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Optimal guaranteed cost control of uncertain 2-D discrete state-delayed systems described by the Roesser model via memory state feedback

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Abstract

This paper is concerned with the problem of optimal guaranteed cost control via memory state feedback for a class of uncertain two-dimensional (2-D) discrete state-delayed systems described by the Roesser model with norm-bounded uncertainties. A linear matrix inequality (LMI)-based sufficient condition for the existence of memory state feedback guaranteed cost controllers is established and a parameterized representation of such controllers (if they exist) is given in terms of feasible solutions to a certain LMI. Furthermore, a convex optimization problem with LMI constraints is formulated to select the optimal guaranteed cost controllers that minimize the upper bound of the closed-loop cost function. The proposed method yields better results in terms of least upper bound of the closed-loop cost function as compared with a previously reported result.

Keywords

2-D discrete systems, guaranteed cost control, linear matrix inequality, memory state feedback, Roesser model, state-delayed systems

Introduction

The potential applications of two-dimensional (2-D) discrete systems in image data processing, seismographic data processing, thermal processes, gas absorption, water stream heating, and so forth, (Bose, 1982; Bracewell, 1995; Kazorek, 1985; Lu and Antoniou, 1992) have received considerable attention in recent years. The stability properties of 2-D discrete systems described by the Roesser model (Roesser, 1975) have been investigated extensively (Agathoklis et al., 1989; Anderson et al., 1986; Bauer and Jury, 1990; Bauer and Ralev, 1998; El-Agizi and Fahmy, 1979; Kar, 2008a, 2008b; Kar and Singh, 1997, 2000, 2001, 2005; Leclerc and Bauer, 1994; Liu and Michel, 1994; Lodge and Fahmy, 1981; Singh, 2007; Tzafestas et al., 1992; Xiao and Hill, 1996, 1999). The stability margin of 2-D discrete systems has been studied in Agathoklis (1988), Agathoklis et al. (1982) and Swamy et al. (1981). In Du et al. (2001), the solutions for the H_∞ control and robust stabilization problems for 2-D systems in Roesser model using the 2-D system bounded realness property have been presented. The design methods for the H_2 and mixed H_2/H_∞ control of 2-D systems in Roesser model have been developed in Yang et al. (2006). A solution to robust optimal H_∞ control problem for an uncertain 2-D discrete system described by the Fornasini–Marchesini (FM) second model using Asymmetric Lyapunov Matrix has been presented in Vidyarthi et. al. (2017). In Gao and Wang (2017), the passivity issues of discrete-time 2-D switched systems described by the Roesser model under a state-dependent switching law have been discussed.

In the recent past, the guaranteed cost control problem for 2-D discrete uncertain systems has attracted several researchers (Dhawan and Kar, 2007a, 2007b, 2007c, 2010, 2011a, 2011b; Guan et al., 2001; Tiwari and Dhawan, 2012a). The aim of guaranteed cost control is to design a controller such that the closed-loop system is asymptotically stable and the closed-loop cost function value is not more than a specified upper bound for all admissible uncertainties.

It is well known that shift-delays are frequently the main source of instability and performance degradation of many control systems. Shift-delays correspond to transportation time or computation time and must be taken into account in a realistic system design. Therefore, the control problem for 2-D discrete shift-delayed systems has received huge attention. In Paszke et. al. (2004), Linear Matrix Inequality (LMI)-based sufficient conditions for robust stability and stabilization of uncertain 2-D discrete state-delayed systems have been developed. Robust output feedback guaranteed cost control problem for uncertain 2-D discrete state-delayed systems described by the FM second has been studied in Peng et. al.

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(2007). The problem of stability analysis for 2-D discrete state-delayed systems in the general model (GM) has been considered in Xu et. al. (2007) and sufficient condition for the stability has been derived via Lyapunov approach. A delay-dependent approach to investigate the H_∞ control problem for 2-D discrete state-delayed systems described by the FM second model has been presented in Xu and Yu (2009a). The problem of delay–dependent guaranteed cost control via memoryless state feedback for uncertain 2-D discrete state-delayed systems described by the FM second model has been studied in Xu and Yu (2009b). In Ye et. al. (2009), the problem of robust guaranteed cost control via memoryless state feedback for a class of 2-D discrete shift delayed systems in the FM second model setting has been considered and a sufficient condition for the existence of such controllers has been derived. Several technical errors that have occurred in the main results of Ye et. al. (2009) were corrected in Tiwari and Dhawan (2012b). A solution to the guaranteed cost control problem via memory state feedback control laws for a class of uncertain 2-D discrete state-delayed systems described by the FM second model has been presented in Tiwari and Dhawan (2012c) and it has been shown with the help of illustrative examples that the criterion given in Tiwari and Dhawan (2012c) provides less conservative results than the approach given in Ye et. al. (2009). In Xu et al. (2013), the problem of delay-dependent H_∞ control for 2-D state-delayed systems described by the Roesser model has been addressed. The problem of delay-dependent stability analysis for a class of 2-D discrete switched systems described by the Roesser model with state delays has been investigated in Huang and Xiang (2013). In Huang and Xiang (2014a), the problem of robust reliable control for a class of uncertain 2-D discrete switched systems with state delays and actuator faults represented by the Roesser model has been considered and a reliable state feedback controller has been designed to guarantee the exponential stability and reliability for the underlying systems. The problem of robust H_∞ control for a class of 2-D discrete state-delayed systems with sector nonlinearity described by a model of Roesser type has been discussed in Huang and Xiang (2014b). In Ghous et. al. (2015), the problem of state feedback H_∞ stabilization of discrete 2-D switched delay systems with actuator saturation represented by the FM second model has been studied. The problems of stability and positive observation for positive 2-D discrete-time systems in the Roesser model with multiple delays have been considered in Shen and Wang (2017). In Xu et al. (2010), the guaranteed cost control problem for uncertain 2-D discrete state-delayed systems described by the Roesser model has been considered and an optimal guaranteed cost controller design method via memoryless state feedback has been established. It is worth mentioning here that if we design the optimal guaranteed cost controllers via memory state feedback with feedback provisions on current state and the past history of the state for the guaranteed cost control problem considered in Xu et. al. (2010), we will certainly achieve an improved performance (Tiwari and Dhawan, 2012c).

With this motivation, we consider the problem of optimal guaranteed cost control via memory state feedback for the uncertain 2-D discrete state-delayed systems described by the Roesser model. The approach adopted here is as follows: we first establish the criterion for the existence of memory state feedback guaranteed cost controllers in terms of feasible solution to a certain LMI. Further, a convex optimization problem is introduced to select the optimal guaranteed cost controllers that minimize the upper bound of the closed-loop cost function. The paper is organized as follows. In Section 2, we formulate the problem of guaranteed cost control via memory state feedback for a class of uncertain 2-D discrete state-delayed system described by the Roesser model. Some useful related results are also recalled in this section. In Section 3, an LMI-based criterion for the design of optimal guaranteed cost controllers that minimize the upper bound on the closed-loop cost function is presented. In Section 4, a comparison of the proposed criterion with previously reported criterion (Xu et. al., 2010) is made. It is observed that the least upper bound of closed-loop cost function obtained via our approach is significantly smaller than that arrived at via previously reported approach (Xu et. al., 2010). Finally, the conclusion of the paper is presented in Section 5.

