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On the Renormalization of Dirac Fields in Metric-Affine and Teleparallel Gravity

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Abstract

Metric-Affine theories of Gravity (MAGs) are a broad class of alternative theories of gravitation that generalize the structure of spacetime itself. Metrics and connections are now independent variables on a general differentiable manifold, representing different degrees of freedom a priori unrelated. Depending on the imposed constraints and symmetry requirements, it is then possible to retrieve a number of known gravitational theories, the most notable being General Relativity (GR) and the Teleparallel Equivalent of GR (TEGR). Whereas the first assumes vanishing torsion and can therefore be entirely described in terms of curvature through the metric tensor, TEGR allows connections to be torsionful and independent from the metric, but assumes zero curvature. As such, it can be formulated as a gauge theory of the translation group, with torsion playing the role of field strength, bringing gravity back to be conceived as a *force* in the Yang-Mills sense, rather than a geometrical effect.

Among the properties that make TEGR an interesting framework, the most striking is, as the name suggests, its equivalence with General Relativity. Their actions only differ by a boundary term, making the two theories dynamically equivalent at classical level. Because this equivalence is in principle expected to be spoiled upon quantization, we aim to study the loop corrections that quantum fields bring to the background geometry. In order to do this, we adopt the Heat Kernel method to calculate the one-loop effective action of a massive, uncharged Dirac spinor non-minimally coupled to gravity through torsion. This will be done in the more general MAG scenario for a number of reasons, the main being generality. Taking the limits of GR and TEGR is then a matter of imposing the aforementioned assumptions, and a direct comparison of the respective renormalized actions can be made.

*"Téchne was born not as an expression of the human spirit,
but as a remedy for its biological shortcomings."*

Contents

Notation

- Greek letters μ, ν, .. are *spacetime* indices (on the manifold), and are raised and lowered by the metric $g_{\mu\nu}$
- Latin letters a, b, .. are *frame* indices (on the tangent space) and are raised and lowered by the Minkowski metric

$$\eta_{ab} = \mathrm{diag}(+, -, -, -)$$

  Other indices such as internal and gauge ones are conveniently defined in each section.
- Quantities reported in bold refer to their indexed counterpart, so as not to write their components in a cumbersome way: $\mathbf{\Gamma}$ is the connection appearing in covariant derivatives.
- On the same note, some contractions are reported without indices, such as $A_\mu B^\mu \equiv AB$.
- Quantities with a '$\circ$' on top of them refer to their expression in General Relativity:
  $\overset{\circ}{\nabla}$ is the covariant derivative with $\mathbf{\Gamma} = \overset{\circ}{\mathbf{\Gamma}}$ the Levi-Civita connection.
- Quantities with a '$\bullet$' on top of them refer to their expression in Teleparallel Gravity:
  $\overset{\bullet}{\mathcal{D}}$ is the covariant derivative with $\mathbf{\Gamma} = \overset{\bullet}{\mathbf{\Gamma}}$ the Weitzenböck connection.
- [..] and (..) represent the antisymmetric and symmetric part of a symbol with respect to the indices in their argument. $\gamma^{[a}\gamma^{b]} = \frac{1}{2}(\gamma^a\gamma^b - \gamma^b\gamma^a)$, $\overset{\circ}{\Gamma}{}^\rho{}_{[\mu\nu]} = 0$, etc.

1 Introduction

All four currently known fundamental interactions manifest themselves in *spacetime*, a pseudo-Riemannian differentiable manifold M which can be entirely described by the pair $(\boldsymbol{g}, \boldsymbol{\Gamma})$: a metric and a connection that set the notions of distance and parallel transport.

The electromagnetic, electroweak, and strong forces are described by *gauge* theories of their underlying symmetries, and the corresponding matter fields live in principal bundles, internal spaces with a copy of the group at each point of spacetime. These make up the *Standard Model* of particle physics, a staggeringly effective framework for the description of ordinary matter.

Gravity, on the other hand, is set in the tangent bundle TM.[1] At each point p on M there is a tangent space T_pM and a set of bases from its atlas. This set forms a bundle which has a tangent space itself, the *tangent bundle of frames*. Here, a *solder* 1-form can be defined as the map that relates each one of these to a tangent vector space, which we identify with Minkowski spacetime $\mathbb{M}$.[2] It is what allows us to say that every point on the manifold has a tangent space that is locally indistinguishable from flat space.

The soldering tool with which this is done is tetrads. These structures glue together the coordinate charts of M to a preferred basis of $\mathbb{M}$, allowing for computations to be carried out in a much simpler way by relating internal (principal) and external (manifold) indices to each other.[3] They allow for new contractions of tensors that would otherwise be absent in the gauge realm and give us access to *torsion*, a tensor identified with the antisymmetric part of connections that can be intuitively seen as the "cracking" of infinitesimal parallelograms, and that modifies fundamental relations such as the Bianchi identities and the commutator of covariant derivatives, providing new coupling possibilities to fields as well.

Depending on the approach, connections have to satisfy a number of constraints. In General Relativity (GR), gravity is conceived as a replacement of the Minkowski metric $\eta_{\mu\nu}$ with a Riemannian metric $g_{\mu\nu}$, in principle, defined everywhere. The most natural choice is then to pick up the Levi-Civita connection, which is the only symmetric and metric-compatible one, therefore uniquely determined by $\boldsymbol{g}$. This formulation has the metric tensor as the only fundamental field of the theory, making the gravitational interaction a purely geometrical effect.[4]

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1. Formally the disjoint union of tangent spaces of M.
2. A very comprehensible view is given in Sec. 3.2 of [1]
3. The tetrad formalism is notoriously inevitable when coupling fermions to gravity [2]
4. In General Relativity one never talks about *force* of gravity, inertial observers follow geodesic paths dictated by the mass-energy distribution that curves spacetime. This design is physically motivated by noting that when torsion vanishes (or when it is fully antisymmetric),

One can then envisage alternative (or *extended*) theories for gravitation that start with the same structures, but impose different restrictions on connections. In *Metric-Affine* theories (MAG for short), $\boldsymbol{g}$ and $\boldsymbol{\Gamma}$ are both dynamical fields, a priori independent, and the Lagrangian density is a scalar built with their curvature invariants. This approach has become more popular in recent years as working on the gravity side of Einstein's equations, rather than on the matter sector, may open new possibilities to explain the missing energy content of the observed universe [4]. A comprehensive introduction to some formulations like *New General Relativity* [5], *Poincarè Gauge Gravity* [6], *Einstein Cartan* theory [7] and others are given in the brilliant review [8].

Our main interest is the *Teleparallel Equivalent of General Relativity* (TEGR) limit of MAG. Here, the only conditions are that the connection be compatible with the metric, $\nabla_\alpha g_{\mu\nu} = 0$, and that curvature vanishes, $R^\alpha_{\beta\mu\nu} = 0$. The spacetime M is thus a flat, twisted manifold called *Weitzenböck* spacetime in which gravity emerges as the gauge field of the translation group, making it a force in the Yang-Mills sense with torsion playing the role of field strength, replacing geodesics with force equations analogous to the ones for $U(1)$. The generators of the translation group are partial derivatives on the tangent bundle, and since they couple the same way to every field, the universality of gravitation comes as a natural consequence of the coupling prescription with matter. Furthermore, because of its formulation, inertial and gravitational effects can be attributed to the spin connection (from Lorentz transformations) and a gauge potential (of translations) separately. In General Relativity both are embedded in the Levi-Civita connection, and its local vanishing (free fall) means that such effects compensate each other in the same object, only allowing the Energy-Momentum Tensor (EMT) of the gravitational field to be a pseudotensor. In TEGR these contributions can be separated, and a purely tensorial EMT for gravity can be defined. Another point in favor of Teleparallel theory is that it is locally equivalent to General Relativity. Their actions only differ by a boundary term, resulting in the same equations of motion and making them dynamically equivalent at the classical level. The effect of this boundary is mainly relevant in black hole thermodynamics as it may spoil the equivalence between black hole energy and entropy between the two frameworks. Some work has been done on the subject, with Ref. [9] showing that the boundary term is equal to the Gibbons-Hawking-York one at semiclassical level, and Ref. [10] that later calculated it for a general MAG theory.

The aim of this work is then to present the framework of TEGR and show the effect that quantum fields coupled to gravity have on the background geometry.

affine geodesics (or straightest lines) given by the condition of parallel transport on tangent vectors, coincide with the *metrical* ones (shortest paths) found by minimizing the spacetime interval[3].

We will compute the one-loop part of the effective action for a massive, uncharged fermion field non-minimally coupled to the vector and axial components of the torsion tensor. This will be done by using the Heat Kernel method, a very general and versatile tool to compute these logarithmically divergent terms [11, 12]. The resulting effective action will depend on a cutoff scale, but can be easily converted to the Minimal-Subtraction scheme to explicitly write the necessary counterterms to the bare action. In order to simplify the theory, the bare fields are not dynamical and are assumed to act as a background, so only bare couplings will get renormalized. The corresponding beta functions are therefore constants that parametrically depend on the non-minimal coupling values.

Because we start the calculation in terms of general MAG quantities, the result can easily be cast in any of its limits, providing the interesting cases of GR and TEGR. This allows us to compare the two theories at one-loop level, once the respective constraints on connections are imposed.

The thesis is organized as to give the reader a clear path through the tools used for the final computation. In Chapter 2, the basic geometrical tools are given, such as the notions of connection, metric structure, and covariant derivative without symmetry requirements. In particular, the main objects of interest are defined: torsion, tetrads, and Lorentz connections.

Chapter 3 is devoted to a review of Gauge Theory and how gravity can be encompassed within it. The case of GR is studied and the main differences between the two are set. Next, the Teleparallel Equivalent of GR is introduced, putting together the work of previous sections in order to construct a suitable theory of gravity.

In Chapter 4, the functional formalism in Euclidean space is reported with the aim of studying the connected part of Green's functions, followed by the definition of effective action. Expanding around a background field we then relate the one-loop part of the action to functional determinants, whose calculation requires the introduction of yet another technique: the heat kernel. Based on the aforementioned work by Schwinger and DeWitt, this tool was further developed by Barvinsky and Vilkovisky [13] for the computation of any covariant loop graph in the effective action. The only caveat is that the operator in the action be of second order, which is not the case for spin $\frac{1}{2}$ particles, and will require us to manipulate it accordingly.

In Chapter 5 a direct calculation of the one-loop effective action is completed. Starting from the fermionic action coupled to gravity, a suitable second order differential operator is constructed in terms of GR terms plus torsional corrections, as this is the form for which expansion coefficients were calculated. From this, the appropriate renormalization procedure is carried out with the goal of finding correlations in the renormalizability of GR and TEGR.

2 Geometrical setting

Adopting a bottom-up approach, we define all necessary structures for a proper introduction of Teleparallel Gravity. These include the basic notion of connection on spacetime, tetrad fields, connections on tangent space, torsion and curvature tensors, and all their properties under Lorentz transformations. The main implications of the TEGR approach will be briefly discussed in terms of trajectories, with emphasis on the similarities with gauge theories.

2.1 Connections on manifolds

Consider a pseudo-Riemannian manifold M, call it *spacetime*. At each point x of this spacetime there is a *tangent space* T_xM of the same dimensionality which locally looks like pseudo-Euclidean space, and which we identify with Minkowski space $\mathbb{M}$. In this context, the special relativistic notion of inertial observer is replaced by the notion of general coordinate systems, or charts. In the region where two charts intersect, each spacetime point is mapped by two (or more) different coordinate systems, $\{x\}$ and $\{x'\}$, which are related by a *general coordinate transformation* (GCT). These correspond to diffeomorphisms: bijective, differentiable maps with differentiable inverse, thus characterized by a non-vanishing Jacobian determinant.

Under the GCT $x^\mu \to x'^\mu$, the coordinate differentials are related by

$$dx^\mu = (J^{-1})^\mu{}_\nu dx'^\nu \,, \tag{2.1}$$

where $\boldsymbol{J}$ is the Jacobian matrix of the transformation

$$J^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu} \,. \tag{2.2}$$

In these terms, a *contravariant* spacetime vector is an object that transforms like a coordinate differential does (eq. (2.1)). This consequently means that its differential transforms as

$$dA^\mu = (J^{-1})^\mu{}_\nu dA'^\nu + \frac{\partial^2 x^\mu}{\partial x'^\rho \partial x'^\nu} A'^\nu dx'^\rho \,. \tag{2.3}$$

The second term, which takes into account possible variations of A due to its parallel transportation from x to $x + dx$, prevents it from transforming linearly under GCT, thus breaking covariance.[5] In order to recover diffeomorphism symmetry we can

[5] An object with well-defined behavior under point-dependent (*local*) transformations is said to be *covariant*. Ordinary derivatives and differentials of such objects are not necessarily covariant themselves, and require a *connection* to be so. In QFTs, gauge potentials are connections. When "promoting" transformations from global to local, derivatives act on the transformation coefficients and the arising terms need to be accounted for.

define the covariant differential operator

$$DA^\mu = dA^\mu + \Gamma^\mu{}_{\alpha\beta} A^\alpha dx^\beta \,, \tag{2.4}$$

where the new coefficients Γ are components of a new compensating field called *connection*, or *gauge* field, depending on the framework, which transforms in such a way as to make DA^μ covariant [14]:

$$DA^\mu = (J^{-1})^\mu{}_\nu (DA^\nu)' \,. \tag{2.5}$$

Explicitly, this means

$$\begin{aligned}(J^{-1})^\mu{}_\nu dA'^\nu + \frac{\partial^2 x^\mu}{\partial x'^\alpha \partial x'^\beta} dx'^\alpha A'^\beta + \Gamma^\mu{}_{\lambda\sigma} (J^{-1})^\lambda{}_\alpha (J^{-1})^\sigma{}_\beta dx'^\alpha A'^\beta& \\ = (J^{-1})^\mu{}_\nu (dA'^\nu + \Gamma'^\nu{}_{\alpha\beta} dx'^\alpha A'^\beta)&\end{aligned} \tag{2.6}$$

or, equating the coefficients, that the connection transforms as

$$\Gamma'^\mu{}_{\alpha\beta} = J^\mu{}_\nu (J^{-1})^\lambda{}_\alpha (J^{-1})^\sigma{}_\beta \Gamma^\nu{}_{\lambda\sigma} + \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial^2 x^\nu}{\partial x'^\alpha \partial x'^\beta} \,, \tag{2.7}$$

which is clearly not a tensor under GCT[6]. This was expected because of the non-vectorial nature of dA. It is in fact said to transform *affinely*

We can now define the *covariant derivative* of A^μ as the limit for $dx^\alpha \to 0$ of the ratio between DA^μ and dx^α,

$$\nabla_\alpha A^\mu = \partial_\alpha A^\mu + \Gamma^\mu{}_{\alpha\beta} A^\beta \,. \tag{2.8}$$

Assuming that covariant derivatives reduce to ordinary ones when acting on scalars, $\nabla_\mu \phi = \partial_\mu \phi$, a straightforward application of the Leibniz rule to saturated contractions of tensors readily provides their covariant derivative.

We can now relate a general connection to the metric through ∇:

$$\begin{aligned} \nabla_\alpha g_{\mu\nu} \equiv \quad N_{\mu\nu\alpha} &= +\partial_\alpha g_{\mu\nu} - \Gamma^\beta{}_{\alpha\mu} g_{\beta\nu} - \Gamma^\beta{}_{\alpha\nu} g_{\mu\beta} \\ -\nabla_\mu g_{\nu\alpha} \equiv -N_{\nu\alpha\mu} &= -\partial_\mu g_{\nu\alpha} + \Gamma^\beta{}_{\mu\nu} g_{\beta\alpha} + \Gamma^\beta{}_{\mu\alpha} g_{\nu\beta} \\ -\nabla_\nu g_{\alpha\mu} \equiv -N_{\alpha\mu\nu} &= -\partial_\nu g_{\alpha\mu} + \Gamma^\beta{}_{\nu\alpha} g_{\beta\mu} + \Gamma^\beta{}_{\nu\mu} g_{\alpha\beta} \,, \end{aligned} \tag{2.9}$$

conveniently defining the tensors $\boldsymbol{N}$, called *non-metricity*, to be symmetric in their first two indices. Multiplying them by $\frac{1}{2}$ and adding them together we get (lowering the connection indices with the metric tensor)

$$\begin{aligned}\frac{1}{2}(\partial_\alpha g_{\mu\nu} - \partial_\mu g_{\nu\alpha} - \partial_\nu g_{\alpha\mu}) + \Gamma_{\alpha(\mu\nu)} - \Gamma_{\nu[\alpha\mu]} - \Gamma_{\mu[\alpha\nu]}& \\ = \frac{1}{2}(N_{\mu\nu\alpha} - N_{\nu\alpha\mu} - N_{\alpha\mu\nu}) \,.&\end{aligned} \tag{2.10}$$

[6] A general tensor would transform with a J and a J^{-1} for each upper and lower index respectively

Introducing the *torsion* of a connection as its antisymmetric part

$$T^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\nu\mu} \equiv 2\Gamma^{\alpha}{}_{[\mu\nu]}\,, \tag{2.11}$$

and writing $\mathbf{\Gamma}$ as $\Gamma_{\alpha(\mu\nu)} + \Gamma_{\alpha[\mu\nu]}$, together with Eq. (2.10) we get

$$\begin{aligned}\Gamma_{\alpha\mu\nu} = \Gamma_{\alpha(\mu\nu)} + T_{\alpha\mu\nu} = &\frac{1}{2}(\partial_\alpha g_{\nu\mu} - \partial_\mu g_{\nu\alpha} + \partial_\mu g_{\alpha\nu}) \\ &+ \frac{1}{2}(T_{\alpha\mu\nu} + T_{\mu\nu\alpha} + T_{\nu\mu\alpha}) + \\ &+ \frac{1}{2}(N_{\alpha\mu\nu} - N_{\nu\alpha\mu} - N_{\alpha\mu\nu}) \qquad .\end{aligned} \tag{2.12}$$

Recasting the first index to a contravariant one and renaming the remaining ones for convenience, we get the decomposition

$$\Gamma^{\rho}{}_{\mu\nu} = \mathring{\Gamma}^{\rho}{}_{\mu\nu} + K^{\rho}{}_{\mu\nu} + W^{\rho}{}_{\mu\nu}\,, \tag{2.13}$$

where

$$\mathring{\Gamma}^{\rho}{}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha}(\partial_\mu g_{\nu\alpha} - \partial_\nu g_{\alpha\mu} + \partial_\alpha g_{\mu\nu})\,, \tag{2.14}$$

are the *Christoffel symbols*, the usual connection of GR,

$$K^{\rho}{}_{\mu\nu} = \frac{1}{2}(T^{\rho}{}_{\mu\nu} + T_{\mu\nu}{}^{\rho} + T_{\nu\mu}{}^{\rho})\,, \tag{2.15}$$

is the *contortion* tensor, and

$$W^{\rho}{}_{\mu\nu} = \frac{1}{2}(N_{\mu\nu}{}^{\rho} - N^{\rho}{}_{\mu\nu} + N_{\nu}{}^{\rho}{}_{\mu})\,, \tag{2.16}$$

is the *disformation* tensor. This shows that the contributions to a connection can come from a combination of the partial derivatives of the metric, from torsion, and from the covariant derivatives of the metric tensor. Furthermore, $\mathring{\mathbf{\Gamma}}$ and $\boldsymbol{W}$ only contribute to the symmetric part of a connection, whereas $\boldsymbol{K}$ has both an antisymmetric part (torsion) and a non-vanishing symmetric one. This means that in general, the Levi-Civita connection is not simply equal to the symmetric part of connections, as it would require the vanishing of torsion, and therefore contortion. This suggests a classification of connections based on the constraints we impose on their components. For example, in the context of General Relativity one enforces metric compatibility at the dynamical level

$$N^{\rho}{}_{\mu\nu} = 0\,, \tag{2.17}$$

so $W^\rho{}_{\mu\nu} = 0$, and connection symmetry

$$T^\rho{}_{\mu\nu} = 0\,, \tag{2.18}$$

so $K^\rho{}_{\mu\nu} = 0$. This leaves the Christoffel symbols (2.14) as the only contribution to the total connection, so the whole spacetime structure is described by the metric as the only field, and universality of gravitation is automatically embedded because all fields with tensorial representation couple the same way to it.

For the sake of generality, it is wise and convenient to consider connections as additional structures on a manifold. We will work with *metric compatible* connections, for which we impose $\nabla_\rho g_{\mu\nu} = 0$, which means that equation (2.13) reduces to

$$\Gamma^\rho{}_{\mu\nu} = \mathring{\Gamma}^\rho{}_{\mu\nu} + K^\rho{}_{\mu\nu}\,. \tag{2.19}$$

These relations hold in all frames, particularly in the ones living in tangent space, where greek indices will be replaced by latin ones with tetrads, as we shall see in the next section.

Before continuing, it is crucial that we point out the aforementioned modifications that the torsion tensor brings to well-known geometrical identities. In the derivation of Riemann's curvature tensor, one usually faces the expansion of the commutator

$$\begin{aligned}
[\nabla_\mu, \nabla_\nu]V^\alpha &= \nabla_\mu(\nabla_\nu V^\alpha) - \nabla_\nu(\nabla_\mu V^\alpha) \\
&= (\partial_\mu \Gamma^\alpha{}_{\beta\nu} - \partial_\nu \Gamma^\alpha{}_{\beta\mu} + \Gamma^\alpha{}_{\rho\mu}\Gamma^\rho{}_{\beta\nu} - \Gamma^\alpha{}_{\rho\nu}\Gamma^\rho{}_{\beta\mu})V^\beta \\
&+ (\Gamma^\rho{}_{\mu\nu} - \Gamma^\rho{}_{\nu\mu})(\partial_\rho V^\alpha + \Gamma^\alpha{}_{\sigma\rho}V^\sigma) \\
&= R^\alpha{}_{\beta\mu\nu}V^\beta + T^\beta{}_{\mu\nu}\nabla_\beta V^\alpha\,.
\end{aligned} \tag{2.20}$$

By definition, working in GR, the last term vanishes, so the commutator defines the Riemann tensor. We shall keep this and see how it enters the minimal-coupling prescription for fermions. Plugging the decomposition (2.19) into the definition of the Riemann tensor components in covariant indices, we also find a relation between the GR curvature tensor and the torsionful case, which we express in terms of GR covariant derivatives for later convenience

$$\begin{aligned}
R_{\alpha\beta\mu\nu} &= \partial_\mu \Gamma_{\alpha\beta\nu} - \partial_\nu \Gamma_{\alpha\beta\mu} + \Gamma_{\alpha\rho\mu}\Gamma^\rho{}_{\beta\nu} - \Gamma_{\alpha\rho\nu}\Gamma^\rho{}_{\beta\mu} \\
&= \mathring{R}_{\alpha\beta\mu\nu} + \mathring{\nabla}_\alpha K_{\mu\nu\beta} - \mathring{\nabla}_\beta K_{\mu\nu\alpha} + K_{\mu\rho\beta}K_\nu{}^\rho{}_\alpha - K_{\mu\rho\alpha}K_\nu{}^\rho{}_\beta\,.
\end{aligned} \tag{2.21}$$

A direct computation of cyclic permutations of appropriate indices shows that the first Bianchi identity becomes

$$\begin{aligned}
&R^\alpha{}_{\beta\mu\nu} + R^\alpha{}_{\nu\beta\mu} + R^\alpha{}_{\mu\nu\beta} \\
&= \nabla_\beta T^\alpha{}_{\mu\nu} + \nabla_\nu T^\alpha{}_{\beta\mu} + \nabla_\mu T^\alpha{}_{\nu\beta} \\
&+ T^\alpha{}_{\rho\beta}T^\rho{}_{\mu\nu} + T^\alpha{}_{\rho\nu}T^\rho{}_{\beta\mu} + T^\alpha{}_{\rho\mu}T^\rho{}_{\nu\beta}\,.
\end{aligned} \tag{2.22}$$

and the second one

$$\begin{aligned} &\nabla_\rho R^\alpha{}_{\beta\mu\nu} + \nabla_\nu R^\alpha{}_{\beta\rho\mu} + \nabla_\mu R^\alpha{}_{\beta\nu\rho} \\ &= R^\alpha{}_{\beta\mu\lambda} T^\lambda{}_{\nu\rho} + R^\alpha{}_{\beta\nu\lambda} T^\lambda{}_{\rho\mu} + R^\alpha{}_{\beta\rho\lambda} T^\lambda{}_{\mu\nu} \,, \end{aligned} \tag{2.23}$$

Taking $T^\alpha{}_{\mu\nu} = 0$, both right hand sides of (2.23) and (2.22) vanish, returning the standard Bianchi identities of GR.

