Lasso Regression

Nipun Batra

IIT Gandhinagar

July 30, 2025

random, round-robin

No step-size to choose!

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Converges for Lasso objective

where:

$$\hat{y}_i^{(-j)} = \theta_0 x_i^0 + \ldots + \theta_d x_i^d$$

is $\hat{y}_i$ without θ_j

Coordinate Descent for Unregularised regression

$$\text{Set } \frac{\partial \operatorname{RSS}\left(\theta_j\right)}{\partial \theta_j} = 0$$

$$\theta_j = \sum_{i=1}^{n} \frac{\left(y_i - \left(\theta_0 x_i^0 + \ldots + \theta_d x_i^d\right)\right)\left(x_i^j\right)}{\left(x_i^j\right)^2} = \frac{\rho_j}{z_j}$$

$$\rho_j = \sum_{i=1}^{n} x_i^j \left(y_i - \hat{y}_i^{(-j)}\right) \quad \text{and} \quad z_j = \sum_{i=1}^{n} \left(x_i^j\right)^2$$

z_j is the squared of ℓ_2 norm of the j^{th} feature

Coordinate Descent for Lasso Regression

$$\text{Minimise} \underbrace{\sum_{i=1}^{n} \epsilon^2 + \delta^2 \left\{ |\theta_0| + |\theta_1| + \dots \left|\theta_j\right| + \dots |\theta_d| \right\}}_{\text{LASSO OBJECTIVE}}$$

$$\frac{\partial}{\partial \theta_j}(\text{LASSO OBJECTIVE}) = -2\rho_j + 2\theta_j z_j + \delta^2 \frac{\partial}{\partial \theta_j} \left|\theta_j\right|$$

$$\frac{\partial}{\partial \theta_j} \left|\theta_j\right| = \begin{cases} 1 & \theta_j > 0 \\ [-1, 1] & \theta_j = 0 \\ -1 & \theta_j < 0 \end{cases}$$

Coordinate Descent for Lasso Regression

- **Case 1:** $\theta_j > 0$

$$-2\rho_j + 2\theta_j z_j + \delta^2 = 0$$

$$\theta_j = \frac{\rho_j - \frac{\delta^2}{2}}{z_j}$$

$$\rho_j > \frac{\delta^2}{2} \Rightarrow \theta_j = \frac{\rho_j - \frac{\delta^2}{2}}{z_j}$$

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- **Case 2:** $\theta_j < 0$

$$\rho_j < \frac{\delta^2}{2} \Rightarrow \theta_j = \frac{\rho_j + \delta^2/2}{z_j} \tag{1}$$

Coordinate Descent for Lasso Regression

- **Case 3:** $\theta_j = 0$

$$\frac{\partial}{\partial \theta_j}(\text{LASSO OBJECTIVE}) = -2\rho_j + 2\theta_j z_j + \delta^2 \underbrace{\frac{\partial}{\partial \theta_j}\left|\theta_j\right|}_{\text{[-1,1]}}$$

$$\in \underbrace{[-2\rho_j - \delta^2, -2\rho_j + \delta^2]}_{\{0\}\text{ lies in this range}}$$

$$-2\rho_j - \delta^2 \leq 0 \text{ and } -2\rho_j + \delta^2 \geq 0$$

$$-\frac{\delta^2}{2} \leq \rho_j \leq \frac{\delta^2}{2} \Rightarrow \theta_j = 0$$

Summary of Lasso Regression

$$\theta_j = \begin{bmatrix} \frac{\rho_j + \frac{\delta^2}{2}}{z_j} & \text{if} & \rho_j < -\frac{\delta^2}{2} \\ 0 & \text{if} & -\frac{\delta^2}{2} \leq \rho_j \leq \frac{\delta^2}{2} \\ \frac{\rho_j - \frac{\delta^2}{2}}{z_j} & \text{if} & \rho_j > \frac{\delta^2}{2} \end{bmatrix} \tag{2}$$