Notations

Throughout the paper, the following notations are used: R^n denotes real vector space of dimension n, $R^{n\times m}$ is the set of $n\times m$ real matrices, the superscript T stands for matrix transposition, $\mathbf{0}$ denotes null matrix or null vector of appropriate dimension, $\boldsymbol{I}$ is the identity matrix of appropriate dimension, diag{ ….} stands for a block diagonal matrix, $\boldsymbol{G} > \mathbf{0}$ (respectively, $\boldsymbol{G} < \mathbf{0}$) denotes a matrix $\boldsymbol{G}$ that is real symmetric and positive (respectively, negative) definite, $\lambda_{\max}(\boldsymbol{G})$ stands for maximum eigenvalue of matrix $\boldsymbol{G}$ and $\boldsymbol{G}=\boldsymbol{G}_1 \oplus \boldsymbol{G}_2$ stands for the direct sum, that is $\boldsymbol{G}=\begin{bmatrix}\boldsymbol{G}_1 & \mathbf{0}\\ \mathbf{0} & \boldsymbol{G}_2\end{bmatrix}$.

Problem formulation and preliminaries

This paper deals with the problem of optimal guaranteed cost control via memory state feedback for a class of uncertain 2-D discrete state-delayed systems represented by the Roesser model (Roesser, 1975). Specifically, the system under consideration is given by

$$\begin{bmatrix}\boldsymbol{x}^h(i+1,j)\\ \boldsymbol{x}^v(i,j+1)\end{bmatrix} = (\boldsymbol{A}+\Delta\boldsymbol{A})\begin{bmatrix}\boldsymbol{x}^h(i,j)\\ \boldsymbol{x}^v(i,j)\end{bmatrix} + (\boldsymbol{A}_d+\Delta\boldsymbol{A}_d)\begin{bmatrix}\boldsymbol{x}^h(i-d^h,j)\\ \boldsymbol{x}^v(i,j-d^v)\end{bmatrix} + (\boldsymbol{B}+\Delta\boldsymbol{B})\boldsymbol{u}(i,j) \tag{1a}$$

where $\boldsymbol{x}^h(i,j)\in R^n$ and $\boldsymbol{x}^v(i,j)\in R^m$ are the horizontal and vertical state, respectively, $\boldsymbol{u}(i,j)\in R^q$ is the control input. The matrices $\boldsymbol{A}\in R^{(n+m)\times(n+m)}$, $\boldsymbol{A}_d\in R^{(n+m)\times(n+m)}$ and $\boldsymbol{B}\in R^{(n+m)\times q}$ are known constant matrices representing the nominal plant; d^h and d^v are constant positive integers representing delays along horizontal direction and vertical direction, respectively. The matrices $\Delta\boldsymbol{A}$, $\Delta\boldsymbol{A}_d$ and $\Delta\boldsymbol{B}$ represent parameter uncertainties that are assumed to be of the form

$$[\Delta\boldsymbol{A}\quad \Delta\boldsymbol{A}_d\quad \Delta\boldsymbol{B}] = \boldsymbol{D}\,\boldsymbol{F}(i,j)[\boldsymbol{E}_a\quad \boldsymbol{E}_d\quad \boldsymbol{E}_b] \tag{1b}$$

In the above, $\boldsymbol{D}\in R^{(n+m)\times g}$, $\boldsymbol{E}_a\in R^{h\times(n+m)}$, $\boldsymbol{E}_d\in R^{h\times(n+m)}$ and $\boldsymbol{E}_b\in R^{h\times q}$ can be regarded as known structural matrices of uncertainty and $\boldsymbol{F}(i,j) \in R^{g\times h}$ is an unknown matrix representing parameter uncertainty that satisfies

$$\boldsymbol{F}^T(i,j)\boldsymbol{F}(i,j) \leq \boldsymbol{I}(\text{or equivalently}, \|\boldsymbol{F}(i,j)\| \leq 1 \tag{1c}$$

It is assumed that the system (1a) has a finite set of initial conditions (Xu et. al., 2010), that is, there exist two positive integers L_1 and L_2 such that

$$\left.\begin{array}{l} \boldsymbol{x}^v(i,j) = \boldsymbol{0} \quad \forall\ i \geq L_1, \quad j = -d^v, \ -d^v+1, \cdots, 0 \\ \boldsymbol{x}^h(i,j) = \boldsymbol{0} \quad \forall\ j \geq L_2, \quad i = -d^h, \ -d^h+1, \cdots, 0 \end{array}\right\}, \tag{1d}$$

and the initial conditions are arbitrary, but belong to the set (Xu et. al., 2010)

$$\begin{aligned} S = \{&\boldsymbol{x}^v(i,l) \in R^n : \boldsymbol{x}^v(i,l) = \Theta \boldsymbol{N}_i, \quad \boldsymbol{N}_i^T\boldsymbol{N}_i < \boldsymbol{I}, \\ &l = -d^v, \ -d^v+1, \cdots, 0, \quad 0 \leq i < L_1\} \\ &\cup \{\boldsymbol{x}^h(l,j) \in R^n : \boldsymbol{x}^h(l,j) = \Theta \boldsymbol{N}_j, \quad \boldsymbol{N}_j^T\boldsymbol{N}_j < \boldsymbol{I}, \\ &l = -d^h, \ -d^h+1, \cdots, 0, \quad 0 \leq j < L_2\}, \end{aligned} \tag{1e}$$

where Θ is a given matrix.

Associated with the uncertain system (1a) is the cost function

$$\begin{aligned} J = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} [&\boldsymbol{u}^T(i,j)\ \boldsymbol{W}\ \boldsymbol{u}(i,j) + \boldsymbol{x}^{hT}(i,j)\boldsymbol{Q}_h\boldsymbol{x}^h(i,j) + \boldsymbol{x}^{vT}(i,j) \\ &\boldsymbol{Q}_v\boldsymbol{x}^v(i,j) + \boldsymbol{x}^{hT}(i-d^h,j)\boldsymbol{Q}_{dh}\boldsymbol{x}^h(i-d^h,j) + \boldsymbol{x}^{vT}(i,j-d^v) \\ &\boldsymbol{Q}_{dv}\boldsymbol{x}^v(i,j-d^v)] \end{aligned}, \tag{2a}$$

where

$$\boldsymbol{0} < \boldsymbol{W} = \boldsymbol{W}^T \in R^{q\times q} \tag{2b}$$

$$\begin{gathered} \boldsymbol{0} < \boldsymbol{Q}_h \in R^{n\times n}, \ \boldsymbol{0} < \boldsymbol{Q}_v \in R^{m\times m}, \ \boldsymbol{0} < \boldsymbol{Q}_{dh} \in R^{n\times n}, \\ \boldsymbol{0} < \boldsymbol{S}_{dv} \in R^{m\times m} \end{gathered} \tag{2c}$$

