Logistic Regression

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IIT Gandhinagar

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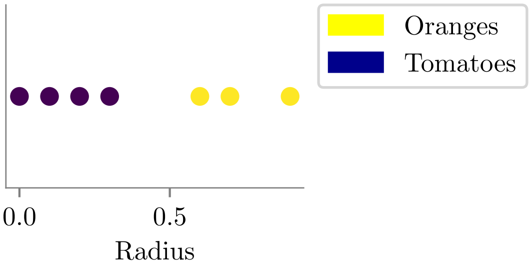
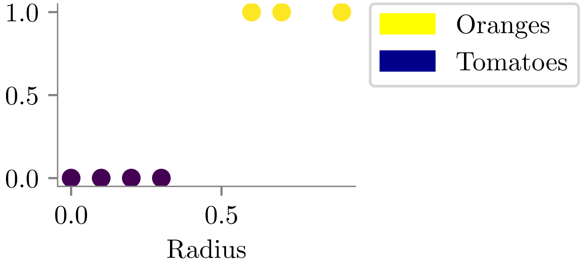
Classification Technique



Classification Technique



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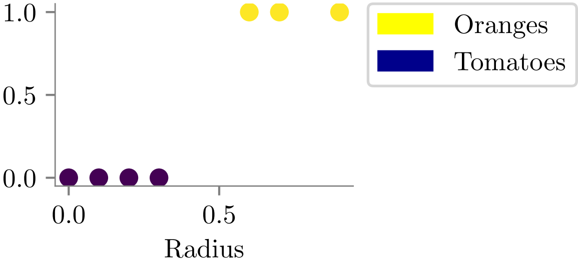
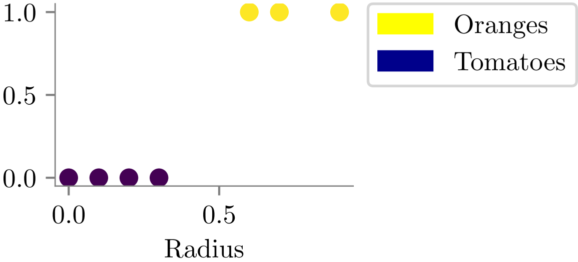


Aim: Probability(Tomatoes | Radius) ? or

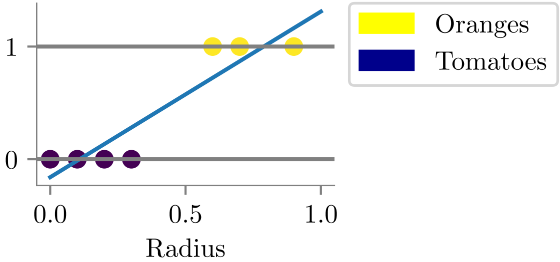
Classification Technique



Aim: Probability(Tomatoes | Radius) ? or

$\boxed{\text{More generally, } P(y = 1|X = x)?}$

Idea: Use Linear Regression



$$P(X = Orange|Radius) = \theta_0 + \theta_1 \times Radius$$

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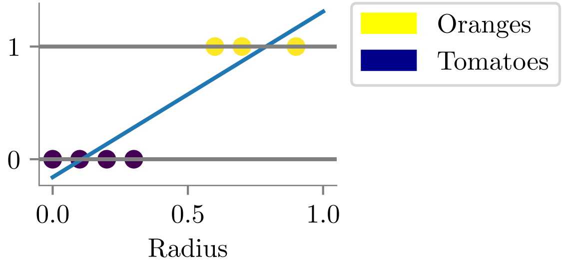


$$P(X = Orange|Radius) = \theta_0 + \theta_1 \times Radius$$

Generally,

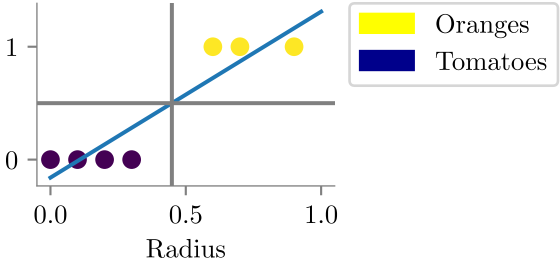
$$P(y = 1|x) = X\theta$$

Idea: Use Linear Regression

Prediction:
If $\theta_0 + \theta_1 \times \mathit{Radius} > 0.5 \rightarrow$ Orange
Else $\rightarrow$ Tomato

