

Probability and Statistics

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Textbooks

- ❑ **Probability & Statistics for Engineers & Scientists**, Ninth Edition, Ronald E. Walpole, Raymond H. Myer
- ❑ **Elementary Statistics: Picturing the World**, 6th Edition, Ron Larson and Betsy Farber
- ❑ **Elementary Statistics**, 13th Edition, Mario F. Triola

Three things cannot be long hidden: the sun, the moon, and the truth

Buddha

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- ❑ **Elementary Statistics: Picturing the World**, 6th Edition, Ron Larson and Betsy Farber
- ❑ **Elementary Statistics**, 13th Edition, Mario F. Triola

Reference books

- ❑ **Probability and Statistical Inference, Ninth Edition,** Robert V. Hogg, Elliot A. Tanis, Dale L. Zimmerman
- ❑ **Probability Demystified,** Allan G. Bluman
- ❑ **Schaum's Outline of Probability,** Second Edition, Seymour Lipschutz, Marc Lipson
- ❑ **Python for Probability, Statistics, and Machine Learning,** José Unpingco
- ❑ **Practical Statistics for Data Scientists: 50 Essential Concepts,** Peter Bruce and Andrew Bruce
- ❑ **Think Stats: Probability and Statistics for Programmers,** Allen Downey

References

Readings for these lecture notes:

Probability & Statistics for Engineers & Scientists, Ninth Edition, Ronald E. Walpole, Raymond H. Myer

Elementary Statistics, 10th Edition, Mario F. Triola

<http://mathworld.wolfram.com/CircularPermutation.html>

<https://www.zero-factorial.com/whatis.html>

<http://mathworld.wolfram.com/CircularPermutation.html>

<http://doubleroot.in/lessons/permutations-combinations/circular-permutations-examples/#.WbeY04-cHIU>

<http://www.onlinemathlearning.com/combinations.html>

These notes contain material from the above resources.

Counting Sample Points

- ❑ In many cases, we shall be able to solve a probability problem by counting the number of points in the **sample space without actually listing each element**.
- ❑ The **fundamental principle of counting**, often referred to as the **multiplication rule**, is stated in Rule 2.1.

Rule 2.1: If an operation can be performed in n_1 ways, and if for each of these ways a second operation can be performed in n_2 ways, then the two operations can be performed together in $n_1 n_2$ ways.

□ **Example** : How many **sample points** are there in the **sample space** when a **pair of dice** is thrown once?

Solution :

□ The first die gives us, $n_1 = 6$ ways.

□ For each of these **6 ways**, the **second die** gives us, $n_2 = 6$ ways.

□ Therefore, the pair of dice give us $n_1 n_2 = (6)(6) = 36$ possible ways.

Example : If a **22-member club** needs to elect a **chair** and a **treasurer**, how many **different ways** can these **two** to be elected?

Solution :

□ For the **chair position**, we have $n_1 = 22$ ways

□ For the **treasurer** position, for each of those **21 possibilities**, we have $n_2 = 21$ ways

□ Total number of ways = $n_1 \times n_2 = 22 \times 21 = 462$

□ **Rule 2.2:** If an operation can be performed in n_1 ways, and **if for each of these a second operation** can be performed in n_2 ways, and for each of the **first two a third operation** can be performed in n_3 ways, and so forth, then the sequence of k operations can be performed in $n_1 n_2 \cdots n_k$ ways.

□ **Example:** Sam is going to assemble a computer by himself. He has the choice of chips from **two brands**, a **hard drive** from **four**, **memory** from **three**, and an **accessory bundle** from **five** local stores. How many **different ways** can Sam order the parts?

Solution :

$$n_1 = 2 \text{ (No of brands)}$$

$$n_2 = 4 \text{ (No of hard drives)}$$

$$n_3 = 3 \text{ (No of memory sticks)}$$

$$n_4 = 5 \text{ (No of accessory bundles)}$$

$$\square \text{Total number of ways} = n_1 \times n_2 \times n_3 \times n_4 = 2 \times 4 \times 3 \times 5 = 120$$

Permutation

Permutation: A **permutation** is an arrangement of all or part of a set of objects.