Suppose the system states are available for feedback, the purpose of this paper is to develop a procedure to design a memory state feedback control law

$$\boldsymbol{u}(i,j) = \boldsymbol{K}_1 \begin{bmatrix} \boldsymbol{x}^h(i,j) \\ \boldsymbol{x}^v(i,j) \end{bmatrix} + \boldsymbol{K}_2 \begin{bmatrix} \boldsymbol{x}^h(i-d^h,j) \\ \boldsymbol{x}^v(i,j-d^v) \end{bmatrix} \tag{3}$$

where $\boldsymbol{K}_1, \ \boldsymbol{K}_2 \in R^{q\times(n+m)}$ are the stabilizing control law matrices to be determined, for the system (1) and the cost function (2), such that the closed-loop system

$$\begin{aligned} \begin{bmatrix} \boldsymbol{x}^h(i+1,j) \\ \boldsymbol{x}^v(i,j+1) \end{bmatrix} &= [(\boldsymbol{A} + \Delta\boldsymbol{A}) + (\boldsymbol{B} + \Delta\boldsymbol{B})\boldsymbol{K}_1] \begin{bmatrix} \boldsymbol{x}^h(i,j) \\ \boldsymbol{x}^v(i,j) \end{bmatrix} \\ &+ [(\boldsymbol{A}_d + \Delta\boldsymbol{A}_d) + (\boldsymbol{B} + \Delta\boldsymbol{B})\boldsymbol{K}_2] \begin{bmatrix} \boldsymbol{x}^h(i-d^h,j) \\ \boldsymbol{x}^v(i,j-d^v) \end{bmatrix} \end{aligned} \tag{4}$$

is asymptotically stable and the closed-loop cost function

$$J = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \xi_{ij}{}^T \begin{bmatrix} \boldsymbol{Q} + \boldsymbol{K}_1^T\boldsymbol{W}\boldsymbol{K}_1 & \boldsymbol{K}_1^T\boldsymbol{W}\boldsymbol{K}_2 \\ \boldsymbol{K}_2^T\boldsymbol{W}\boldsymbol{K}_1 & \boldsymbol{Q}_d + \boldsymbol{K}_2^T\boldsymbol{W}\boldsymbol{K}_2 \end{bmatrix} \xi_{ij} \tag{5a}$$

where

$$\boldsymbol{\xi}_{ij} = \begin{bmatrix} \boldsymbol{x}^{hT}(i,j) & \boldsymbol{x}^{vT}(i,j) & \boldsymbol{x}^{hT}(i-d^h,j) & \boldsymbol{x}^{vT}(i,j-d^v) \end{bmatrix}^T \tag{5b}$$

and

$$\boldsymbol{Q} = \boldsymbol{Q}_h \oplus \boldsymbol{Q}_v; \ \boldsymbol{Q}_d = \boldsymbol{Q}_{dh} \oplus \boldsymbol{Q}_{dv} \tag{5c}$$

satisfies $J \leq J^*$, where J^* is some specified constant.

Definition 1: (Xu et. al., 2010). Consider the system (1) and cost function (2), if there exist a control law $\boldsymbol{u}^*(i,j)$ and a positive scalar J^* such that for all admissible uncertainties, the closed-loop system (4) is asymptotically stable and the closed-loop value of the cost function (5) satisfies $J \leq J^*$, then J^* is said to be a guaranteed cost and $\boldsymbol{u}^*(i,j)$ is said to be a guaranteed cost control law for the uncertain system (1).

The following well-known lemmas are needed in the proof of our main results.

Lemma 1: (Xie et al., 2002). Given matrices $\boldsymbol{A}$, $\boldsymbol{D}$, $\boldsymbol{F}$, $\boldsymbol{E}$ of appropriate dimensions and with $\boldsymbol{A}$ symmetric, then

$$\boldsymbol{A} + \boldsymbol{D}\boldsymbol{F}\boldsymbol{E} + \boldsymbol{E}^T\boldsymbol{F}^T\boldsymbol{D}^T < \boldsymbol{0} \tag{6}$$

for all $\boldsymbol{F}$ satisfying $\boldsymbol{F}^T\boldsymbol{F} \leq \boldsymbol{I}$, if and only if there exists a scalar $\varepsilon > 0$ such that

$$\boldsymbol{A} + \varepsilon\boldsymbol{D}\boldsymbol{D}^T + \varepsilon^{-1}\boldsymbol{E}^T\boldsymbol{E} < \boldsymbol{0} \tag{7}$$

Lemma 2: (Boyd et al., 1994). For real matrices $\boldsymbol{T}$, $\boldsymbol{L}$, $\boldsymbol{U}$ of appropriate dimensions, where $\boldsymbol{T} = \boldsymbol{T}^T$ and $\boldsymbol{U} = \boldsymbol{U}^T > \boldsymbol{0}$, then $\boldsymbol{T} + \boldsymbol{L}^T\boldsymbol{U}\boldsymbol{L} < \boldsymbol{0}$ if and only if

$$\begin{bmatrix} \boldsymbol{T} & \boldsymbol{L}^T \\ \boldsymbol{L} & -\boldsymbol{U}^{-1} \end{bmatrix} < \boldsymbol{0} \ \text{ or equivalently} \begin{bmatrix} -\boldsymbol{U}^{-1} & \boldsymbol{L} \\ \boldsymbol{L}^T & \boldsymbol{T} \end{bmatrix} < \boldsymbol{0} \tag{8}$$

In this context, it may be mentioned that a method to design a robust optimal guaranteed cost controller via memoryless state feedback for uncertain 2-D discrete state-delayed systems described by the Roesser model has been presented in Xu et. al. (2010). Here, it may be noted that the cost function given in Xu et. al. (2010) does not consider the matrices $\boldsymbol{Q}_{dh}$ and $\boldsymbol{Q}_{dv}$ corresponding to the delayed states $\boldsymbol{x}^h(i-d^h,j)$ and $\boldsymbol{x}^v(i,j-d^v)$, respectively (see equation (6), Xu et. al., 2010). One of the main results for system (1) and cost function (2) with $\boldsymbol{Q}_{dh} = \boldsymbol{Q}_{dv} = \boldsymbol{0}$ may be stated as follows (see equation (28); Xu et. al., 2010).

Theorem 1: (Xu et. al., 2010). Consider system (1) and cost function (2) with $\boldsymbol{Q}_{dh} = \boldsymbol{Q}_{dv} = \boldsymbol{0}$, if the following optimization problem

minimize $[L_2(\beta_1 + d^h\gamma_1) + L_1(\beta_2 + d^v\gamma_2)]$

$$\text{s.t.}\begin{cases}(i).\begin{bmatrix}-\bar{P}+\bar{R} & 0 & \bar{P}A^T+N^TB^T & \bar{P} & N^T & \bar{P}E_a^T+N^TE_b^T\\ * & -\bar{R} & \bar{P}A_d^T & 0 & 0 & \bar{P}E_d^T\\ * & * & \varepsilon DD^T-\bar{P} & 0 & 0 & 0\\ * & * & * & -Q^{-1} & 0 & 0\\ * & * & * & * & -W^{-1} & 0\\ * & * & * & * & * & -\varepsilon I\end{bmatrix}<0,\\ (ii).\begin{bmatrix}\beta_1 I & \Theta^T\\ \Theta & \bar{P}_h\end{bmatrix}>0,\\ (iii).\begin{bmatrix}\gamma_1 I & \Theta^T\\ \Theta & \bar{R}_h\end{bmatrix}>0,\\ (iv).\begin{bmatrix}\beta_2 I & \Theta^T\\ \Theta & \bar{P}_v\end{bmatrix}>0,\\ (v).\begin{bmatrix}\gamma_2 I & \Theta^T\\ \Theta & \bar{R}_v\end{bmatrix}>0\end{cases}\tag{9}$$

has a feasible solution $\beta_1>0,\ \beta_2>0,\ \gamma_1>0,\ \gamma_2>0,\ \varepsilon>0$, $(n+m)\times(n+m)$ positive definite symmetric matrices $\bar{P}=\mathrm{diag}\{\bar{P}_h,\ \bar{P}_v\}$, $\bar{R}=\mathrm{diag}\{\bar{R}_h,\ \bar{R}_v\}$ (where $\bar{P}_h,\ \bar{R}_h\in R^{n\times n}$ and $\bar{P}_v,\ \bar{R}_v\in R^{m\times m}$), $q\times(n+m)$ matrix N, then the control law $K=N\bar{P}^{-1}$ is an optimal guaranteed cost control law which ensures the minimization of the guaranteed cost (see equation (26), Xu et. al., 2010)

$$J\le J^*=[L_2(\beta_1+d^h\gamma_1)+L_1(\beta_2+d^v\gamma_2)]\tag{10}$$

Design of optimal guaranteed cost controllers via memory state feedback

In this section, we will first present a sufficient condition for the existence of memory state feedback guaranteed cost controllers for the system (1) and cost function (2), and then give a parameterized representation of such controllers (if they exist) in terms of the feasible solutions to a certain LMI.