Problem:
Range of $X\theta$ is $(-\infty, \infty)$
But $P(y = 1| \ldots) \in [0, 1]$

Idea: Use Linear Regression



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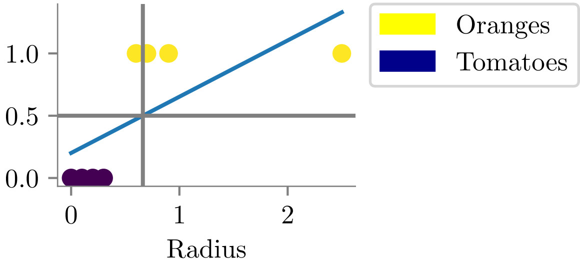
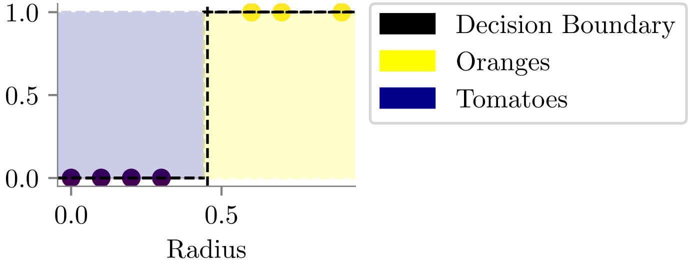


Linear regression for classification gives a poor prediction!

Ideal boundary



- Have a decision function similar to the above (but not so sharp and discontinuous)

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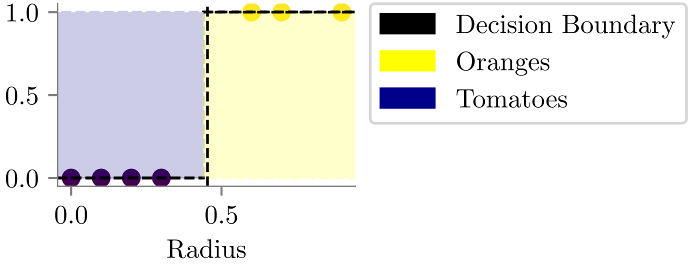
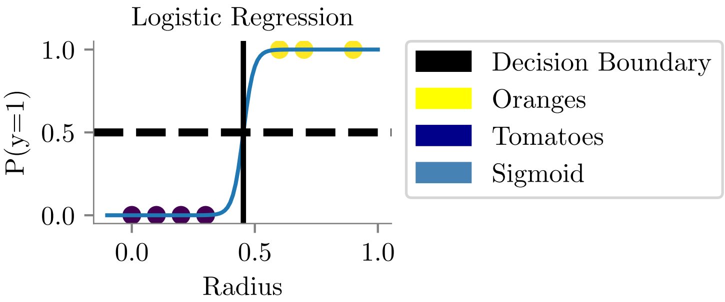


- Have a decision function similar to the above (but not so sharp and discontinuous)
- Aim: use linear regression still!

