Probability and Statstics

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Textbooks

- **Probability & Statistics for Engineers & Scientists**, Ninth Edition, Ronald E. Walpole, Raymond H. Myer

- **Elementary Statistics: Picturing the World,** 6th Edition, Ron Larson and Betsy Farber

- **Elementary Statistics**, 13th Edition, Mario F. Triola

Reference books

- **Probability and Statistical Inference, Ninth Edition,** Robert V. Hogg, Elliot A. Tanis, Dale L. Zimmerman

- **Probability Demystified**, Allan G. Bluman

- **Practical Statistics for Data Scientists: 50 Essential Concepts,** Peter Bruce and Andrew Bruce

- **Schaum's Outline of Probability,** Second Edition, Seymour Lipschutz, Marc Lipson

- **Python for Probability, Statistics, and Machine Learning,** José Unpingco

References

Readings for these lecture notes:

- **Schaum's Outline of Probability, Second Edition (Schaum's Outlines)**

  by by Seymour Lipschutz, Marc Lipson

- **Probability & Statistics for Engineers & Scientists**, Ninth Edition, Ronald E. Walpole, Raymond H. Myer
- **Introduction to Probability** SECOND EDITION Dimitri P. Bertsekas and John N. Tsitsiklis

These notes contain material from the above resources.

Recall: Mutually Exclusive or Disjoint

Two events A and B are **mutually exclusive, or disjoint,** if **$A \cap B$ = { }** or **∅**

OR

Two events A and B are **mutually exclusive**, **or disjoint,** if **$A \cap B$ = ∅**, that is, if A and B have no elements in common.

OR

Events ***A*** and ***B*** are **disjoint** (or **mutually exclusive**) if they cannot occur at the same time. (That is, disjoint events do not overlap.)

Addition Rule I

Addition Rule I: When two events are **mutually exclusive**,

P(A or B) = P(A) + P(B)

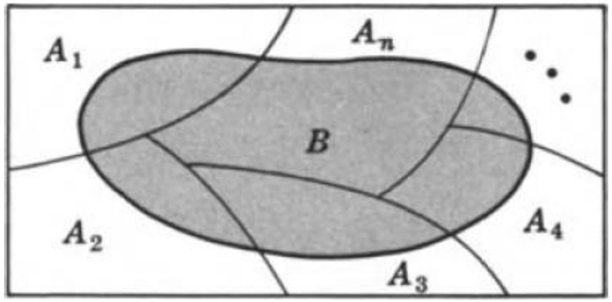
OR

$P(A \cup B) = P(A) + P(B)$

PARTITIONS AND BAYES' THEOREM

Suppose the events $A_1, A_2, \ldots, A_n$ form a partition of a sample space S; that is, the events A_i are **mutually exclusive** and their union is S. Now let B be any other event. Then

$$\mathbf{B} = S \cap B = (A_1 \cup A_2 \cup \cdots \cup A_n) \cap B$$
$$= (A_1 \cap B) \cup (A_2 \cap B) \cup \ldots (A_n \cap B)$$



where the $A_i \cap B$ are also **mutually exclusive**.

$$\therefore P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \ldots + P(A_n \cap B)$$

Using the **multiplication theorem**, we get

$$\therefore P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \ldots + P(A_n)P(B|A_n) \ldots\ldots\ldots\ldots(1)$$

Total probability

$$P(B) = \sum_{i=1}^{n} (A_i \cap B) = \sum_{i=1}^{n} P(A_i)P(B|A_i)$$

The above formula is sometimes called the **theorem of total probability** or the **rule of elimination**.

On the other hand, for any *i*, the **conditional probability** of A_i given B is defined by

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Substitute value of **P(B)** from (1) and

$P(A_i \cap B) = P(A_i)P(B|A_i)$ in (2), we get

$$\therefore P(A_i|B) = \frac{P(A_i)P(B|A_i)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \ldots + P(A_n)P(B|A_n)}$$

Bayes' Theorem

Suppose $A_1, A_2, \ldots, A_n$ is a partition of **S** and B is any event. Then for any i,

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \ldots + P(A_n)P(B|A_n)}$$

or

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{i=1}^{n} P(A_i)P(B|A_i)}$$

What is Bayes' Law?