Theorem 2: $u(i,j)=K_1\begin{bmatrix}x^h(i,j)\\ x^v(i,j)\end{bmatrix}+K_2\begin{bmatrix}x^h(i-d^h,j)\\ x^v(i,j-d^v)\end{bmatrix}$ is a memory state feedback guaranteed cost control law if there exist positive definite symmetric matrices $P=(P_h\oplus P_v)\in R^{(n+m)\times(n+m)}$ and $S=(S_h\oplus S_v)\in R^{(n+m)\times(n+m)}$ such that for all the admissible parameter uncertainties, the following matrix inequality holds

$$\begin{bmatrix}A_C^TPA_C-P+S+K_1^TWK_1+Q & A_C^TPA_D+K_1^TWK_2\\ A_D^TPA_C+K_2^TWK_1 & A_D^TPA_D-S+K_2^TWK_2+Q_d\end{bmatrix}<0\tag{11a}$$

where

$$A_C=[(A+\Delta A)+(B+\Delta B)K_1]=A+BK_1+DF(i,j)(E_a+E_bK_1)\tag{11b}$$

and

$$A_D=[(A_d+\Delta A_d)+(B+\Delta B)K_2]=A_d+BK_2+DF(i,j)(E_d+E_bK_2)\tag{11c}$$

Proof: Consider the following 2-D Lyapunov function for system (4) (Xu et. al., 2010)

$$\begin{aligned}v(x(i,j))&=v^h(x^h(i,j))+v^v(x^v(i,j))\\ &=x^{hT}(i,j)P_hx^h(i,j)+\sum_{l=i-d^h}^{i-1}x^{hT}(l,j)S_hx^h(l,j)\\ &\quad+x^{vT}(i,j)P_vx^v(i,j)+\sum_{k=j-d^v}^{j-1}x^{vT}(i,k)S_vx^v(i,k)\end{aligned}\tag{12}$$

Then the difference of $v(x(i,j))$ along the trajectory of system (4) is given by

$$\begin{aligned}\Delta v(x(i,j))&=v^h(x^h(i+1,j))+v^v(x^v(i,j+1))\\ &\quad-v^h(x^h(i,j))-v^v(x^v(i,j))\\ &=x^{hT}(i+1,j)P_hx^h(i+1,j)+\sum_{l=i-d^h+1}^{i}x^{hT}(l,j)S_hx^h(l,j)\\ &\quad+x^{vT}(i,j+1)P_vx^v(i,j+1)+\sum_{k=j-d^v+1}^{j}x^{vT}(i,k)S_vx^v(i,k)\\ &\quad-x^{hT}(i,j)P_hx^h(i,j)-\sum_{l=i-d^h}^{i-1}x^{hT}(l,j)S_hx^h(l,j)\\ &\quad-x^{vT}(i,j)P_vx^v(i,j)-\sum_{k=j-d^v}^{j-1}x^{vT}(i,k)S_vx^v(i,k)\\ &=\begin{bmatrix}x^h(i,j)\\ x^v(i,j)\end{bmatrix}^T(A_C^TPA_C-P+S)\begin{bmatrix}x^h(i,j)\\ x^v(i,j)\end{bmatrix}\\ &\quad+\begin{bmatrix}x^h(i,j)\\ x^v(i,j)\end{bmatrix}^T(A_C^TPA_D)\begin{bmatrix}x^h(i-d^h,j)\\ x^v(i,j-d^v)\end{bmatrix}\\ &\quad+\begin{bmatrix}x^h(i-d^h,j)\\ x^v(i,j-d^v)\end{bmatrix}^T(A_D^TPA_C)\begin{bmatrix}x^h(i,j)\\ x^v(i,j)\end{bmatrix}\\ &\quad+\begin{bmatrix}x^h(i-d^h,j)\\ x^v(i,j-d^v)\end{bmatrix}^T(A_D^TPA_D-S)\begin{bmatrix}x^h(i-d^h,j)\\ x^v(i,j-d^v)\end{bmatrix}\\ &=\xi_{ij}^T\begin{bmatrix}A_C^TPA_C-P+S+K_1^TWK_1+Q & A_C^TPA_D+K_1^TWK_2\\ A_D^TPA_C+K_2^TWK_1 & A_D^TPA_D-S+K_2^TWK_2+Q_d\end{bmatrix}\xi_{ij}\\ &\quad-\xi_{ij}^T\begin{bmatrix}Q+K_1^TWK_1 & K_1^TWK_2\\ K_2^TWK_1 & Q_d+K_2^TWK_2\end{bmatrix}\xi_{ij}.\end{aligned}\tag{13}$$

From condition (11), it follows that

$$\xi_{ij}^T\begin{bmatrix}Q+K_1^TWK_1 & K_1^TWK_2\\ K_2^TWK_1 & Q_d+K_2^TWK_2\end{bmatrix}\xi_{ij}<-\Delta v(x(i,j))\tag{14}$$