2.2 Frames and tetrad fields

We now want to locally characterize the geometry of spacetime by introducing at each point a set of four covariant vectors $\{h_a\}$ called *general frames*, or *linear frames*, that act as an orthonormal basis of tangent space and relate the spacetime and Minkowski metric by

$$g(h_a, h_b) = \eta_{ab} \,. \tag{2.24}$$

It then naturally follows the definition of *co-frames* $\{h^a\}$, their dual, from the relation

$$h^a(h_b) = \delta^a_b \,. \tag{2.25}$$

As with every basis, it can be written as a linear combination of any other, so

$$h_a = h_a{}^\mu \partial_\mu \quad \text{and} \quad h^a = h^a{}_\mu dx^\mu \,, \tag{2.26}$$

or

$$\partial_\mu = h^a{}_\mu h_a \quad \text{and} \quad dx^\mu = h_a{}^\mu h^a \,, \tag{2.27}$$

which together with the dual identities $dx^\nu(\partial_\mu) = \delta^\nu_\mu$ and (2.25) give

$$h^a{}_\mu h_a{}^\nu = \delta^\nu_\mu \quad \text{and} \quad h^a{}_\mu h_b{}^\mu = \delta^a_b \,. \tag{2.28}$$

Combining these equations with (2.24) we find

$$\eta_{ab} = g_{\mu\nu} h_a{}^\mu h_b{}^\nu \quad \text{or} \quad g_{\mu\nu} = \eta_{ab} h^a{}_\mu h^b{}_\nu \,. \tag{2.29}$$

The frame components $h^a{}_\mu$ and $h_a{}^\mu$ are called *tetrads*, or *tetrad fields*. Although they have a covariant and a contravariant index they are not tensors in both indices, but rather sets of four linearly independent four-vectors living in tangent space. An important comment is due: the spacetime metric $\boldsymbol{g}$ has 10 independent degrees of freedom (because of the symmetry constraint) while the tetrads have 16. Looking at equation (2.24), we see that tetrads fields are only determined up to local Lorentz transformations (LLTs) defined in (2.5) because

$$\eta_{ab} = \Lambda^c{}_a \Lambda^d{}_b \eta_{cd} \,, \tag{2.30}$$

so the six extra degrees of freedom are compatible with the six parameters of the Lorentz group under which the metric is invariant, and correspond to the natural freedom of choosing a frame, or basis, in Minkowski space.

General frames also satisfy the commutation relation

$$[h_a, h_b] = f^c{}_{ab} h_c \,, \tag{2.31}$$

with

$$f^c{}_{ab} = h_a{}^\mu h_b{}^\nu (\partial_\nu h^c{}_\mu - \partial_\mu h^c{}_\nu) \,, \tag{2.32}$$

their *structure coefficients*, or *coefficients of anholonomy*[7], which are found using (2.26).

Perhaps, the most crucial aspect of tetrad fields is their ability to convert internal with external indices of tensors and vice versa. On account of the soldered form of the tangent bundle, they can relate tangent-space tensors with spacetime tensors components as in

$$\Phi^{\mu\nu..}{}_{\alpha\beta..} = h_m{}^\mu h_n{}^\nu .. \; \Phi^{mn..}{}_{ab..} \; h^a{}_\alpha h^b{}_\beta .. \;\; , \tag{2.33}$$

$$\Phi^{mn..}{}_{ab..} = h^m{}_\mu h^n{}_\nu .. \; \Phi^{\mu\nu..}{}_{\alpha\beta..} \; h_a{}^\alpha h_b{}^\beta .. \;\; . \tag{2.34}$$

This can be easily seen by the change of basis (2.27).[8].Note that this is only a property of tensors. Not all indexed quantities, such as connections, can convert their Greek and Latin indices like this.

2.3 Tangent space connections

Thanks to the projection of curved indices into flat ones, we can shift from the diffeomorphisms of the manifold to the Lorentz transformations of tangent space. This, however, is a space defined pointwise, so the corresponding Lorentz transformations are automatically local. The power of this approach is that we are already in local Minkowski space, where all quantum fields are defined.[9]

Connections coming from the linear group $GL(4, \mathbb{R})$ and its subgroups, such as the Lorentz group $SO(1, 3)$, are called *linear*. A *Lorentz*, or *spin*, connection is a

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[7] Anholonomy is the property by which a differential form is not the differential of anything. In other words, its integration is not path independent. Heat, work, and the angular velocity of a generic rigid body in Euclidean are typical examples of anholonomic variables in classical mechanics.

[8] Taking the simple example $\boldsymbol{T} = T_\mu dx^\mu = T_\mu (h_a{}^\mu h^a) \equiv (T_\mu h_a{}^\mu) h^a = T_a h^a$, equations (2.33) and (2.34) are immediate.

[9] As anticipated, a standard issue arises when coupling spinors to gravity: one has to face the fact that there are no spinorial representations of the diffeomorphism group, and the tetrad formalism is inevitable.

1-form with values in the algebra of the Lorentz group

$$\omega_\mu = \frac{1}{2}\omega^a{}_{b\mu} S_a{}^b \,, \tag{2.35}$$

with S_{ab} a representation of the generators, and is thus concerned with the spin content of the field. We can then define the Fock-Ivanenko covariant derivative [1]

$$\mathcal{D}_\mu = \partial_\mu - \frac{i}{2}\omega^a{}_{b\mu} S_a{}^b \,, \tag{2.36}$$

which in the vector representation $(S_{ab})^c{}_d = i(\eta_{bd}\delta^c_a - \eta_{ad}\delta^c_b)$ reads

$$\mathcal{D}_\mu v^a = \partial_\mu v^a + \omega^a{}_{b\mu} v^b \,. \tag{2.37}$$

This derivative will be used when acting on Lorentz indices unless otherwise specified. For every spin connection, there is a corresponding *general linear* connection of spacetime

$$\Gamma^\rho{}_{\nu\mu} = h_a{}^\rho \partial_\mu h^a{}_\nu + h_a{}^\rho \omega^a{}_{b\mu} h^b{}_\nu = h_a{}^\rho \mathcal{D}_\mu h^a{}_\nu \,, \tag{2.38}$$

which lies in $GL(4,\mathbb{R})$. The relation is easily invertible using (2.28):

$$\omega^a{}_{b\mu} = h^a{}_\nu \partial_\mu h_b{}^\nu + h^a{}_\nu \Gamma^\nu{}_{\rho\mu} h_b{}^\rho = h^a{}_\nu \nabla_\mu h_b{}^\nu \,, \tag{2.39}$$

where

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu{}_{\rho\mu} v^\rho \,, \tag{2.40}$$

is the usual covariant derivative that only acts on spacetime indices. These equations can be neatly rewritten as

$$\partial_\mu h_b{}^\nu + \Gamma^\nu{}_{\rho\mu} h_b{}^\rho - \omega^a{}_{b\mu} h_a{}^\nu = 0 \,, \tag{2.41}$$

that is, the covariant derivative (with respect to both indices) of the tetrad field vanishes identically. Equation (2.41), also known as *tetrad postulate*,[10] is a necessary condition if we require that the notion of parallel transport coincides using either $\boldsymbol{\Gamma}$ or $\boldsymbol{\omega}$. This shows there is a functional relation $\boldsymbol{\Gamma} \to \boldsymbol{\omega}(\boldsymbol{h})$ which, together with $\boldsymbol{g} \to \boldsymbol{h}$, makes the two pairs of geometrical objects $\{\boldsymbol{g}, \boldsymbol{\Gamma}\}$ and $\{\boldsymbol{h}, \boldsymbol{\omega}\}$ completely equivalent for a consistent description of gravity on both M and T_xM, with the latter being the go-to when considering fermions, as repeatedly said.

Using these, we can relate the spacetime and Fock-Ivanenko covariant derivatives of a vector field

$$\mathcal{D}_\mu \phi^a = h^a{}_\rho \nabla_\mu \phi^\rho \,, \tag{2.42}$$

which is what we expect from (2.34) given the tensorial nature of $\boldsymbol{\mathcal{D}}$ and $\boldsymbol{\nabla}$.

[10] There are many geometrical details that make this a non-trivial statement for mathematicians, which are animatedly discussed in [15], following a paper from Evans [16], claiming the postulate to be a fundamental identity of differential geometry.

2.4 Curvature and torsion

As shown in Chapter 2, metric compatible connections can be decomposed as the sum of Christoffel symbols and contortion given by Eq.(2.19). Formally, the curvature of a connection is a 2-form with values in the Lie algebra of $SO(p,q)$

$$\boldsymbol{R} = \frac{1}{4} R^{a}{}_{b\mu\nu}\, S_{a}{}^{b}\, dx^{\mu} \wedge dx^{\nu} \,, \tag{2.43}$$

with components

$$R^{a}{}_{b\mu\nu} = \partial_{\mu}\omega^{a}{}_{b\nu} - \partial_{\nu}\omega^{a}{}_{b\mu} + \omega^{a}{}_{\alpha\mu}\omega_{\alpha b\nu} - \omega^{a}{}_{\alpha\nu}\omega_{\alpha b\mu} \,, \tag{2.44}$$

and its torsion is a 2-form assuming values in the Lie algebra of the translation group, whose generators we can define as $P_a = \partial_a$,

$$\boldsymbol{T} = -\frac{1}{2} T^{a}{}_{\mu\nu}\, P_{a}\, dx^{\mu} \wedge dx^{\nu} \,, \tag{2.45}$$

with components

$$T^{a}{}_{\mu\nu} = \partial_{\nu} h^{a}{}_{\mu} - \partial_{\mu} h^{a}{}_{\nu} + \omega^{a}{}_{c\nu} h^{c}{}_{\mu} - \omega^{a}{}_{c\mu} h^{c}{}_{\nu} \quad . \tag{2.46}$$

By contracting latin indices with $h_{a}{}^{\mu}$, or by putting (2.39) in them, we get the usual spacetime indexed versions

$$R^{\rho}{}_{\lambda\mu\nu} = \partial_{\mu}\Gamma^{\rho}{}_{\lambda\nu} - \partial_{\nu}\Gamma^{\rho}{}_{\lambda\mu} + \Gamma^{\rho}{}_{\alpha\mu}\Gamma^{\alpha}{}_{\lambda\nu} - \Gamma^{\rho}{}_{\alpha\nu}\Gamma^{\alpha}{}_{\lambda\mu} \,, \tag{2.47}$$

and

$$T^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\rho}{}_{\nu\mu} \,. \tag{2.48}$$

Rather than converting to spacetime indices, we can keep contracting with tetrads to cast the connection decomposition (2.19) in Lorentz indices. First, we see that

$$R^{a}{}_{bcd} = h_{c}(\omega^{a}{}_{bd}) - h_{d}(\omega^{a}{}_{bc}) + \omega^{a}{}_{ec}\omega_{ebd} - \omega^{a}{}_{ed}\omega_{ebc} \,, \tag{2.49}$$

and

$$T^{a}{}_{bc} = \omega^{a}{}_{bc} - \omega^{a}{}_{cb} - f^{a}{}_{cb} \,. \tag{2.50}$$

By taking the sum of three permutations of (2.50), the spin connection (2.39) has the same decomposition of (2.19)

$$\omega^{a}{}_{bc} = \overset{\circ}{\omega}{}^{a}{}_{bc} + \Omega^{a}{}_{bc} \,, \tag{2.51}$$

in which

$$\overset{\circ}{\omega}{}^{a}{}_{bc} = \frac{1}{2}(f_b{}^a{}_c + f_c{}^a{}_b - f^a{}_{bc})\,, \tag{2.52}$$

is the symmetric spin connection of General Relativity, obtained from the tetrad postulate by only using the Levi-Civita connection, and

$$\Omega^a{}_{bc} = \frac{1}{2}(T_b{}^a{}_c + T_c{}^a{}_b - T^a{}_{bc})\,, \tag{2.53}$$

is the latin indexed contortion tensor.

A useful identity we will extensively use is the torsion decomposition

$$T_{abc} = \frac{1}{3}(\eta_{ac}T_b - \eta_{ab}T_c) - \frac{1}{3!}\varepsilon_{dabc}\theta^d + \kappa_{abc}\,, \tag{2.54}$$

where T_d and θ_d are the vector and axial components of torsion

$$T_d = T^a{}_{da} \qquad \theta_d = \varepsilon_d{}^{abc}T_{abc}\,, \tag{2.55}$$

and κ_{abc} is the fully tensorial, traceless hook-antisymmetric remainder [17]

$$\kappa^a{}_{[bc]} = \kappa^a{}_{bc}, \qquad \kappa_{[abc]} = 0\,. \tag{2.56}$$

This last tensor will be omitted from here on because it vanishes in all contractions we consider. Notice that we have used the Minkowski metric because we are in tangent space.

2.5 Local Lorentz transformations

Local transformations act on the tangent space coordinates x^a, on which they depend. Under Local Lorentz transformations (LLT), these coordinates transform as

$$x'^a = \Lambda^a{}_b(x)x^b\,, \tag{2.57}$$

so for tetrad frames, which have a vector index, we have

$$h'^a = \Lambda^a{}_b(x)h^b \quad \text{and} \quad h'_a = \Lambda_a{}^b(x)h_b\,. \tag{2.58}$$

As stated before, at each point of spacetime, Eq.(2.29) determines the tetrad fields up to a transformation of the six-parameter Lorentz group in the tangent space. This means there exists another tetrad basis $\{h'_a\}$ that satisfies

$$g_{\mu\nu} = \eta_{cd}h'^c{}_\mu h'^d{}_\nu\,. \tag{2.59}$$

Contracting with $h_a{}^\mu$ and $h_b{}^\nu$

$$\eta_{ab} = \eta_{cd}\left(h'^c{}_\mu h_a{}^\mu\right)\left(h'^d{}_\nu h_b{}^\nu\right)\,, \tag{2.60}$$

we see that the transformation

$$h'^c{}_\mu h_a{}^\mu \equiv \Lambda^c{}_a(x)\,, \tag{2.61}$$

belongs to the vector representation of the Lorentz group, so each tetrad component[11] transforms as

$$h'^a{}_\mu = \Lambda^a{}_b(x) h^b{}_\mu\,, \tag{2.62}$$

under LLTs. From this, the spin connection (2.39) transforms as

$$\omega'^a{}_{b\mu} = \Lambda^a{}_c \omega^c{}_{d\mu} \Lambda_b{}^d + \Lambda^a{}_c \partial_\mu \Lambda_b{}^c\,. \tag{2.63}$$

Plugging this into (2.44) and (2.46), we get

$$R'^a{}_{b\mu\nu} = \Lambda^a{}_c \Lambda_b{}^d R^c{}_{d\mu\nu}\,, \tag{2.64}$$

for curvature, and

$$T'^a{}_{\mu\nu} = \Lambda^a{}_c T^c{}_{\mu\nu}\,, \tag{2.65}$$

for torsion. These results are obtainable by seeing that spacetime indices can be cast into tangent space with one tetrad, Eq.(2.34), and this transforms with one Lorentz matrix under LLT itself, Eq.(2.62), so each latin index transforms covariantly under LLT. In turn, this confirms that spacetime-indexed quantities like $R^\alpha{}_{\beta\mu\nu}$ are local Lorentz invariants.

2.6 Inertia

Let us momentarily work in the context of Special Relativity (SR), where no gravitation is present, and $g_{\mu\nu} = \eta_{\mu\nu}$. Tetrads relating internal and external Minkowski metrics as in (2.29)

$$\eta_{ab} = e_a{}^\mu e_b{}^\nu \eta_{\mu\nu} \quad \text{or} \quad \eta_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab}\,, \tag{2.66}$$

are called *trivial*, and will be denoted with the letter e. Within these, a class of frames e'_a is *inertial*, or *holonomic*, if

$$f'^c{}_{ab} = e'_a{}^\mu e'_b{}^\nu (\partial_\nu e'^c{}_\mu - \partial_\mu e'^c{}_\nu) = 0\,, \tag{2.67}$$

which in the case of SR is a global condition: equation (2.67) implies $de'^a = 0$ (from Cartan's structure equations), so that $e'^a = dx^a$ is a locally exact differential form for some x^a, and is valid everywhere for frames belonging to this class. They have components

$$e'^a{}_\mu = \partial_\mu x'^a\,, \tag{2.68}$$

[11] Recall that tetrads are to be regarded as sets of Lorentz four-vectors

with $x'^a(x)$ is a spacetime dependent Lorentz vector. In cartesian coordinates, they are simply $e'^a{}_\mu = \delta^a_\mu$.

Under LLT, combining (2.57) and (2.62), we get the explicit form

$$e^a{}_\mu = \partial_\mu x^a + \Lambda^a{}_c \partial_\mu \Lambda_b{}^c x^b = \partial_\mu x^a + \overset{\bullet}{\omega}{}^a{}_{b\mu} x^b \equiv \overset{\bullet}{\mathcal{D}}_\mu x^a \,, \tag{2.69}$$

where $\overset{\bullet}{\omega}{}^a{}_{b\mu}$ is a connection that only depends on Lorentz transformations, and therefore encodes all inertial effects present in the transformed frame. Within each class, Eq.(2.69) tells us there is an infinity-fold of frames related by *global* Lorentz transformations.

We can also get to $\overset{\bullet}{\omega}{}^a{}_{b\mu}$ by performing a LT on the *Weizenböck* connection

$$\overset{\bullet}{\omega}{}'^a{}_{b\mu} = 0 \,. \tag{2.70}$$

This is, as we shall see, a sort of *gauge fixing* procedure on connections that explicitly breaks Lorentz invariance, which will be recovered by using the appropriate spin connection in TEGR. It is clear from Eq.(2.69) that the frame $e^a{}_\mu$ is no longer holonomic. In fact, from Eq.(2.31) we get

$$f^c{}_{ab} = \overset{\bullet}{\omega}{}^c{}_{ba} - \overset{\bullet}{\omega}{}^c{}_{ab} \,, \tag{2.71}$$

with $\overset{\bullet}{\omega}{}^c{}_{ab} = e_b{}^\mu \overset{\bullet}{\omega}{}^c{}_{a\mu}$. A similar procedure to Eq.(2.10) gives ω in terms of f

$$\overset{\bullet}{\omega}{}^c{}_{ab} = \frac{1}{2}(f_a{}^c{}_b + f_b{}^c{}_a - f^c{}_{ab}) \,. \tag{2.72}$$

This connection is purely inertial by construction, and using (2.64) and (2.65) we see that it always has vanishing curvature and torsion

$$\overset{\bullet}{R}{}^a{}_{b\mu\nu} = 0 \,, \tag{2.73}$$

$$\overset{\bullet}{T}{}^a{}_{\mu\nu} = 0 \,, \tag{2.74}$$

which is of course to be expected in the framework of Special Relativity.

In the presence of gravitation, tetrads are anholonomic by definition. At any point p, we can still find coordinates x^a such that $h^a = dx^a$ holds locally. The difference with inertial frames is that in this case, the holonomicity is only valid in a neighborhood of p, and we must find new coordinates if we move away from it.

2.7 Trajectories in Minkowski space

The free particle in the class of inertial frames $e'^a{}_\mu$ is described by the equations of motion

$$\frac{du'^a}{d\sigma} = 0\,, \tag{2.75}$$

where u^a is the particle four-velocity in the frame (notice the tangent space index), and

$$d\sigma^2 = \eta_{\mu\nu} dx^\mu dx^\nu = \eta_{ab} dx^a dx^b\,, \tag{2.76}$$

is the Minkowski invariant interval, with σ the *proper time* (when (2.76) is timelike).

In a anholonomic frame $e^a{}_\mu$, obtained by LLT of e', equation (2.75) becomes

$$\frac{du^a}{d\sigma} + \overset{\bullet}{\omega}{}^a{}_{b\mu} u^b u^\mu = 0\,, \tag{2.77}$$

where $u^b = \Lambda^b{}_a u'^a$ and

$$u^\mu = e_a{}^\mu u^a = \frac{dx^\mu}{d\sigma}\,, \tag{2.78}$$

is the usual spacetime four-velocity. Because the anholonomy of the new frame is only related to inertia, the spacetime metric $\eta_{\mu\nu}$ is still Minkowski, but expressed through a more general x-dependent form (owed to the LLT). In terms of holonomic velocity, we can write (2.77) as

$$\frac{du^\rho}{d\sigma} + \overset{\bullet}{\gamma}{}^\rho{}_{\mu\nu} u^\mu u^\nu = 0\,, \tag{2.79}$$

where the connection γ is the spacetime indexed version of the spin connection (Eq. (2.38) in Minkowski space)

$$\overset{\bullet}{\gamma}{}^\rho{}_{\mu\nu} = e_c{}^\rho \partial_\nu e^c{}_\mu + e_c{}^\rho \overset{\bullet}{\omega}{}^c{}_{b\nu} e^b{}_\mu \equiv e_c{}^\rho \overset{\bullet}{\mathcal{D}}_\nu e^c{}_\mu\,, \tag{2.80}$$

and in the inertial frame $e'^a{}_\mu$ such that $\overset{\bullet}{\omega}{}'^a{}_{b\mu} = 0$, we have $\overset{\bullet}{\gamma}{}'^\rho{}_{\mu\nu} = e'_c{}^\rho \partial_\nu e'^c{}_\mu$, which vanishes in cartesian coordinates ($e'^a{}_\mu = \delta^a_\mu$) and the EOM (2.79) assumes the familiar form

$$\frac{du'^\rho}{d\sigma} = 0\,, \tag{2.81}$$

in spacetime indices. This is how inertial effects, coming from LLTs, affect the EOM of a free particle.

The addition of gravitation is now only dependent on the approach. As we shall see, because of the coupling prescription of TEGR, only torsion will be non-vanishing, with inertial effects separated from gravitational ones

3 Gauge theory and gravity

The goal is to now apply the gauge paradigm to a theory of gravitation. We first introduce some fundamental aspects of the framework and present a summary of well-established results in order to discuss possible compatibility issues with gravity.

Gauge theories, which successfully describe the electromagnetic, electroweak, and strong interactions, are built upon local (point dependent) symmetries of *internal* spaces under a set of transformations belonging to group.[12] To encode the gauging methods in the manifold structure we proceed as follows. At every point p of spacetime, assign a copy of the gauge group G[13] and call it a *fiber* F. *Fiber bundles* are then product spaces of a base manifold M and F, with the *principal bundle* being the one with G itself, and *associated bundles* the ones where a representation of G is substituted.[14] A *bundle projection* π is a map that connects all points of the fiber over p to p itself. A *section* σ is then the map from a neighborhood of p to the bundle. We can start to see how this formulation encodes the transformation properties of fields into the fabric of spacetime.

In this sense, source fields, which live in vector spaces carrying representations of G, are local sections of the fiber bundle

$$\Psi_V \ : \ V \to \pi^{-1}(V) \sim V \times F \,, \tag{3.1}$$

where V is an open set of spacetime and $\sim$ indicates equality up to diffeomorphisms called trivializations defined by

$$\varphi_V \ : \ \pi^{-1}(V) \to V \times F \,. \tag{3.2}$$

These local sections can be expressed as

$$\Psi_V \ : \ x^\mu \to \varphi_V^{-1}(x^\mu, x^a) \,, \tag{3.3}$$

where $\{x^a(x^\mu)\}$ are *internal* coordinates of points within the fiber attached to the spacetime point with coordinates x^μ. In Yang-Mills theories, where fibers are finite-dimensional vector spaces given by unitary representations of gauge groups, local sections, or fields, carry the internal index of a multiplet space

$$\Psi = \Psi(x^\mu, x^a) \equiv \Psi^a(x^\mu) \,, \tag{3.4}$$

12 The proton and neutron live in the space of isotopic spin as doublets of SU(2), of which there is a copy at each point of spacetime.

13 This comes in parallel with the fact that relativistic fields are sets of infinitely many degrees of freedom, one degree for each point of spacetime.

14 So for each principal bundle, there are infinitely many associated bundles, one for each representation.

so a are the *components* of the field.

Because YM theories are embedded in flat spacetime and other Latin letters will be used for group indices, only in this chapter must Greek letters be employed as Minkowski indices.

3.1 Traditional Yang-Mills in Minkowski space

Because source fields are expressed as representations of groups themselves, they live in the associated bundle and obey gauge transformations of the form

$$\Psi'^i(x) = U^i{}_j(x)\Psi^j(x)\,, \tag{3.5}$$

where $U^i{}_j(x)$ are the entries of the matrix representing the gauge transformation $U(x^\mu)$ at the spacetime point p of coordinates x^μ. Group elements have the form

$$U^i{}_j(x) = \left[e^{\varepsilon^A(x)T_A}\right]^i{}_j\,, \tag{3.6}$$

where T_A are generators of the transformation in the corresponding representation, and $\varepsilon^A(x)$ are position-dependent group parameters. The index A runs from 1 to n, the dimension of the group, whereas i, j run up to the dimension of the representation.