Formula: $P(A|B) = \dfrac{P(B|A)P(A)}{P(B)}$

Bayes' Law is a formula for determining conditional probabilities.

Components:

$P(A|B)$: Posterior probability

$P(B|A)$: Likelihood

$P(A)$: Prior probability

$P(B)$: Marginal likelihood

Scenario: Classifying Emails as Spam or Not Spam

You are tasked with classifying an email as spam or not spam using Bayes' Law.

Given:

Prior: P(Spam) = 0.2

Likelihood: P(Word = 'win' | Spam) = 0.8

Marginal likelihood: P(Word = 'win') = 0.3

Objective

Calculate the **posterior probability** of the **email being spam**, given that the word **'win'** appears in the email.

Calculate: P(Spam | Word = 'win')

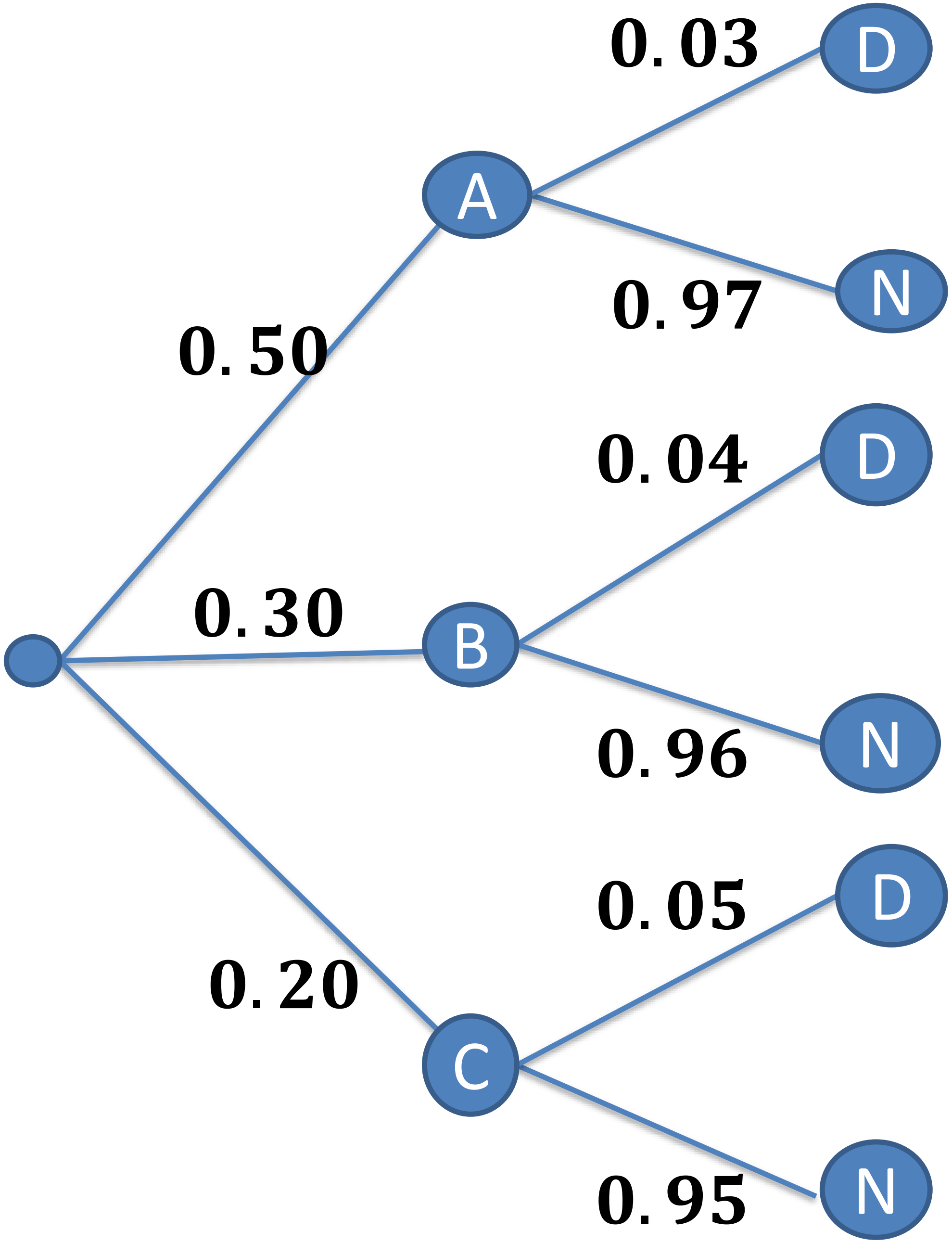
$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$$P(Spam \mid Word='win') = \frac{P(Word='win' | Spam) \times P(Spam)}{P(Word='win')}$$

