Linear Regression

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IIT Gandhinagar

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$$\textit{weight}_i \approx \theta_0 + \theta_1 \cdot \textit{height}_i$$

$$\hat{\mathbf{y}}_{n \times 1} = \mathbf{X}_{n \times d} \boldsymbol{\theta}_{d \times 1}$$

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- θ_0 - Bias Term/Intercept Term
- θ_1 - Slope

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Demand = f(# occupants, Temperature)

Demand = Base Demand + K_1 * # occupants + K_2 * Temperature

$$\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_N \end{bmatrix}_{N\times 1} = \begin{bmatrix} 1 & x_{1,1} & x_{1,2} & \dots & x_{1,M} \\ 1 & x_{2,1} & x_{2,2} & \dots & x_{2,M} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & x_{N,1} & x_{N,2} & \dots & x_{N,M} \end{bmatrix}_{N\times(M+1)} \begin{bmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_M \end{bmatrix}_{(M+1)\times 1}$$

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$$

$$
\boxed{\hat{Y} = X\theta}
$$

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To Learn: θ

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Objective: minimize $\epsilon_1^2 + \epsilon_2^2 + \cdots + \epsilon_N^2$

Objective: Minimize $\epsilon^T \epsilon$

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Other transformations: $\log(x), x_1 \times x_2$

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$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 + \cdots + \alpha_i \mathbf{v}_i$$

where $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_i \in \mathbb{R}$

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If we stack the vectors $v_1, v_2, \dots, v_i$ as columns of a matrix V, then the span of $v_1, v_2, \dots, v_i$ is given as $V\alpha$ where $\alpha \in \mathbb{R}^i$

Can we obtain a point (x, y) s.t. $x = 3y$?

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No

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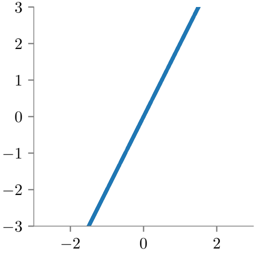
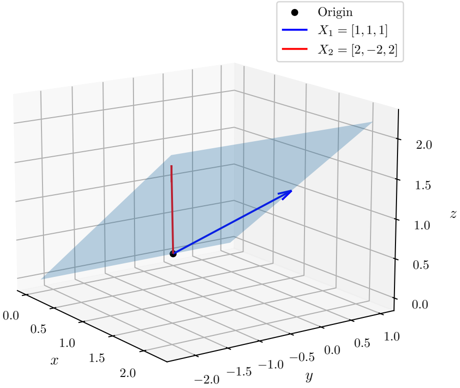
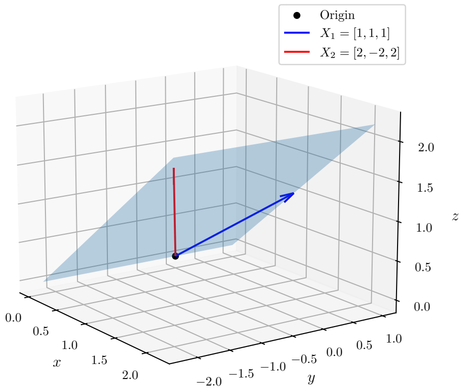
Span of the above set is along the line $y = 2x$




Origin
$X_1 = [1, 1, 1]$
$X_2 = [2, -2, 2]$
x
y
z
0.0
0.5
1.0
1.5
2.0
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
0.0
0.5
1.0
1.5
2.0



The span is the plane $z = x$ or $x_3 = x_1$

This condition arises when the $|X^T X| = 0$.

$$X = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 4 \\ 1 & 3 & 6 \end{bmatrix} \tag{1}$$

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The matrix X is not full rank.

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Can we use the direct encoding?
Then this implies that S>W>E>N

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Is it S $=$ 1 - (Is it N $+$ Is it W $+$ Is it E)

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This introduces dependencies between them, and this can cause confusion in classifiers.

Encoding

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Is Female	height
1	...
1	...
1	...
0	...
0	...

Is Female	height
1	5
1	5.2
1	5.4
0	5.8
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θ_1 is chosen based on 5-5.9, 5.2-5.9, 5.4-5.9 $\theta_1 =$ Avg. female height (5+5.2+5.4)/3 - Avg. male height(5.9)

$$x_i = \begin{cases} 1 & \text{if } i \text{ th person is female} \\ -1 & \text{if } i \text{ th person is male} \end{cases}$$

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$$y_i = \theta_0 + \theta_1 x_i + \epsilon_i = \begin{cases} \theta_0 + \theta_1 + \epsilon_i & \text{if } i \text{ th person is female} \\ \theta_0 - \theta_1 + \epsilon_i & \text{if } i \text{ th person is male.} \end{cases}$$

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Now, θ_0 can be interpreted as average person height. θ_1 as the amount that female height is above average and male height is below average.

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What are the assumptions behind linear regression?

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- **Feature Engineering**: Basis expansion enables non-linear modeling
- **Foundation**: Building block for more complex models