OR

An arrangement of a set of **n objects** in a **given order** is called a ***permutation*** of the objects (taken all at a **time**).

Example: Consider the three **letters a , b , and c** .

The possible **permutations** are **abc , acb , bac , bca , cab , and cba** .

Permutation

□ **Definition** For any **non-negative integer n** , **$n!$** , called “ **n factorial**,” is defined as **$n! = n(n - 1) \cdots (2)(1)$** , with special case **$0! = 1$** .

□ **Theorem 2.1:** The number of permutations of **n objects** is **$n!$** .

□ **Example** The number of permutations of the **four letters a, b, c** , and **d** will be **$4! = 24$** .

Why 0! one?

The idea of the factorial (in simple terms) is used to compute the number of permutations (combinations) of arranging a set of **n numbers**.

n	Number of permutations (n!)	Visual examples
1	1	{1}
2	2	{1, 2}, {2, 1}
3	6	{1, 2, 3}, {1, 3, 2}, {2, 1, 3}, {2, 3, 1}, {3, 1, 2}, {3, 2, 1}
⋮	⋮	⋮
0	1	{ }

Why 0! one?

$$n! = n \times (n - 1)!$$

$$\Rightarrow (n - 1)! = \frac{n!}{n}$$

Substitute 1, we get

$$\Rightarrow (1 - 1)! = \frac{1!}{1}$$

$$\Rightarrow 0! = 1$$

Permutations Rule (When items are all different)

□ **Theorem 2.2:** The number of permutations of **n** distinct objects taken **r** at a time is ${}_nP_r = \frac{n!}{(n-r)!}$, where **$r \leq n$**

Permutations Rule (When items are all different)

1. There are n *different* items available. (This rule **does not apply** if some of the items are identical to others.)
2. We **select r** of the n items (without replacement).
3. We consider **rearrangements** of the **same items** to be **different sequences**. (The permutation of **ABC** is different from **CBA** and is counted separately)

Permutations Rule (When items are all different)

□ If the preceding **requirements are satisfied**, the number of **permutations (or sequences)** of **r items** selected from **n different** available items **(without replacement)** is ${}_nP_r = \frac{n!}{(n-r)!}$, where **$r \leq n$** .

Example : In one year, **three awards (research, teaching, and service)** will be given to a class of **25** graduate students in a statistics department. If each student can receive at **most one award**, how many possible selections are there?

Solution : Since the awards are **distinguishable**, it is a permutation problem. The total number of sample points is

$${}_{25}P_3 = \frac{25!}{(25-3)!} = \frac{25!}{22!} = 25 \times 24 \times 23 = 13800$$

OR

$$\text{Total number of ways} = 25 \times 24 \times 23 = 13800$$

Example Clinical Trial of New Drug When testing a new drug, Phase I involves **only 8 volunteers**, and the objective is to assess the drug's safety. To be very cautious, you plan to treat the **8 subjects in sequence**, so that any particularly adverse effect can allow for stopping the treatments before any other subjects are treated. If **10 volunteers** are available and **8 of them are to be selected**, how **many different sequences of 8 subjects** are possible?

Solution We have $n = 10$ different subjects available, and we plan to **select** $r = 8$ of them **without replacement**. The number of different sequences of arrangements is found as shown

$${}_{10}P_8 = \frac{10!}{(10-8)!} = \frac{10!}{2!} = 1814400$$

Or

$$\text{Total number of ways} = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 = 1814400$$

Example Find the number of ways of forming four-digit codes in which no digit is repeated.