$$P(Spam|Word='win') = \frac{0.80 \times 0.20}{0.30}$$

$$= 0.533$$

Example: Three machines ***A, B*** and ***C*** produce respectively **50%**, **30%** and **20%** of the total number of items of a factory. The percentages of **defective output** of these machines are **3%**, **4%** and **5%.** If an item is selected at random, find the probability that the item is **defective**.


0.50
0.30
0.20
A
B
C
0.03
0.97
0.04
0.96
0.05
0.95
D
N
D
N
D
N

Solution:

Let **D** be the event that the item is **defective**

$\because$ $P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \ldots + P(A_n)P(B|A_n)$

$\therefore$ $\boldsymbol{P(D) = P(A)P(D|A) + P(B)P(D|B) + P(C)P(D|C)}$

$= (0.50)(0.03) + (0.30)(0.04) + (0.20)(0.05)$

$= \mathbf{0.037}$ **(or 3.7%)**

Example: Consider the factory in the preceding example. Suppose an item is selected at random and is found to be **defective**. Find the probability that the item was produced by **machine A**; that is, find **$P(A|D)$.**

Solution:

By Bayes' theorem,

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{i=1}^{n} P(A_i)P(B|A_j)}$$

$$P(A|D) = \frac{P(A)P(D|A)}{P(A)P(D|A) + P(B)P(D|B) + P(C)P(D|C)}$$

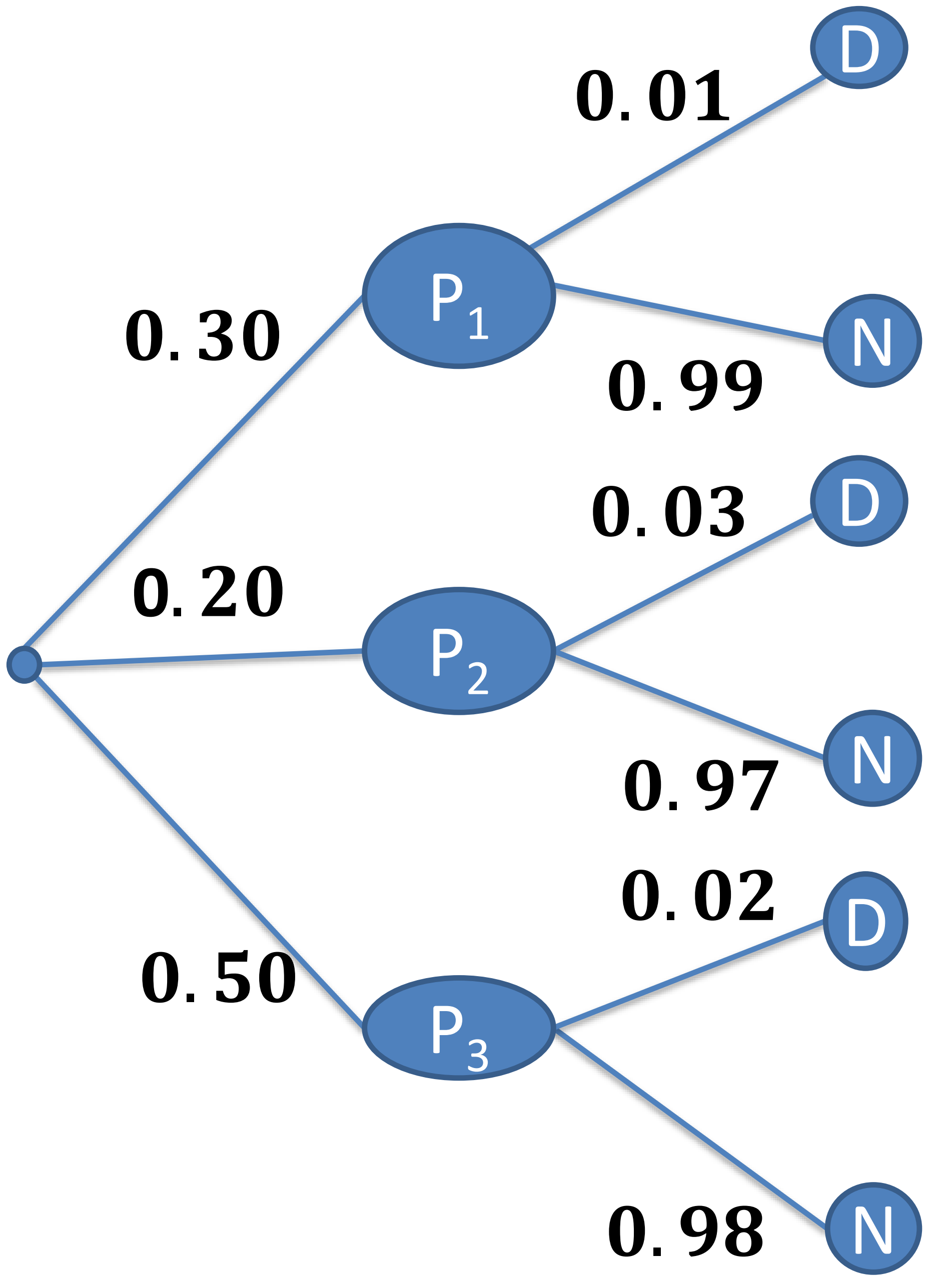
$$P(A|D) = \frac{(0.50)(0.03)}{(0.50)(0.03) + (0.30)(0.04) + (0.20)(0.05)}$$

$$= \frac{15}{37} \text{ (or 0.4054)}$$

Example: A manufacturing firm employs **three** analytical plans for the design and development of a particular product. For cost reasons, **all three** are used at varying times. In fact, plans **1**, **2**, and **3** are used for **30%**, **20%**, and **50%** of the products, respectively. The defect rate is different for the three procedures as follows:

$$P(D|P_1) = 0.01,\ P(D|P_2) = 0.03,\ P(D|P_3) = 0.02,$$

where $P(D|P_j)$ is the probability of a defective product, given **plan *j***. If a random product was observed and found to be **defective**, which **plan** was **most likely** used and thus responsible?


0.30
0.20
0.50
P1
P2
P3
0.01
0.99
0.03
0.97
0.02
0.98
D
N
D
N
D
N

Solution: Given

$P(P_1) = 0.30$, $P(P_2) = 0.20$, and $P(P_3) = 0.50$,

$P(D|P_1) = 0.01$, $P(D|P_2) = 0.03$, $P(D|P_3) = 0.02$

We have to find **$P(P_j|D)$ for $j = 1, 2, 3$**.