Summing both sides of the above inequality over $i,j=0\to\infty$ yields

$$
\begin{aligned}
J &< -\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Delta v(\boldsymbol{x}(i,j)) \\
&= -\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} v^h(\boldsymbol{x}^h(i+1,j)) + v^v(\boldsymbol{x}^v(i,j+1)) - v^h(\boldsymbol{x}^h(i,j)) - v^v(\boldsymbol{x}^v(i,j)) \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \left[\boldsymbol{x}^{hT}(i,j)\,\boldsymbol{P}_h\,\boldsymbol{x}^h(i,j) - \boldsymbol{x}^{hT}(i+1,j)\,\boldsymbol{P}_h\,\boldsymbol{x}^h(i+1,j) + \boldsymbol{x}^{hT}(i-d^h,j)\,\boldsymbol{S}_h\,\boldsymbol{x}^h(i-d^h,j) - \boldsymbol{x}^{hT}(i,j)\,\boldsymbol{S}_h\,\boldsymbol{x}^h(i,j)\right] \\
&+ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \left[\boldsymbol{x}^{vT}(i,j)\,\boldsymbol{P}_v\,\boldsymbol{x}^v(i,j) - \boldsymbol{x}^{vT}(i,j+1)\,\boldsymbol{P}_v\,\boldsymbol{x}^v(i,j+1) + \boldsymbol{x}^{vT}(i,j-d^v)\,\boldsymbol{S}_v\,\boldsymbol{x}^v(i,j-d^v) - \boldsymbol{x}^{vT}(i,j)\,\boldsymbol{S}_v\,\boldsymbol{x}^v(i,j)\right] \\
&= \sum_{j=0}^{\infty} \boldsymbol{x}^{hT}(0,j)\,\boldsymbol{P}_h\,\boldsymbol{x}^h(0,j) + \sum_{j=0}^{\infty}\left[\boldsymbol{x}^{hT}(-d^h,j)\,\boldsymbol{S}_h\,\boldsymbol{x}^h(-d^h,j) + \boldsymbol{x}^{hT}(-d^h+1,j)\,\boldsymbol{S}_h\,\boldsymbol{x}^h(-d^h+1,j) + \cdots\cdots + \boldsymbol{x}^{hT}(-1,j)\right. \\
&\left.\boldsymbol{S}_h\,\boldsymbol{x}^h(-1,j)\right] + \sum_{i=0}^{\infty} \boldsymbol{x}^{vT}(i,0)\,\boldsymbol{P}_v\,\boldsymbol{x}^v(i,0) + \sum_{i=0}^{\infty}\left[\boldsymbol{x}^{vT}(i,-d^v)\,\boldsymbol{S}_v\,\boldsymbol{x}^v(i,-d^v) + \boldsymbol{x}^{vT}(i,-d^v+1)\right. \\
&\left.\boldsymbol{S}_v\,\boldsymbol{x}^v(i,-d^v+1) + \cdots\cdots + \boldsymbol{x}^{vT}(i,-1)\,\boldsymbol{S}_v\,\boldsymbol{x}^v(i,-1)\right] = \sum_{j=0}^{L_2-1} \boldsymbol{x}^{hT}(0,j)\,\boldsymbol{P}_h\,\boldsymbol{x}^h(0,j) + \sum_{j=0}^{L_2-1}\left[\boldsymbol{x}^{hT}(-d^h,j)\right. \\
&\left.\boldsymbol{S}_h\,\boldsymbol{x}^h(-d^h,j) + \boldsymbol{x}^{hT}(-d^h+1,j)\,\boldsymbol{S}_h\,\boldsymbol{x}^h(-d^h+1,j) + \cdots\cdots + \boldsymbol{x}^{hT}(-1,j)\,\boldsymbol{S}_h\,\boldsymbol{x}^h(-1,j)\right] + \sum_{i=0}^{L_1-1} \boldsymbol{x}^{vT}(i,0)\,\boldsymbol{P}_v\,\boldsymbol{x}^v(i,0) \\
&+ \sum_{i=0}^{L_1-1}\left[\boldsymbol{x}^{vT}(i,-d^v)\,\boldsymbol{S}_v\,\boldsymbol{x}^v(i,-d^v) + \boldsymbol{x}^{vT}(i,-d^v+1)\,\boldsymbol{S}_v\,\boldsymbol{x}^v(i,-d^v+1) + \cdots\cdots + \boldsymbol{x}^{vT}(i,-1)\,\boldsymbol{S}_v\,\boldsymbol{x}^v(i,-1)\right] \\
&\le L_2\,\lambda_{\max}(\Theta^T\boldsymbol{P}_h\,\Theta) + d^h L_2\,\lambda_{\max}(\Theta^T\boldsymbol{S}_h\,\Theta) + L_1\,\lambda_{\max}(\Theta^T\boldsymbol{P}_v\,\Theta) + d^v L_1\,\lambda_{\max}(\Theta^T\boldsymbol{S}_v\,\Theta)
\end{aligned}
\tag{15}
$$

where use has been made of (5), (1d) and (1e) and the relation $\lim_{i+j\to\infty} \boldsymbol{x}(i,j) = \boldsymbol{0}$. It follows from Definition 1 that the result of Theorem 2 is true. This completes the proof of the Theorem 2.

The following theorem establishes that the problem of determining guaranteed cost control for system (4) and the cost function (5) can be recast to an LMI feasibility problem.

Theorem 3: Consider system (1) and cost function (2), then there exists a memory state feedback control law $\boldsymbol{u}(i,j) = \boldsymbol{K}_1\begin{bmatrix}\boldsymbol{x}^h(i,j)\\ \boldsymbol{x}^v(i,j)\end{bmatrix} + \boldsymbol{K}_2\begin{bmatrix}\boldsymbol{x}^h(i-d^h,j)\\ \boldsymbol{x}^v(i,j-d^v)\end{bmatrix}$ that solves the addressed guaranteed cost control problem if there exist a positive scalar ε, $(n+m)\times(n+m)$ positive definite symmetric matrices $\boldsymbol{X} = \boldsymbol{X}_h \oplus \boldsymbol{X}_v$, $\boldsymbol{M} = \boldsymbol{M}_h \oplus \boldsymbol{M}_v$ and $q\times(n+m)$ matrices $\boldsymbol{Y}, \boldsymbol{Z}$ such that the following LMI is feasible

$$
\begin{bmatrix}
-\boldsymbol{X}+\varepsilon\boldsymbol{D}\boldsymbol{D}^T & \boldsymbol{A}\boldsymbol{X}+\boldsymbol{B}\boldsymbol{Y} & \boldsymbol{A}_d\boldsymbol{M}+\boldsymbol{B}\boldsymbol{Z} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
(\boldsymbol{A}\boldsymbol{X}+\boldsymbol{B}\boldsymbol{Y})^T & -\boldsymbol{X} & \boldsymbol{0} & (\boldsymbol{E}_a\boldsymbol{X}+\boldsymbol{E}_b\boldsymbol{Y})^T & \boldsymbol{X} & \boldsymbol{Y}^T & \boldsymbol{X} & \boldsymbol{0} \\
(\boldsymbol{A}_d\boldsymbol{M}+\boldsymbol{B}\boldsymbol{Z})^T & \boldsymbol{0} & -\boldsymbol{M} & (\boldsymbol{E}_d\boldsymbol{M}+\boldsymbol{E}_b\boldsymbol{Z})^T & \boldsymbol{0} & \boldsymbol{Z}^T & \boldsymbol{0} & \boldsymbol{M} \\
\boldsymbol{0} & (\boldsymbol{E}_a\boldsymbol{X}+\boldsymbol{E}_b\boldsymbol{Y}) & (\boldsymbol{E}_d\boldsymbol{M}+\boldsymbol{E}_b\boldsymbol{Z}) & -\varepsilon\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
\boldsymbol{0} & \boldsymbol{X} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{M} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
\boldsymbol{0} & \boldsymbol{Y} & \boldsymbol{Z} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{W}^{-1} & \boldsymbol{0} & \boldsymbol{0} \\
\boldsymbol{0} & \boldsymbol{X} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{Q}^{-1} & \boldsymbol{0} \\
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{M} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{Q}_d^{-1}
\end{bmatrix} < \boldsymbol{0}
\tag{16}
$$

In this situation, the suitable stabilizing control law matrices are given by

$$\boldsymbol{K}_1 = \boldsymbol{Y}\boldsymbol{X}^{-1} \tag{17a}$$

and

$$\boldsymbol{K}_2 = \boldsymbol{Z}\boldsymbol{M}^{-1} \tag{17b}$$

Moreover, closed-loop cost function satisfies the bound

$$
\begin{aligned}
J \le \hat{J}^* &= L_2\,\lambda_{\max}(\Theta^T\boldsymbol{X}_h^{-1}\,\Theta) + d^h L_2\,\lambda_{\max}(\Theta^T\boldsymbol{M}_h^{-1}\,\Theta) \\
&+ L_1\,\lambda_{\max}(\Theta^T\boldsymbol{X}_v^{-1}\,\Theta) + d^v L_1\,\lambda_{\max}(\Theta^T\boldsymbol{M}_v^{-1}\,\Theta)
\end{aligned}
\tag{18}
$$

Proof: Matrix inequality (11a) can be written as

$$\begin{bmatrix} \boldsymbol{A}_C^T \\ \boldsymbol{A}_D^T \end{bmatrix} \boldsymbol{P}[\boldsymbol{A}_C \quad \boldsymbol{A}_D] + \begin{bmatrix} -\boldsymbol{P} + \boldsymbol{S} + \boldsymbol{K}_1^T \boldsymbol{W} \boldsymbol{K}_1 + \boldsymbol{Q} & \boldsymbol{K}_1^T \boldsymbol{W} \boldsymbol{K}_2 \\ \boldsymbol{K}_2^T \boldsymbol{W} \boldsymbol{K}_1 & -\boldsymbol{S} + \boldsymbol{K}_2^T \boldsymbol{W} \boldsymbol{K}_2 + \boldsymbol{Q}_d \end{bmatrix} < \boldsymbol{0} \tag{19}$$

