Logistic Regression

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July 30, 2025

Aim: Probability(Tomatoes | Radius) ? or

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$\boxed{\text{More generally, P}(y = 1|\mathbf{X} = \mathbf{x})\text{?}}$

Generally,

$$P(y = 1|\mathbf{x}) = \mathbf{X}\boldsymbol{\theta}$$

$\sigma(z) \to 1$

$\sigma(z) \to 1$
$z \to -\infty$

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$z \to -\infty$
$\sigma(z) \to 0$

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$$\sigma(z) \to 1$$
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$$\sigma(z) \to 0$$
$$z = 0$$
$$\sigma(z) = 0.5$$

$$P(y=0|X) = 1 - P(y=1|X) = 1 - \frac{1}{1+e^{-\mathbf{X}\boldsymbol{\theta}}} = \frac{e^{-\mathbf{X}\boldsymbol{\theta}}}{1+e^{-\mathbf{X}\boldsymbol{\theta}}}$$

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$$\therefore \frac{P(y=1|X)}{1-P(y=1|X)} = e^{\mathbf{X}\boldsymbol{\theta}} \implies \mathbf{X}\boldsymbol{\theta} = \log\frac{P(y=1|X)}{1-P(y=1|X)}$$

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- No guarantee gradient descent finds global optimum
- This is why we need cross-entropy loss instead!

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Why?

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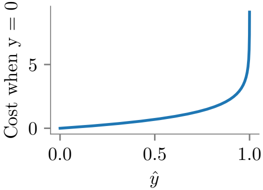
$$J(\theta) = -y_i \log \hat{y}_i - (1 - y_i) \log(1 - \hat{y}_i)$$

First, assume y_i is 0, then if $\hat{y}_i$ is 0, the loss is 0; but, if $\hat{y}_i$ is 1, the loss tends towards infinity!



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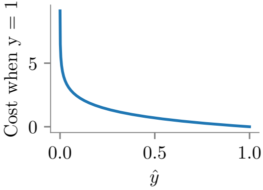
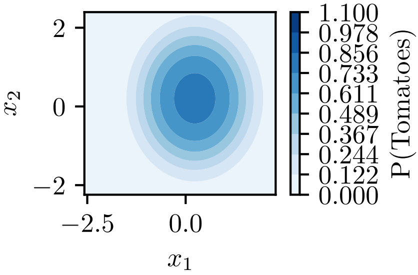
Now, assume y_i is 1, then if $\hat{y}_i$ is 0, the loss is huge; but, if $\hat{y}_i$ is 1, the loss is zero!



Bias!



2
0
−2
−2.5
0.0
x_1
x_2
1.100
0.978
0.856
0.733
0.611
0.489
0.367
0.244
0.122
0.000
P(Tomatoes)

How would you learn a classifier? Or, how would you expect the classifier to learn decision boundaries?

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2. Use one-vs.-one on Binary Logistic Regression

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2. Use one-vs.-one on Binary Logistic Regression
3. Extend Binary Logistic Regression to Multi-Class Logistic Regression

1. Learn P(setosa (class 1)) = $\mathcal{F}(\mathbf{X}\boldsymbol{\theta}_1)$

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1. Question: What could be an $\mathcal{F}$?
2. Property: $\sum_{i=1}^{3} \mathcal{F}(\mathbf{X}\boldsymbol{\theta}_i) = 1$
3. Also $\mathcal{F}(z) \in [0, 1]$
4. Also, $\mathcal{F}(z)$ has squashing proprties: $R \mapsto [0, 1]$

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Tends to zero

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= $-(0 \times \log(0.1) + 1 \times \log(0.4) + 0 \times \log(0.1))$
High number! Huge penalty for misclassification!

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Extend to K-class:

$$J(\theta) = -\left\{ \sum_{i=1}^{N} \sum_{k=1}^{K} y_i^k \log(\hat{y}_i^k) \right\}$$

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How does regularization help in logistic regression?

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- **Cross-Entropy Loss**: Appropriate for classification problems
- **No Closed Form**: Requires iterative optimization (gradient descent)
- **Regularization**: L1/L2 help prevent overfitting