By Bayes' theorem,

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{i=1}^{n} P(A_i)P(B|A_j)}$$

$$P(P_1|D) = \frac{P(P_1)P(D|P_1)}{P(P_1)P(D|P_1) + P(P_2)P(D|P_2) + P(P_3)P(D|P_3)}$$

$$= \frac{(0.30)(0.01)}{(0.30)(0.01)+(0.20)(0.03) + (0.50)(0.02)}$$

$$= \frac{0.003}{0.019}$$

$$= \mathbf{0.158}$$

$$P(P_2|D) = \frac{P(P_2)P(D|P_2)}{P(P_1)P(D|P_1) + P(P_2)P(D|P_2) + P(P_3)P(D|P_3)}$$

$$= \frac{(0.20)(0.03)}{(0.30)(0.01)+(0.20)(0.03) + (0.50)(0.02)} = \mathbf{0.316}$$

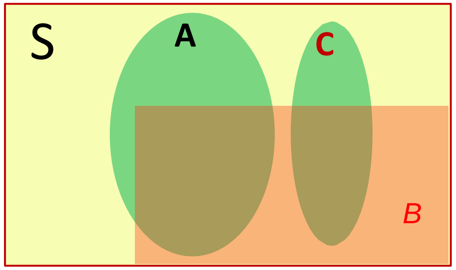
$$P(P_3|D) = \frac{P(P_3)P(D|P_3)}{P(P_1)P(D|P_1) + P(P_2)P(D|P_2) + P(P_3)P(D|P_3)}$$

$$= \frac{(0.50)(0.02)}{(0.30)(0.01)+(0.20)(0.03) + (0.50)(0.02)} = \mathbf{0.526.}$$

The conditional probability of a defect given **plan 3** is the largest of the **three**; thus a defective for a random product is most likely the result of the use of **plan 3**

Conditional probabilities share properties of ordinary probabilities

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ assuming } P(B) > 0$$



I. $P(A \mid B) \geq 0$

II. $P(S \mid B) = \frac{P(S \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

III. $P(B \mid B) = \frac{P(B \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

Conditional probabilities share properties of ordinary probabilities

If **A ∩ C = Ø** then $P(A \cup C \mid B) = P(A \mid B) + P(C \mid B)$

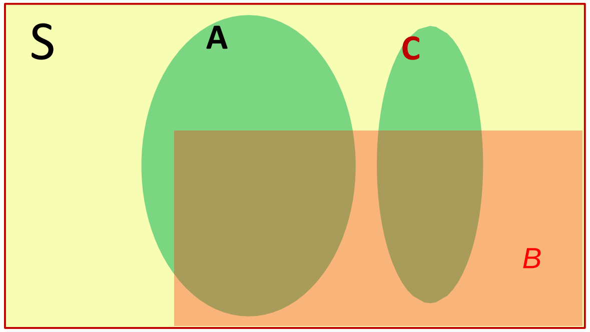
$$P(A \cup C \mid B) = \frac{P((A \cup C) \cap B)}{P(B)}$$

$$P(A \cup C \mid B) = \frac{P((A \cap B) \cup (C \cap B))}{P(B)}$$

$$P(A \cup C \mid B) = \frac{P((A \cap B) + (C \cap B)\ )}{P(B)}$$

$$= \frac{P(A \cap B)}{P(B)} + \frac{P\,(C \cap B)}{P(B)}$$

$$= P(A|B) + P(C|B)$$



Conditional probabilities share properties of ordinary probabilities

Since conditional probabilities satisfy all of **the probability axioms, any formula or theorem that we** ever derive for ordinary probabilities will remain true for conditional probabilities as well.

Model based on conditional probabilities

- Let us now examine what conditional probabilities are good for.
- They are used to **revise a model** when we get **new information**, but there is another way in which they arise.
- We can use **conditional probabilities** to **build a multi-stage model** of a probabilistic experiment.

