Logistic Regression: From Linear to Classification

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July 30, 2025

Outline

1. From Regression to Classification
2. The Logistic Function
3. Odds and Log-Odds
4. Maximum Likelihood Estimation
5. Summary
6. Deriving Cost Function via Maximum Likelihood Estimation

The Problem with Linear Regression for Classification

Question: Can we use linear regression for classification?

Example: Email Spam Detection

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What We Really Want

Predict **probabilities** $P(y = 1|x)$ where $0 \leq P \leq 1$

Pop Quiz: Why Not Linear Regression?

Quick Quiz 1

What's wrong with using linear regression $\hat{y} = \mathbf{X}\boldsymbol{\theta}$ for binary classification?

a) It's too slow to compute

Answer: b) Linear regression can predict negative values or values > 1, which aren't valid probabilities!

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Answer: b) Linear regression can predict negative values or values > 1, which aren't valid probabilities!

The Sigmoid/Logistic Function

Solution: Transform linear output to probability using the **sigmoid function**

Logistic Regression Model

$$P(y = 1|\mathbf{x}) = \sigma(\mathbf{x}^T\boldsymbol{\theta}) = \frac{1}{1 + e^{-\mathbf{x}^T\boldsymbol{\theta}}}$$

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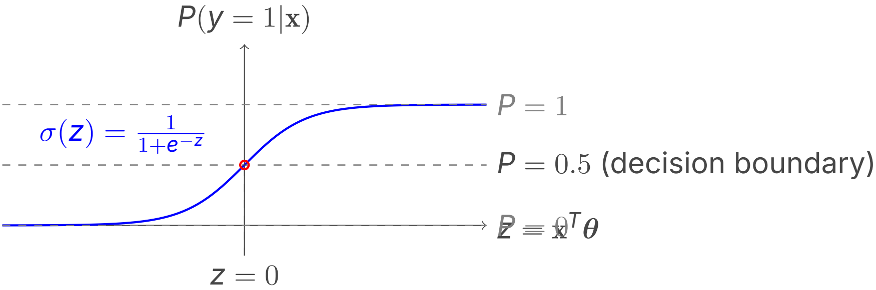
Interpretation

Visualizing the Sigmoid Function



Decision rule: Predict class 1 if $P(y = 1|\mathbf{x}) > 0.5$, else class 0

Pop Quiz: Sigmoid Properties

Quick Quiz 2

What is the value of the sigmoid function at $z = 0$?

a) $\sigma(0) = 0$

Answer: b) $\sigma(0) = \frac{1}{1+e^{-0}} = \frac{1}{1+1} = 0.5$ (decision boundary)

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Understanding the Log-Odds

From sigmoid to interpretable quantities:

$$P(y = 0|\mathbf{x}) = 1 - P(y = 1|\mathbf{x}) = \frac{e^{-\mathbf{x}^T\boldsymbol{\theta}}}{1 + e^{-\mathbf{x}^T\boldsymbol{\theta}}} \tag{1}$$

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Better Approach: Maximum Likelihood

Design a loss function that creates a **convex** optimization problem

Cross-Entropy Loss

Likelihood for one sample:

$$P(y|\mathbf{x}, \boldsymbol{\theta}) = P(y = 1|\mathbf{x})^y \cdot P(y = 0|\mathbf{x})^{1-y} = p^y(1 - p)^{1-y}$$

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- Natural choice for binary classification

Pop Quiz: Cross-Entropy

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For a true positive example ($y = 1$), what happens to the cross-entropy loss as $p \to 0$?

a) Loss approaches 0

Answer: b) Loss $= -\log p \to +\infty$ as $p \to 0$ (heavily penalizes confident wrong predictions!)

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- **Extensions:** Multinomial logistic regression for multi-class problems

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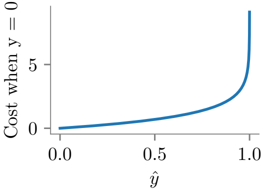
First, assume y_i is 0, then if $\hat{y}_i$ is 0, the loss is 0; but, if $\hat{y}_i$ is 1, the loss tends towards infinity!



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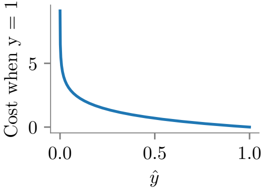
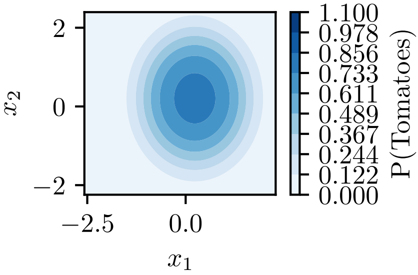
Now, assume y_i is 1, then if $\hat{y}_i$ is 0, the loss is huge; but, if $\hat{y}_i$ is 1, the loss is zero!



Bias!



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0
−2
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0.978
0.856
0.733
0.611
0.489
0.367
0.244
0.122
0.000
P(Tomatoes)

How would you learn a classifier? Or, how would you expect the classifier to learn decision boundaries?

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3. Extend Binary Logistic Regression to Multi-Class Logistic Regression

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2. Property: $\sum_{i=1}^{3} \mathcal{F}(\mathbf{X}\boldsymbol{\theta}_i) = 1$
3. Also $\mathcal{F}(z) \in [0, 1]$
4. Also, $\mathcal{F}(z)$ has squashing proprties: $R \mapsto [0, 1]$

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Tends to zero

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= $-(0 \times \log(0.1) + 1 \times \log(0.4) + 0 \times \log(0.1))$
High number! Huge penalty for misclassification!

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Extend to K-class:

$$J(\theta) = -\left\{ \sum_{i=1}^{N} \sum_{k=1}^{K} y_i^k \log(\hat{y}_i^k) \right\}$$

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How does regularization help in logistic regression?

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- **Regularization**: L1/L2 help prevent overfitting