The corresponding first-order infinitesimal variation, for $||\varepsilon^A(x)|| \ll 1$, of the source field would then be

$$\delta\Psi(x) = \Psi'(x) - \Psi(x) = \varepsilon^A(x)T_A\Psi(x)\,. \tag{3.7}$$

From the general definition of covariant derivative of a field

$$D_\mu\Psi(x) = \partial_\mu\Psi(x) - A^B{}_\mu\frac{\delta\Psi(x)}{\delta\varepsilon^B(x)}\,, \tag{3.8}$$

and plugging Eq.(3.7) into it, we get

$$D_\mu\Psi(x) = \partial_\mu\Psi(x) - A^B{}_\mu T_B\Psi(x)\,, \tag{3.9}$$

where $A^B{}_\mu$ are the components of the *gauge field*, a 1-form assuming values in the Lie algebra of the gauge group

$$A = J_B A^B{}_\mu dx^\mu\,, \tag{3.10}$$

with J_B the generators in the adjoint representation

$$[J_A]^C{}_B = f^C{}_{AB}\,, \tag{3.11}$$

equal to the structure coefficients of the group given by

$$\big[T_A, T_B\big] = f^C{}_{AB} T_C \,. \tag{3.12}$$

Following the same steps from Eq.(2.3) and requiring that $D\Psi$ transforms just as Ψ under gauge transformations, we find that for D_μ to be covariant

$$D'_\mu \Psi'(x) = U(x) D_\mu \Psi(x) \,, \tag{3.13}$$

the gauge field $A_\mu \equiv A^B{}_\mu T_B$ needs to transform as

$$A'_\mu = U(x) A_\mu U^{-1}(x) + U(x) \partial_\mu U^{-1}(x) \,, \tag{3.14}$$

which, just as Eq.(2.63), shows that it does not transform covariantly, but rather *affinely*.[15] It is in fact a connection.

Taking the infinitesimal version of (3.14), we see that

$$\begin{aligned} \delta A^B{}_\mu(x) &= A'^B{}_\mu(x) - A^B{}_\mu(x) \\ &= -\partial_\mu \varepsilon^B(x) - f^B{}_{CD} A^C{}_\mu \varepsilon^D(x) = -D_\mu \varepsilon^B(x) \,. \end{aligned} \tag{3.15}$$

The curvature of such a connection is then the 2-form defined as

$$F = \frac{1}{2} J_B F^B{}_{\mu\nu} dx^\mu \wedge dx^\nu \,, \tag{3.16}$$

whose components

$$F^B{}_{\mu\nu} = \partial_\mu A^B{}_\nu - \partial_\nu A^B{}_\mu + f^B{}_{CD} A^C{}_\mu A^D{}_\nu \,, \tag{3.17}$$

can be contracted with the generators to construct the *field strength*

$$F_{\mu\nu} \equiv \big[D_\mu, D_\nu\big] = F^B{}_{\mu\nu} T_B \,, \tag{3.18}$$

which transforms covariantly under local gauge transformations (3.6)

$$F'_{\mu\nu} = U(x) F_{\mu\nu} U^{-1}(x) \,. \tag{3.19}$$

From Eq.(3.17) follows the identity

$$D_\rho F^A{}_{\mu\nu} + D_\nu F^A{}_{\rho\mu} + D_\mu F^A{}_{\nu\rho} = 0 \,, \tag{3.20}$$

which is a Bianchi identity, a generalization of the first two Maxwell equations to the non-abelian case. Not coming from a Lagrangian, these are not dynamical equations. To be so, they must come from the gauge Lagrangian

$$\mathcal{L}_{gauge} = -\frac{1}{4} \gamma_{AB} F^A{}_{\mu\nu} F^{B\mu\nu} \,, \tag{3.21}$$

[15] Similarly to the affine connection (2.6), which is notoriously non-covariant.

where γ_{AB} is the invariant, symmetric tensor

$$\gamma_{AB} = Tr(J_A J_B) = f^C{}_{AD} f^D{}_{BC} , \tag{3.22}$$

called Cartan-Killing metric, used to raise, lower, and contract internal space indices.

For non-semisimple groups, those with invariant abelian subgroups such as the Poincaré group, Eq.(3.22) is not a metric since it is degenerate and not invertible. Some exceptions happen to exist where another gauge invariant metric can be found, and a Lagrangian of the form (3.21) can be constructed. This is the case of some abelian groups, where structure coefficients identically vanish. In the gauge theory for $U(1)$, electromagnetism, $\gamma = 1$, and in teleparallel gravity, the gauge theory of the translation group, $\gamma_{ab} = \eta_{ab}$.

The complete gauged Lagrangian is then

$$\mathcal{L} = \mathcal{L}_s[\Psi, D_\mu \Psi] + \mathcal{L}_{gauge} , \tag{3.23}$$

where the source Lagrangian $\mathcal{L}_s$ is obtained from the free one by the *minimal coupling* prescription, in which ordinary derivatives are replaced by covariant ones, allowing for interaction terms between the source and the gauge fields. The second one, the gauge Lagrangian, contains the kinetic term and self interactions of the gauge fields. The corresponding equations of motion are the Yang-Mills equations

$$\begin{aligned} \partial_\mu F^{B\mu\nu} + f^B{}_{CD} A^C{}_\mu F^{D\mu\nu} &= -\frac{\partial \mathcal{L}_s}{\partial A_{B\nu}} \\ \equiv D_\mu F^{B\mu\nu} &= \mathcal{J}^{B\nu} \end{aligned} , \tag{3.24}$$

where $\mathcal{J}^{B\nu}$ is the source current which, due to the antisymmetry of $\boldsymbol{F}$ in its spacetime indices, is covariantly conserved

$$D_\nu \mathcal{J}^{B\nu} = 0 . \tag{3.25}$$

This is not a true conservation law because the connection term in the covariant derivative prevents us from writing it as a time derivative of a distribution plus a divergence.[16] Identifying the second term of (3.24) with the gauge field current

$$j^{B\nu} = -f^B{}_{CD} A^C{}_\mu F^{D\mu\nu} , \tag{3.26}$$

we can write

$$\partial_\nu \left(\mathcal{J}^{B\nu} + j^{B\nu} \right) = \partial_\nu \partial_\mu F^{B\mu\nu} = 0 , \tag{3.27}$$

[16] In the case of partial derivatives, $\partial_\mu A^\mu = \partial_t A^0 + \bar{\nabla} \cdot \bar{A}$

which is now a true conservation law: the *total* current (source plus gauge fields) is conserved. Coming from the covariant equation (3.25), Eq.(3.27) is covariant itself, thus physically meaningful. In fact, they conserve the *gauge charges*

$$q^B = \int d^3x \left(\mathcal{J}^{B0} + j^{B0}\right), \tag{3.28}$$

which give, for a test particle of mass m and charge q^B, the equations of motion

$$\frac{du^\mu}{d\sigma} - \gamma_{BC}\frac{q^B}{mc^2}F^{C\mu}{}_\nu u^\nu = 0\,. \tag{3.29}$$

3.2 Gravity sector: General Relativity

In the context of gravity formulations, the diffeomorphism group constitutes the cornerstone symmetry on which the theory is built. Following the steps at the beginning 3, consider the set of linear bases connected by $GL(4,\mathbb{R})$. At each point p of spacetime $\mathbb{R}$, any particular basis on $T_p\mathbb{R}$ is then obtained by a transformation (group member) acting on another one. This means that starting from a basis in tangent space, each group element at that point p corresponds to another basis. Therefore, we have just substituted a copy of G at every point of spacetime: the bundle $\mathbb{BR}$ of bases is now the principal bundle. This seemingly redundant fact is a marvel of differential geometry with tetrads: being invertible, they determine the LLT that relates them, and all physics is encoded in the principal bundle of frames.

This miraculous relationship is possible thanks to *soldering*. If a base b on $T_p\mathbb{R}$ is a point in $\mathbb{BR}$, b and all its companions (related by G) are taken to the point p by the bundle projection π. The *solder* 1-form θ then relates each tangent space $T_b\mathbb{BR}$ of the bundle of frames to Minkowski space $\mathbb{M}$. The existence of such a mapping is cardinal: it lets us define a vector-space isomorphism that locally takes $\mathbb{M}$ into $T_p\mathbb{R}$, identifying the tangent space with Minkowski space. As it turns out, soldering is also a prerequisite for torsion: the torsion tensor of a connection is just the covariant derivative (with respect to that connection) of θ (see equation (2.46)). Thus, torsion is non-existent in internal gauge theories such as Yang-Mills, where the solder form is not defined.

In the framework of GR, gravity is encoded in the Riemannian metric $g_{\mu\nu}$ that replaces the Minkowski one in the presence of mass-energy content, and plays the role of the basic field defined everywhere. With this, we have to impose the metric-compatibility condition (2.17), and the torsionless condition (2.18). The only connection compatible with such requirements is then the Levi-Civita connection, whose components are the Christoffel symbols[17]

$$\overset{\circ}{\Gamma}{}^\rho{}_{\mu\nu} = \frac{1}{2}g^{\rho\alpha}(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu})\,, \tag{3.30}$$

[17] Recall we are using the "$\circ$" notation for all quantities related to the Levi-Civita connection

which in turn depend on the metric, so $g_{\mu\nu}$ is the only fundamental field describing gravitational interactions. This defines the usual GR Riemann tensor

$$\mathring{R}^\alpha{}_{\beta\mu\nu} = \partial_\mu \mathring{\Gamma}^\alpha{}_{\beta\nu} + \mathring{\Gamma}^\alpha{}_{\rho\mu}\mathring{\Gamma}^\rho{}_{\beta\nu} - (\mu \leftrightarrow \nu)\,, \tag{3.31}$$

and its contractions: the Ricci tensor

$$\mathring{R}_{\mu\nu} = \mathring{R}^\alpha{}_{\mu\alpha\nu}\,, \tag{3.32}$$

and the scalar curvature

$$\mathring{R} = g^{\mu\nu}\mathring{R}_{\mu\nu}\,. \tag{3.33}$$

In GR, the curvature tensor is just the field strength tensor of the theory (from (2.20))

$$[\mathring{\nabla}_\mu, \mathring{\nabla}_\nu]V^\alpha = \mathring{R}^\alpha{}_{\beta\mu\nu}V^\beta + (\cancel{\mathring{\Gamma}^\beta{}_{\mu\nu} - \mathring{\Gamma}^\beta{}_{\nu\mu}})\mathring{\nabla}_\beta V^\alpha\,, \tag{3.34}$$

by virtue of the Christoffel symbols' symmetry with respect to their lower indices. With standard use of Noether's theorem machinery, and after minimally coupling fields to the metric via minimal substitution, the General Coordinate Transformation symmetry of a source Lagrangian $\mathcal{L}_s$ yields the covariant equation

$$\mathring{\nabla}_\mu \mathcal{T}^{\mu\nu} = 0\,, \tag{3.35}$$

where

$$\mathcal{T}^{\mu\nu} = -\frac{1}{2\sqrt{-g}}\frac{\delta\mathcal{L}_s}{\delta g_{\mu\nu}}\,, \tag{3.36}$$

is the symmetric Hilbert energy-momentum tensor.

On the other hand, the second Bianchi identity can be rearranged, using contractions of the curvature tensor, to read

$$\mathring{\nabla}_\mu(\mathring{R}^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\mathring{R}) = 0\,, \tag{3.37}$$

whose argument is often called the *Einstein tensor*. These are the equations of motion for the Einstein-Hilbert Lagrangian

$$\mathring{\mathcal{L}}_g = -\frac{1}{16\pi}\sqrt{-g}\mathring{R}\,, \tag{3.38}$$

when taking its variation with respect to the metric. It is thus natural to perceive the equations

$$\mathring{R}^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\mathring{R} = 8\pi\mathcal{T}^{\mu\nu}\,, \tag{3.39}$$

as the equations of motion of the total gravitational Lagrangian

$$\mathcal{L} = \overset{\circ}{\mathcal{L}}_g + \mathcal{L}_s \,, \tag{3.40}$$

with respect to the metric field $g_{\mu\nu}$. The correspondence principle with Newton's theory of gravity (static weak field limit) is used to determine $k = \frac{8\pi G}{c^4}$, so that we finally get *Einstein's field equations*

$$\overset{\circ}{R}^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \overset{\circ}{R} = \frac{8\pi G}{c^4} \mathcal{T}^{\mu\nu} \,. \tag{3.41}$$

Given that the interaction is entirely described by the metric through the Riemann tensor, it is now geometrized: gravity does not act as a force, but rather as a deformation of spacetime in which free-falling objects follow its natural curvature.

In Minkowski spacetime, a free massive point particle would obey the equation

$$u^\mu \partial_\mu u^\alpha = \frac{du^\alpha}{d\tau} = 0 \,, \tag{3.42}$$

where u^μ is its four-velocity and τ is the invariant interval $d\tau^2 = \eta_{\mu\nu} dx^\mu dx^\nu$. Under minimal coupling prescription it reads

$$u^\mu \overset{\circ}{\nabla}_\mu u^\alpha = \frac{du^\alpha}{ds} + \overset{\circ}{\Gamma}{}^\alpha{}_{\mu\nu} u^\mu u^\nu = 0 \,, \tag{3.43}$$

with $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. Known as *geodesic equations*, they impose $\overset{\circ}{a}{}^\nu = 0$, confirming the geometric nature of this description.

Because the Levi-Civita connection fully depends on $\boldsymbol{g}$, it is neither a fundamental field in the classical sense nor a pure spacetime connection. As can be seen from Eq. (3.43), $\boldsymbol{\Gamma}$ encodes both inertial and gravitational effects that can distort Minkowskian trajectories. The power of the equivalence principle lies in the possibility of finding a local inertial frame in which these two compensate and the connection vanishes. As will be seen shortly, TEGR distinguishes between the two contributions, and the gravitational sector can be represented as a translational-valued gauge potential.

Some differences with YM theories are immediately obvious:

- In YM, interactions are mediated by gauge fields, which are connections on the fibers of the tangent bundle (they live in copies of gauge assigned at each point of tangent space). In GR, the set of diffeomorphisms is not the gauge group of gravity, and the connection is defined on the manifold (spacetime) via the metric only.

- Gauge Lagrangians are quadratic in the field strength tensor, which is an invariant linear in the derivatives of the fields $\rightarrow \mathcal{L}_{YM}[\Psi, \partial\Psi]$, whereas the Einstein-Hilbert Lagrangian linearly depends on $\overset{\circ}{R}$, which is an invariant that also contains second derivatives of $\boldsymbol{g}$

- While gauge fields represent the mediator of a force, the metric is not a force carrier. The connection term that appears in (3.43) is a correction for parallel transporting basis vectors in curved space. A massive point particle with gauge charge q^A in a gravitational field would follow the minimally coupled version of (3.29) to

$$\frac{du^\alpha}{ds} + \overset{\circ}{\Gamma}{}^\alpha{}_{\mu\nu} u^\mu u^\nu = \gamma_{BC} \frac{q^B}{mc^2} F^{C\alpha}{}_\nu u^\nu \,. \tag{3.44}$$

3.3 Gravity sector: Teleparallel Equivalent of GR

As anticipated, Teleparallel Gravity is a gauge theory of the translation group living in Minkowski tangent space. Therefore, a translation of some coordinate x^a on $T_pM = \mathbb{M}$ is

$$x'^a = x^a + \varepsilon^a(x^\mu)\,, \tag{3.45}$$

whose infinitesimal variation is

$$\delta x^a = \varepsilon^b \, P_b x^a\,, \tag{3.46}$$

with P_b the group generators

$$P_b = \partial_b\,, \tag{3.47}$$

which obviously commute with each other, so all structure coefficients identically vanish:

$$[P_a, P_b] = 0 \;\longrightarrow\; f^c{}_{ab} = 0\,. \tag{3.48}$$

Thanks to the presence of soldering, the tangent bundle in which these gauge transformations happen is isomorphic to $\mathbb{M}$, and allows for the presence of tetrad fields.

Under infinitesimal tangent space translations, the source fields then transform as

$$\delta\Psi\big(x^a(x^\mu)\big) = \varepsilon^a(x^\mu)\partial_a \Psi\big(x^a(x^\mu)\big)\,, \tag{3.49}$$

which is the variation of Ψ at fixed x^a and x^μ.

As it is customary in gauge theories, making this a local transformation breaks the covariance of derivative terms

$$\delta(\partial_\mu \Psi) = \varepsilon^a \partial_a(\partial_\mu \Psi) + (\partial_\mu \varepsilon^a)\partial_a \Psi\,, \tag{3.50}$$

and a gauge potential is introduced to restore it, whose components

$$B_\mu = B^a{}_\mu P_a \,, \tag{3.51}$$

belong to the Lie algebra of translations. It is thus natural to define the translational covariant derivative

$$h_\mu \Psi = \partial_\mu \Psi + B^a{}_\mu \partial_a \Psi \,, \tag{3.52}$$

such that it follows (3.50)

$$\delta(h_\mu \Psi) = \varepsilon^a \partial_a (h_\mu \Psi) \,, \tag{3.53}$$

as long as the field components transform as

$$\delta B^a{}_\mu = -\partial_\mu \varepsilon^a \,, \tag{3.54}$$

under translations. The coupling prescription is then straightforwardly given by the replacement of partial derivatives with covariant ones

$$\partial_\mu \Psi \to h_\mu \Psi \,. \tag{3.55}$$

Because of the soldering property of the bundle we can use the chain rule between internal and external indices, and the covariant derivative (3.52) can be written as

$$h_\mu \Psi = \frac{\partial x^a}{\partial x_\mu} \frac{\partial \Psi}{\partial x^a} + B^a{}_\mu \partial_a \Psi = h^a{}_\mu \partial_a \Psi \,, \tag{3.56}$$

with $h^a{}_\mu$ a tetrad field given by

$$h^a{}_\mu = \partial_\mu x^a + B^a{}_\mu \,. \tag{3.57}$$

This equation is physically meaningful if the tetrad is nontrivial, which implies that the gauge field satisfies $B^a{}_\mu \neq \partial_\mu \varepsilon^a$. If this were not true, $h^a{}_\mu$ would just be the gauge transformed version of (2.68). The translational coupling prescription is then[18] [1]

$$\partial_\mu \Psi = e^a{}_\mu \partial_a \Psi \to h^a{}_\mu \partial_a \Psi = h_\mu \Psi \,, \tag{3.58}$$

which simply requires the replacement of a trivial tetrad with a nontrivial one

$$e^a{}_\mu \to h^a{}_\mu \,. \tag{3.59}$$

This change also affects the metric, as can be seen by the simple substitution

$$\eta_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab} \to h^a{}_\mu h^b{}_\nu \eta_{ab} = g_{\mu\nu} \,, \tag{3.60}$$

[18] Because the generators P_a are derivatives which act on tangent-space arguments, every source field will couple equally to them and consequently their translational gauge potentials. Therefore universality of gravitation is still present in the theory

and is solely a consequence of the translational coupling prescription.

From equation (3.57) it is clear we are working in a Lorentz frame with vanishing spin connection,[19] so, in order to explicitly include inertial effects, we can perform a Local Lorentz transformation

$$x^a \to \Lambda^a{}_b(x) x^b \ , \quad B^a{}_\mu \to \Lambda^a{}_b(x) B^b{}_\mu \, , \tag{3.61}$$

so that the translational covariant derivative (3.52) is expressed as

$$\begin{aligned} h^a{}_\mu &= \partial_\mu x^a + \overset{\bullet}{\omega}{}^a{}_{b\mu} x^b + B^a{}_\mu \\ &= \overset{\bullet}{\mathcal{D}}_\mu x^a + B^a{}_\mu \\ &\equiv e^a{}_\mu + B^a{}_\mu \, , \end{aligned} \tag{3.62}$$

where the connection

$$\overset{\bullet}{\omega}{}^a{}_{b\mu} = \Lambda^a{}_c(x) \partial_\mu \Lambda_b{}^c(x) \, , \tag{3.63}$$

is purely inertial by definition. Because of this construction, inertial contributions appear only through the Lorentz part of the trivial tetrad $e^a{}_\mu = \overset{\bullet}{\mathcal{D}}_\mu x^a$, and gravitation is entirely encoded in the gauge potential $B^a{}_\mu$. In this set of frames, its transformation (3.54) becomes

$$\delta B^a{}_\mu = -\overset{\bullet}{\mathcal{D}}_\mu \varepsilon^a \, , \tag{3.64}$$

which directly implies that $h^a{}_\mu$ is invariant under the joint gauge transformations (3.46) and (3.64)

$$\delta h^a{}_\mu = 0 \, . \tag{3.65}$$

So what is the strength tensor of this theory? First, take the commutator of the newly defined covariant derivatives h_μ,

$$\begin{aligned} [h_\mu, h_\nu]\Psi &= (\partial_\mu B^a{}_\nu - \partial_\nu B^a{}_\mu + \overset{\bullet}{\omega}{}^a{}_{b\mu} B^b{}_\nu - \overset{\bullet}{\omega}{}^a{}_{b\nu} B^b{}_\mu)\Psi \\ &= (\overset{\bullet}{\mathcal{D}}_\mu B^a{}_\nu - \overset{\bullet}{\mathcal{D}}_\nu B^a{}_\mu)\partial_a \Psi \, , \end{aligned} \tag{3.66}$$

where ∂_a are the generators we called P_a. As we have seen in the previous section, the torsion tensor vanishes for trivial tetrads (which are the base to which we are adding the coupling prescription for gravitation)

$$[\overset{\bullet}{\mathcal{D}}_\mu, \overset{\bullet}{\mathcal{D}}_\nu] V^a = \overset{\bullet}{\mathcal{D}}_\mu (\overset{\bullet}{\mathcal{D}}_\nu V^a) - \overset{\bullet}{\mathcal{D}}_\mu (\overset{\bullet}{\mathcal{D}}_\mu V^a) = 0 \, . \tag{3.67}$$

[19] There are no inertial contributions to Eq. (3.57), it could be rewritten as $h^a{}_\mu = e'^a{}_\mu + B^a{}_\mu$ where e' is the tetrad (2.68)

Adding this term to the right of (3.66) we get

$$\dot{\mathcal{D}}_\mu(\dot{\mathcal{D}}_\nu V^a + B^a{}_\nu) - \dot{\mathcal{D}}_\nu(\dot{\mathcal{D}}_\mu V^a + B^a{}_\mu) = \dot{\mathcal{D}}_\mu h^a{}_\nu - \dot{\mathcal{D}}_\nu h^a{}_\mu = \dot{T}^a{}_{\mu\nu}\,, \tag{3.68}$$

where we have used the definitions of tetrad field (3.62) and of torsion (2.46). Using our definition of translational covariant derivative (3.62), this means

$$[h_\mu, h_\nu]\Psi = \dot{T}^a{}_{\mu\nu}\partial_a\Psi\,. \tag{3.69}$$

The torsion tensor is thus seen to play the role of translational field strength in Teleparallel Gravity,[20] is gauge invariant,[21] and encodes all gravitational field dynamics in the gauge potential $B^a{}_\mu$ which, we recall, is a 1-form with values in the Lie algebra of the translation group

$$\boldsymbol{B} = B^a{}_\mu\partial_a dx^\mu\,. \tag{3.70}$$

In turn, this also means that in the absence of gravity, the fundamental Lorentz connection is purely inertial and only depends on LLTs, confirming the distinct nature of these degrees of freedom and the vanishing of curvature in TEGR.

3.3.1 Equivalence with GR

Being a gauge theory of the translation group, the dynamical action is

$$\dot{S} = \frac{1}{16\pi}\int \eta_{ab}\dot{T}^a \wedge \star\dot{T}^b\,, \tag{3.71}$$

where

$$\dot{T}^a = \frac{1}{2}\dot{T}^a{}_{\mu\nu}dx^\mu \wedge dx^\nu \quad ; \qquad \star\dot{T}^a = \frac{1}{2}\big(\star\dot{T}^a{}_{\mu\nu}\big)dx^\mu \wedge dx^\nu\,, \tag{3.72}$$

are the torsion 2-form and its dual, defined as[22]

$$\star\dot{T}^a{}_{\mu\nu} = \frac{h}{2}\varepsilon_{\mu\nu\alpha\beta}\mathcal{S}^{a\alpha\beta}\,, \tag{3.73}$$

[20] This is analogous to the case of equation (2.20) for a scalar field rather than a vector, as the left-most covariant derivatives would only act on the partial derivatives to their right, leaving the antisymmetric part of $\boldsymbol{\Gamma}$ behind.

[21] This is automatically expected because the translation group is abelian, which means that all structure coefficients vanish making field strengths trivially gauge invariant.