Model based on conditional probabilities

Example Radar Detection. If an **aircraft is present** in a certain area, a **radar detects** it and generates **an alarm** signal with **probability 0.99**. If an **aircraft is not present** the radar generates a **(false) alarm**, with probability **0.10**. We assume that an **aircraft is present** with probability **0.05**. What is the probability of **no aircraft presence** and a **false alarm**? What is the probability of **aircraft presence** and **no detection?**

Model based on conditional probabilities

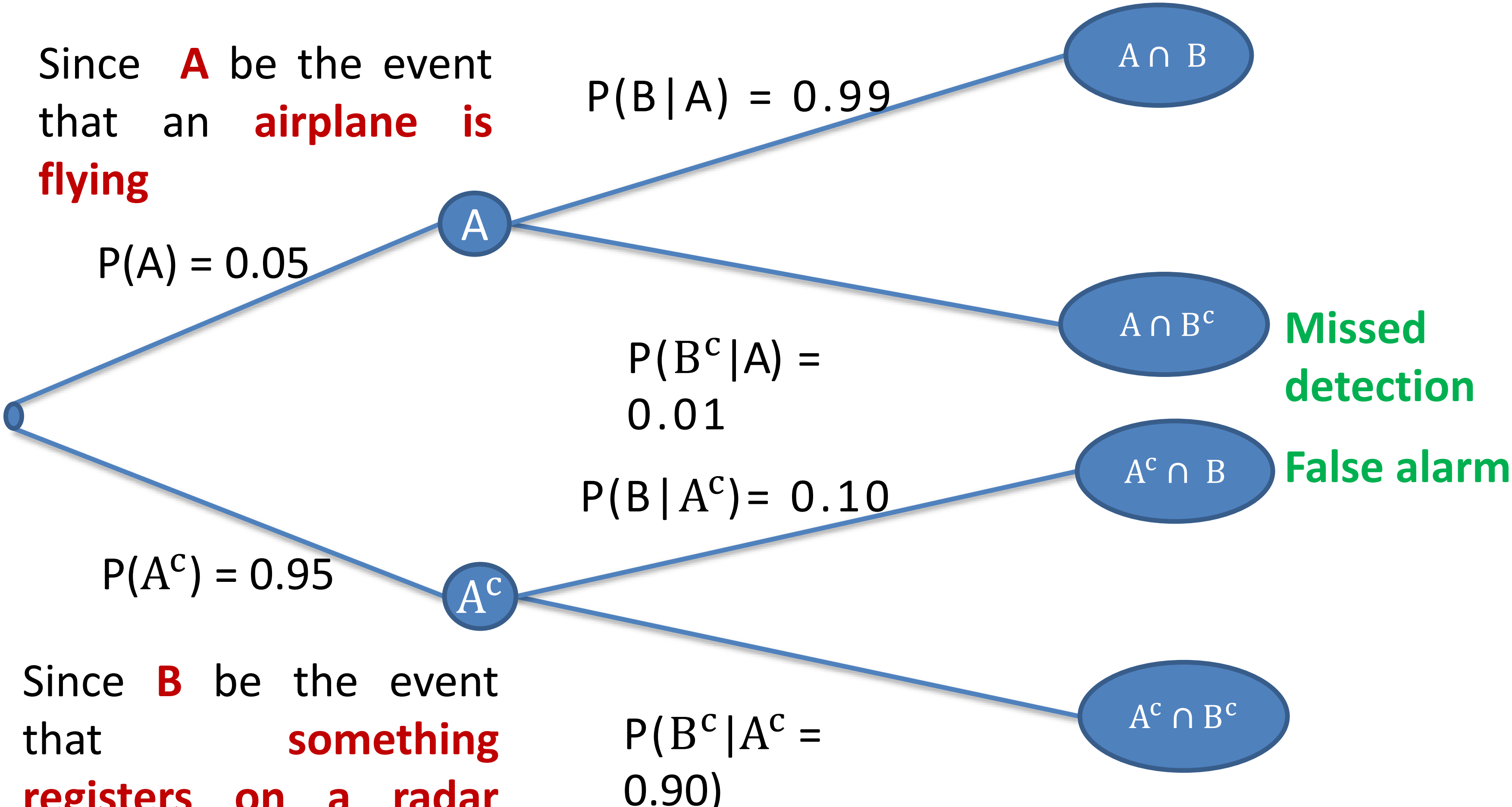
Let **A** be the event that an **airplane is flying**

Let $\mathbf{A^c}$ be the event that an **airplane is not flying**

Let **B** be the event that **something registers on a radar screen**.