Solution

To form a four-digit code with no repeating digits, you need to select 4 digits from a group of 10, so $n = 10$ and $r = 4$.

$${}_{10}P_4 = \frac{10!}{(10-4)!} = \frac{10!}{6!} = 5040$$

Or

$$\text{Total number of ways} = 10 \times 9 \times 8 \times 7 = 5040$$

Example How many 4-digit numbers are there with no digit repeated?

Solution

$$\begin{aligned}\text{Total number of ways} &= 9 \times 9 \times 8 \times 7 \\ &= 4536\end{aligned}$$

Example Forty-three race cars started the **2013 Daytona 500**. How many ways can the cars finish first, second, and third?

Solution

You need to select three race cars from a group of 43, so $n = 43$ and $r = 3$. Because the order is important, the number of ways the cars can finish first, second, and third is

$${}_{43}P_3 = \frac{43!}{(43 - 3)!} = \frac{43!}{40!} = 74046.$$

Or

$$\text{Total number of ways} = 43 \times 42 \times 41 = 74046.$$

Theorem 2.3: The number of **permutations** of n **objects** arranged in a circle is $(n - 1)!$.

Example In how many ways can **6 people** be seated at a **round table**?

Solution

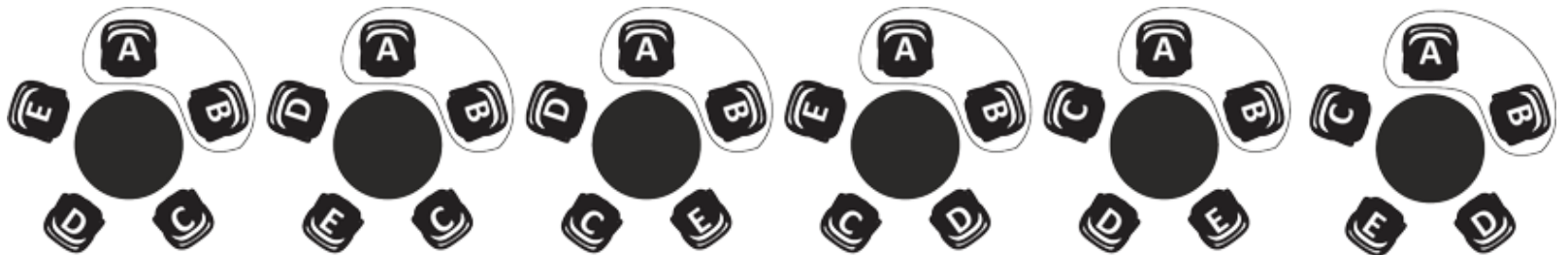
Here $n = 6$ (total number of people)

The total number of ways = $(6 - 1)! = 120$

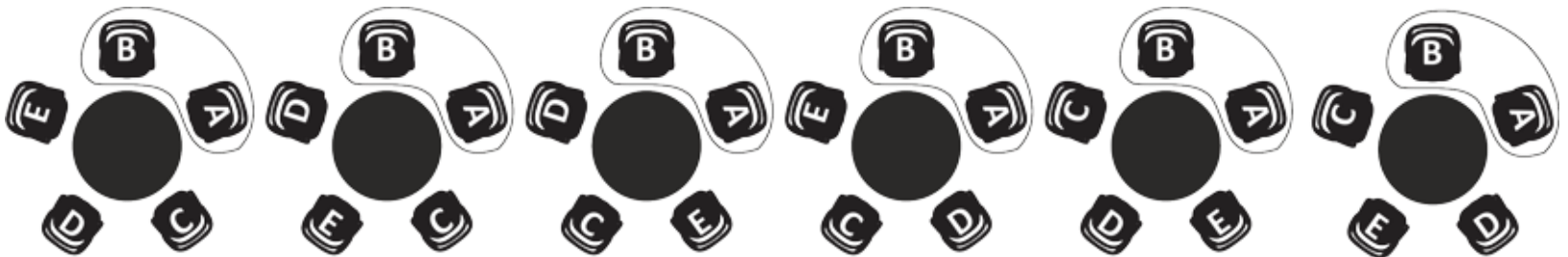
Example Find the number of ways in which **5 people A, B, C, D, E** can be seated at **a round table**, such that

- a. A and B** must always sit together.
- b. C and D** must not sit together.