[22] As previously mentioned, the soldered character of the tangent bundle allows for more contractions, so the dual field strength is more general than in YM. This choice is such that $\star\star\dot{T} \propto \dot{T}$. A full derivation can be found in [1].

where we have called $\det h^b{}_\rho \equiv h$ and

$$\mathcal{S}^{a\mu\nu} = h^a{}_\rho \big(\overset{\bullet}{K}{}^{\mu\nu\rho} + g^{\rho\mu} \overset{\bullet}{T}{}^{\sigma\nu}{}_\sigma - g^{\rho\nu} \overset{\bullet}{T}{}^{\sigma\mu}{}_\sigma \big) \,, \tag{3.74}$$

is called *superpotential*, with $\boldsymbol{K}$ the contortion tensor of the Weitzenböck connection. Using the identity

$$dx^\mu \wedge dx^\nu \wedge dx^\alpha \wedge dx^\beta = -\varepsilon^{\mu\nu\alpha\beta} h d^4x \,, \tag{3.75}$$

the TEGR action can be written as

$$\overset{\bullet}{S} = \frac{1}{32\pi} \int h d^4x \overset{\bullet}{T}{}_{\alpha\mu\nu} \mathcal{S}^{\alpha\mu\nu} \,, \tag{3.76}$$

which, remembering that $\overset{\bullet}{K}{}^\mu{}_{\nu\mu} = \overset{\bullet}{T}{}^\mu{}_{\mu\nu}$, can be further manipulated to finally read

$$\overset{\bullet}{S} = \frac{1}{16\pi} \int h d^4x \left(\overset{\bullet}{K}{}_{\alpha\mu\nu} \overset{\bullet}{K}{}^{\nu\mu\alpha} - \overset{\bullet}{K}{}^\alpha{}_{\mu\alpha} \overset{\bullet}{K}{}^{\beta\mu}{}_\beta \right) , \tag{3.77}$$

or, in terms of torsion,

$$\overset{\bullet}{S} = \frac{1}{16\pi} \int h d^4x \left(\frac{1}{4} \overset{\bullet}{T}{}_{\alpha\mu\nu} \overset{\bullet}{T}{}^{\alpha\mu\nu} + \frac{1}{2} \overset{\bullet}{T}{}_{\alpha\mu\nu} \overset{\bullet}{T}{}^{\nu\mu\alpha} - \overset{\bullet}{T}{}^\alpha{}_{\mu\alpha} \overset{\bullet}{T}{}^{\beta\mu}{}_\beta \right) . \tag{3.78}$$

Recalling the torsion tensor's role, the first term is simply the analog Lagrangian of gauge theory (3.21). The other two contributions, absent in YM, are understood as a direct consequence of the soldering character of the tangent bundle. These are the novel contractions we have so cryptically anticipated. Furthermore, because all terms are contractions of tensors, they can all be covariantly cast into latin indices with tetrads, making Lorentz invariance manifest.

To show the TEGR-GR equivalence, we start by inserting the decomposition of metric-compatible connections

$$\overset{\bullet}{\Gamma}{}^\alpha{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^\alpha{}_{\mu\nu} + \overset{\bullet}{K}{}^\alpha{}_{\mu\nu} \,, \tag{3.79}$$

into the definition of curvature

$$\begin{aligned} \overset{\bullet}{R}{}^\alpha{}_{\beta\mu\nu} &= \partial_\mu \overset{\bullet}{\Gamma}{}^\alpha{}_{\beta\nu} - \partial_\nu \overset{\bullet}{\Gamma}{}^\alpha{}_{\beta\mu} + \overset{\bullet}{\Gamma}{}^\alpha{}_{\rho\mu} \overset{\bullet}{\Gamma}{}^\rho{}_{\beta\nu} - \overset{\circ}{\Gamma}{}^\alpha{}_{\rho\nu} \overset{\bullet}{\Gamma}{}^\rho{}_{\beta\mu} \\ &= \overset{\circ}{R}{}^\alpha{}_{\beta\mu\nu} + \overset{\bullet}{Q}{}^\alpha{}_{\beta\mu\nu} \,, \end{aligned} \tag{3.80}$$

where we have collected the Levi-Civita contributions into the usual Riemann tensor of GR, and the contortion ones into

$$\begin{aligned} \overset{\bullet}{Q}{}^\alpha{}_{\beta\mu\nu} = {} & \partial_\mu \overset{\bullet}{K}{}^\alpha{}_{\beta\nu} - \partial_\nu \overset{\bullet}{K}{}^\alpha{}_{\beta\mu} + \overset{\bullet}{\Gamma}{}^\alpha{}_{\rho\mu} \overset{\bullet}{K}{}^\rho{}_{\beta\nu} - \overset{\bullet}{\Gamma}{}^\alpha{}_{\rho\nu} \overset{\bullet}{K}{}^\rho{}_{\beta\mu} \\ & - \overset{\bullet}{\Gamma}{}^\rho{}_{\beta\mu} \overset{\bullet}{K}{}^\alpha{}_{\rho\nu} + \overset{\bullet}{\Gamma}{}^\rho{}_{\beta\nu} \overset{\bullet}{K}{}^\alpha{}_{\rho\mu} - \overset{\bullet}{K}{}^\alpha{}_{\rho\mu} \overset{\bullet}{K}{}^\rho{}_{\beta\nu} + \overset{\bullet}{K}{}^\alpha{}_{\rho\nu} \overset{\bullet}{K}{}^\rho{}_{\beta\mu} \,. \end{aligned} \tag{3.81}$$

As it has hopefully become clear by now, Weitzenböck spacetime is Riemann-flat, so $\overset{\bullet}{R}{}^{\alpha}{}_{\beta\mu\nu} = 0$, which translates to

$$\overset{\bullet}{Q}{}^{\alpha}{}_{\beta\mu\nu} = -\overset{\circ}{R}{}^{\alpha}{}_{\beta\mu\nu} \,. \tag{3.82}$$

Plugging (3.81) into the previous equation and taking the standard contractions that lead to the Ricci scalar, this can be written as

$$\begin{aligned} \overset{\bullet}{Q} = g^{\mu\nu}\overset{\bullet}{Q}{}^{\alpha}{}_{\mu\alpha\nu} &= \left(\overset{\bullet}{K}{}_{\alpha\mu\nu}\overset{\bullet}{K}{}^{\nu\mu\alpha} - \overset{\bullet}{K}{}^{\alpha}{}_{\mu\alpha}\overset{\bullet}{K}{}^{\beta\mu}{}_{\beta}\right) + \frac{2}{h}\partial_\mu\left(h\overset{\bullet}{T}{}^{\nu\mu}{}_{\nu}\right) \\ &= -\overset{\circ}{R}\,. \end{aligned} \tag{3.83}$$

The first term is exactly the Lagrangian density that appears in the TEGR action (3.77), which is now seen to be equal to $\overset{\circ}{R}$, the Lagrangian density of the Hilbert action, up to a total derivative. In formulae,

$$\overset{\bullet}{\mathcal{L}} = -\frac{1}{16\pi}\overset{\circ}{R} - \partial_\mu\left(\frac{h}{8\pi}\overset{\bullet}{T}{}^{\nu\mu}{}_{\nu}\right), \tag{3.84}$$

which means that the two theories are equal at classical level. This equivalence is in general expected to break after quantization and will be investigated in the 5.

3.3.2 Coupling prescription and Equations of Motion

We have so far only talked about the gravitational prescription $e^a{}_\mu \to h^a{}_\mu$ coming from invariance under local translations, which amounts to the standard $\eta_{\mu\nu} \to g_{\mu\nu}$. Because any physical theory must also be invariant under Local Lorentz Transformations,[23] we could have first introduced the *inertial* prescription

$$\partial_\mu \to \mathcal{D}_\mu = \partial_\mu + \frac{i}{2} w^a{}_{b\mu} S_a{}^b \,, \tag{3.85}$$

with the Fock-Ivanenko covariant derivative defined in (2.36) and $S_a{}^b$ the appropriate Lorentz generators. Note that, differently from the gravitational one, this prescription is not universal, as it depends on the spin of each field through its Lorentz representation. By use of the general covariance principle, it is clear that we must do this first:[24] moving to an anholonomic frame (with inertial effects)

[23] All measurable physical processes must be independent of the inertial frame in which they occur.

[24] In the passage from Special to General Relativity, one has to introduce a connection that necessarily encodes inertial properties of the frame, and can later invoke the equivalence principle (inertial and gravitational effects are indistinguishable) to replace it with a true gravitational connection. This is what [1] call *active* covariance principle, which allows the TEGR connection to represent pure gravity.

and then adding a term that compensates inertia with gravity. Working in latin indices, the first step is

$$\partial_c \to \mathcal{D}_c = h_c - \frac{i}{2}\big(w^a{}_{bc} - K^a{}_{bd}\big) S_a{}^b \,, \tag{3.86}$$

after which we add the translational coupling

$$e^a{}_\mu \mathcal{D}_a \to h^a{}_\mu \mathcal{D}_a \,, \tag{3.87}$$

which, using (3.58), means that the full gravitational prescription equates to

$$\partial_\mu \to \overset{\bullet}{\mathcal{D}}_\mu = \partial_\mu - \frac{i}{2}\big(\overset{\bullet}{w}{}^a{}_{b\mu} - \overset{\bullet}{K}{}^a{}_{b\mu}\big) S_a{}^b \,. \tag{3.88}$$

By the very definition of contortion (2.51),

$$\overset{\circ}{w}{}^a{}_{b\mu} = \overset{\bullet}{w}{}^a{}_{b\mu} - \overset{\bullet}{K}{}^a{}_{b\mu} \,, \tag{3.89}$$

we can see that, in TEGR, the coupling prescription is equivalent to the one of General Relativity

$$\overset{\bullet}{\mathcal{D}}_\mu = \overset{\circ}{\mathcal{D}}_\mu \,. \tag{3.90}$$

Finding the equations of motion for a test particle is now just a matter of variational calculus. Let us start with no gravitational field to further stress the properties of the theory. Recalling the definition of four velocity, we can write the spacetime interval as

$$d\sigma = u_\mu dx^\mu = \big(e^a{}_\mu u_a\big)\big(e_b{}^\mu e^b\big) = U_a e^a = u_a(dx^a + \overset{\bullet}{w}{}^a{}_{b\mu} x^b dx^\mu) \,, \tag{3.91}$$

where we have used (2.26), (2.28), and (2.69). The action for a massive, free particle is then

$$S = -m \int d\sigma = -m \int u_a \Big(dx^a + \overset{\bullet}{w}{}^a{}_{b\mu} x^b dx^\mu\Big) \,, \tag{3.92}$$

of which we have to take the variation under coordinate change

$$x^\mu \to x^\mu + \delta x^\mu \,, \tag{3.93}$$

that implies

$$\delta x^a = \partial_\mu x^a \delta x^\mu \quad ; \qquad \delta \overset{\bullet}{w}{}^a{}_{b\mu} = \partial_\nu \overset{\bullet}{w}{}^a{}_{b\mu} dx^\nu \,, \tag{3.94}$$

and

$$\delta S = m \int e^b{}_\nu \Big(\frac{du_b}{d\sigma} + \overset{\bullet}{w}{}^a{}_{b\mu} u_a u^\mu\Big) d\sigma \delta x^\nu \,. \tag{3.95}$$

Imposing stationary solutions ($\delta S = 0$), the equations of motion for a free particle in the absence of gravitation are

$$\frac{du_b}{d\sigma} - \overset{\bullet}{w}{}^a{}_{b\mu} u_a u^\mu = 0\,, \tag{3.96}$$

and correctly coincide with (2.77), previously found by means of LLT.

If the particle does interact with gravity, we only need to replace e by h

$$S = -m \int u_a h^a = -m \int u_a \Big(dx^a + \overset{\bullet}{w}{}^a{}_{b\mu} x^b dx^\mu + B^a{}_\mu dx^\mu \Big)\,, \tag{3.97}$$

and consider the translational potential variation

$$\delta B^a{}_\mu = \partial_\nu B^a{}_\mu \delta x^\nu\,, \tag{3.98}$$

which gives a cumulative variation

$$\delta S = m \int \Big[h^b{}_\nu \Big(\frac{du_b}{d\sigma} + \overset{\bullet}{w}{}^a{}_{b\mu} u_a u^\mu \Big) - \overset{\bullet}{T}{}^a{}_{\nu\mu} u_a u^\mu \Big] d\sigma \delta x^\nu\,. \tag{3.99}$$

Contracting $\overset{\bullet}{T}$'s second index with the tetrad, and using the identity

$$\overset{\bullet}{T}{}^a{}_{b\mu} u_a u^\mu = -\overset{\bullet}{K}{}^a{}_{b\mu} u_a u^\mu\,, \tag{3.100}$$

the final equations of motion for a free, massive particle in a gravitational field are[25]

$$\frac{du_b}{ds} - \overset{\bullet}{w}{}^a{}_{b\mu} u_a u^\mu = -\overset{\bullet}{K}{}^a{}_{b\mu} u_a u^\mu\,, \tag{3.101}$$

which look exactly like equations (3.29), or (3.96) with a force term, with contortion having the role of *force of gravity*. Moving everything to the left side and using (3.89), we find that they are exactly equal to the usual geodesic equations of GR, which was expected from (3.84). Once again we can see the separation of inertia and gravitation: the inertial effects of the frame, represented by $\overset{\bullet}{\boldsymbol{w}}$ on the left-hand side and non-covariant by nature, are still geometrized in the sense of General Relativity. On the other hand, gravitational effects are accounted for by the contortion tensor $\overset{\bullet}{\boldsymbol{K}}$, so they are now described as a covariant force.

[25] The minus sign is due to the velocity components being covariant rather than contravariant.

4 Functional formalism

4.1 Effective action

In the presence of a source current J linearly coupled to a scalar field ϕ, the generating functional integral Z reads

$$Z[J;g] = \int [D\phi] e^{\frac{i}{\hbar}(S[\phi;g]+\int d^4x\sqrt{-g}J\phi)}\,, \tag{4.1}$$

where $D\phi$ is some functional measure. After the standard Wick rotation $t \to -i\tau_E$, which turns oscillatory path integrals into damped ones, we can relate Z to the Euclidean generating functional of connected Green's functions W

$$Z[J;g] = \int [D\phi] e^{-\frac{1}{\hbar}S_E[\phi;g]+\int d^4x\sqrt{-g}J\phi} = e^{\frac{1}{\hbar}W[J;g]}\,, \tag{4.2}$$

where S_E is the Euclidean action. Throughout this section, we will omit the subscript E, the dependence of the functionals from $\boldsymbol{g}$ and $\sqrt{-g}$ from the invariant infinitesimal volume element for a clearer view, working on the Euclidean manifold from now on. Here, $g_{\mu\nu}$ enters an external independent field on which functionals depend parametrically.

Taking the variation of W with respect to the source,

$$\begin{aligned}\frac{\delta W[J]}{\delta J(x)} &= \frac{1}{Z[J]}\frac{\delta Z[J]}{\delta J(x)} = \frac{\langle 0|T\phi e^{\int d^4x\sqrt{-g}\phi(x)J(x)}|0\rangle}{\langle 0|Te^{\int d^4x\sqrt{-g}\phi(x)J(x)}|0\rangle} \\ &= \bar{\phi}(x)\,,\end{aligned} \tag{4.3}$$

we find the source-dependent mean field $\bar{\phi}(x)$ in the presence of J, where T is the *time-ordered product* of operators in the quantum amplitude. Assuming its invertibility, this equation lets us express the source through the mean field

$$J = J[\bar{\phi}](x) \equiv J(x)\,. \tag{4.4}$$

This way, we define the *effective action* Γ as the Legendre transform of W

$$\Gamma[\bar{\phi}] = -W[J] + \int d^4x\sqrt{-g}J(x)\bar{\phi}(x)\,. \tag{4.5}$$

Differentiating this with respect to $\bar{\phi}$

$$\begin{aligned}\frac{\delta\Gamma[\bar{\phi}]}{\delta\bar{\phi}(x)} &= \int d^4y\sqrt{-g}\frac{\delta W[J]}{\delta J(y)}\frac{\delta J(y)}{\delta\bar{\phi}(x)} - J(x) - \int d^4y\sqrt{-g}\bar{\phi}\frac{\delta J(y)}{\delta\bar{\phi}(x)} \\ &= -J(x)\,,\end{aligned} \tag{4.6}$$

we see that Γ acts as the generating functional of the 1-particle irreducible Green functions given by the source current [18]: in quantum theory, $\Gamma[\bar{\phi}]$ plays the same role that $S[\phi]$ does in classical theory, and will have the general structure

$$\Gamma[\bar{\phi}] = S[\bar{\phi}] + \text{ \textit{quantum corrections}}. \tag{4.7}$$

Taking W from (4.5) and plugging it into Eq. (4.2) we see that the effective action has a compact equation:

$$\exp\left[-\frac{1}{\hbar}\Gamma[\bar{\phi}]\right] = \int [D\phi]\exp\left[-\frac{1}{\hbar}\left(S[\phi] - \int d^4x\sqrt{-g}\frac{\delta\Gamma[\bar{\phi}]}{\delta\bar{\phi}(x)}(\phi-\bar{\phi})\right)\right]. \tag{4.8}$$

If we split the field ϕ between background and quantum corrections

$$\phi = \bar{\phi} + \sqrt{\hbar}\varphi\,, \tag{4.9}$$

we can expand the action in powers of the perturbation φ

$$\begin{aligned} S\left[\bar{\phi}+\sqrt{\hbar}\varphi\right] &= S[\bar{\phi}] \\ &+ \sum_{n=1}^{\infty}\frac{\hbar^{n/2}}{n!}\int d^4x_1...d^4x_n\frac{\delta^n S[\phi]}{\delta\varphi(x_1)...\delta\varphi(x_n)}\bigg|_{\phi=\bar{\phi}}\varphi(x_1)...\varphi(x_n) \equiv \\ &= S[\bar{\phi}] + \sum_{n=1}^{\infty}\frac{\hbar^{n/2}}{n!}S_n[\bar{\phi}]\varphi^n\,, \end{aligned} \tag{4.10}$$

and plugging it into Eq.(4.8) we get[26]

$$\begin{aligned} &\exp\left[-\frac{1}{\hbar}\left(\Gamma[\bar{\phi}]-S[\bar{\phi}]\right)\right] \\ &\quad = \int[D\varphi]\exp\left[-\sum_{n=1}^{\infty}\frac{\hbar^{n/2-1}}{n!}S_n[\bar{\phi}]\varphi^n + \frac{1}{\sqrt{\hbar}}\int d^4x\sqrt{-g}\frac{\delta\Gamma[\bar{\phi}]}{\delta\bar{\phi}(x)}\varphi\right]. \end{aligned} \tag{4.11}$$

Taking the effective action's formal expansion in powers of $\hbar$ around the background field $\bar{\phi}$

$$\Gamma = \sum_{n=0}^{\infty}\hbar^n\Gamma_n = S + \hbar\Gamma_1 + O(\hbar^2)\,, \tag{4.12}$$

where Γ_n is the n-loop contribution to the classical action S, we compare the coefficients by their powers of $\hbar$ and find that the one-loop contribution to the functional integral (4.2) is

$$\exp\left[-\frac{1}{\cancel{\hbar}}\cancel{\hbar}\Gamma_1[\bar{\phi}]\right] = \int[D\varphi]\exp\left[\frac{1}{2}S_2[\bar{\phi}]\varphi^2\right]. \tag{4.13}$$

[26] Note that the only non-vanishing contributions to the functional integral come from even n, so there are no fractional powers of $\hbar$ in the effective action

Following the definition of S_n in Eq.(4.10) and assuming boundary conditions that let us integrate the derivative terms in the action by parts without any boundary terms left, we find

$$\exp\Big[-\Gamma_1[\bar{\phi}]\Big] = \int [D\varphi]\exp\Big[-\frac{1}{2}\int d^dx\sqrt{-g}\varphi\Delta\varphi\Big] = \frac{1}{\sqrt{\det\Delta}}\,, \tag{4.14}$$

where Δ is a covariant laplacian built by contracting covariant derivatives and the last equality comes from the well known functional determinant. The one-loop effective action is then

$$\Gamma_1 = \frac{1}{2}\ln\det\Delta = \frac{1}{2}\mathrm{Tr}\ln\Delta\,, \tag{4.15}$$

with

$$\Delta = \frac{\delta^2 S}{\delta\varphi(x)\delta\varphi(y)}\Big|_{\bar{\phi}}\delta(x,y)\,, \tag{4.16}$$

and the uppercase Trace implies the presence of an integral over every spacetime coordinate on which its argument depends, as well as a summation over internal indices. This is the one-loop effective action, and it describes quantum effects due to the interactions of φ at one-loop level.

In the simple case of a massive scalar field in a potential, the classical action is

$$S[\varphi] = \int d^4x\sqrt{-g}\big(-\frac{1}{2}\varphi\Box\varphi - \frac{1}{2}m^2\varphi^2 - V(\varphi)\big)\,, \tag{4.17}$$

and so the one-loop effective action reads

$$\Gamma_1[\varphi] = S[\varphi] + \frac{1}{2}\hbar\mathrm{Tr}\,\ln\big[-\Box - m^2 - V''(\varphi)\big]\,. \tag{4.18}$$

4.2 Heat kernel method

We are now interested in the divergences of the effective action. To study them, we start by defining a heat equation whose solutions let us compute the effective action. Using the covariant laplacian (4.16), define the *heat equation*

$$\frac{d\phi}{d\tau} + \Delta\phi = 0 \quad , \qquad \phi(x,y;\tau=0) = \delta(x,y)\,, \tag{4.19}$$

which describes the diffusion process of ϕ on the manifold M with metric $\boldsymbol{g}$ occurring in external proper-time τ with dimension of length squared.[27] The *heat kernel*

[27] Usually denoted by s, this is Schwinger's *proper time*. This is clearer if one derives equation (4.30) by means of functional variations and the Schwinger-DeWitt technique, where the Schwinger representation is used

$K_\Delta(x,y;\tau)$ is a function on $M\times M\times\mathbb{R}$ and a solution to the heat equation (4.19) with initial condition

$$K_\Delta(x,y;0)=\delta(x,y)\,, \tag{4.20}$$

so it has dimensions of an inverse volume. It is a unique solution if Δ is an elliptic operator [12], which is our case This way, given $\phi(\tau=0)$, the solution of the heat equation is

$$\phi(x,\tau)=\int d^dy\sqrt{g(y)}K_\Delta(x,y;\tau)\phi(y,0)\,. \tag{4.21}$$

This means (together with (4.20)) that

$$\frac{dK_\Delta(x,y;\tau)}{d\tau}+\Delta K_\Delta(x,y;\tau)=0\,. \tag{4.22}$$

The heat kernel can thus be formally expressed as the expectation value

$$K_\Delta(x,y;\tau)=\langle x|\,e^{-\tau\Delta}\,|y\rangle\ . \tag{4.23}$$

Given the Laplacian eigenvalue problem

$$\delta\phi_n=\lambda_n\phi_n\,, \tag{4.24}$$

we obtain the spectral decomposition

$$K_\Delta(x,y;\tau)=\sum_n\phi_n(x)\phi_n(y)e^{-\tau\lambda_n}\,, \tag{4.25}$$

in its spectral decomposition. Let us look at the trace

$$\mathrm{Tr}K_\Delta(\tau)=\int d^dx\sqrt{-g}K_\Delta(x,x;\tau)=\sum_n e^{-\tau\lambda_n}\,, \tag{4.26}$$

and show how we can relate this trace to the Riemann zeta function. First, note that from Euler's $\Gamma(s)=\int_0^\infty d\tau\,\tau^{s-1}e^{-\tau}$ we have

$$\lambda_n^{-s}=\frac{1}{\Gamma(s)}\int_0^\infty d\tau\,\tau^{s-1}e^{-\lambda_n\tau}\,. \tag{4.27}$$

From the definition $\zeta(s)=\sum_{n=1}^\infty n^{-s}$ and using Eq.(4.26), we can define the Minakshisundaram–Pleijel zeta function for the operator Δ

$$\zeta_\Delta(s)=\sum_n\lambda_n^{-s}=\frac{1}{\Gamma(s)}\int_0^\infty dt\,\tau^{s-1}\mathrm{Tr}K_\Delta(\tau)\,. \tag{4.28}$$

With this in mind, we can rewrite Eq.(4.15) as

$$\Gamma_1 = \frac{1}{2}\mathrm{Tr}\ln\Delta = -\frac{1}{2}\sum_n \frac{d}{ds}\lambda_n^{-s} = -\frac{d}{ds}\zeta_\Delta(s)\big|_{s=0}\,, \tag{4.29}$$

and inserting Eq.(4.28) into it, we see that the one-loop effective action

$$\Gamma_1 = -\frac{1}{2}\int_0^\infty d\tau\,\tau^{-1}\mathrm{Tr}K_\Delta(\tau)\,, \tag{4.30}$$

can be expressed in terms of the trace of the heat kernel. This expression is always divergent because

$$\Gamma_1^s = -\frac{1}{2}\mu^{2s}\int_0^\infty d\tau\,\tau^{s-1}\mathrm{Tr}K_\Delta(\tau) = -\frac{1}{2}\mu^{2s}\Gamma(s)\zeta_\Delta(s)\,, \tag{4.31}$$

and taking the limit for $s \to 0$ we see that

$$\Gamma_1^{s\to 0} = \Gamma_1 \propto \frac{1}{s} - \gamma_E + O(s)\,, \tag{4.32}$$

where μ is a mass parameter that gives the effective action the proper dimension[28] and γ_E is the Euler-Mascheroni constant. We thus see that Γ_1 always has a pole. Because the proper-time parameter τ has inverse squared mass dimension, the upper and lower ends of integration in Eq. (4.30) correspond to low and high energies respectively. As such, to make the divergences explicit, a cutoff scale Λ_{UV} and a mass scale $\mu < \Lambda_{UV}$ are introduced so that the integral can be split into

$$\int_0^\infty = \int_{1/\Lambda_{UV}^2}^{1/\mu^2} + \int_{1/\mu^2}^\infty\,. \tag{4.33}$$

If Δ is elliptic in Euclidean space and its eigenvalues are strictly positive (see Eq. (4.25)), the second part is convergent. To study the UV regime it is then enough to look for an early proper-time asymptotic expansion.