Let $\mathbf{B^c}$ be the event that **something does not registers on a radar screen**.

Since **A** be the event that an **airplane is flying**



Since **B** be the event that **something registers on a radar screen**

$P(B|A) = \dfrac{P(A \cap B)}{P(A)}$

$P(A \cap B)$ $= P(A) \times P(B|A)$

$= 0.05 \times 0.99$ = **0.0495 or 4.95 %**

$P(B)$ $= P(A \cap B) + P(A^c \cap B)$ **(total probability)**

$= P(A) \times P(B|A) + P(A^c) \times P(B|A^c)$

$= 0.05 \times 0.99 + 0.95 \times 0.10$ **= 0.1445 or 14.45%**

$P(A|B)$ $= \dfrac{P(A \cap B)}{P(B)}$

$= \dfrac{0.0495}{0.1445}$ **= 0.3426 or 34.26%**

P(not present , false alarm) = $P(A^c \cap B)$

$= P(A^c)P(B|A^c)$

$= 0.95 \times 0.10$

= **0.095 or 9.5 % ans**

P(present , no detection) **= $P(A \cap B^c)$**

$= P(A)P(B^c\,|A)$

$= 0.05 \times 0.01$

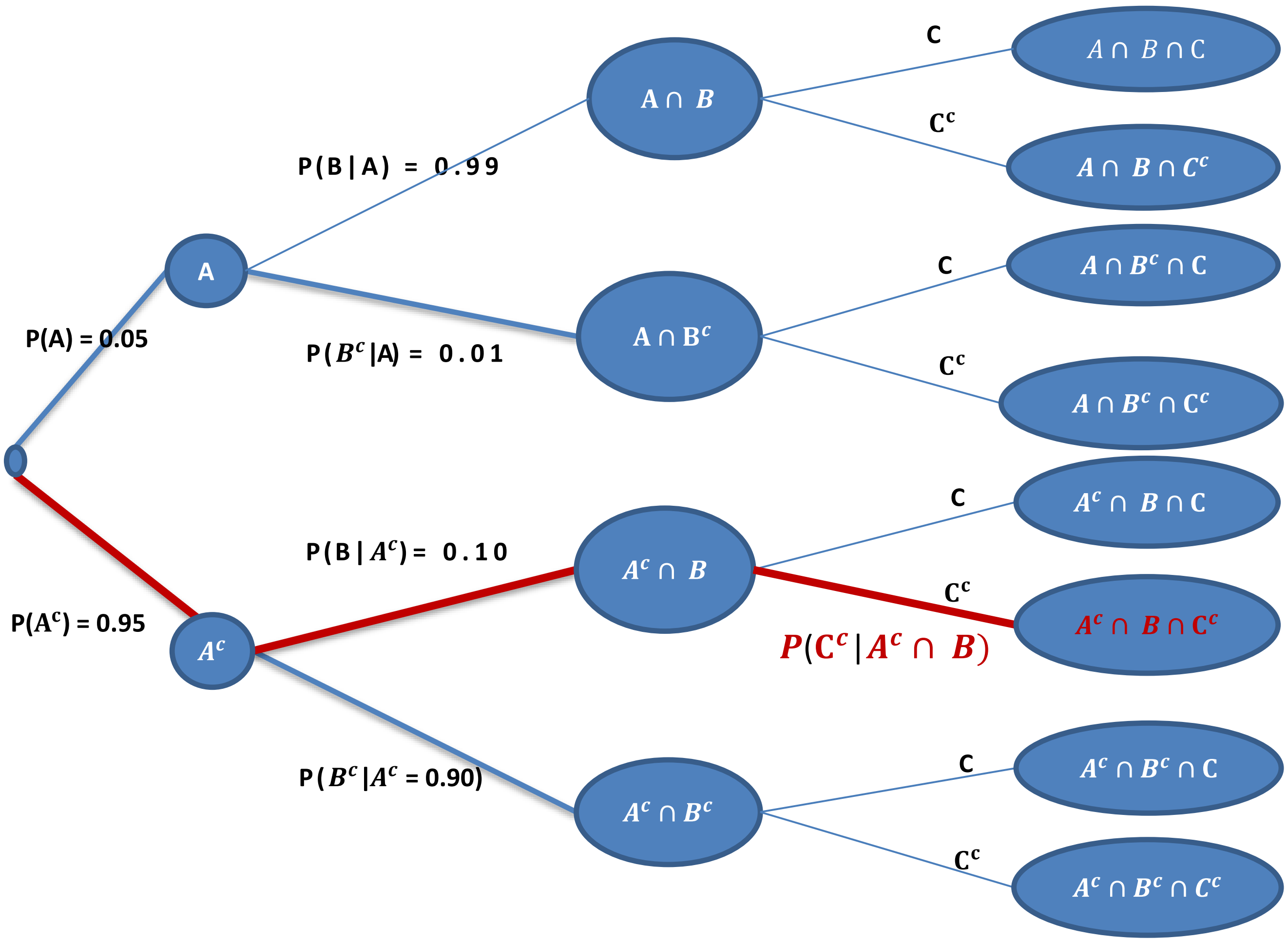
= **0.0005 or 0.05 % ans**

The multiplication rule

$$\because P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$\Rightarrow P(A \cap B) = P(A) \times P(B|A)$$

$$\Rightarrow P(A \cap B) = P(B) \times P(A|B)$$



P(A) = 0.05
P(A^c) = 0.95
A
A^c
P(B | A) = 0.99
P(B^c |A) = 0.01
P(B | A^c)= 0.10
P(B^c |A^c = 0.90)
A ∩ B
A ∩ B^c
A^c ∩ B
A^c ∩ B^c
C
C^c
A ∩ B ∩ C
A ∩ B ∩ C^c
A ∩ B^c ∩ C