□ **Solution a.** If we wish to seat **A and B** together in all arrangements, we can consider these **two as one unit**, along with 3 **others**. So effectively we've to arrange **4 people** in a circle. The number of ways = **$(4 - 1)! = 6$**



But in each of these arrangements, **A** and **B** can themselves interchange places in **2 ways**.



Therefore, the total number of ways will be **$6 \times 2 = 12$** .

5 people A, B, C, D, E

- ❑ **b.** The total number of ways will be $(5 - 1)!$ or **24**.
- ❑ Similar to **a.** above, the number of cases in which **C** and **D** are seated together, will be **12**.
- ❑ Therefore the required number of ways = **24 - 12 = 12**.

Permutations when repetition is allowed

- ❑ So far we have considered **permutations** of **distinct objects**. That is, all the objects were completely different or distinguishable.
- ❑ Obviously, if the letters **b** and **c** are both equal to **x**, then the **6 permutations** of the letters **a, b, and c** become **axx, axx, xax, xax, xxa, and xxa**, of which only **3** are distinct.
- ❑ Therefore, with **3 letters, 2 being the same**, we have **$3!/2! = 3$** distinct permutations.

Permutations Rule (When some items are identical to others)

□ **Theorem 2.4:** The number of **distinct permutations** of **n things** of which **n_1** are of one kind, **n_2** of a second kind, . . . , **n_k** of a **k^{th}** kind is

$$\frac{n!}{n_1!n_2! \cdots n_k!}$$

where $n_1 + n_2 + n_3 + \dots + n_k = n$.

Permutations Rule (When some items are identical to others)

Requirements

1. There are n items available, and some items are identical to others.
2. We select all of the n items (without replacement).
3. We consider rearrangements of distinct items to be different sequences.

If the preceding requirements are satisfied, and if there are n_1 alike, n_2 alike, \dots , n_k alike, the number

$$\frac{n!}{n_1!n_2! \cdots n_k!}$$

Example A building contractor is planning to develop a subdivision. The subdivision is to consist of **6 one-story houses**, **4 two-story houses**, and **2 split-level houses**. In how many **distinguishable** ways can the houses be arranged?

Solution

The total number of arrangements = $\frac{12!}{6! 4! 2!}$

$$= 13,860$$

distinguishable ways

Example Calculate the number of distinguishable permutations of the letters **AAAABBC**.

Solution

The total number of arrangements = $\frac{7!}{4! 2! 1!}$

$$= 105$$

distinguishable ways

□ **Example** In a college football training session, the defensive coordinator needs to have **10 players** standing in a row. Among these **10 players**, there are **1 freshman**, **2 sophomores**, **4 juniors**, and **3 seniors**. How many **different ways** can they be arranged in a row if only their **class level** will be distinguished?

□Solution

$n = 10$ (Total number of players)

$n_1 = 1$ (Total number of freshman)

$n_2 = 2$ (Total number of sophomores)

$n_3 = 4$ (Total number of juniors)

$n_4 = 3$ (Total number of seniors)

$$\begin{aligned}\text{The total number of arrangements} &= \frac{10!}{1! 2! 4! 3!} \\ &= 12,600.\end{aligned}$$

Theorem 2.5: The number of ways of **partitioning** a set of **n objects** into **r cells** with **n_1 elements** in the first cell, **n_2 elements** in the second, and so forth, is

$$\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}$$

where **$n_1 + n_2 + \cdots + n_r = n$**

Example : In how many ways can **7 graduate** students be assigned to **1 triple** and **2 double** hotel rooms during a conference?