It follows from Lemma 2 that the above inequality is equivalent to

$$\begin{bmatrix} -\boldsymbol{P}^{-1} & \boldsymbol{A}_C & \boldsymbol{A}_D \\ \boldsymbol{A}_C^T & -\boldsymbol{P} + \boldsymbol{S} + \boldsymbol{K}_1^T \boldsymbol{W} \boldsymbol{K}_1 + \boldsymbol{Q} & \boldsymbol{K}_1^T \boldsymbol{W} \boldsymbol{K}_2 \\ \boldsymbol{A}_D^T & \boldsymbol{K}_2^T \boldsymbol{W} \boldsymbol{K}_1 & -\boldsymbol{S} + \boldsymbol{K}_2^T \boldsymbol{W} \boldsymbol{K}_2 + \boldsymbol{Q}_d \end{bmatrix} < \boldsymbol{0} \tag{20}$$

By substituting $\boldsymbol{A}_C$ and $\boldsymbol{A}_D$ from (11b) and (11c) respectively, (20) can be rewritten as

$$\begin{bmatrix} -\boldsymbol{P}^{-1} & \boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}_1 & \boldsymbol{A}_d + \boldsymbol{B}\boldsymbol{K}_2 \\ (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}_1)^T & -\boldsymbol{P} + \boldsymbol{S} + \boldsymbol{K}_1^T \boldsymbol{W} \boldsymbol{K}_1 + \boldsymbol{Q} & \boldsymbol{K}_1^T \boldsymbol{W} \boldsymbol{K}_2 \\ (\boldsymbol{A}_d + \boldsymbol{B}\boldsymbol{K}_2)^T & \boldsymbol{K}_2^T \boldsymbol{W} \boldsymbol{K}_1 & -\boldsymbol{S} + \boldsymbol{K}_2^T \boldsymbol{W} \boldsymbol{K}_2 + \boldsymbol{Q}_d \end{bmatrix} + \begin{bmatrix} \boldsymbol{D} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{F}(i,j)\,[\boldsymbol{0} \quad (\boldsymbol{E}_a + \boldsymbol{E}_b\boldsymbol{K}_1) \quad (\boldsymbol{E}_d + \boldsymbol{E}_b\boldsymbol{K}_2)] + [\boldsymbol{0} \quad (\boldsymbol{E}_a + \boldsymbol{E}_b\boldsymbol{K}_1) \quad (\boldsymbol{E}_d + \boldsymbol{E}_b\boldsymbol{K}_2)]^T \boldsymbol{F}^T(i,j) \begin{bmatrix} \boldsymbol{D} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}^T < \boldsymbol{0} \tag{21}$$

In the light of Lemma 1 and using Lemma 2, (21) can be rearranged as

$$\begin{bmatrix} -\boldsymbol{P}^{-1} + \varepsilon \boldsymbol{D}\boldsymbol{D}^T & \boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}_1 & \boldsymbol{A}_d + \boldsymbol{B}\boldsymbol{K}_2 & \boldsymbol{0} \\ (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}_1)^T & -\boldsymbol{P} + \boldsymbol{S} + \boldsymbol{K}_1^T \boldsymbol{W} \boldsymbol{K}_1 + \boldsymbol{Q} & \boldsymbol{K}_1^T \boldsymbol{W} \boldsymbol{K}_2 & (\boldsymbol{E}_a + \boldsymbol{E}_b\boldsymbol{K}_1)^T \\ (\boldsymbol{A}_d + \boldsymbol{B}\boldsymbol{K}_2)^T & \boldsymbol{K}_2^T \boldsymbol{W} \boldsymbol{K}_1 & -\boldsymbol{S} + \boldsymbol{K}_2^T \boldsymbol{W} \boldsymbol{K}_2 + \boldsymbol{Q}_d & (\boldsymbol{E}_d + \boldsymbol{E}_b\boldsymbol{K}_2)^T \\ \boldsymbol{0} & \boldsymbol{E}_a + \boldsymbol{E}_b\boldsymbol{K}_1 & \boldsymbol{E}_d + \boldsymbol{E}_b\boldsymbol{K}_2 & -\varepsilon \boldsymbol{I} \end{bmatrix} < \boldsymbol{0} \tag{22}$$

Premultiplying and postmultiplying (22) by the matrix

$$\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P}^{-1} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{S}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}, \tag{23}$$

one obtains

$$\begin{bmatrix} -\boldsymbol{X} + \varepsilon \boldsymbol{D}\boldsymbol{D}^T & \boldsymbol{A}\boldsymbol{X} + \boldsymbol{B}\boldsymbol{Y} & \boldsymbol{A}_d\boldsymbol{M} + \boldsymbol{B}\boldsymbol{Z} & \boldsymbol{0} \\ (\boldsymbol{A}\boldsymbol{X} + \boldsymbol{B}\boldsymbol{Y})^T & -\boldsymbol{X} & \boldsymbol{0} & (\boldsymbol{E}_a\boldsymbol{X} + \boldsymbol{E}_b\boldsymbol{Y})^T \\ (\boldsymbol{A}_d\boldsymbol{M} + \boldsymbol{B}\boldsymbol{Z})^T & \boldsymbol{0} & -\boldsymbol{M} & (\boldsymbol{E}_d\boldsymbol{M} + \boldsymbol{E}_b\boldsymbol{Z})^T \\ \boldsymbol{0} & (\boldsymbol{E}_a\boldsymbol{X} + \boldsymbol{E}_b\boldsymbol{Y}) & (\boldsymbol{E}_d\boldsymbol{M} + \boldsymbol{E}_b\boldsymbol{Z}) & -\varepsilon \boldsymbol{I} \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & (\boldsymbol{X}\boldsymbol{M}^{-1}\boldsymbol{X} + \boldsymbol{Y}^T\boldsymbol{W}\boldsymbol{Y} + \boldsymbol{X}\boldsymbol{Q}\boldsymbol{X}) & \boldsymbol{Y}^T\boldsymbol{W}\boldsymbol{Z} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Z}^T\boldsymbol{W}\boldsymbol{Y} & (\boldsymbol{Z}^T\boldsymbol{W}\boldsymbol{Z} + \boldsymbol{M}\boldsymbol{Q}_d\boldsymbol{M}) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} < \boldsymbol{0}, \tag{24}$$

where

$$\boldsymbol{X} = \boldsymbol{P}^{-1} \tag{25}$$

$$\boldsymbol{Y} = \boldsymbol{K}_1 \boldsymbol{P}^{-1} = \boldsymbol{K}_1 \boldsymbol{X} \tag{26}$$

$$\boldsymbol{M} = \boldsymbol{S}^{-1} \tag{27}$$

and

$$\boldsymbol{Z} = \boldsymbol{K}_2 \boldsymbol{S}^{-1} = \boldsymbol{K}_2 \boldsymbol{M} \tag{28}$$

The equivalence of (24) and (16) follows trivially from Lemma 2. Using (25) and (27), the bound of the cost function can be easily obtained from (15). This completes the proof of Theorem 3.

Remark 1: Note that (16) is linear in the variables ε, $\boldsymbol{X}$, $\boldsymbol{M}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$, which can be easily solved using Matlab LMI Toolbox (Boyd et al., 1994; Gahinet et al., 1995).

Theorem 3 provides a parameterized representation of a set of guaranteed cost controllers (if they exist) in terms of the feasible solutions to the LMI (16). The following theorem presents a method of selecting the optimal guaranteed cost controllers that minimize the guaranteed cost in (18).