A ∩ B^c ∩ C^c
A^c ∩ B ∩ C
A^c ∩ B ∩ C^c
A^c ∩ B^c ∩ C
A^c ∩ B^c ∩ C^c
P(C^c | A^c ∩ B)

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$\mathbf{P(A^c \cap B \cap C^c)} = \mathbf{P((A^c \cap B) \cap C^c)}$$
$$= \mathbf{P(A^c \cap B) \times P(C^c | A^c \cap B)}$$
$$= P(\mathbf{A^c}) \times P(B|\mathbf{A^c}) \times \mathbf{P(C^c | A^c \cap B)}$$

$$\mathbf{P(A_1 \cap A_2 \cap A_3 \ldots \cap A_n) = (A_1) \prod_{i=2}^{n} P(A_i | A_1 \cap A_2 \ldots \cap A_{i-1})}$$

- **And this is the most general version of the multiplication** rule and allows you to calculate the probability of several events happening by multiplying probabilities and conditional probabilities.

Bayes' rule and inference

- Bayesian inference
  - initial beliefs $P(A_i)$ on possible causes of an observed event ***B***
  - model of the world under each A_i: $P(B \mid A_i)$

$$A_i \xrightarrow[P(B \mid A_i)]{\text{model}} B$$

– **draw conclusions about causes**

$$B \xrightarrow[P(A_i \mid B)]{\text{Inference}} A_i$$

Bayesian method

- Using **Bayes' rule**, a statistical methodology called the **Bayesian approach** has attracted a lot of attention in **applications**.

- An introduction to the **Bayesian method** will be discussed in coming lectures.