Linear Regression: From Intuition to Mathematics

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IIT Gandhinagar

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Outline

The Linear Relationship Hypothesis

Fundamental assumption: Many relationships can be approximated as linear

Height-Weight Example

$$\textit{weight}_i \approx \theta_0 + \theta_1 \cdot \textit{height}_i$$

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- How do we find the best line?

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Key questions:

- How do we find the best line?
- What makes one line better than another?
- How do we extend to multiple variables?

Pop Quiz: Linear Relationships

Quick Quiz 1

Which of these is NOT a good candidate for linear regression?

a) Predicting house price from square footage

Answer: c) Classification problems need different approaches than regression!

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b) Predicting salary from years of experience
c) Classifying emails as spam or not spam

Answer: c) Classification problems need different approaches than regression!

Matrix Form of Linear Regression

General Form

$$\hat{\mathbf{y}}_{n\times 1} = \mathbf{X}_{n\times d}\boldsymbol{\theta}_{d\times 1}$$

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- θ_0 - **Bias/Intercept**: Where line crosses y-axis
- θ_1 - **Slope**: How much y changes per unit x

Multiple Linear Regression

Real-world problems often involve multiple features

IITGN Water Demand Prediction

Goal: Predict daily water consumption

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- Average temperature

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$$\text{Demand} = \theta_0 + \theta_1 \times \text{occupants} + \theta_2 \times \text{temperature}$$

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General Form

$$y = \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_d x_d$$

Pop Quiz: Multiple Regression

Quick Quiz 2

In the water demand model: Demand $= 100 + 2 \times$ occupants $+ 3 \times$ temperature, if occupants = 50 and temperature = 25°C, what is the predicted demand?

a) 175 units

Answer: b) $100 + 2(50) + 3(25) = 100 + 100 + 75 = 275$ units

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b) 275 units
c) 375 units

Answer: b) $100 + 2(50) + 3(25) = 100 + 100 + 75 = 275$ units

Matrix Form: The Complete Picture

$$
\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_N \end{bmatrix}_{N\times 1}
=
\begin{bmatrix} \color{red}{1} & x_{1,1} & x_{1,2} & \dots & x_{1,M} \\ \color{red}{1} & x_{2,1} & x_{2,2} & \dots & x_{2,M} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \color{red}{1} & x_{N,1} & x_{N,2} & \dots & x_{N,M} \end{bmatrix}_{N\times(M+1)}
\begin{bmatrix} \color{red}{\theta_0} \\ \theta_1 \\ \vdots \\ \theta_M \end{bmatrix}_{(M+1)\times 1}
$$

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$$\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_N \end{bmatrix}_{N\times 1} = \begin{bmatrix} 1 & x_{1,1} & x_{1,2} & \dots & x_{1,M} \\ 1 & x_{2,1} & x_{2,2} & \dots & x_{2,M} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & x_{N,1} & x_{N,2} & \dots & x_{N,M} \end{bmatrix}_{N\times(M+1)} \begin{bmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_M \end{bmatrix}_{(M+1)\times 1}$$

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Compact Notation

$$\hat{\mathbf{y}} = \mathbf{X}\boldsymbol{\theta}$$

Key insight: Column of 1's in $\mathbf{X}$ allows us to include the intercept term θ_0!

$$Y = X\theta + \epsilon$$

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To Learn: θ

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To Learn: θ
Objective: minimize $\epsilon_1^2 + \epsilon_2^2 + \cdots + \epsilon_N^2$

Objective: Minimize $\epsilon^T \epsilon$

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Other transformations: $\log(x), x_1 \times x_2$

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$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \cdots + \alpha_i\mathbf{v}_i$$

where $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_i \in \mathbb{R}$

The span of $v_1, v_2, \ldots, v_i$ is denoted by SPAN$\{v_1, v_2, \ldots, v_i\}$

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If we stack the vectors $v_1, v_2, \ldots, v_i$ as columns of a matrix V, then the span of $v_1, v_2, \ldots, v_i$ is given as $V\alpha$ where $\alpha \in \mathbb{R}^i$

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No

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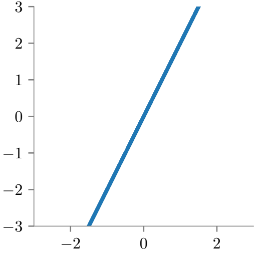
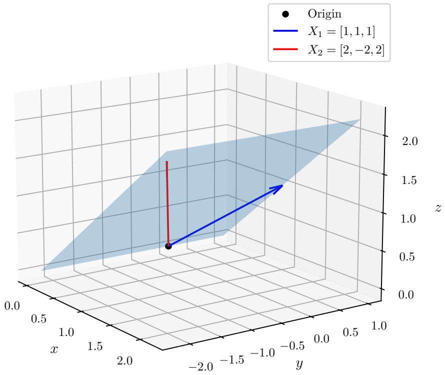
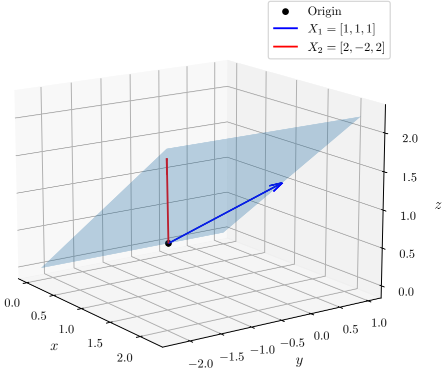
Span of the above set is along the line $y = 2x$





Origin
$X_1 = [1, 1, 1]$
$X_2 = [2, -2, 2]$
x
y
z
0.0
0.5
1.0
1.5
2.0
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
0.0
0.5
1.0
1.5
2.0



The span is the plane $z = x$ or $x_3 = x_1$

This condition arises when the $|X^T X|$ = 0.

$$X = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 4 \\ 1 & 3 & 6 \end{bmatrix} \tag{1}$$

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The matrix X is not full rank.

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Can we use the direct encoding?
Then this implies that S$>$W$>$E$>$N

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Is it S = 1 - (Is it N + Is it W + Is it E)

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This introduces dependencies between them, and this can cause confusion in classifiers.

Encoding

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Is Female	height
1	...
1	...
1	...
0	...
0	...

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1	5.2
1	5.4
0	5.8
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θ_1 is chosen based on 5-5.9, 5.2-5.9, 5.4-5.9 θ_1 = Avg. female height (5+5.2+5.4)/3 - Avg. male height(5.9)

$$x_i = \begin{cases} 1 & \text{if } i \text{ th person is female} \\ -1 & \text{if } i \text{ th person is male} \end{cases}$$

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Now, θ_0 can be interpreted as average person height. θ_1 as the amount that female height is above average and male height is below average.

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What are the assumptions behind linear regression?

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- **Feature Engineering**: Basis expansion enables non-linear modeling
- **Foundation**: Building block for more complex models