Theorem 4: Consider system (1) and cost function (2), then there exists an optimal memory state feedback control law $\boldsymbol{u}(i,j) = \boldsymbol{K}_1 \begin{bmatrix} \boldsymbol{x}^h(i,j) \\ \boldsymbol{x}^v(i,j) \end{bmatrix} + \boldsymbol{K}_2 \begin{bmatrix} \boldsymbol{x}^h(i-d^h,j) \\ \boldsymbol{x}^v(i,j-d^v) \end{bmatrix}$ if the following optimization problem

minimize $[L_2(\alpha + d^h\beta) + L_1(\gamma + d^v\delta)]$

$$\text{s.t.} \begin{cases} (i). \quad (16), \\ (ii). \begin{bmatrix} -\alpha\boldsymbol{I} & \Theta^T \\ \Theta & -\boldsymbol{X}_h \end{bmatrix} < \boldsymbol{0}, \\ (iii). \begin{bmatrix} -\beta\boldsymbol{I} & \Theta^T \\ \Theta & -\boldsymbol{M}_h \end{bmatrix} < \boldsymbol{0}, \\ (iv). \begin{bmatrix} -\gamma\boldsymbol{I} & \Theta^T \\ \Theta & -\boldsymbol{X}_v \end{bmatrix} < \boldsymbol{0}, \\ (v). \begin{bmatrix} -\delta\boldsymbol{I} & \Theta^T \\ \Theta & -\boldsymbol{M}_v \end{bmatrix} < \boldsymbol{0} \end{cases} \tag{29}$$

has a feasible solution $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\delta > 0$, $\varepsilon > 0$, $(n+m)\times(n+m)$ positive definite symmetric matrices $\boldsymbol{X} = \boldsymbol{X}_h \oplus \boldsymbol{X}_v$, $\boldsymbol{M} = \boldsymbol{M}_h \oplus \boldsymbol{M}_v$ and $q \times (n+m)$ matrices $\boldsymbol{Y}, \boldsymbol{Z}$. In this situation, the optimal memory state feedback guaranteed cost controllers are given by $\boldsymbol{K}_1 = \boldsymbol{Y}\boldsymbol{X}^{-1}$ and $\boldsymbol{K}_2 = \boldsymbol{Z}\boldsymbol{M}^{-1}$, which ensure the minimization of the guaranteed cost in (18).

Proof: By Theorem 3, the controllers (17) constructed in terms of any feasible solution ε, $\boldsymbol{X}$, $\boldsymbol{M}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are the guaranteed cost controllers of system (1). To obtain the optimum value of the upper bound of guaranteed cost, the terms $\lambda_{\max}(\Theta^T \boldsymbol{X}_h^{-1} \Theta)$, $\lambda_{\max}(\Theta^T \boldsymbol{M}_h^{-1} \Theta)$, $\lambda_{\max}(\Theta^T \boldsymbol{X}_v^{-1} \Theta)$ and $\lambda_{\max}(\Theta^T \boldsymbol{M}_v^{-1} \Theta)$ in (18) are changed to $\lambda_{\max}(\Theta^T \boldsymbol{X}_h^{-1} \Theta) < \alpha \Leftrightarrow \Theta^T \boldsymbol{X}_h^{-1} \Theta < \alpha \boldsymbol{I}$, $\lambda_{\max}(\Theta^T \boldsymbol{M}_h^{-1} \Theta) < \beta \Leftrightarrow \Theta^T \boldsymbol{M}_h^{-1} \Theta < \beta \boldsymbol{I}$, $\lambda_{\max}(\Theta^T \boldsymbol{X}_v^{-1} \Theta) < \gamma \Leftrightarrow \Theta^T \boldsymbol{X}_v^{-1} \Theta < \gamma \boldsymbol{I}$

and $\lambda_{\max}(\Theta^T \boldsymbol{M}_v^{-1}\Theta) < \delta \Leftrightarrow \Theta^T \boldsymbol{M}_v^{-1}\Theta < \delta \boldsymbol{I}$ respectively, which, in turn, implies the constraints (ii), (iii), (iv) and (v) in (29). Thus, the minimization of $[L_2(\alpha + d^h\beta) + L_1(\gamma + d^v\delta)]$ implies the minimization of the guaranteed cost in (18). This completes the proof of Theorem 4.

Remark 2: The optimization problem given by (29) is an LMI eigenvalue problem (Boyd et. al., 1994; Gahinet et. al., 1995), which provides a procedure to design the optimal guaranteed cost controllers via memory state feedback.

To illustrate the above theorem, consider an uncertain 2-D discrete state-delayed system represented by (1) and (2) with

$$\begin{aligned}
&\boldsymbol{A} = \begin{bmatrix} 0.6 & 1 \\ 0 & 0.2 \end{bmatrix}, \boldsymbol{A}_d = \begin{bmatrix} 0.3 & 0.8 \\ 0 & 0.7 \end{bmatrix} \boldsymbol{B} = \begin{bmatrix} 0.04 \\ 0.1 \end{bmatrix}, \\
&\boldsymbol{E}_a = [0.006 \quad -0.006], \boldsymbol{E}_d = [0.001 \quad -0.001], \\
&\boldsymbol{E}_b = 0.02, \boldsymbol{D} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \boldsymbol{Q} = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.02 \end{bmatrix}, \\
&\boldsymbol{Q}_d = \begin{bmatrix} 0.003 & 0 \\ 0 & 0.003 \end{bmatrix}, \boldsymbol{W} = 0.3, \Theta = 0.1, \\
&L_1 = L_2 = 2, d^h = 2, d^v = 3.
\end{aligned} \tag{30}$$

We wish to construct the optimal guaranteed cost controllers for this system. It is found using the LMI toolbox in Matlab (Boyd et. al., 1994; Gahinet et. al., 1995) that the optimization problem (29) is feasible for the present example, and the optimal solution is given by

$$\begin{aligned}
&\boldsymbol{X} = \begin{bmatrix} 2.5070 & 0 \\ 0 & 0.0206 \end{bmatrix}, \boldsymbol{M} = \begin{bmatrix} 7.9981 & 0 \\ 0 & 0.0310 \end{bmatrix}, \\
&\boldsymbol{Y} = [-0.0435 \quad -0.0052], \boldsymbol{N} = [-0.0296 \quad -0.0286], \\
&\varepsilon = 0.0020, \alpha = 0.0040, \beta = 0.0013, \gamma = 0.4857, \delta = 0.3226.
\end{aligned} \tag{31}$$

By Theorem 4, the optimal memory state feedback guaranteed cost controllers for this system are

$$\boldsymbol{K}_1 = [-0.0173 \quad -0.2516], \boldsymbol{K}_2 = [-0.0037 \quad -0.9231] \tag{32}$$

and the least upper bound of the corresponding closed-loop cost function is

$$\hat{J}^* = 1.6294 \tag{33}$$

Comparison

It is worth comparing Theorem 1 with Theorem 4. Note that, if we set $\boldsymbol{Q}_d = \boldsymbol{0}$ and $\boldsymbol{K}_2 = \boldsymbol{0}$, then (29) reduces to (9). Therefore, Theorem 1 is subsumed by Theorem 4.

