak{L}(\boldsymbol{x}, \boldsymbol{\lambda})$$

But, our original problem was minimizing $J(\boldsymbol{x})$, which is equivalent to:

$$\min_{\boldsymbol{x} \in \mathbb{R}^d} \max_{\boldsymbol{\lambda} \geqslant \mathbf{0}} \mathfrak{L}(\boldsymbol{x}, \boldsymbol{\lambda})$$

Using the Minimax inequality, we can write:

$$\min_{\boldsymbol{x} \in \mathbb{R}^d} \max_{\boldsymbol{\lambda} \geqslant \mathbf{0}} \mathfrak{L}(\boldsymbol{x}, \boldsymbol{\lambda}) \geqslant \max_{\boldsymbol{\lambda} \geqslant \mathbf{0}} \min_{\boldsymbol{x} \in \mathbb{R}^d} \mathfrak{L}(\boldsymbol{x}, \boldsymbol{\lambda})$$

Revisiting the Lagrange multipliers

$$J(\boldsymbol{x}) = \max_{\boldsymbol{\lambda} \geqslant 0} \mathfrak{L}(\boldsymbol{x}, \boldsymbol{\lambda})$$

But, our original problem was minimizing $J(\boldsymbol{x})$, which is equivalent to:

$$\min_{\boldsymbol{x} \in \mathbb{R}^d} \max_{\boldsymbol{\lambda} \geqslant \mathbf{0}} \mathfrak{L}(\boldsymbol{x}, \boldsymbol{\lambda})$$

Using the Minimax inequality, we can write:

$$\min_{\boldsymbol{x} \in \mathbb{R}^d} \max_{\boldsymbol{\lambda} \geqslant \mathbf{0}} \mathfrak{L}(\boldsymbol{x}, \boldsymbol{\lambda}) \geqslant \max_{\boldsymbol{\lambda} \geqslant \mathbf{0}} \min_{\boldsymbol{x} \in \mathbb{R}^d} \mathfrak{L}(\boldsymbol{x}, \boldsymbol{\lambda})$$

We can write the dual objective as a function of λ as $\mathfrak{D}(\boldsymbol{\lambda}) = \min_{\boldsymbol{x} \in \mathbb{R}^d} \mathfrak{L}(\boldsymbol{x}, \boldsymbol{\lambda})$

Revisiting the Lagrange multipliers

- Primal objective:

$$\begin{array}{lll} \min_{\boldsymbol{x}} & f(\boldsymbol{x}) & \\ \text{subject to} & g_i(\boldsymbol{x}) \leqslant 0 \quad \text{for all} & i = 1, \ldots, m \end{array}$$

Revisiting the Lagrange multipliers

- Primal objective:

$$
\begin{array}{cccc}
\min_{\boldsymbol{x}} & f(\boldsymbol{x}) & & \\
\text{subject to} & g_i(\boldsymbol{x}) \leqslant 0 & \text{for all} & i = 1, \ldots, m
\end{array}
$$

- Or, primal objective $= J(x) \geq \max_{\boldsymbol{\lambda}} \mathfrak{D}(\boldsymbol{\lambda})$

Revisiting the Lagrange multipliers

- Primal objective:

$$\begin{array}{lll} \min_{\boldsymbol{x}} & f(\boldsymbol{x}) & \\ \text{subject to} & g_i(\boldsymbol{x}) \leqslant 0 \quad \text{for all} & i = 1, \dots, m \end{array}$$

- Or, primal objective $= J(x) \geq \max_{\boldsymbol{\lambda}} \mathfrak{D}(\boldsymbol{\lambda})$
- Or, primal objective (in terms of x) $\geq$ dual objective (in terms of λ)

Revisiting the Lagrange multipliers

- Primal objective:

$$\begin{array}{ll} \min_{\boldsymbol{x}} & f(\boldsymbol{x}) \\ \text{subject to} & g_i(\boldsymbol{x}) \leqslant 0 \quad \text{for all} \quad i = 1, \ldots, m \end{array}$$

- Or, primal objective $= J(x) \geq \max_{\boldsymbol{\lambda}} \mathfrak{D}(\boldsymbol{\lambda})$
- Or, primal objective (in terms of x) $\geq$ dual objective (in terms of λ)
- For SVM like formulations, primal objective is the same as dual objective (strong duality)

Revisiting the Lagrange multipliers

- Primal objective:

$$\begin{array}{ll} \min_{\boldsymbol{x}} & f(\boldsymbol{x}) \\ \text{subject to} & g_i(\boldsymbol{x}) \leqslant 0 \quad \text{for all} \quad i = 1, \ldots, m \end{array}$$

- Or, primal objective $= J(x) \geq \max_{\boldsymbol{\lambda}} \mathfrak{D}(\boldsymbol{\lambda})$
- Or, primal objective (in terms of x) $\geq$ dual objective (in terms of λ)
- For SVM like formulations, primal objective is the same as dual objective (strong duality)
- For some problems, there is a "daulity-gap" between the two objectives