In flat spacetime, where $\Box = \partial^2$, we can resort to Fourier space to solve Eq.(4.19)

$$\frac{d}{d\tau}\tilde{K}_\Delta(\vec{q},\vec{y};\tau) + q^2\tilde{K}_\Delta(\vec{q},\vec{y};\tau) = 0\,, \tag{4.34}$$

with initial condition $\tilde{K}_\Delta(\vec{q},\vec{y};0) = e^{-i\vec{q}\cdot\vec{y}}$, so

$$K_\Delta(\vec{x},\vec{y},\tau) = \int \frac{d\vec{q}}{(2\pi)^d} e^{-q^2\tau + i\vec{q}\cdot(\vec{x}-\vec{y})} = \frac{1}{(4\pi\tau)^{d/2}} e^{-\frac{|\vec{x}-\vec{y}|^2}{4\tau}}\,, \tag{4.35}$$

[28] In the regularization procedure, quantization is carried out in d dimensions. The resulting divergent momentum integrals are then analytically continued to a non-integer spacetime dimension $d \to 4 - \varepsilon$. In this step, couplings do not retain their original mass dimensions, and the mass parameter μ compensates for these changes.

and

$$\mathrm{Tr}K_\Delta(\tau) = \int d^d x \frac{\sqrt{-g}}{(4\pi\tau)^{d/2}} = \frac{V}{(4\pi\tau)^{d/2}}\,, \tag{4.36}$$

with V the infinite volume.

If we return to curved spacetime, we expect the trace to reduce to (4.36) for $\tau \to 0$ because any manifold locally looks like Euclidean space,[29] so

$$\mathrm{Tr}K_\Delta(\tau) \approx \frac{1}{(4\pi\tau)^{d/2}} \int d^d x \sqrt{-g}\left[b_0(\Delta) + \tau b_2(\Delta) + \tau^2 b_4(\Delta) + ..\right], \tag{4.37}$$

where the coefficients $b_n(\Delta)$, called HAMIDEW[30] coefficients, are invariant scalars constructed with curvatures and covariant derivatives.[31] Consequently, they strongly depend on the metric structure and connection of the theory.

Plugging this early time expansion in the one-loop effective action (4.30) gives the *master formula*

$$\begin{aligned}\Gamma_1 = &-\frac{1}{2(4\pi)^{d/2}} \int d^d x \sqrt{-g} \times \\ &\times \int_{1/\Lambda_{UV}^2}^{1/\mu^2} d\tau \left[\tau^{-\frac{d}{2}-1} b_0 + \tau^{-\frac{d}{2}} b_2 + ... + \tau^{-1} b_d + \text{positive powers}\right] \\ = &-\frac{1}{2(4\pi)^{d/2}} \int d^d x \sqrt{-g}\left[\frac{\Lambda_{UV}^d}{d/2} b_0 + \frac{\Lambda_{UV}^{d-2}}{\frac{d}{2}-1} b_2 + ... + \ln\frac{\Lambda_{UV}^2}{\mu^2} b_d + \text{finite terms}\right].\end{aligned} \tag{4.38}$$

As will be seen, these coefficients contain tensorial structures that already appear in the gravitational action. These will renormalize the constants of the theory. Other terms, the ones not present at first, tell us that new couplings must be added to renormalize the theory. Because we choose to work with background fields, these do not need to be introduced at the start.

In general, b_0 is a constant so the first term is proportional to the vacuum energy, b_2 is linear in invariants, so in the case of GR it is proportional to R and renormalizes Newton's constant in the Hilbert action, and b_4 will contain contractions of covariant tensors which need the introduction of counterterms. This framework is easily generalized by taking a generic differential operator containing both spacetime (or tangent space) and gauge connection. These methods are illustrated below.

---

[29] It can be easily understood by using Riemann normal coordinates, which exactly reproduce this correspondence with flatness at a point.

[30] Owed to Hadamard, Minakshisundaram, Seeley(not in the acronym) and De-Witt.

[31] One can infer their functional form via dimensional analysis, but many algorithms and techniques to compute them are known in literature: [12]. Because curvature represents deviations from flatness, it is easy to see why curvature invariants enter the expression.

4.3 Coefficients for a Generic Field

Let Ψ be a quantum field with mass and spin on a pseudo-Riemannian manifold M. Following the reasoning at the beginning of Chapter 3, the metric $\boldsymbol{g}$ of signature (p,q) defines a bundle of frames S associated with the Lorentz group $SO(p,q)$ and a G-bundle associated to a Gauge group G. In other words, at every point of spacetime it transforms under a representation $\boldsymbol{\sigma}$ of $SO(p,q)$ and under a representation $\boldsymbol{T}$ of the Gauge group. Ψ should be thought of as a section of the bundle $S \otimes V$, so it carries both Lorentz $(i,j,...)$ and gauge $(I,J,...)$. We can then define the full covariant derivative

$$
\begin{aligned}
D_\mu \Psi_i^I &= \partial_\mu \Psi_i^I + \overset{\circ}{\omega}{}^{ab}{}_\mu [S_{ab}]_i{}^j \Psi_j^I + A^B{}_\mu [T_B]^I{}_J \Psi_i^J \\
&= \overset{\circ}{\nabla}_\mu \Psi_i^I + A_\mu{}^I{}_J \Psi_i^J \,,
\end{aligned}
\tag{4.39}
$$

where $\boldsymbol{\omega}$ is the torsion-free spin connection of equation (2.39) with indices on S, and A is a gauge connection with internal indices in V. It is crucial that the spacetime connection be the Levi-Civita one, as extra contributions on the spacetime side, like torsion, would modify the form of the coefficients we are about to show. It is possible to find such new coefficients, but it is much faster to accept $\overset{\circ}{\nabla}$ and introduce modifications elsewhere. This is exactly what will be done in the next chapter.

With the usual assumption on boundary terms that lets us freely integrate derivatives by parts, it is clear that the quadratic part of the action for Ψ can be written as

$$
S[\Psi;\Gamma,A] = \frac{1}{2}\int d^dx \sqrt{-g}\, G^{ij}_{IJ} \Psi_i^I (\Delta\Psi)_j^J \,,
\tag{4.40}
$$

where G^{ij}_{IJ} is a metric in $S \otimes V$, which for us is just the identity, and Δ is a second order differential operator of the form

$$
\Delta = -g^{\mu\nu} D_\mu D_\nu - \mathbf{E} \,,
\tag{4.41}
$$

which we call *covariant laplacian*, where $\mathbf{E}$ is an endomorphism on $S \otimes V$ with component structure ${E_i}^{jI}{}_J$. The HAMIDEW coefficients for this operator read[32]

$$
\begin{aligned}
b_0(x,x) &= \mathrm{Tr}\mathbf{1} \\
b_2(x,x) &= \frac{1}{6}R + \mathrm{Tr}\mathbf{E} \\
b_4(x,x) &= \frac{1}{180}\Big(\mathring{R}_{\alpha\beta\mu\nu}\mathring{R}^{\alpha\beta\mu\nu} - \mathring{R}_{\mu\nu}\mathring{R}^{\mu\nu} + \frac{5}{2}\mathring{R}^2 + 6D_\mu D^\mu \mathring{R}\Big)\mathrm{Tr}\mathbf{1} + \\
&\quad + \frac{1}{2}\mathrm{Tr}\mathbf{E}^2 + \frac{1}{6}\mathring{R}\ \mathrm{Tr}\mathbf{E} + \frac{1}{12}\mathrm{Tr}\Omega_{\mu\nu}\Omega^{\mu\nu} + \frac{1}{6}D_\mu D^\mu \mathrm{Tr}\mathbf{E} \qquad ,
\end{aligned}
\tag{4.42}
$$

where $\mathbf{1}$ is the identity of $S \otimes V$ so $\mathrm{Tr}\mathbf{1}$ is the product of the dimensions of S and V, R is the spacetime curvature

$$[\mathring{\nabla}_\mu, \mathring{\nabla}_\nu]V^\alpha = \mathring{R}^\alpha{}_{\beta\mu\nu}V^\beta\,, \tag{4.43}$$

and Ω is the *gauge bundle curvature* of the full connection

$$[D_\mu, D_\nu]\Psi = \Omega_{\mu\nu}\Psi\,, \tag{4.44}$$

which, for a gauge theory without gravity, would be equal to the field strength tensor. For a theory of pure gravitation, it would be equal to the Riemann tensor.

[32] Because they appear in a trace (Eq.(4.37)) they are calculated at the same point $b(x,x)$. Furthermore, the last terms (and all terms of higher order) proportional to Δ are total derivatives and will be neglected in calculations.

5 Dirac fields

Let us set the stage for our calculation. First, recall the free Dirac spinor action in Minkowski spacetime

$$S = \int d^4x \big(i\bar{\psi}\gamma^a\partial_a\psi - m\bar{\psi}\psi \big)\,, \tag{5.1}$$

where the *gamma* matrices γ^a satisfy the Clifford algebra

$$\big\{\gamma^a, \gamma^b\big\} = 2\eta^{ab}\,, \tag{5.2}$$

with which the generators of the spinor representation of the Lorentz group are built

$$\sigma^{ab} = \frac{i}{4}\big[\gamma^a, \gamma^b\big] \equiv \frac{i}{2}\gamma^{ab}\,, \tag{5.3}$$

such that the action (5.1) is invariant under *global* LT given by $U = e^{-\frac{i}{2}\epsilon_{ab}\sigma^{ab}}$ with ϵ_{ab} the transformation parameters.

We also recall some useful identities involving the gamma matrices in four dimensions that will be later used:

$$\gamma^c\gamma^{ab} = \gamma^{abc} + \eta^{ca}\gamma^b - \eta^{cb}\gamma^a\,, \tag{5.4}$$

with

$$\gamma^{abc} \equiv \gamma^{[a}\gamma^b\gamma^{c]} = -\frac{i}{3!}\varepsilon^{abcd}\gamma_d\gamma^5\,, \tag{5.5}$$

and of course

$$\gamma^5 = \frac{i}{4!}\varepsilon_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d\,, \tag{5.6}$$

$$\gamma^{ab}\gamma_a{}^c = 2\gamma^{bc} - 3\eta^{bc} \quad ; \quad \gamma^{ab}\gamma_{ab} = 12\,, \tag{5.7}$$

and finally

$$\varepsilon_{abcd}\gamma^{ab} = -2i\gamma^{cd}\gamma^5\,, \tag{5.8}$$

which will be used for the interaction with the axial part of torsion.

Coupling the field to a gravitational background is then just a matter of minimal coupling $\boldsymbol{\eta} \to \boldsymbol{g}$, $\partial_a \to \nabla_\mu = \partial_\mu + \frac{1}{4}w_{ab\mu}\gamma^{ab}$, relating Dirac indices to spacetime ones as

$$\big\{\gamma^\mu, \gamma^\nu\big\} = 2g^{\mu\nu} = 2h_a{}^\mu h_b{}^\nu\eta^{ab}\,, \tag{5.9}$$

from which we add $\gamma^a \to \gamma^\mu = h_a{}^\mu\gamma^a$ to the minimal coupling procedure.
The action (5.1) then becomes

$$S = \int d^4x\sqrt{-g}\big(i\bar{\psi}\gamma^\mu\nabla_\mu\psi - m\bar{\psi}\psi \big)\,. \tag{5.10}$$

Due to the identities (2.28), contractions between pairs of spacetime indices are the same as their Lorentz counterpart.[33] The coupling of spinors to gravity, as well as to other half-integer spin fields, requires the use of frame fields and cannot be performed with the metric alone. We will therefore work with Latin indices only.

5.1 Non-minimal coupling to Gravity

The goal is to now apply the heat kernel techniques of section 4.2 to an uncharged Dirac field non-minimally coupled to gravity. In order to apply the master formula (4.38), we would need to express the Dirac operator in terms of $\overset{\circ}{\nabla}$, which we call *Einstein-Cartan* (EC) basis $\mathcal{L}[\overset{\circ}{\nabla}]$, rather than in terms of ∇, the *metric affine gravity* (MAG) basis $\mathcal{L}[\nabla]$. The latter, though, is more interesting and general, so we aim at using it.

Since spinors have mass dimension $\frac{3}{2}$, if we want interaction terms to have dimensionless coupling constants,[34] we only have one mass dimension left: torsion. We thus add the following terms to the Lagrangian density in the MAG basis with care to preserve the Hermicity of the action

$$q_1 = ic_1\overline{\psi}\not{T}\psi \ , \quad q_2 = c_2\overline{\psi}\not{\theta}\gamma^5\psi \, , \tag{5.11}$$

where T^a and θ^a are the vector and axial parts of the torsion (see (2.54)) and

$$\overset{\circ}{q}_1 = ia_1\overline{\psi}\not{T}\psi \ , \quad \overset{\circ}{q}_2 = a_2\overline{\psi}\not{\theta}\gamma^5\psi \, , \tag{5.12}$$

in the EC basis. As expected, both couplings c_i and a_i are real and dimensionless. A relationship between the two bases will be computed for the sake of completeness and can be found in Appendix A. Let us work starting from MAG.

Looking at the complete action

$$S = \int d^4x\sqrt{-g}\overline{\psi}\big(i\not{\nabla} - m + ic_1\not{T} + c_2\not{\theta}\gamma^5\big)\psi \, , \tag{5.13}$$

the Hermitian operator Δ which defines the effective action (see (4.16))

$$\Delta = i\gamma^c\nabla_c - m + ic_1\gamma^cT_c + c_2\gamma^c\theta_c\gamma^5 \, , \tag{5.14}$$

is of first-order in covariant derivatives, and therefore not elliptic, which is a requirement for the Heat kernel. A way to fix this is to exploit some properties

[33] $\gamma^\mu V_\mu = h_a{}^\mu\gamma^a h^b{}_\mu V_b = \delta^b_a\gamma^a V_b = \gamma^a V_a$.

[34] This choice is justified upon renormalization grounds. *Marginal* couplings, those with zero mass dimensions, can give rise to a renormalizable theory, and their asymptotic behavior is generally more interesting.

of the logarithm, which for matrices are valid only under trace operation. As presented in [19], If we introduce a factor $\mathbb{1} = \gamma^5\gamma^5$ and use $\gamma^c\gamma^5 = -\gamma^5\gamma^c$ to move one of them through the operator in equation (4.15), we can then bring the second one next to it thanks to Tr $\ln(AB) =$ Tr $\ln A +$ Tr $\ln B$, so that

$$\begin{aligned} \text{Tr } \ln\Delta &= \text{Tr } \ln \big(i\gamma^c\nabla_c - m + ic_1\gamma^c T_c + c_2\gamma^c\theta_c\gamma^5\big)\gamma^5\gamma^5 = \\ &= \text{Tr } \ln\gamma^5\gamma^5\big(-i\gamma^c\nabla_c - m - ic_1\gamma^c T_c - c_2\gamma^c\theta_c\gamma^5\big) \\ &\equiv \text{Tr } \ln\Delta^\dagger\,, \end{aligned} \tag{5.15}$$

with

$$\Delta^\dagger = -i\gamma^c\nabla_c - m - ic_1\gamma^c T_c - c_2\gamma^c\theta_c\gamma^5\,. \tag{5.16}$$

We can thus make it into a second order differential operator and calculate the HAMIDEW coefficients from

$$\text{Tr } \ln\Delta = \frac{1}{2}\text{Tr } \ln\Delta\Delta^\dagger\,. \tag{5.17}$$

This result can also be found by noting that in even dimensional spacetime γ^c and $-\gamma_c$ give equivalent representations of the Clifford algebra (5.2), justifying the reasoning behind equation (5.15) and therefore Eq.(5.17). A rigorous treatment is given by DeWitt in [20].

The goal is, again, to expand the operator products so that we end up with a covariant laplacian only containing Christoffel connections on spacetime, plus an endomorphism. This is not an easy task, and depending on the route, it is easy to end up in an iterative loop of adding/removing connections. We use the strategy of [21], where a similar computation is carried out. In the article, a general second-order operator

$$\eta^{ab}\tilde{\nabla}_a\tilde{\nabla}_b + X^a\tilde{\nabla}_a + Y\,, \tag{5.18}$$

is considered on the V bundle, where $\tilde{\nabla}$ is a covariant derivative containing both a spacetime and a gauge connection with no particular constraints, and therefore does not satisfy the requirements for which (4.42) are valid. A modification of the HAMIDEW coefficients is then provided. To find this, the author explicitly separates $\mathring{\nabla}$ from $\tilde{\nabla}$ and introduces a new, auxiliary connection on the gauge bundle. [35] This will be defined so as to effectively cancel the linear term in (5.18). Rather than using the results directly, we pedagogically reproduce such technique.

[35] This connection is auxiliary in the sense that it is not related to any symmetry of the action, and only serves computational purposes. One can start with a vanishing connection $A' = 0$ on the gauge bundle and then perform a gauge transformation that brings $A' \to A$ in accordance with the requirements.

To better visualize how operators get manipulated, we first split our torsionful covariant derivative

$$\begin{aligned}\Delta_{\frac{1}{2}} &= m + i\not{\nabla} + c_1\not{T} + c_2\not{\theta}\gamma^5 \\ &= m + i\not{\mathring{\nabla}} + \frac{i}{4}\not{\Omega} + ic_1\not{T} + c_2\not{\theta}\gamma^5\,,\end{aligned} \tag{5.19}$$

where $\not{\Omega} = \gamma^c\Omega_c = \gamma^c\gamma^{ab}\Omega_{abc}$ is the contortion contracted with the generators (see Eq. (2.35)). To effectively "shift" the last three terms out of ∇, we introduce a new connection A_c on the gauge bundle and require the aforementioned compensation. The new covariant derivative, on which all further results are based, is

$$\hat{\nabla}_c = \mathring{\nabla}_c - \alpha A_c\,. \tag{5.20}$$

Substituting for $\mathring{\nabla}$ in (5.19), we fix

$$\alpha A_c = -\frac{1}{4}\Omega_c - c_1 T_c + ic_2\theta_c\gamma^5\,, \tag{5.21}$$

so that equation (5.19) becomes

$$\begin{aligned}\Delta_{\frac{1}{2}} &= m + i\hat{\not{\nabla}} + i\alpha\not{A} + \frac{i}{4}\not{\Omega} + c_1\not{T} + c_2\not{\theta}\gamma^5 \\ &= m + i\hat{\not{\nabla}}\,.\end{aligned} \tag{5.22}$$

Note that the constant α is implicitly fixed by equation (5.21) in the sense that starting with whatever value of it, the product αA_c must satisfy (5.21), and under the field rescaling $A \to kA$, the coupling must also be redefined as $\alpha \to \frac{\alpha}{k}$.

Squaring slashed operators, we have to follow the rule

$$\begin{aligned}\not{A}\not{B} &= \gamma^c\gamma^d A_c B_d = \left(\frac{1}{2}\{\gamma^c,\gamma^d\} + \frac{1}{2}[\gamma^c,\gamma^d]\right)A_c B_d \\ &= A_c B^c + \frac{1}{2}\gamma^{cd}[A_c, B_d]\,,\end{aligned} \tag{5.23}$$

and be careful not to automatically discard the commutators of quantities that have a vector index, such as Ω_c, because this contains gamma matrices, which do not indeed commute. With this in mind, the final quadratic operator in (5.17) reads

$$\begin{aligned}\Delta_{\frac{1}{2}}\Delta^\dagger{}_{\frac{1}{2}} &= (m + i\hat{\not{\nabla}})(m - i\hat{\not{\nabla}}) = m^2 + \hat{\not{\nabla}}\hat{\not{\nabla}} \\ &= m^2 + \hat{\nabla}_c\hat{\nabla}^c + \frac{1}{2}\gamma^{cd}[\hat{\nabla}_c, \hat{\nabla}_d]\,,\end{aligned} \tag{5.24}$$

which now has the required functional form (4.41).

5.2 HAMIDEW coefficients and Effective Action

We start by calculating the commutator

$$
\begin{aligned}
[\hat{\nabla}_c, \hat{\nabla}_d] &= [\mathring{\nabla}_c + \alpha A_c, \hat{\nabla}_d] \\
&= [\mathring{\nabla}_c, \mathring{\nabla}_d + \alpha A_d] + \alpha [A_c, \mathring{\nabla}_d + \alpha A_d] \\
&= [\mathring{\nabla}_c, \mathring{\nabla}_d] + \alpha^2 [A_c, A_d] + \alpha [\mathring{\nabla}_c, A_d] + \alpha [A_c, \mathring{\nabla}_d] \\
&= \frac{1}{4} \mathring{R}_{abcd} \gamma^{ab} + \alpha^2 [A_c, A_d] \\
&+ \alpha (\mathring{\nabla}_c A_d - \cancel{A_d \mathring{\nabla}_c} + \cancel{A_d \mathring{\nabla} c} + \cancel{A_c \mathring{\nabla}_d} - \mathring{\nabla}_d A_c - \cancel{A_c \mathring{\nabla}_d}) \\
&= \frac{1}{4} \mathring{R}_{abcd} \gamma^{ab} + \alpha (\mathring{\nabla}_c A_d - \mathring{\nabla}_d A_c) + \alpha^2 [A_c, A_d] \,,
\end{aligned}
\tag{5.25}
$$

where, remembering it acts on Ψ, we have used the (crucial) Leibniz rule in the last row. Because the last two terms are antisymmetric and contracted with γ^{cd}, we can keep the first of each one and cancel the $\frac{1}{2}$ in front of the commutator. In formulae, $\frac{1}{2}\gamma^{cd}(A_c B_d - A_d B_c) = \gamma^{cd} A_c B_d$. This is a recurring phenomenon throughout the calculation and will be implicitly used from now on.

