

KKT Conditions

Nipun Batra

July 18, 2025

IIT Gandhinagar

KKT Conditions

Used for constrained optimization of the form

Minimize $f(x)$, where $x \in \mathbb{R}^k$
such that

$$h_i(x) = 0, \forall i = 1, \dots, m \text{ (m equalities)}$$
$$g_j(x) \leq 0, \forall j = 1, \dots, n \text{ (n inequalities)}$$

Step 1

- Create a new function for minimization,

$$L(x, \lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_n) = f(x) + \sum_{i=1}^m \lambda_i h_i(x) + \sum_{j=1}^n \mu_j g_j(x)$$

where,

$\lambda_1 - \lambda_m$ are multipliers for the m equalities

$\mu_1 - \mu_n$ are multipliers for the n inequalities

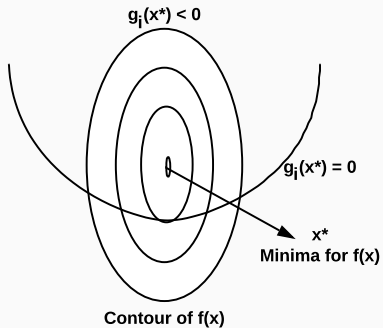
Step 2

- Minimize $L(x, \lambda, \mu)$ w.r.t. $x \implies \nabla_x L(x, \lambda, \mu) = 0$
Gives k equations

Step 3

- Minimize $L(x, \lambda, \mu)$ w.r.t. $\lambda \implies \nabla_{\lambda} L(x, \lambda, \mu) = 0$
Gives m equations

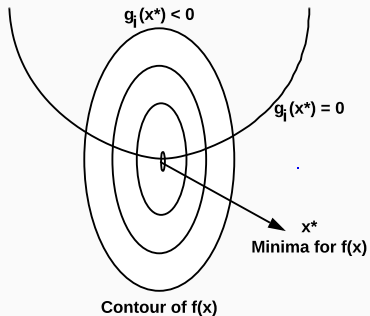
Step 4



Contour of $f(x)$

$$g_i(x^*) \leq 0$$

$$\mu_i = 0$$

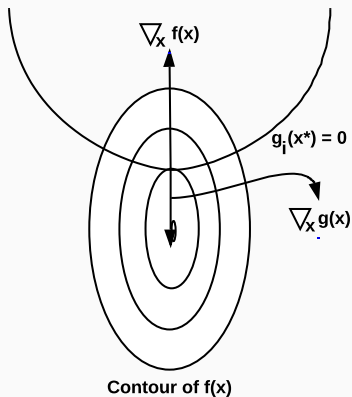


Contour of $f(x)$

$$g_i(x^*) = 0$$

In both cases, $\mu_i g_i(x^*) = 0$

Constraint on μ_i 's



$$\min_x L(x, \lambda, \mu) \implies \nabla_x f(x) + \nabla_x \mu_i g_i(x) = 0$$

$$\mu_i = \frac{\nabla_x f(x)}{\nabla_x \mu_i g_i(x)} = +ve$$

Stationarity (For minimization)

$$\nabla_x f(x) + \sum_{i=1}^m \nabla_x \lambda_i h_i(x) + \sum_{i=1}^n \nabla_x \mu_i g_i(x) = 0$$

Stationarity (For minimization)

$$\nabla_x f(x) + \sum_{i=1}^m \nabla_x \lambda_i h_i(x) + \sum_{i=1}^n \nabla_x \mu_i g_i(x) = 0$$

Equality Constraints

$$\nabla_\lambda f(x) + \sum_{i=1}^m \nabla_\lambda \lambda_i h_i(x) + \sum_{i=1}^n \nabla_\lambda \mu_i g_i(x) = 0$$

$$\sum_{i=1}^m \nabla_\lambda \lambda_i h_i(x) = 0$$

KKT Conditions

Stationarity (For minimization)

$$\nabla_x f(x) + \sum_{i=1}^m \nabla_x \lambda_i h_i(x) + \sum_{i=1}^n \nabla_x \mu_i g_i(x) = 0$$