Idea: Use Linear Regression

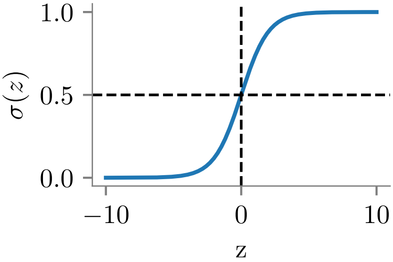


Question. Can we still use Linear Regression?
Answer. Yes! Transform $\hat{y} \to [0, 1]$

Logistic / Sigmoid Function

$\hat{y} \in (-\infty, \infty)$
$\phi =$ Sigmoid / Logistic Function (σ)
$\phi(\hat{y}) \in [0, 1]$

$$\sigma(z) = \frac{1}{1 + e^{-z}}$$



Logistic / Sigmoid Function

$z \to \infty$

Logistic / Sigmoid Function

$z \to \infty$
$\sigma(z) \to 1$

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Logistic / Sigmoid Function

$z \to \infty$
$\sigma(z) \to 1$
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$\sigma(z) \to 0$
$z = 0$
$\sigma(z) = 0.5$

$$P(y=0|X) = 1 - P(y=1|X) = 1 - \frac{1}{1+e^{-\mathbf{X}\boldsymbol{\theta}}} = \frac{e^{-\mathbf{X}\boldsymbol{\theta}}}{1+e^{-\mathbf{X}\boldsymbol{\theta}}}$$

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$$\therefore \frac{P(y=1|X)}{1-P(y=1|X)} = e^{\mathbf{X}\boldsymbol{\theta}} \implies \mathbf{X}\boldsymbol{\theta} = \log \frac{P(y=1|X)}{1-P(y=1|X)}$$

Why? Squared loss + sigmoid creates non-convex surface:

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- No guarantee gradient descent finds global optimum
- This is why we need cross-entropy loss instead!

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Why?

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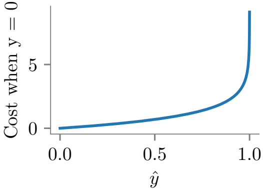
First, assume y_i is 0, then if $\hat{y}_i$ is 0, the loss is 0; but, if $\hat{y}_i$ is 1, the loss tends towards infinity!



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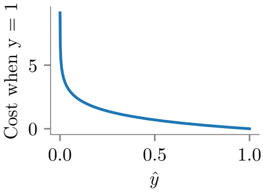
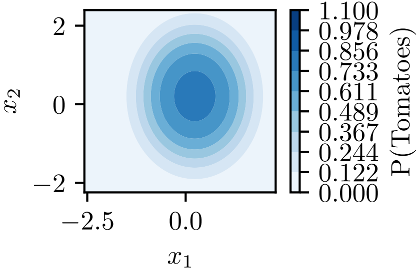
Now, assume y_i is 1, then if $\hat{y}_i$ is 0, the loss is huge; but, if $\hat{y}_i$ is 1, the loss is zero!



Bias!



2
0
−2
−2.5
0.0
x_1
x_2
1.100
0.978
0.856
0.733
0.611
0.489
0.367
0.244
0.122
0.000
P(Tomatoes)

How would you learn a classifier? Or, how would you expect the classifier to learn decision boundaries?

1. Use one-vs.-all on Binary Logistic Regression

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2. Use one-vs.-one on Binary Logistic Regression

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2. Use one-vs.-one on Binary Logistic Regression
3. Extend Binary Logistic Regression to Multi-Class Logistic Regression

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2. Property: $\sum_{i=1}^{3} \mathcal{F}(\mathbf{X}\boldsymbol{\theta}_i) = 1$
3. Also $\mathcal{F}(z) \in [0, 1]$
4. Also, $\mathcal{F}(z)$ has squashing proprties: $R \mapsto [0, 1]$

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Tends to zero

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= $-(0 \times \log(0.1) + 1 \times \log(0.4) + 0 \times \log(0.1))$
High number! Huge penalty for misclassification!

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Extend to K-class:

$$J(\theta) = -\left\{ \sum_{i=1}^{N} \sum_{k=1}^{K} y_i^k \log(\hat{y}_i^k) \right\}$$

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How does regularization help in logistic regression?

Key Takeaways

- **Probabilistic Model**: Outputs probabilities via sigmoid function

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- **Cross-Entropy Loss**: Appropriate for classification problems
- **No Closed Form**: Requires iterative optimization (gradient descent)
- **Regularization**: L1/L2 help prevent overfitting