The baseline for the rest of the calculation is then

$$
\Delta_{\frac{1}{2}} \Delta^{\dagger}{}_{\frac{1}{2}} = m^2 + \hat{\Box} + \frac{1}{8} \gamma^{cd} \gamma^{ab} \mathring{R}_{abcd} + \alpha \gamma^{cd} (\mathring{\nabla}_c A_d) + \alpha^2 \gamma^{cd} A_c A_d \,. \tag{5.26}
$$

In the last term, T and θ inside of A commute with every vector-indexed object, so the only possible contribution is due to Ω. Taking the definition of contortion, and substituting the torsion tensor decomposition (2.54) we find

$$
\begin{aligned}
\Omega_{abc} &= \frac{1}{2} (T_{abc} + T_{bca} + T_{cba}) \\
&= \frac{1}{2} \Bigg[\frac{1}{3} (\eta_{ac} T_b - \cancel{\eta_{ab} T_c} - \frac{1}{3!} \varepsilon_{dabc} \theta^d \\
&+ \quad \frac{1}{3} (\cancel{\eta_{ba} T_c} - \eta_{bc} T_a) - \frac{1}{3!} \cancel{\varepsilon_{dbca} \theta^d} \\
&+ \quad \frac{1}{3} (\eta_{ca} T_b - \eta_{bc} T_a) - \frac{1}{3!} \cancel{\varepsilon_{dcba} \theta^d} \Bigg] \\
&= \quad \frac{1}{3} (\eta_{ac} T_b - \eta_{bc} T_a) - \frac{1}{12} \varepsilon_{dabc} \theta^d \,,
\end{aligned}
\tag{5.27}
$$

which shows that the contraction of γ with Ω is

$$\begin{aligned}\gamma^{ab}\Omega_{abc} &= \frac{2}{3}\gamma^{ab}\eta_{ab}T_b - \frac{1}{12}\theta^d\varepsilon_{dabc}\gamma^{ab} \\ &= \frac{2}{3}\gamma_{cd}(T^d - \frac{i}{4}\gamma^5\theta^d) \\ &= \frac{2}{3}\gamma_{cd}B^d \equiv \Omega_c\,. \end{aligned} \tag{5.28}$$

With this in mind, let us write the components of A in a sensible way for later:

$$\alpha A_c = -\frac{1}{4}\Omega_c - c_1 T_c + ic_2\theta_c\gamma^5 \tag{5.29}$$

$$= -\frac{1}{6}\gamma_{cd}B^d + Q_c \tag{5.30}$$

Using this, the final expression for the last term of Eq. (5.26) is then

$$\begin{aligned}\gamma^{cd}A_cA_d &= \frac{1}{36}\gamma^{cd}\gamma_{ca}\gamma_{db}B^aB^b \\ &= \frac{1}{36}(2\gamma^d{}_a\gamma_{db} - 3\delta^d{}_a\gamma_{db})B^aB^b \\ &= \frac{1}{36}[2(2\gamma_{ab} - 3\eta_{ab}) - 3\gamma_{ab}]B^aB^b \\ &= \frac{1}{36}[\cancel{\gamma_{ab}} - 6\eta_{ab}]B^aB^b \\ &= -\frac{1}{6}B^2\,, \end{aligned} \tag{5.31}$$

so we get

$$\begin{aligned}\Delta_{\frac{1}{2}}\Delta^+{}_{\frac{1}{2}} &= \hat{\Box} + m^2 + \frac{1}{8}\gamma^{cd}\overset{\circ}{R}_{abcd}\gamma^{ab} - \frac{1}{6}B^2 \\ &\quad + \gamma^{cd}\overset{\circ}{\nabla}_c(Q_d - \frac{1}{6}\gamma_{da}B^a)\,. \end{aligned} \tag{5.32}$$

Moving γ^{cd} to the right of the covariant derivative in the last row, and considering that

$$\begin{aligned}\gamma^{cd}(a\eta_{df} + b\gamma_{df}) &= a\gamma^c{}_f - b(2\gamma^c{}_f - 3\delta^c{}_f) \\ &= (a - 2b)\gamma^c{}_f + 3b\delta^c{}_f\,, \end{aligned} \tag{5.33}$$

we finally arrive at the desired functional form (4.41)

$$\Delta_{\frac{1}{2}}\Delta^\dagger{}_{\frac{1}{2}} = \hat{\Box} + E \tag{5.34}$$

with[36]

$$E = m^2 - \frac{1}{4}\overset{\circ}{R} - \frac{1}{6}B^2 \tag{5.35}$$

$$- \frac{1}{2}\overset{\circ}{\nabla}_c B^c + \gamma^{cd}\overset{\circ}{\nabla}_c(Q_d + \frac{1}{3}B_d) \tag{5.36}$$

and, we recall,

$$B_d = T_d - \frac{i}{4}\gamma^5\theta_d \tag{5.37}$$

$$Q_d = -c_1 T_d + ic_2\gamma^5\theta_d \tag{5.38}$$

are the contributions from the minimal and non-minimal couplings respectively. We are almost ready to start calculating the coefficients (4.42).

Before continuing, let us clear a few rules for the next step. Because we are going to take the trace of our operators, every term linear in γ^{cd} vanishes,[37] so not all mixed terms will make it to the final expression. This is also true for most terms in θ because they carry a γ^5 with them, so we do not expect any contribution with an odd number of them to appear. Albeit being true, we will convert b_4 from its form in terms of $\overset{\circ}{\nabla}$ and GR curvatures, in terms of ∇ and MAG curvatures to obtain a more general, malleable result. As computations will show, the conversion can make these odd terms appear without a γ^5 because they come from the contortion terms of Eq. (2.21), which do not contain any gamma matrices, yet they cancel out in the final expression. With this in mind, all terms in B and Q that vanish when tracing will be omitted in order to keep expressions tidy.

Let us first compute the trace of the endomorphism as an example: looking at (5.35) it is immediate to see that the last term vanishes because it is either proportional to γ^{ab} or $\gamma^{ab}\gamma^5$. The first one of the last row is proportional to B. Within this, there is a contribution proportional to the identity due to T, and a contribution proportional to γ^5 due to θ that vanishes. The quadratic term in B will explicitly contribute $T^2 - \frac{1}{16}\theta^2 - \frac{i}{2}\gamma^5 T\theta$, of which, again, we only keep the first two. In the end, we have

$$\text{tr } E = 4m^2 - \overset{\circ}{R} - \frac{2}{3}T^2 + \frac{1}{24}\theta^2 - 2\overset{\circ}{\nabla}_c T^c \tag{5.39}$$

For the calculation of b_4 (see (4.42)), we also need the trace of E^2. By looking at (5.35), and with the now standard tracing considerations, it is immediately clear

36 Some useful identities and contractions of curvature with gamma matrices can be found in [22]

37 Tr γ^{cd}= Tr $(\gamma^c\gamma^d - \gamma^d\gamma^c)$=Tr $(\gamma^c\gamma^d - \gamma^c\gamma^d)$=0 because of the cyclic property of the trace.

that its square will be of the form $E = A + \gamma B \longrightarrow E^2 = A^2 + \gamma\gamma BB$. In fact,

$$\begin{aligned}
E^2 &= m^4 + \frac{1}{16}\mathring{R}^2 + \frac{1}{36}B^4 + \frac{1}{4}(\mathring{\nabla}_c B^c)^2 \\
&\quad - m^2\left(\frac{1}{2}\mathring{R} + \frac{1}{3}B^2 + \mathring{\nabla}_c B^c\right) + \frac{1}{4}\mathring{R}\left(\frac{1}{3}B^2 + \mathring{\nabla}_c B^c\right) + \frac{1}{6}B^2\mathring{\nabla}_c B^c \\
&\quad + \gamma^{ab}\gamma^{cd}\left[\frac{1}{9}(\mathring{\nabla}_a B_b)(\mathring{\nabla}_c B_d) + \frac{2}{3}(\mathring{\nabla}_a B_b)(\mathring{\nabla}_c Q_d) + (\mathring{\nabla}_a Q_b)(\mathring{\nabla}_c Q_d)\right].
\end{aligned} \tag{5.40}$$

The last necessary operator is the field strength of the full covariant derivative $\hat{\nabla}$, $F_{cd} = [\hat{\nabla}_c, \hat{\nabla}_d]$, which we have already calculated in (5.25). The difference is that it will not get contracted with a γ^{cd}, but rather squared and then traced. A straightforward computation gives

$$\begin{aligned}
F_{cd} &= \frac{1}{4}\gamma^{ab}\mathring{R}_{abcd} + 2\alpha\mathring{\nabla}_{[c}A_{d]} + 2\alpha^2 A_{[c}A_{d]} \\
&= \frac{1}{4}\gamma^{ab}\mathring{R}_{abcd} - \frac{1}{36}B^2 \\
&\quad - \frac{1}{18}B^a\gamma_{a[c}B_{d]} + 2\mathring{\nabla}_{[c}\left(Q_{d]} - \frac{1}{6}\gamma_{d]a}B^a\right).
\end{aligned} \tag{5.41}$$

Squaring this, we find

$$\begin{aligned}
F_{cd}F^{cd} &= -\frac{1}{216}B^4 + \frac{1}{18}B^2\mathring{\nabla}_a B^a - \frac{1}{18}B^a B^b\mathring{\nabla}_a B_b \\
&\quad - \frac{1}{6}(\mathring{\nabla}_a B_b)(\mathring{\nabla}^a B^b) + 2(\mathring{\nabla}_a Q_b)(\mathring{\nabla}^a Q^b) - 2(\mathring{\nabla}_a Q_b)(\mathring{\nabla}^b Q^a) \\
&\quad + \gamma^{ab}\gamma^{cd}\Bigg[\frac{1}{16}\mathring{R}_{abef}\mathring{R}_{cd}{}^{ef} - \frac{1}{72}B^2\mathring{R}_{abcd} + \frac{1}{36}B_a B^e\mathring{R}_{ebcd} \\
&\qquad + \frac{1}{6}(\mathring{\nabla}_e B_a)\mathring{R}^e{}_{bcd} + \frac{1}{54}B_a B_c(\mathring{\nabla}_b B_d) + \frac{1}{18}(\mathring{\nabla}_c B_a)(\mathring{\nabla}_b B_d)\Bigg].
\end{aligned} \tag{5.42}$$

Taking the trace of the expressions (5.40) and (5.42) is then straightforward: terms proportional to the identity get multiplied by 4, and the remaining ones need us to work with the trace identity

$$\operatorname{tr}\gamma^a\gamma^b\gamma^c\gamma^d = 4(\eta^{ab}\eta^{cd} - \eta^{ac}\eta^{bd} + \eta^{ad}\eta^{bc}), \tag{5.43}$$

which in our case is significant for

$$\begin{aligned}
\operatorname{tr}\gamma^{ab}\gamma^{cd} &= \frac{1}{4}\operatorname{tr}(\gamma^a\gamma^b\gamma^c\gamma^d - \gamma^a\gamma^b\gamma^d\gamma^c - \gamma^b\gamma^a\gamma^c\gamma^d + \gamma^b\gamma^a\gamma^d\gamma^c) \\
&= \cancel{\eta^{ab}\eta^{cd}} - \cancel{\eta^{ab}\eta^{dc}} - \cancel{\eta^{ba}\eta^{cd}} + \cancel{\eta^{ba}\eta^{dc}} \\
&\quad - \eta^{ac}\eta^{bd} + \eta^{ad}\eta^{bc} + \eta^{bc}\eta^{ad} - \eta^{bd}\eta^{ac} \\
&\quad + \eta^{ad}\eta^{bc} - \eta^{ac}\eta^{bd} - \eta^{bd}\eta^{ac} + \eta^{bc}\eta^{ad} \\
&= 4(\eta^{ad}\eta^{bc} - \eta^{ac}\eta^{bd}).
\end{aligned} \tag{5.44}$$

Substituting this in (5.40) and (5.42) finally results in

$$
\begin{aligned}
\mathrm{tr}\ E^2 = 4m^4 &+ \frac{1}{4}\mathring{R}^2 + \frac{1}{9}B^4 + (\mathring{\nabla}_c B^c)^2 \\
&- 2m^2\mathring{R} - \frac{4}{3}m^2B^2 - 4m^2\mathring{\nabla}_c B^c \\
&+ \frac{1}{3}\mathring{R}B^2 + \mathring{R}\mathring{\nabla}_c B^c + \frac{2}{3}B^2\mathring{\nabla}_c B^c \\
&- \frac{4}{9}(\mathring{\nabla}_a B_b)(\mathring{\nabla}^a B^b) + \frac{4}{9}(\mathring{\nabla}_a B_b)(\mathring{\nabla}^b B^a) \\
&- \frac{8}{3}(\mathring{\nabla}_a B_b)(\mathring{\nabla}^a Q^b) + \frac{8}{3}(\mathring{\nabla}_a B_b)(\mathring{\nabla}^b Q^a) \\
&- 4(\mathring{\nabla}_a Q_b)(\mathring{\nabla}^a Q^b) + 4(\mathring{\nabla}_a Q_b)(\mathring{\nabla}^b Q^a)\,,
\end{aligned}
\tag{5.45}
$$

and

$$
\begin{aligned}
\mathrm{tr}\ F^2 = &\frac{1}{27}B^4 + 16(\mathring{\nabla}_c Q_d)^2 + \frac{4}{3}(\mathring{\nabla}_c B_d)^2 \\
&- \frac{1}{2}\mathring{R}_{abcd}\mathring{R}^{abcd} - \frac{4}{3}\mathring{R}^{cd}\mathring{\nabla}_c B_d \\
&- \frac{2}{9}\mathring{R}^{cd}B_cB_d + \frac{1}{9}\mathring{R}B^2 \\
&+ \frac{4}{27}(B^cB^d\mathring{\nabla}_c B_d + \frac{1}{2}B^2\mathring{\nabla}_c B^c)\,.
\end{aligned}
\tag{5.46}
$$

Although we are tempted to collect the quadratic terms in $\mathring{\nabla}$ as products of gauge field strengths, these mix non-trivially in the final expression, so it is easier to group them at the very end.

Before plugging B and Q and taking their powers, let us illustrate what terms would have survived the trace. We already know from the calculation of (5.39) that $B^2 = T^2 - \frac{1}{16}\theta^2 - \frac{i}{2}\gamma^5 T\theta$. In general, any further multiplication by B will only include T times whatever is proportional to the identity, and $\gamma^5\theta$ times the terms which already have a γ^5 in them so to have even powers of it. By omitting indices to only highlight their structure, so not to focus on what contractions are going on:

$$
\begin{aligned}
B &= T - \frac{i}{4}\cancel{\gamma^5\theta} \\
BB &= TT - \frac{1}{16}\theta\theta - \frac{i}{2}\cancel{\gamma^5 T\theta} \\
BBB &= TTT - \frac{1}{16}T\theta\theta - \frac{1}{16}\theta T\theta - \frac{1}{16}\theta\theta T \\
B^4 &= T^4 + \frac{1}{256}\theta^4 - \frac{1}{8}T^2\theta^2 - \frac{1}{4}(T\theta)^2\,.
\end{aligned}
\tag{5.47}
$$

The same reasoning applies to Q.

Substituting these, we explicitly get the contribution that T and θ bring to the expansion, multiplied by the appropriate prefactors

$$\begin{aligned}
\frac{1}{2}\text{tr } E^2 = &\frac{1}{2}(\frac{1}{2}\overset{\circ}{R} - 2m^2)^2 + \frac{1}{2}(\overset{\circ}{\nabla}_c T^c)^2 - \frac{1}{32}(\overset{\circ}{\nabla}_c \theta^c)^2 \\
&+ (\frac{1}{2}\overset{\circ}{R} - 2m^2)\Big[\frac{1}{3}(T^2 - \frac{1}{16}\theta^2) + \overset{\circ}{\nabla}_c T^c\Big] \\
&+ \frac{1}{18}\Big[T^4 + \frac{1}{256}\theta^4 - \frac{1}{8}T^2\theta^2 - \frac{1}{4}(T\theta)^2\Big] \\
&+ \frac{1}{3}(T^2 - \frac{1}{16}\theta^2)\overset{\circ}{\nabla}_c T^c - \frac{1}{24}T_d\theta^d\overset{\circ}{\nabla}_c\theta^c \\
&- 2(c_1 - \frac{1}{3})^2(\overset{\circ}{\nabla}{}^c T^d)\Big[\overset{\circ}{\nabla}_c T_d - \overset{\circ}{\nabla}_d T_c\Big] \\
&+ 2(c_2 - \frac{1}{12})^2(\overset{\circ}{\nabla}{}^c \theta^d)\Big[\overset{\circ}{\nabla}_c \theta_d - \overset{\circ}{\nabla}_d \theta_c\Big]\,,
\end{aligned} \tag{5.48}$$

and

$$\begin{aligned}
\frac{1}{12}\text{tr } F^2 = &-\frac{1}{24}\overset{\circ}{R}_{abcd}\overset{\circ}{R}{}^{abcd} - \frac{1}{9}\overset{\circ}{R}{}^{cd}\overset{\circ}{\nabla}_c T_d \\
&- \frac{1}{54}(\overset{\circ}{R}{}^{cd} - \frac{1}{2}\overset{\circ}{R}\eta^{cd})\Big[T_c T_d - \frac{1}{16}\theta_c\theta_d\Big] \\
&- \frac{1}{648}\Big[T^4 + \frac{1}{256}\theta^4 - \frac{1}{8}T^2\theta^2 - \frac{1}{4}(T\theta)^2\Big] \\
&+ \Big[\frac{2}{3}c_1^2 - \frac{1}{27}\Big](\overset{\circ}{\nabla}_c T_d)^2 - \frac{2}{3}c_1^2(\overset{\circ}{\nabla}_c T_d)(\overset{\circ}{\nabla}{}^d T^c) - \frac{1}{54}(\overset{\circ}{\nabla}_a T^a)^2 \\
&+ \Big[\frac{1}{432} - \frac{2}{3}c_2^2\Big](\overset{\circ}{\nabla}_c \theta_d)^2 + \frac{2}{3}c_2^2(\overset{\circ}{\nabla}_c \theta_d)(\overset{\circ}{\nabla}{}^d \theta^c) - \frac{1}{864}(\overset{\circ}{\nabla}_a \theta^a)^2 \\
&- \frac{1}{81}\Big[T^2\overset{\circ}{\nabla}_a T^a - T^a T^b\overset{\circ}{\nabla}_a T_b - \frac{1}{16}\theta^2\overset{\circ}{\nabla}_a\theta^a \\
&\qquad + \frac{1}{4}T\theta\overset{\circ}{\nabla}_a\theta^a - \frac{1}{16}\theta^a T^b(\overset{\circ}{\nabla}_a\theta_b + \overset{\circ}{\nabla}_b\theta_a)\Big]\,.
\end{aligned} \tag{5.49}$$

We now have all the ingredients to find the HAMIDEW coefficients, and subsequently the effective action. In order not to be too repetitive with results, these are directly shown in the action itself.

First, a consistency requirement needs to be checked: because the operator (5.34) has $\overset{\circ}{\nabla}$ as a spacetime covariant derivative, the coefficients of curvature invariants need to coincide with the GR limit, in which the spinor is integrated whilst coupled to pure metric gravity, where we set $T = \theta = 0$. We therefore only need to collect terms where only curvature appears from (5.39) times $\overset{\circ}{R}$, (5.48), (5.49), and add them to the ones that are already present in (4.42). We get

$$
\begin{aligned}
b_2|_{\overset{\circ}{R}} &= (\frac{4}{6}-1)\overset{\circ}{R} = -\frac{1}{2}\overset{\circ}{R} \\
b_4|_{\overset{\circ}{R}} &= (\frac{4}{180}\frac{5}{2}+\frac{1}{8}-\frac{1}{6}) = \frac{1}{72}\overset{\circ}{R}{}^2 \\
b_4|_{\overset{\circ}{R}_{ab}} &= (-\frac{4}{180}) = -\frac{1}{45}\overset{\circ}{R}{}^2_{ab} \\
b_4|_{\overset{\circ}{R}_{abcd}} &= (\frac{4}{180}-\frac{1}{24}) = -\frac{7}{360}\overset{\circ}{R}{}^2_{abcd}\,,
\end{aligned}
\tag{5.50}
$$

which correctly coincide with the known literature [11, 12] for spinors coupled to General Relativity.

The one-loop effective action of an uncharged, massive Dirac spinor non-minimally coupled to a torsionful gravitational field, with UV-cutoff Λ_{UV}, is

$$
\begin{aligned}
\Gamma^1_{\rm div} = &-\frac{1}{2(4\pi)^2}\int d^4x\sqrt{-g}\times \\
\times\Bigg\{ &+2\Lambda^4_{UV}+\Lambda^2_{UV}\Big(4m^2-\frac{1}{3}\overset{\circ}{R}-\frac{2}{3}(T^2-\frac{1}{4}\theta^2-2\overset{\circ}{\nabla}_cT^c)\Big) \\
&+\ln(\frac{\Lambda^2_{UV}}{\mu^2})\Big[-\frac{7}{360}\overset{\circ}{R}{}^2_{abcd}-\frac{1}{45}\overset{\circ}{R}{}^2_{cd}+\frac{1}{72}\overset{\circ}{R}{}^2-2m^4-\frac{1}{3}m^2\overset{\circ}{R} \\
&\qquad+\frac{1}{54}\Big(\frac{7}{2}\overset{\circ}{R}\eta^{cd}-36m^2\eta^{cd}-\overset{\circ}{R}{}^{cd}\Big)(T_cT_d-\frac{1}{16}\theta_c\theta_d) \\
&\qquad+\frac{1}{3}\Big(\frac{1}{2}\overset{\circ}{R}\eta^{cd}-6m^2\eta^{cd}-\frac{1}{3}\overset{\circ}{R}{}^{cd}\Big)\overset{\circ}{\nabla}_cT_d \\
&\qquad+\frac{35}{648}\Big(T^4+\frac{1}{256}\theta^4-\frac{1}{8}T^2\theta^2-\frac{1}{4}(T\theta)^2\Big) \\
&\qquad-\frac{4}{3}\Big(c_1^2-c_1+\frac{7}{36}\Big)(\overset{\circ}{\nabla}{}^cT^d)\Big[\overset{\circ}{\nabla}_cT_d-\overset{\circ}{\nabla}_dT_c\Big] \\
&\qquad-\frac{1}{27}(\overset{\circ}{\nabla}_cT_d)(\overset{\circ}{\nabla}{}^dT^c)+\frac{13}{27}(\overset{\circ}{\nabla}_cT^c)^2 \\
&\qquad+\frac{1}{3}\Big(4c_1^2-c_1+\frac{7}{144}\Big)(\overset{\circ}{\nabla}{}^c\theta^d)\Big[\overset{\circ}{\nabla}_c\theta_d-\overset{\circ}{\nabla}_d\theta_c\Big] \\
&\qquad-\frac{1}{432}(\overset{\circ}{\nabla}_c\theta_d)(\overset{\circ}{\nabla}{}^d\theta^c)+\frac{13}{432}(\overset{\circ}{\nabla}_c\theta^c)^2 \\
&\qquad-\frac{1}{81}(T^2-\frac{1}{16}\theta^2)(\overset{\circ}{\nabla}_cT^c)-\frac{7}{162}(T\theta)(\overset{\circ}{\nabla}_c\theta^c) \\
&\qquad+\frac{28}{81}(\overset{\circ}{\nabla}_cT^c)(T^2-\frac{1}{16}\theta^2)+\frac{1}{1296}\theta^aT^b(\overset{\circ}{\nabla}_c\theta_d+\overset{\circ}{\nabla}_d\theta_c)\Big]\Bigg\}\,.
\end{aligned}
\tag{5.51}
$$

First, notice that not all present terms are actually independent: we must remove all terms proportional to a total derivative and use the Bianchi identities of GR when integration by parts is possible. Because it only applies to terms in the fifth row, the result will not be shown, as it tantamounts to the substitution $\frac{1}{6} \to \frac{1}{9}$ in the first term, and a vanishing of the second two.

We now want the dimensional regularisation version, which we will use for the actual renormalization. Rather than starting over in this framework, the result can easily be found by remembering that only logarithmic divergences survive the regularisation procedure and appear as simple poles with the same coefficient. This means leaving out the first two quartic and quadratic divergences of (5.51) and swapping

$$\ln \frac{\Lambda_{UV}^2}{\mu^2} \longrightarrow \frac{\mu^{-\epsilon}}{\epsilon} \,, \tag{5.52}$$

which carries over from the regularization procedure $d = 4 - \epsilon$. In formulae,

$$\begin{aligned}\Gamma^1_{\text{div}} = &-\frac{1}{2(4\pi)^2} \int d^4x \sqrt{-g} \times \\ &\times \frac{\mu^{-\epsilon}}{\epsilon} \left\{ -\frac{7}{360} \mathring{R}^2_{abcd} - \frac{1}{45} \mathring{R}^2_{cd} + \frac{1}{72} \mathring{R}^2 + ... + \frac{1}{1296} \theta^a T^b (\mathring{\nabla}_c \theta_d + \mathring{\nabla}_d \theta_c) \right\} .\end{aligned} \tag{5.53}$$

where the ellipsed terms are the ones multiplied by $\ln \frac{\Lambda_{UV}^2}{\mu^2}$ in (5.51).

To obtain a more general, easy-to-use result, we cast this effective action in MAG terms. This means changing the curvatures back from $\mathring{R}$ to R (Eq. (2.21)) and covariant derivatives from $\mathring{\nabla}$ to ∇ (Eq. (2.19)) through contractions of contortion and its decomposition (Eq. (5.27)). This way, all our results will easily provide useful limits depending on their framework of interest.