Equality Constraints

$$\nabla_\lambda f(x) + \sum_{i=1}^m \nabla_\lambda \lambda_i h_i(x) + \sum_{i=1}^n \nabla_\lambda \mu_i g_i(x) = 0$$

$$\sum_{i=1}^m \nabla_\lambda \lambda_i h_i(x) = 0$$

Inequality Constraints (Complementary Slackness)

$$\mu_i g_i(x) = 0 \forall i = 1, \dots, n$$

$$\mu_i \geq 0$$

Example

Minimize $x^2 + y^2$ such that,

$$x^2 + y^2 \leq 5$$

$$x + 2y = 4$$

$$x, y \geq 0$$

Example

$$f(x, y) = x^2 + y^2$$

Example

$$f(x, y) = x^2 + y^2$$

$$h(x, y) = x + 2y - 4$$

Example

$$f(x, y) = x^2 + y^2$$

$$h(x, y) = x + 2y - 4$$

$$g_1(x, y) = x^2 + y^2 - 5$$

Example

$$f(x, y) = x^2 + y^2$$

$$h(x, y) = x + 2y - 4$$

$$g_1(x, y) = x^2 + y^2 - 5$$

$$g_2(x, y) = -x$$

Example

$$f(x, y) = x^2 + y^2$$

$$h(x, y) = x + 2y - 4$$

$$g_1(x, y) = x^2 + y^2 - 5$$

$$g_2(x, y) = -x$$

$$g_3(x, y) = -y$$

Example

$$f(x, y) = x^2 + y^2$$

$$h(x, y) = x + 2y - 4$$

$$g_1(x, y) = x^2 + y^2 - 5$$

$$g_2(x, y) = -x$$

$$g_3(x, y) = -y$$

$$L(x, y, \lambda, \mu_1, \mu_2, \mu_3) =$$

$$x^2 + y^2 + \lambda(x + 2y - 4) + \mu_1(x^2 + y^2 - 5) + \mu_2(-x) + \mu_3(-y)$$

Example

Stationarity

$$\nabla_x L(x, y, \lambda, \mu_1, \mu_2, \mu_3) = 0$$

$$\implies 2x + \lambda + 2\mu_1 x - \mu_2 = 0 \dots\dots\dots (1)$$

$$\nabla_y L(x, y, \lambda, \mu_1, \mu_2, \mu_3) = 0$$

$$\implies 2y + 2\lambda + 2\mu_1 y - \mu_3 = 0 \dots\dots\dots (2)$$

Example

Stationarity

$$\nabla_x L(x, y, \lambda, \mu_1, \mu_2, \mu_3) = 0$$

$$\implies 2x + \lambda + 2\mu_1 x - \mu_2 = 0 \dots\dots\dots (1)$$

$$\nabla_y L(x, y, \lambda, \mu_1, \mu_2, \mu_3) = 0$$

$$\implies 2y + 2\lambda + 2\mu_1 y - \mu_3 = 0 \dots\dots\dots (2)$$

Equality Constraint

$$x + 2y = 4 \dots\dots\dots (3)$$

Example

Stationarity

$$\nabla_x L(x, y, \lambda, \mu_1, \mu_2, \mu_3) = 0$$

$$\implies 2x + \lambda + 2\mu_1 x - \mu_2 = 0 \dots\dots\dots (1)$$

$$\nabla_y L(x, y, \lambda, \mu_1, \mu_2, \mu_3) = 0$$

$$\implies 2y + 2\lambda + 2\mu_1 y - \mu_3 = 0 \dots\dots\dots (2)$$

Equality Constraint

$$x + 2y = 4 \dots\dots\dots (3)$$

Slackness

$$\mu_1(x^2 + y^2 - 5) = 0 \dots\dots\dots (4)$$

$$\mu_2 x = 0 \dots\dots\dots (5)$$

$$\mu_3 y = 0 \dots\dots\dots (6)$$

Example

From (6), $\mu_3 = 0$ or $y = 0$

But if, $y = 0$, then $x = 4$ according to (3) . This violates (1).

Hence, $y \neq 0$ and $\mu_3 = 0$

Example

From (6), $\mu_3 = 0$ or $y = 0$

But if, $y = 0$, then $x = 4$ according to (3) . This violates (1).

Hence, $y \neq 0$ and $\mu_3 = 0$

From (5), $\mu_1 = 0$ or $x = 0$

If $x = 0$, $y = 2$, which implies $x^2 + y^2 = 4 (\leq 5)$

Since $(x,y) = (0,2)$ gives smaller $x^2 + y^2$ terms than 5,

Using (4), $\mu_1 = 0$

Example

From (6), $\mu_3 = 0$ or $y = 0$

But if, $y = 0$, then $x = 4$ according to (3) . This violates (1).

Hence, $y \neq 0$ and $\mu_3 = 0$

From (5), $\mu_1 = 0$ or $x = 0$

If $x = 0$, $y = 2$, which implies $x^2 + y^2 = 4 (\leq 5)$

Since $(x,y) = (0,2)$ gives smaller $x^2 + y^2$ terms than 5,

Using (4), $\mu_1 = 0$

On further solving we get,

$$x = 0.8$$

$$y = 1.6$$