To illustrate the improvement realized from the present approach, we now apply our proposed method to the control of thermal processes in heat exchangers which can be expressed by the following first-order partial differential equation with time (Kaczorek, 1985; Xu et. al., 2010).

$$\begin{aligned}
\frac{\partial T(x,t)}{\partial x} &= -\frac{\partial T(x,t)}{\partial t} - a_0\, T(x,t) - a_1\, T(x - x_d, t) \\
&\quad - a_2\, T(x, t-\tau) + b\, \mathrm{u}(x,t)
\end{aligned} \tag{34}$$

where $T(x,t)$ is the temperature at space $x \in [0, x_f]$ and time $t \in [0, \infty)$, $\mathrm{u}(x,t)$ is input function, τ is the time delay, x_d is the space delay, a_0, a_1, a_2 and b are the real coefficients. Taking

$$T(i,j) = T(i\,\Delta x, j\,\Delta t),\ u(i,j) = u(i\,\Delta x, j\,\Delta t) \tag{35a}$$

$$\frac{\partial T(x,t)}{\partial x} \approx \frac{T(i,j) - T(i-1,j)}{\Delta x},\ \frac{\partial T(x,t)}{\partial t} \approx \frac{T(i,j+1) - T(i,j)}{\Delta t} \tag{35b}$$

it is easy to verify that (34) can be expressed in the following form

$$\begin{aligned}
T(i,j+1) &= \left(1 - \frac{\Delta t}{\Delta x} - a_0\,\Delta t\right) T(i,j) + \frac{\Delta t}{\Delta x}\, T(i-1,j) \\
&\quad - a_1 \Delta t T(i - x_d, j) - a_2 \Delta t T(i, j-\tau) + b\,\Delta t\, u(i,j).
\end{aligned} \tag{36}$$

Denote $x^h(i,j) = T(i-1,j)$, $x^v(i,j) = T(i,j)$, $d^h = \mathrm{int}(x_d/\Delta x)$ and $d^v = \mathrm{int}((\tau/\Delta t) + 1)$ (where $\mathrm{int}(\cdot)$ is the integer function), it is easy to verify that (36) can be converted into the following Roesser model

$$\begin{aligned}
\begin{bmatrix} \boldsymbol{x}^h(i+1,j) \\ \boldsymbol{x}^v(i,j+1) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ \frac{\Delta t}{\Delta x} & 1 - \frac{\Delta t}{\Delta x} - a_0\,\Delta t \end{bmatrix} \begin{bmatrix} \boldsymbol{x}^h(i,j) \\ \boldsymbol{x}^v(i,j) \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 \\ -a_1\,\Delta t & -a_2\,\Delta t \end{bmatrix} \begin{bmatrix} \boldsymbol{x}^h(i-d^h,j) \\ \boldsymbol{x}^v(i,j-d^v) \end{bmatrix} + \begin{bmatrix} 0 \\ b\,\Delta t \end{bmatrix} \boldsymbol{u}(i,j).
\end{aligned} \tag{37}$$

Now, consider the problem of optimal guaranteed cost control for a system characterized by (37) with $a_0 = 1$, $a_1 = 0.3$, $a_2 = 0.4$, $b = 1$, $\Delta t = 0.1$, $\Delta x = 0.4$, $d^h = 1$, $d^v = 1$ and the initial state satisfies the condition (1d) and (1e) with

$$L_1 = 10,\ L_2 = 10,\ \Theta = 0.5 \tag{38}$$

It is also assumed that the above system is subjected to the parameter uncertainties of the form (1b)-(1c) with

$$\boldsymbol{E}_a = [0 \quad 0.2], \boldsymbol{E}_d = [0.06 \quad 0.08], \boldsymbol{D} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \tag{39}$$

Associated with the uncertain system (37)–(39), the cost function is given by (2) with

$$\boldsymbol{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \boldsymbol{Q}_d = \boldsymbol{0}, \boldsymbol{W} = 1 \tag{40}$$

We wish to design the optimal memory state feedback guaranteed cost controllers for this system. To this end, we apply our proposed method (Theorem 4) to find the same. It is found using the LMI toolbox in Matlab (Boyd et. al., 1994; Gahinet et. al., 1995) that the optimization problem (29) is feasible for the present example and the optimal solution is given by

Table 1. A comparison between least upper bounds $\hat{J}^*$ and J^* for different values of d^h and d^v.

d^h	d^v	$\hat{J}^*$	K_1	K_2	J^*	K
1	1	47.7165	[-0.4419 -1.180]	[0.0433 -0.0837]	515.1	[-0.3071 -1.0781]
3	3	52.2651	[-0.445 -1.190]	[0.0436 -0.0843]	1402.4	[-0.3459 -1.1120]
5	5	56.4952	[-0.4496 -1.2009]	[0.0443 -0.0853]	2290	[-0.3469 -1.1319]
7	7	60.5161	[-0.4543 -1.2135]	[0.0447 -0.0857]	3177	[0.3427 -1.1445]

$$\boldsymbol{X} = \begin{bmatrix} 0.2319 & 0 \\ 0 & 0.0724 \end{bmatrix}, \boldsymbol{M} = \begin{bmatrix} 2.5431 & 0 \\ 0 & 1.7840 \end{bmatrix},$$

$$\boldsymbol{Y} = [-0.0558 \quad -0.0611], \boldsymbol{Z} = [-0.0937 \quad -0.0879],$$

$$\varepsilon = 0.1249, \alpha = 1.0782, \beta = 0.0983, \gamma = 3.4551, \delta = 0.1401 .$$

By Theorem 4, the optimal memory state feedback guaranteed cost controllers for this system are

$$\boldsymbol{K}_1 = [-0.2406 \quad -0.8450], \boldsymbol{K}_2 = [-0.0369 \quad -0.0493] \quad (41)$$

and the least upper bound of the corresponding closed-loop cost function is

$$\hat{J}^* = 47.7165 \quad (42)$$

We now apply Theorem 1 for the system described by (37)–(40). It is found that the optimization problem (9) is feasible for the present example and the optimal solution is obtained as

$$\bar{\boldsymbol{P}} = \begin{bmatrix} 0.1702 & 0 \\ 0 & 0.0469 \end{bmatrix}, \bar{\boldsymbol{R}} = \begin{bmatrix} 0.0273 & 0 \\ 0 & 0.0071 \end{bmatrix},$$

$$\boldsymbol{N} = [-0.0516 \quad -0.0508], \varepsilon = 0.0827, \beta_1 = 1.4688,$$

$$\beta_2 = 5.3272, \gamma_1 = 9.1699, \gamma_2 = 35.4223 . \quad (43)$$

By Theorem 1, the optimal guaranteed cost controller is

$$\boldsymbol{K} = [-0.3071 \quad -1.0781] \quad (44)$$

and the least upper bound of the corresponding closed-loop cost function is

$$J^* = 515.1 \quad (45)$$

The same example was also considered in Xu et. al. (2010) and the corresponding results for different values of d^h and d^v are tabulated in Table 1 (Xu et. al., 2010). For comparison purpose, we have also listed the least upper bounds of the closed-loop cost functions $\hat{J}^*$ obtained via Theorem 4 and J^* obtained via Theorem 1for the present system with different values of d^h and d^v in Table 1. From Table 1, it is clear that the least upper bound of the closed-loop cost function obtained via Theorem 4 is significantly smaller than that obtained via Theorem 1. In other words, our proposed method (Theorem 4) provides less stringent results over Theorem 1 for the present system.

Conclusion

A solution to the optimal guaranteed cost control problem via memory state feedback for a class of uncertain 2-D discrete state-delayed systems in Roesser model setting has been presented. An LMI-based sufficient condition for the existence of memory state feedback guaranteed cost controllers has been derived. Further, a convex optimization problem has been introduced to select the optimal guaranteed cost controllers that minimize the upper bound of the closed-loop cost function. A comparison of the proposed method with the criterion given in Xu et. al. (2010) is made and it is found that the proposed method leads to less stringent result than that obtained via Xu et. al. (2010).

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