A hefty calculation gives

$$
\begin{aligned}
\Gamma^{1\,\mathrm{MAG}}_{\mathrm{div}} = {} & \frac{1}{2(4\pi)^2}\int d^4x\sqrt{-g}\frac{\mu^{-\epsilon}}{\epsilon}\times \\
\times\Bigg\{ & -2m^2+\frac{m^2}{3}R-\frac{1}{72}R^2+\frac{1}{45}R^{ab}R_{ab}+\frac{7}{360}R^{abcd}R_{abcd} \\
& +\frac{2}{9}m^2T^2-\frac{2473}{9720}T^4+\frac{97}{1620}RT^2+\frac{1}{54}R^{ab}T_aT_b \\
& -\frac{41}{270}R\nabla_aT^a+\frac{11}{135}R^{ab}\nabla_aT_b+\frac{13}{15}T^2\nabla_aT^a+\frac{4}{81}T^aT^b\nabla_aT_b \\
& -\left(\frac{4}{3}(c_1^2-c_1)-\frac{116}{405}\right)(\nabla_aT_b)(\nabla_aT_b) \\
& +\left(\frac{4}{3}(c_1^2-c_1)+\frac{2}{9}\right)(\nabla_aT_b)(\nabla^bT^a)-\frac{296}{405}(\nabla_aT^a)^2 \\
& -\frac{1}{18}m^2\theta^2-\frac{7}{18432}\theta^4+\frac{7}{1728}R\theta^2+\frac{7}{12960}R^{ab}\theta_a\theta_b \\
& -\left(\frac{4}{3}c_2^2-\frac{1}{3}c_2+\frac{19}{1080}\right)(\nabla_a\theta_b)(\nabla^a\theta^b) \\
& +\left(\frac{4}{3}c_2^2-\frac{1}{3}c_2-\frac{23}{1620}\right)(\nabla_a\theta_b)(\nabla^b\theta^a)+\frac{383}{12960}(\nabla_a\theta^a)^2 \\
& -\left[\frac{1457}{46656}+\frac{1}{27}(c_1^2+4c_2^2-c_1-c_2)\right]T^2\theta^2+\frac{671}{18440}\theta^2\nabla_aT^a \\
& +\left[\frac{121}{14580}+\frac{1}{27}(c_1^2+4c_2^2-c_1-c_2)\right](T\theta)^2+\frac{1}{2430}\theta^a\theta^b\nabla_aT_b \\
& -\left[\frac{5}{432}+\frac{2}{9}(4c_2^2-c_2)\right]\theta^aT^b(\nabla_a\theta_b) \\
& +\left[\frac{179}{19440}+\frac{2}{9}(4c_2^2-c_2)\right]\theta^aT^b(\nabla_b\theta_a)-\frac{293}{19440}T\theta\nabla_a\theta^a \\
& +\varepsilon^{abcd}\left[\frac{7}{1080}R^e{}_{bcd}(\nabla_e\theta_a+\frac{1}{3}\theta_aT_e)+\frac{7}{810}R_{bcd}{}^e\theta_aT_e+\frac{11}{1620}\nabla_a\theta_b\nabla_cT_d\right],
\end{aligned}
\tag{5.54}
$$

where we have incorporated the outermost minus sign of (5.53).

The next step is to reduce the number of terms by manipulating some tensorial structures and incorporating them elsewhere. We start with the last row, where the aforementioned linear terms in θ appear. From the first Bianchi identity (2.23), which we recall being,

$$
R^a{}_{[bcd]}=\nabla_{[b}T^a{}_{cd]}+T^a{}_{e[b}T^e{}_{cd]}\,, \tag{5.55}
$$

we see that contractions with totally antisymmetric symbols are trivial. As we can see, this is linear in torsion, and thus in θ, restoring the "even-ness" in θ of the action. Contracting the antisymmetrized indices with the Levi-Civita tensor we see that

$$\begin{aligned}\epsilon^{acde} R^{b}{}_{cde} = & \frac{2}{4}\theta^a T^b - \frac{1}{9}T^a\theta^b - \frac{1}{9}\eta^{ab}T\theta \\ & + \frac{1}{3}\nabla^b\theta^a - \frac{1}{3}\eta^{ab}\nabla_c\theta^c - \frac{2}{3}\epsilon^{abcd}\nabla_c T_d\,,\end{aligned} \tag{5.56}$$

and

$$\begin{aligned}\epsilon^{acde} R_{cde}{}^{b} = & -\frac{2}{3}\theta^a T^b + \frac{1}{3}T^a\theta^b + \frac{1}{6}\eta^{ab}T\theta \\ & + \frac{1}{3}\nabla^b\theta^a + \frac{1}{6}\eta^{ab}\nabla_c\theta^c + \frac{2}{3}\epsilon^{abcd}\nabla_c T_d\,.\end{aligned} \tag{5.57}$$

Further progress can be made by integrating some terms by parts. In the presence of torsion, this is a non-trivial task. As [23] shows in the appendix, moving ∇_a comes at the cost of an extra contribution proportional to T_a, namely

$$\int \sqrt{-g}X^{ab}(\nabla_a Y_b) = -\int \sqrt{-g}Y_b(\nabla_a - T_a)X^{ab}\,, \tag{5.58}$$

because the torsionful covariant derivative ∇ acts nontrivially on $\sqrt{-g}$. First, we use this on ∇_c in the very last term of (5.54) to get

$$\begin{aligned}\epsilon^{abcd}(\nabla_a\theta_b)(\nabla_c T_d) &= -\epsilon^{abcd}T_d\nabla_c(\nabla_a\theta_b) - \cancel{\epsilon^{abcd}T_d T_c(\nabla_a\theta_b)} \\ &= -\frac{1}{2}\epsilon^{abcd}T_d R_{beca}\theta^e - \frac{1}{2}\epsilon^{abcd}T_d T^{e}{}_{ca}(\nabla_e\theta_b) \\ &= -\frac{1}{2}\epsilon^{abcd}R_{beca}T_d\theta^e + \frac{1}{12}\epsilon^{abcd}\epsilon^{fe}{}_{c}a T_d\theta_f(\nabla_a\theta_b) \\ &= -\frac{1}{2}\epsilon^{abcd}R_{beca}T_d\theta^e + \frac{1}{6}T\theta(\nabla_a\theta^a) - \frac{1}{6}T^a\theta^b(\nabla_a\theta_b) \\ &= \frac{1}{3}T\theta\nabla_a\theta^a - \frac{1}{6}\theta^a T^b\nabla_a\theta_b + \frac{1}{12}\theta^2\nabla_a T^a \\ &\quad - \frac{1}{18}(T\theta)^2 + \frac{5}{36}T^2\theta^2\,,\end{aligned} \tag{5.59}$$

where we have let the Levi-Civita tensor select the antisymmetric part of all contractions. We have also used Eq.(2.20) for the second equality, the torsion decomposition (2.54) in the third, and simply plugged in equation (5.56) in the last step.

Other terms that can be removed with this integration method are

$$\begin{aligned}T^aT^b(\nabla_aT_b) &= -T_b(\nabla_a - T_a)(T^aT^b)\\ &= -T^2\nabla_aT^a - T^bT^a\nabla_aT_b + T^4\\ &= -\frac{1}{2}T^2\nabla_aT^a + \frac{1}{2}T^2\,,\end{aligned}\tag{5.60}$$

$$\begin{aligned}\theta^a\theta^b(\nabla_aT_b) &= -T_b\nabla_a(\theta^a\theta^b) + T_bT_a(\theta^a\theta^b)\\ &= -T\theta\nabla_a\theta^a - T^b\theta^a\nabla_a\theta_b + (T\theta)^2\,,\end{aligned}\tag{5.61}$$

$$\begin{aligned}T^a\theta^b\nabla_a\theta^b &= -\theta_b\nabla_a(T^a\theta^b) + \theta_bT_aT^a\theta^b\\ &= -\theta^2\nabla_aT^a - T^a\theta^b(\nabla_a\theta_b) + T^2\theta^2\\ &= -\frac{1}{2}\theta^2\nabla_aT^a + \frac{1}{2}T^2\theta^2\,,\end{aligned}\tag{5.62}$$

which finally reduces the number of independent components from 35 to 27.

The final resulting effective action is

$$\Gamma^{1\,\mathrm{MAG}}_{\mathrm{div}} = \int d^4x\sqrt{-g}\frac{\mu^{-\epsilon}}{2(4\pi)^2\varepsilon}\Big[\mathcal{L}_{\mathrm{vac}} + \mathcal{L}_T + \mathcal{L}_\theta + \mathcal{L}_{\mathrm{int}}\Big]\,,\tag{5.63}$$

with Lagrangian densities

$$\mathcal{L}_{\mathrm{vac}} =-2m^2 + \frac{m^2}{3}R - \frac{1}{72}R^2 + \frac{1}{45}R^{ab}R_{ab} + \frac{7}{360}R^{abcd}R_{abcd}\,,\tag{5.64}$$

$$\begin{aligned}\mathcal{L}_T =&\frac{97}{1620}RT^2 + \frac{1}{54}R^{ab}T_aT_b - \frac{41}{270}R\nabla_aT^a + \frac{11}{135}R^{ab}\nabla_aT_b\\ &+\frac{2}{9}m^2T^2 - \frac{2233}{9720}T^4 + \frac{341}{405}T^2\nabla_aT^a\\ &+\Big(\frac{4}{3}(c_1^2 - c_1) + \frac{116}{405}\Big)(\nabla_aT_b)(\nabla_aT_b)\\ &-\Big(\frac{4}{3}(c_1^2 - c_1) + \frac{2}{9}\Big)(\nabla_aT_b)(\nabla^bT^a) - \frac{296}{405}(\nabla_aT^a)^2\,,\end{aligned}\tag{5.65}$$

$$\begin{aligned}\mathcal{L}_\theta =&\frac{7}{1728}R\theta^2 + \frac{7}{12960}R^{ab}\theta_a\theta_b - \frac{1}{18}m^2\theta^2 - \frac{7}{18432}\theta^4\\ &-\Big(\frac{1}{3}(4c_2^2 - c_1) + \frac{5}{324}\Big)(\nabla_a\theta_b)(\nabla^a\theta_b)\\ &+\Big(\frac{1}{3}(4c_2^2 - c_1) + \frac{23}{1620}\Big)(\nabla_a\theta_b)(\nabla^b\theta_a) + \frac{71}{2592}(\nabla_a\theta^a)^2\,,\end{aligned}\tag{5.66}$$

$$
\begin{aligned}
\mathcal{L}_{\text{int}} = &- \left[\frac{139}{5184} + \frac{1}{27}(c_1^2 + 4c_2^2 - c_1 - c_2)\right] T^2\theta^2 \\
&+ \left[\frac{2}{243} + \frac{1}{27}(c_1^2 + 4c_2^2 - c_1 - c_2)\right] (T\theta)^2 \\
&+ \left[\frac{1157}{38880} - \frac{1}{9}(4c_2^2 - c_2)\right] \theta^2 \nabla_a T^a \\
&- \left[\frac{239}{19440} + \frac{2}{9}(4c_2^2 - c_2)\right] \theta^a T^b \nabla_a \theta_b - \frac{269}{19440} T\theta \nabla_a \theta^a \,.
\end{aligned}
\tag{5.67}
$$

5.3 Renormalization procedure

The poles in the effective action (5.63) are the divergences that renormalize the bare fields and couplings. Omitting the index B for *bare* on all quantities to prevent useless clutter, their actions would be

$$
S_{\text{vac}}^{\text{bare}} = \int d^4x \sqrt{-g} \left[\Lambda + \kappa_1 R + \kappa_2 R^2 + \kappa_3 R^{ab} R_{ab} + \kappa_4 R^{abcd} R_{abcd}\right] , \tag{5.68}
$$

$$
\begin{aligned}
S_T^{\text{bare}} = \int d^4x \sqrt{-g} \Big[&\alpha_1 R T^2 + \alpha_2 R^{ab} T_a T_b + \alpha_3 R \nabla_a T^a + \alpha_4 R^{ab} \nabla_a T_b \\
&+ \sigma_1 T^2 + \sigma_2 T^4 + \sigma_3 T^2 \nabla_a T^a \\
&+ \sigma_4 (\nabla_a T_b)(\nabla^a T^b) + \sigma_5 (\nabla_a T_b)(\nabla^b T^a) + \sigma_6 (\nabla_a T^a)^2 \Big] ,
\end{aligned}
\tag{5.69}
$$

$$
\begin{aligned}
S_\theta^{\text{bare}} = \int d^4x \sqrt{-g} \Big[&\beta_1 R \theta^2 + \beta_2 R^{ab} \theta_a \theta_b + \rho_1 \theta^2 + \rho_2 \theta^4 \\
&+ \rho_3 (\nabla_a \theta_b)(\nabla^a \theta^b) + \rho_4 (\nabla_a \theta_b)(\nabla^b \theta^a) + \rho_5 (\nabla_a \theta^a)^2 \Big] ,
\end{aligned}
\tag{5.70}
$$

$$
\begin{aligned}
S_{\text{int}}^{\text{bare}} = \int d^4x \sqrt{-g} \Big[&\lambda_1 T^2 \theta^2 + \lambda_2 (T\theta)^2 \\
&+ \lambda_3 \theta^2 \nabla_a T^a + \lambda_4 T\theta \nabla_a \theta^a + \lambda_5 \theta^a T^b \nabla_a \theta_b \Big] ,
\end{aligned}
\tag{5.71}
$$

A very important comment is due here: these actions only contain background fields that make our theory renormalizable, as long as they are added. The only dynamical field is the spinor, which we have integrated out in order to find the corrections it brings to such background geometry. This is why we have kept all terms quadratic in ∇ separate rather than collecting them into a field strength, and expressed T^2 and θ^2 as self-interactions rather than mass terms. As such, only the couplings will get renormalized, with no field rescaling needed.

We start with the renormalization procedure in the $\overline{\text{MS}}$ scheme. With our hypotheses, we only have to introduce the renormalized couplings together with the unit of mass μ to ensure that all quantities have their dimensions fixed at $4-\epsilon$. Remember the origin of these fields. Covariant derivatives on spacetime have mass dimension $[\nabla]=1$, and so do their connections.[38] In our case, we contract their antisymmetric part ($T^a{}_{bc}$) with dimensionless tensors ($\delta^a{}_b$ and ε^{abcd}), so the mass dimension stays $[T]=[\theta]=1$. For the same reason, curvature and its contractions have $[R]=2$ fixed, so all our tensorial contractions have dimension 4.

Although our assumptions do not require us to, we still show the full renormalization procedure for a general theory of interacting fields, and only later apply it to our case. With the dimensional considerations we had just given, the bare fields would be rescaled as

$$\begin{aligned} T_a^{\mathrm{B}} &= Z_T T_a \sim (1+\delta Z_T) T_a \,, \\ \theta_a^{\mathrm{B}} &= Z_\theta \theta_a \sim (1+\delta Z_\theta)\theta_a \,, \end{aligned} \tag{5.72}$$

and all bare couplings as

$$\lambda_{\mathrm{B}} = \mu^{-\epsilon}(\lambda+\delta\lambda)\,. \tag{5.73}$$

A general interaction term with m-th power of T and n-th power of θ would get renormalized as

$$\begin{aligned} g_{\mathrm{B}} T_{\mathrm{B}}^n \theta_{\mathrm{B}}^m &= \mu^{-\epsilon}(g+\delta g)(1+\delta Z_T)^n (1+\delta Z_\theta)^m T^n \theta^m \\ &\sim \mu^{-\epsilon} g T^n \theta^m + \mu^{-\epsilon}\big(\delta g + g(n\delta Z_T + m\delta Z_\theta)\big) T^n\theta^m \,, \end{aligned} \tag{5.74}$$

where we have kept linear terms in perturbations only.

As previously mentioned, we work with $\delta Z_T = \delta Z_\theta = 0$ because T^{B} and θ^{B} are not dynamical, which means

$$\begin{aligned} T_a^{\mathrm{B}} &= T_a \\ \theta_a^{\mathrm{B}} &= \theta_a \\ g_{\mathrm{B}} T_{\mathrm{B}}^n \theta_{\mathrm{B}}^m &= \mu^{-\epsilon}(g+\delta g) T^n \theta^m \,, \end{aligned} \tag{5.75}$$

By defining

$$\frac{1}{2(4\pi)^2}\frac{\mu^{-\epsilon}}{\epsilon} \equiv \frac{1}{\hat{\epsilon}}\,, \tag{5.76}$$

[38] In general, the mass dimension of covariant derivatives on internal spaces can depend on their coupling constant.

and requiring that such terms cancel the poles from (5.64) to (5.67), the counterterms are fixed to:
Vacuum contributions

$$\delta\Lambda = \frac{2}{\hat{\epsilon}}m^4 \tag{5.77a}$$

$$\delta\kappa_1 = \frac{1}{4\hat{\epsilon}}m^2 \tag{5.77b}$$

$$\delta\kappa_{2,3,4} = \frac{1}{\hat{\epsilon}}\left(\frac{7}{360}, \frac{1}{45}, -\frac{1}{72}\right) \tag{5.77c}$$

Vector torsion contributions

$$\delta\alpha_1 = \frac{97}{1620\hat{\epsilon}} \tag{5.78a}$$

$$\delta\alpha_2 = \frac{1}{54\hat{\epsilon}} \tag{5.78b}$$

$$\delta\alpha_3 = -\frac{41}{270\hat{\epsilon}} \tag{5.78c}$$

$$\delta\alpha_4 = \frac{11}{135\hat{\epsilon}} \tag{5.78d}$$

$$\delta\sigma_1 = \frac{2}{9\hat{\epsilon}}m^2 \tag{5.78e}$$

$$\delta\sigma_2 = \frac{2233}{9720\hat{\epsilon}} \tag{5.78f}$$

$$\delta\sigma_3 = \frac{341}{405\hat{\epsilon}} \tag{5.78g}$$

$$\delta\sigma_4 = \frac{1}{\hat{\epsilon}}\left[\frac{116}{405} + \frac{4}{3}(c_1^2 - c_1)\right] \tag{5.78h}$$

$$\delta\sigma_5 = -\frac{1}{\hat{\epsilon}}\left[\frac{2}{9} - \frac{4}{3}(c_1^2 - c_1)\right] \tag{5.78i}$$

$$\delta\sigma_6 = -\frac{296}{405\hat{\epsilon}} \tag{5.78j}$$

Axial torsion contributions

$$\delta\beta_1 = \frac{7}{1728\hat{\epsilon}} \tag{5.79a}$$

$$\delta\beta_2 = \frac{7}{12960\hat{\epsilon}} \tag{5.79b}$$

$$\delta\rho_1 = -\frac{1}{18\hat{\epsilon}}m^2 \tag{5.79c}$$

$$\delta\rho_2 = -\frac{7}{19432\hat{\epsilon}} \tag{5.79d}$$

$$\delta\rho_3 = -\frac{1}{\hat{\epsilon}}\left[\frac{5}{324} + \frac{4}{3}(c_2^2 - \frac{1}{4}c_2)\right] \tag{5.79e}$$

$$\delta\rho_4 = \frac{1}{\hat{\epsilon}}\left[\frac{23}{2592} + \frac{4}{3}(c_2^2 - \frac{1}{4}c_2)\right] \tag{5.79f}$$

$$\delta\rho_5 = \frac{71}{2592\hat{\epsilon}} \tag{5.79g}$$

Vector-axial interaction contributions

$$\delta\lambda_1 = -\frac{1}{\hat{\epsilon}}\left\{\left[\frac{139}{5184} + \frac{1}{27}(c_1^2 + 4c_2^2 - c_1 - c_2)\right]\right. \tag{5.80a}$$

$$\delta\lambda_1 = \frac{1}{\hat{\epsilon}}$$

$$\delta\lambda_2 = \frac{1}{\hat{\epsilon}}\left[\frac{2}{243} + \frac{1}{27}(c_1^2 + 4c_2^2 - c_1 - c_2)\right] \tag{5.80b}$$

$$\delta\lambda_3 = \frac{1}{\hat{\epsilon}}\left[\frac{1157}{38880} + \frac{4}{9}(c_2^2 - \frac{1}{4}c_2)\right] \tag{5.80c}$$

$$\delta\lambda_4 = -\frac{269}{19440\hat{\epsilon}} \tag{5.80d}$$

$$\delta\lambda_5 = -\frac{239}{19440\hat{\epsilon}} \tag{5.80e}$$

which show, as expected, that such counterterms are only parametric functions of the non-minimal couplings c_1 and c_2.

We can greatly simplify the analysis of Renormalization Group functions by considering the general case of a renormalized coupling λ_i

$$\lambda_{i\,\mathrm{B}} = \mu^{\varepsilon\alpha_i}(\lambda_i + \delta\lambda_i)\,. \tag{5.81}$$

where α_i is the factor that keeps dimensions consistent. From the definition of beta function, we can write

$$\begin{aligned}\beta_{\lambda_i} &= \mu\frac{d\lambda_i}{d\mu} = \mu\frac{d}{d\mu}(\mu^{-\varepsilon\alpha_i}\lambda_{i\ \mathrm{B}} - \delta\lambda_i) \\ &= -\varepsilon\alpha_i\mu^{-\varepsilon\alpha_i}\lambda_{i\ \mathrm{B}} - \mu\frac{\partial\lambda_j}{\partial\mu}\frac{\partial}{\partial\lambda_j}\delta\lambda_i \\ &= -\varepsilon\alpha_i(\lambda_i + \delta\lambda_i) - \mu\frac{\partial}{\partial\mu}(\mu^{-\varepsilon\alpha_j}\lambda_{j\ \mathrm{B}})\frac{\partial}{\partial\lambda_j}\delta\lambda_i\,.\end{aligned} \tag{5.82}$$

By using the pole expansion of counterterms

$$\delta\lambda_i = \frac{1}{\varepsilon}\delta\lambda_{i\ 1} + \frac{1}{\varepsilon^2}\delta\lambda_{i\ 2} + \dots\,, \tag{5.83}$$

some terms of the previous expansion will stay of order ε and thus vanish when we finally set $d \to 4$, so $\varepsilon \to 0$. The final expression for the beta function is then

$$\beta_{\lambda_i} = -\alpha_i\delta\lambda_{i\ 1} + \alpha_j\lambda_j\frac{\partial}{\partial\lambda_j}\delta\lambda_{i\ 1}\,, \tag{5.84}$$

where the second term comes from the cross-dependence of couplings $\delta\lambda_i = \delta\lambda_i(\lambda_j)$, and the sum over j is understood. In our case, all counterterms are constant according to Eq. (5.84) and have $\alpha_i = -1$. This fortunately means that

$$\beta_{\lambda_i} = \delta\lambda_{i\ 1} = f(c_1, c_2)\,, \tag{5.85}$$

so all beta functions are parametric constants of $c_1\,, c_2$ just like the respective counterterms, and no further calculation is required. Because of this, we can study the asymptotic behavior of the renormalized couplings by checking appropriate values of c_1 and c_2 and how they affect the sign of the beta functions (5.85). Being a large system of equations, many of which are at a fixed constant value, it is impossible for all couplings to have the same asymptotic properties. We thus only illustrate some interesting cases in which some interaction terms are assumed to vanish.

5.3.1 Behavior of beta functions

The following analysis is preliminary in the sense that fields are still bare and no major physical consequence can be deduced from it. We start by looking at what coupling combinations let all their beta functions be strictly negative, so as to lead to an asymptotically free theory. In the vector-torsion sector, σ_4 and σ_5 can only be separately negative, and with other beta functions being mostly strictly positive, it is not very interesting. On the other hand, a self-interacting, asymptotically free theory is possible for θ alone: the counterterms for the bare terms

$$\rho_1^{\mathrm{B}}\theta^2 + \rho_2^{\mathrm{B}}\theta^4 + \rho_3^{\mathrm{B}}(\nabla_a\theta_b)(\nabla^a\theta^b) + \rho_4^{\mathrm{B}}(\nabla_a\theta_b)(\nabla^b\theta^a) \tag{5.86}$$

yield negative beta functions if

$$\frac{1}{8} - \frac{\sqrt{645}}{360} < c_2 < \frac{1}{8} - \frac{\sqrt{21}}{72} \quad \text{or} \quad \frac{1}{8} + \frac{\sqrt{21}}{72} < c_2 < \frac{1}{8} + \frac{\sqrt{645}}{360}\,. \tag{5.87}$$

By looking at mixed T, θ terms, we find that all couplings can be made simultaneously asymptotically free except for λ_3. We can therefore include the four-legged $T^2\theta^2$ and $(T\theta)^2$) terms, as well as the last two derivative interactions into (5.86)

$$\begin{aligned} &\rho_1^{\mathrm{B}}\theta^2 + \rho_2^{\mathrm{B}}\theta^4 + \rho_3^{\mathrm{B}}(\nabla_a\theta_b)(\nabla^a\theta^b) + \rho_4^{\mathrm{B}}(\nabla_a\theta_b)(\nabla^b\theta^a) \\ &+\lambda_1^{\mathrm{B}}T^2\theta^2 + \lambda_2^{\mathrm{B}}(T\theta)^2 + \lambda_4^{\mathrm{B}}T\theta\nabla_a\theta^a + \lambda_5^{\mathrm{B}}\theta^aT^b\nabla_a\theta_b\,. \end{aligned} \tag{5.88}$$

There are many combinations of c_1 and c_2 for which all beta functions of this theory are negative. One of these is the same as (5.87), but with the added condition

$$\frac{1}{2} - \frac{\sqrt{570}}{90} < c_1 < \frac{1}{2} + \frac{\sqrt{570}}{90}\,. \tag{5.89}$$

If we insist on adding the most amount of couplings and still preserve such behavior, we could only add $R\nabla_aT^a$, the quartic interaction T^4, and the two terms quadratic in covariant derivatives $(\nabla_aT^a)^2$ and $(\nabla_aT_b)^2$, restricting the range of available c_1s to

$$\frac{1}{2} - \frac{\sqrt{285}}{90} < c_1 < \frac{1}{2} + \frac{\sqrt{285}}{90}\,. \tag{5.90}$$

The case of conformal invariance is trivially impossible for more than one coupling at a time, so we do not study it.