Solution :

$$\frac{n!}{n_1!n_2! \cdots n_k!}$$

Here $n = 7$

$$n_1 = 3$$

$$n_2 = 2$$

$$n_3 = 2$$

The total number of possible partitions would be

$$\frac{7!}{3! 2! 2!} = 210$$

Theorem 2.6: The number of combinations of ***n*** distinct objects taken ***r*** at a time is

$${}_nC_r = \frac{n!}{r!(n-r)!}, \text{ where } r \leq n.$$

Combinations Rule

Requirements

1. There are ***n different*** items available.
2. We select ***r*** of the ***n items*** (**without replacement**).
3. We consider rearrangements of the **same items** to **be the same**. (The combination ***ABC*** is the same as ***CBA***.)

If the preceding requirements are satisfied, the number of **combinations** of ***r items*** selected from ***n different items*** is

$${}_nC_r = \frac{n!}{r!(n-r)!}$$

Combination vs. Permutations

❑ The **2-permutations** of the letters **A, B, C, and D** are:

AB, AC, AD, BA, BC, BD, CA, CB, CD, DA, DB, DC.

❑ The combinations of two out of these four letters are:

AB, AC, AD, BC, BD, CD.

(Since the elements of a combination are unordered, **BA** is not viewed as being distinct from **AB**.)

❑ **Example:** In a lottery, each ticket has **5 one-digit numbers 0-9** on it.

a) You win if your ticket has the **digits in any order**. What are your chances of winning?

b) You would win only if your ticket has the digits in **the required order**. What are your chances of winning?

Solution:

There are 10 digits to be taken 5 at a time.

a) The number of ways of **selecting 5** tickets **from 10** is

$${}_{10}C_5 = \frac{10!}{5!(10-5)!} = \frac{10 \times 9 \times 8 \times 7 \times 6}{5!} = 252$$

∴ The chances of winning are **1 out of 252 (0.0040 or 0.3968 %**

□b) Since the order matters, we should use permutation instead of combination.

$${}_{10}P_5 = \frac{10!}{(10-5)!} = \frac{10!}{5!} = 10 \times 9 \times 8 \times 7 \times 6 = \mathbf{30240}$$

Or

$$\text{Total number of ways} = 10 \times 9 \times 8 \times 7 \times 6 = \mathbf{30240}$$

∴ The chances of winning are 1 out of 30240
(**0.0000330668** or **0.0033 %**)

American Powerball jackpot

- **American Powerball Jackpot**
- The Powerball lottery is one of the most famous lotteries in the U.S.
- **Probability of winning the jackpot is extremely low.**
- **Winning Criteria:**
 - **Match all five white balls** (numbered from 1 to 69).
 - **Match the red Powerball** (numbered from 1 to 26).
- **Winning the Jackpot:**
 - A player must correctly choose all **six numbers (5 white + 1 red)** to win.

- Powerball Jackpot Variability:**

- Prize depends on ticket sales and rollovers.

- Minimum starting jackpot:** \$20 million.

- Jackpot increases with each rollover, often reaching hundreds of millions or over a billion dollars.

- Claiming the Jackpot:**

- Annuity Option:**

- Paid in 30 graduated payments over 29 years.
- Payments increase by 5% annually.

- Cash Option:**

- Lump sum, around 60% of the total jackpot before taxes.
- E.g., for a \$500 million jackpot, cash option is around \$300 million before taxes.

- Tax Considerations:**

- Federal and state taxes reduce the final payout.

Probability of Winning the Powerball Jackpot

- **Match Requirement:**

- 5 White Balls (1-69)
- 1 Red Powerball (1-26)

- **Combinations:**

- White Balls: ${}_{69}C_5 = 11,238,513$
- Powerball: ${}_{26}C_1 = 26$

- **Total Possible Outcomes:**

- $11,238,513 \times 26 = 292,201,338$

- **Probability of Winning:**

- $\frac{1}{292,201,338}$ Approximately 1 in 292 million
(or one in 29.2 crore)