The last possibility is that beta functions be greater than 0. In this case, T's couplings dominate the scene: all of them, except for the constant negative ones, satisfy this requirement, allowing us to build a theory with only T

$$\begin{aligned} &\alpha_1^{\mathrm{B}}RT^2 + \alpha_2^{\mathrm{B}}R^{ab}T_aT_b + \alpha_4^{\mathrm{B}}R^{ab}\nabla_aT_b \\ &+\sigma_1^{\mathrm{B}}T^2 + \sigma_3^{\mathrm{B}}T^2\nabla_aT^a \\ &+\sigma_4^{\mathrm{B}}(\nabla_aT_b)(\nabla^aT^b) + \sigma_5^{\mathrm{B}}(\nabla_aT_b)(\nabla^bT^a)\,, \end{aligned} \tag{5.91}$$

with

$$\frac{1}{2} - \frac{\sqrt{6}}{3} < c_1 < \frac{1}{2} - \frac{\sqrt{285}}{90} \quad \text{or} \quad \frac{1}{2} + \frac{\sqrt{285}}{90} < c_1 < \frac{1}{2} + \frac{\sqrt{6}}{3}\,. \tag{5.92}$$

The beta function for $T^2\theta^2$ cannot be made positive, so the largest extension we can make is

$$
\begin{aligned}
&\alpha_1^{\mathrm{B}} R T^2 + \alpha_2^{\mathrm{B}} R^{ab} T_a T_b + \alpha_4^{\mathrm{B}} R^{ab} \nabla_a T_b \\
&+ \sigma_1^{\mathrm{B}} T^2 + \sigma_3^{\mathrm{B}} T^2 \nabla_a T^a \\
&+ \sigma_4^{\mathrm{B}} (\nabla_a T_b)(\nabla^a T^b) + \sigma_5^{\mathrm{B}} (\nabla_a T_b)(\nabla^b T^a) \\
&+ \beta_1^{\mathrm{B}} R \theta^2 + \beta_2^{\mathrm{B}} R^{ab} \theta_a \theta_b + \rho_1^{\mathrm{B}} \theta^2 + \rho_2^{\mathrm{B}} \theta^4 \\
&+ \rho_3^{\mathrm{B}} (\nabla_a \theta_b)(\nabla^a \theta^b) + \rho_4^{\mathrm{B}} (\nabla_a \theta_b)(\nabla^b \theta^a) + \rho_5^{\mathrm{B}} (\nabla_a \theta^a)^2 \\
&+ \lambda_2^{\mathrm{B}} (T\theta)^2 + \lambda_3^{\mathrm{B}} \theta^2 \nabla_a T^a + \lambda_5^{\mathrm{B}} \theta^a T^b \nabla_a \theta_b \,,
\end{aligned}
\tag{5.93}
$$

5.3.2 GR limit

Let us now look at the case in which torsion vanishes completely, both from covariant derivatives and non-minimal couplings to the spinor. This, the limit of General Relativity, is obtained by sending the curvature R_{abcd} to $\mathring{R}_{abcd}$ and the covariant derivatives ∇_a to $\mathring{\nabla}_a$ while canceling all terms with T and θ, so

$$
\mathring{S} = \int \sqrt{-g} d^4 x \frac{1}{\hat{\epsilon}} \mathring{\mathcal{L}}_{\mathrm{vac}} \,, \tag{5.94}
$$

with

$$
\mathring{\mathcal{L}}_{\mathrm{vac}} = -2m^2 + \frac{m^2}{3}\mathring{R} - \frac{1}{72}\mathring{R}^2 + \frac{1}{45}\mathring{R}^{ab}\mathring{R}_{ab} + \frac{7}{360}\mathring{R}^{abcd}\mathring{R}_{abcd} \,, \tag{5.95}
$$

which is a well-known result for minimally coupled fermions [11, 12]. Apart from the necessary higher order curvature terms, this effective action renormalizes the cosmological constant and Newton's constant, already present in the Einstein-Hilbert action. If the fermion were to be massless, this would not be true.

In order to compare it to the TEGR effective action, we have to use Eq. (3.82), which we recall being (2.21) with $R^a{}_{bcd} = 0$. By plugging the torsion decomposition in every contortion tensor of the expression, we find

$$
\begin{aligned}
\mathring{R}^{abcd} = &\frac{1}{9}(\eta^{ad}\eta^{bc} - \eta^{ac}\eta^{bd})(T^2 + \frac{1}{16}\theta^2) \\
&+ \frac{1}{144}(\eta^{ac}\theta^b\theta^d - \eta^{ad}\theta^b\theta^c + \eta^{bd}\theta^a\theta^c - \eta^{bc}\theta^a\theta^d) \\
&+ \frac{1}{3}(\eta^{ac}\nabla^b T^d - \eta^{ad}\nabla^b T^c + \eta^{bd}\nabla^a T^c - \eta^{bc}\nabla^a T^d) \\
&+ \frac{1}{12}\epsilon^{acd}{}_e(\nabla^b\theta^e - \frac{1}{3}T^b\theta^e) - \frac{1}{12}\epsilon^{bcd}{}_e(\nabla^a\theta^e - \frac{1}{3}T^a\theta^e) \,,
\end{aligned}
\tag{5.96}
$$

for the torsionless Riemann tensor,

$$\begin{aligned}\overset{\circ}{R}^{ab} &= \frac{1}{3}\eta^{ab}(\nabla_c T^c - T^2) + \frac{2}{3}\nabla^a T^b \\ &+ \frac{1}{72}(\theta^a\theta^b - \eta^{ab}\theta^2 + 6\epsilon^{abcd}\nabla_c\theta_d)\,,\end{aligned} \tag{5.97}$$

for the Ricci tensor, and

$$\overset{\circ}{R} = -\frac{4}{3}(T^2 + \frac{1}{16}\theta^2) + 2\nabla_a T^a \tag{5.98}$$

for the Ricci scalar. Using the formulas for integration by parts, this means that the effective action (5.95) can be written as

$$\begin{aligned}\overset{\circ}{\Gamma}^{GR} = \int \sqrt{-g}d^4x\frac{1}{\hat{\epsilon}}\Big[&- 2m^4 - \frac{4}{9}m^2T^2 - \frac{11}{1215}T^4 - \frac{1}{72}m^2\theta^2 \\ &+ \frac{11}{405}\Big(T^2\nabla_a T^a - (\nabla_a T^a)^2 + (\nabla_a T_b)(\nabla^a T^b)\Big) \\ &+ \frac{1}{1296}(\nabla_a\theta_b)(\nabla^a\theta^b) + \frac{1}{3240}(\nabla_a\theta_b)(\nabla^b\theta^a) - \frac{7}{2592}(\nabla_a T^a)^2 \\ &+ \frac{1}{19440}\Big(\theta^2\nabla_a T^a + T\theta\nabla_a\theta^a - 44\theta^a T^b\nabla_a\theta_b\Big) \\ &+ \frac{11}{7776}(T\theta)^2 - \frac{1}{1944}T^2\theta^2\Big]\end{aligned} \tag{5.99}$$

5.3.3 TEGR limit

In this case, we simply get rid of all terms in which curvature or one of its contractions appear, yielding the effective action

$$\overset{\bullet}{\Gamma}^{TEGR} = \int \sqrt{-g}d^4x\frac{1}{\hat{\epsilon}}\Big[\overset{\bullet}{\mathcal{L}}_{\text{vac}} + \overset{\bullet}{\mathcal{L}}_T + \overset{\bullet}{\mathcal{L}}_\theta + \overset{\bullet}{\mathcal{L}}_{\text{int}}\Big]\,, \tag{5.100}$$

with

$$\overset{\bullet}{\mathcal{L}}_{\text{vac}} = -\,2m^2\,, \tag{5.101}$$

$$\begin{aligned}\overset{\bullet}{\mathcal{L}}_T =& \frac{2}{9}m^2T^2 - \frac{2233}{9720}T^4 + \frac{341}{405}T^2\nabla_a T^a \\ &+ \Big(\frac{4}{3}(c_1^2 - c_1) + \frac{116}{405}\Big)(\nabla_a T_b)(\nabla_a T_b) \\ &- \Big(\frac{4}{3}(c_1^2 - c_1) + \frac{2}{9}\Big)(\nabla_a T_b)(\nabla^b T^a) - \frac{296}{405}(\nabla_a T^a)^2\,,\end{aligned} \tag{5.102}$$

$$
\begin{aligned}
\overset{\bullet}{\mathcal{L}}_\theta = & -\frac{1}{18}m^2\theta^2 - \frac{7}{18432}\theta^4 \\
& - \left(\frac{1}{3}(4c_2^2 - c_1) + \frac{5}{324}\right)(\nabla_a\theta_b)(\nabla^a\theta_b) \\
& + \left(\frac{1}{3}(4c_2^2 - c_1) + \frac{23}{1620}\right)(\nabla_a\theta_b)(\nabla^b\theta_a) + \frac{71}{2592}(\nabla_a\theta^a)^2 \,,
\end{aligned}
\tag{5.103}
$$

$$
\begin{aligned}
\overset{\bullet}{\mathcal{L}}_{\text{int}} = & -\left[\frac{139}{5184} + \frac{1}{27}(c_1^2 + 4c_2^2 - c_1 - c_2)\right]T^2\theta^2 \\
& + \left[\frac{2}{243} + \frac{1}{27}(c_1^2 + 4c_2^2 - c_1 - c_2)\right](T\theta)^2 \\
& + \left[\frac{1157}{38880} - \frac{1}{9}(4c_2^2 - c_2)\right]\theta^2\nabla_a T^a \\
& - \left[\frac{239}{19440} + \frac{2}{9}(4c_2^2 - c_2)\right]\theta^a T^b\nabla_a\theta_b - \frac{269}{19440}T\theta\nabla_a\theta^a \,,
\end{aligned}
\tag{5.104}
$$

where we have taken the liberty of omitting the usual "•" symbol on the irreducible torsion components.

It is then immediately clear that General Relativity and TEGR cannot retain their equivalence at one-loop level. There are constant terms in (5.99) and (5.100) that are simply not equal, and although we could find some values of the non-minimal couplings c_1 and c_2 for which some terms cancel out, this cannot be consistently achieved for all of them. This is more so true in the case of minimal-coupling $c_1 = c_2 = 0$.

6 Conclusions

We have presented the framework of the Teleparallel Equivalent of General Relativity as the gauge theory of the translation group on a flat (vanishing curvature), twisted (non-zero torsion) manifold called Weitzenböck spacetime. As we have shown, this theory breaks the paradigm that gravity must be a geometrical effect caused by curvature: because TEGR is dynamically equivalent to General Relativity at classical level, describing the gravitational interaction in terms of torsion rather than curvature becomes a matter of convention resulting in the same equations of motion. In these terms, torsion is considered to have already been validated [24] as a candidate for the classical description of gravity. Furthermore, the boundary term that differentiates the TEGR action from Einstein-Hilbert's has been shown to be equal to the Gibbons-Hawking-York one [9], extending the equivalence to manifolds with boundary, and thus to black hole thermodynamics.

Our goal was then to investigate whether such equivalence holds after quantization. We have introduced the Heat Kernel technique to covariantly compute the one-loop contributions to the gravitational effective action due to a massive, uncharged Dirac spinor interacting with the background geometry. Specifically, this was chosen to be non-minimally coupled to the vector and axial components of the torsion tensor, T^μ and θ^μ. The whole calculation has been carried out in the general framework of metric-compatible Metric Affine Gravity, where both the metric $\boldsymbol{g}$ and connection $\boldsymbol{\Gamma}$ are independent, in order to easily provide the two main limits we were interested in, GR and TEGR, and potentially others.

The result that we have found shows that the one-loop effective actions (5.100) and (5.99) are numerically incompatible with each other for both the minimal and non-minimal coupling cases, thus making it impossible to the establish a quantum equivalence of the two theories. This can in principle be recovered by following a more general yet complicated path through renormalization. In order to simplify the calculation, we have assumed the geometric fields to act as a background, so they did not get renormalized: only their couplings did. When considering a broader theory in which T^μ and θ^μ are allowed to be dynamic quantum fields, new divergences will in general arise. It is then possible that, in this case, their contributions to the effective action allow for the necessary cancellations to happen, thus recovering the GR-TEGR equivalence at one-loop order. This is unlikely, but is left for future work.

We have also studied the behavior of the renormalized couplings. Because their beta functions are parametric constants of the non-minimal couplings c_1 and c_2, their asymptotic behavior can be readily found by solving a system of inequalities. We thus immediately excluded the presence of Landau poles and fixed points. It is worth mentioning that, because of their form, only one coupling at a time can be made conformal, so it is not an instance worth discussing. The remaining cases we have shown are restricted to the possibility of having the greatest number of either positive or negative beta functions. The first case is less interesting because the coupling continues to grow at higher energies, and little can be said about it. The latter, which implies asymptotic freedom, sheds some insight into the potential high-energy behavior of our theory. As previously mentioned, the same procedure must be carried out after promoting T^μ and θ^μ to be dynamical, so they too can get renormalized.

If torsion is allowed to propagate, one must also face the issue of experimental corroboration. Although it was not the scope of the thesis, we point out the following, basic consideration. Because it minimally couples to the axial components of fields [25], the effects of torsion depend on the net spin of macroscopic matter, which is usually null. It is therefore expected to be cosmologically relevant only in large, dense structures like neutron stars, in which macroscopic and uncompensated spin can indeed be present [1, 3]. An exhaustive review of possible manifestations of spin-torsion interactions in different regimes can be found in [26, 27] and references therein.

To summarize, the properties that torsionful connections exhibit must not go unnoticed. The Teleparallel Equivalent of General Relativity has been shown to be in accordance with classical GR, yet much closer to Yang-Mills theories in its structure. This suggests that the gauge-like coupling prescription can be carried out so as to lead to a renormalizable theory of quantum fields with gravity. As we have seen, inertial and gravitational effects can be split in TEGR. As Ref.[28] suggests, quantizing a theory of matter fields coupled to the translational gauge fields (pure gravity) and only later imposing Lorentz invariance (inertia), guarantees that the theory is renormalizable at one-loop order, is Lorentz invariant, and has no ghosts. The torsion tensor, now understood as the field strength of pure gravitational interactions, provides novel and interesting insight into the physical aspects of gravity and quantum theory, with new features waiting to be potentially discovered in the near future.

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Appendix

A Change of basis

We have found the one-loop effective action of (5.13) using the full covariant derivative ∇ to obtain the most general result for metric-compatible MAG theories and their limits. As mentioned at the beginning of 5.1, this is not the only basis in which calculations can be carried out. In fact, the master formula (4.38) is expressed in terms of the Einstein-Cartan base, and our computation required some machinery in order to be compatible with it. Recall the non-minimal couplings (5.11) and (5.12)

$$q_1 = ic_1\bar{\psi}\not{T}\psi \ , \quad q_2 = c_2\bar{\psi}\not{\theta}\gamma^5\psi \, , \tag{A.1}$$

and

$$\mathring{q}_1 = ia_1\bar{\psi}\not{T}\psi \ , \quad \mathring{q}_2 = a_2\bar{\psi}\not{\theta}\gamma^5\psi \, . \tag{A.2}$$

We want to look for a relation between the coefficients c_i and a_i from the invariance of the action under change of field coordinates, which at a Lagrangian level reads

$$\mathcal{L}_{MAG} = \mathcal{L}[\nabla] + q_1 + q_2 = \mathcal{L}_{EC} = \mathcal{L}[\mathring{\nabla}] + \mathring{q}_1 + \mathring{q}_2 \, . \tag{A.3}$$

Because T and θ are vectors that do not belong to a particular basis, we only compute the change in the minimal couplings coming from the covariant derivative. The symmetrized Lagrangian density reads

$$\begin{aligned} \mathcal{L}[\nabla] &= \frac{i}{2}\sqrt{-g}\Big[\bar{\psi}\gamma^c(\nabla_c\psi) - (\nabla_c\bar{\psi})\gamma^c\psi + 2im\bar{\psi}\psi\Big] \equiv \\ &\equiv \frac{i}{2}\sqrt{-g}\Big[\quad K_1 \quad - \quad K_2 \quad + \quad M \quad \Big] \quad . \end{aligned} \tag{A.4}$$

It is possible to split the covariant derivative into its GR and torsion contributions. Analogously to (2.19) we have, in all Latin indices,

$$\omega_{abc} = \mathring{\omega}_{abc} + \frac{1}{2}(T_{abc} + T_{bca} + T_{cba}) \equiv \mathring{\omega}_{abc} + \Omega_{abc} \, , \tag{A.5}$$

Also recall the torsion traces

$$T^a{}_{ba} = T_{ab}{}^a = -T^a{}_{ab} \equiv T_b \ , \quad T_{ba}{}^a = 0 \, . \tag{A.6}$$

A.1 K1

Using decomposition (A.5) and identity (5.4) we can write

$$\begin{aligned} K_1 &= \overline{\psi}\gamma^c\nabla_c\psi = \overline{\psi}\gamma^c\overset{\circ}{\nabla}_c\psi + \frac{1}{4}\Omega_{abc}\overline{\psi}\gamma^c\gamma^{ab}\psi \\ &= \overline{\psi}\gamma^c\overset{\circ}{\nabla}_c\psi + \frac{1}{4}\Omega_{abc}\overline{\psi}\gamma^{abc}\psi + \frac{1}{4}\Omega^a{}_{ba}\overline{\psi}\gamma^b\psi - \frac{1}{4}\Omega_{ab}{}^b\overline{\psi}\gamma^a\psi \equiv \\ &\equiv \overline{\psi}\overset{\circ}{\nabla\!\!\!/}\psi + A_1 + A_2 - A_3\,. \end{aligned} \tag{A.7}$$

For A_1, because of the total antisymmetry of γ^{abc}, it is immediate to see that the only surviving term in Ω_{abc} is the first one, as the sum of remaining two is symmetric in bc. This is also valid for the torsion tensor itself. Looking at its decomposition

$$T_{abc} = \frac{1}{3}(\eta_{ac}T_b + \eta_{ab}T_c) - \frac{1}{3!}\varepsilon_{dabc}\theta^d + \mathcal{T}_{abc}\,, \tag{A.8}$$

we see that the only non-vanishing contribution is the totally antisymmetric one. This means that, using identity (5.5)

$$\begin{aligned} A_1 &= \frac{1}{4}\Omega_{abc}\overline{\psi}\gamma^{abc}\psi = -\frac{1}{8}\frac{1}{3!}\varepsilon_{dabc}\theta^d\overline{\psi}(-i\varepsilon^{abcd}\gamma_d\gamma^5)\psi \\ &= \frac{i}{8}\frac{4!}{3!}\theta^d\overline{\psi}\gamma_d\gamma^5\psi = \frac{i}{2}\overline{\psi}\theta\!\!\!/\gamma^5\psi\,. \end{aligned} \tag{A.9}$$

The remaining contributions to K_1 are just a matter of plugging (A.6) into them:

$$A_2 = -A_3 = \frac{1}{4}\overline{\psi}T\!\!\!/\psi\,, \tag{A.10}$$

so in the end, we get

$$K_1 = \overline{\psi}\nabla\!\!\!/\psi = \overline{\psi}\overset{\circ}{\nabla\!\!\!/}\psi + \frac{1}{2}\overline{\psi}T\!\!\!/\psi + \frac{i}{2}\overline{\psi}\theta\!\!\!/\gamma^5\psi\,. \tag{A.11}$$

A.2 K2

This contribution follows the same steps as the previous one, but with a caveat. The kinetic operator of $\overline{\psi}$ reads:

$$K_2 = \left(\nabla_c\overline{\psi}\right)\gamma^c\psi = \left(\overset{\circ}{\nabla}_c\overline{\psi}\right)\gamma^c\psi - \frac{1}{4}\Omega_{abc}\overline{\psi}\gamma^{ab}\gamma^c\psi \tag{A.12}$$

The calculation is therefore identical to the one of K_1, but with a change of sign in the connection term and a flipped product of gamma matrices, which now reads

$$\gamma^{ab}\gamma^c = \gamma^{abc} - \eta^{ca}\gamma^b + \eta^{cb}\gamma^a\,. \tag{A.13}$$

This makes it so we have the same coupling terms with a negative sign on the axial part given by γ^{abc}:

$$K_2 = (\bar{\psi}\not{\nabla}\psi)^\dagger = (\mathring{\nabla}_c\bar{\psi})\gamma^c\psi + \frac{1}{2}\bar{\psi}\not{T}\psi - \frac{i}{2}\bar{\psi}\not{\theta}\gamma^5\psi\,. \tag{A.14}$$

A.3 Coefficient relation

The symmetric Lagrangian (A.4) then reads

$$\begin{aligned}\mathcal{L}[\nabla] &= \frac{i}{2}\sqrt{-g}\Big[\bar{\psi}\gamma^c(\mathring{\nabla}_c\psi) - (\mathring{\nabla}_c\bar{\psi})\gamma^c\psi + i\bar{\psi}\not{\theta}\gamma^5\psi + 2im\bar{\psi}\psi\Big] \equiv \\ &\equiv \mathcal{L}[\mathring{\nabla}] - \frac{\sqrt{-g}}{2}\sqrt{-g}\bar{\psi}\not{\theta}\gamma^5\psi\,.\end{aligned} \tag{A.15}$$

From (A.3),

$$\mathcal{L}[\nabla] + q_1 + q_2 = \mathcal{L}[\mathring{\nabla}] - \frac{\sqrt{-g}}{2}\bar{\psi}\not{\theta}\gamma^5\psi + q_1 + q_2 = \mathcal{L}[\mathring{\nabla}] + \mathring{q}_1 + \mathring{q}_2\,, \tag{A.16}$$

this shows that switching between bases is just a constant shift on coefficients, equal to the transformations

$$a_1 = c_1\ , \quad a_2 = c_2 - \frac{1}{2}\,. \tag{A.17}$$

A note is due: although the coupling between T_c and the vectorial spinor current $\bar{\psi}\gamma^c\psi$ vanishes explicitly when combining K_1 and K_2, its contribution to the equation of motion does not. This can be verified by applying the Euler Lagrange equations to (A.15):

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = -\frac{i}{2}\sqrt{-g}\gamma^\mu\psi\,, \tag{A.18}$$

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = -\frac{i}{2}\sqrt{-g}\gamma^\mu\partial_\mu\psi - \frac{i}{2}\sqrt{-g}(\partial_\mu\gamma^\mu)\psi - \frac{i}{2}(\partial_\mu\sqrt{-g})\gamma^\mu\psi \tag{A.19}$$

Explicitly writing $\gamma^\mu = h_a{}^\mu\gamma^a$ in the last two terms we can write

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = -\frac{i}{2}\sqrt{-g}\Big[\gamma^\mu\partial_\mu\psi + \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}h_a{}^\mu)\gamma^a\psi\Big] \tag{A.20}$$

The second terms is just the spacetime covariant derivative $\partial + \boldsymbol{\Gamma}$ of $\boldsymbol{e}$ which, thanks to the tetrad postulate (2.41), becomes

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}h_a{}^\nu) = \partial_\mu h_a{}^\nu + \Gamma^\nu{}_{\mu\rho}h_a{}^\rho \pm \omega^b{}_{a\mu}h_b{}^\nu = 0 + \omega^b{}_{a\mu}h_b{}^\nu\,. \tag{A.21}$$

Contracting the indices μ and ν as in (A.20) we in fact get

$$-\frac{i}{2}\sqrt{-g}\Big[\gamma^{\mu}\partial_{\mu}\psi+\omega^{b}{}_{ab}\gamma^{a}\psi\Big]=-\frac{i}{2}\sqrt{-g}\Big[\gamma^{\mu}\overset{\circ}{\nabla}_{\mu}\psi+T_{a}\gamma^{a}\psi\Big]\,. \tag{A.22}$$

The remaining axial coupling is straightforwardly found from

$$\frac{\partial\mathcal{L}}{\partial\overline{\psi}}=\frac{i}{2}\sqrt{-g}\Big[\gamma^{\mu}\overset{\circ}{\nabla}_{\mu}\psi-\frac{1}{2}\Omega_{abc}\gamma^{abc}\psi-2im\psi\Big] \tag{A.23}$$

using the same identities implied in (A.9).

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