From U(1) symmetry to Gauge symmetry

(Taking scalar field for example)

First, Let's write out the Lagrangian for an arbitary free charged scalar field:

$$\mathcal{L}_{\mathrm{KG}}^{0}=\partial_{\mu}\phi(x)^{*}\partial^{\mu}\phi(x)-m^{2}\phi(x)^{*}\phi(x) \tag{1}$$

The module of the amplitude of the scalar field remains the same under a rotation in complex plane, so the physical result(transition probability) is also the same. Thus, if we make the scalar field undergo phase shift(to rotate in the complex plane) $\Lambda(x)$, the Lagrangian 1 should remain the same, for we have the same dynamics.

However, one may check the result differs.Since the U(1) symmetry is fundamental(from physical results), there must be something wrong with our Lagrangian, and we need to modify itm introducing a background for the particle to preserve its U(1) symmetry. The Lagrangian we want should:

- Remain the same form for different point x^{μ}
- Should take the form 1 at some specific point

(if you find the above requirements hard to understand, think of gravitational potentials. In consider potential, we require the potential has the same form(otherwise our dynamics differ at different height), and we need to find an null point for the potential for better comparing)

One fancy way to meet the requirements is to use covariant derivative to replace the normal derivatives, to say, we introduces one vector field A^{μ} to cancel the difference the U(1) symmetry introduced in 1. That is, we define:

$$D^{\mu}\equiv\partial^{\mu}+iQA^{\mu} \tag{2}$$

(Here A^{μ} is the stage we introduced. You'll find that is surely the e.m. field)

The lagrangian now reads:

$$\mathcal{L}_{\mathrm{KG}}=\left(D_{\mu}\phi\right)^{*}\left(D^{\mu}\phi\right)-m^{2}\phi^{*}\phi \tag{3}$$

We can prove if the arbitary field A^{μ} undergoes

$$A^{\mu'}=A^{\mu}-\partial^{\mu}\Lambda \tag{4}$$

,the differences is perfectly mapped out, and for another point x'^{μ}, the Lagrangian looks like:

$$\mathcal{L}'_{\mathrm{KG}}=\left(D'_{\mu}\phi'\right)^{*}\left(D'^{\mu}\phi'\right)-m^{2}\phi'^{*}\phi' \tag{5}$$

That looks just the same as equation 3! Thus, our requirement 1 has been met.

For requirement 2, if we choose

$$\partial^{\mu}\Lambda = A^{\mu} - A'^{\mu} \tag{6}$$

at some specific point, we can also have

$$D^{\mu} \equiv \partial^{\mu} + iQA^{\mu} \tag{7}$$

and that meet our demand 2. Good job!

Now, we've succeeded in constructing a Lagrangian to fully preserve the U(1) symmetry of the partical. We next will find that looks like just e.m. field. We may write the Lagrangian as:

$$\begin{aligned}\mathcal{L}_{\mathrm{KG}} &= \left(D_{\mu}\phi\right)^{*}\left(D^{\mu}\phi\right) - m^{2}\phi^{*}\phi \\ &= \left[\left(\partial_{\mu} - iQA_{\mu}\right)\phi^{*}\right]\left(\partial^{\mu} + iQA^{\mu}\right)\phi - m^{2}\phi^{*}\phi \\ &= \partial_{\mu}\phi^{*}\partial^{\mu}\phi + iQ\left(\partial_{\mu}\phi^{*}\right)\phi A^{\mu} - iQA_{\mu}\phi^{*}\partial^{\mu}\phi + Q^{2}A_{\mu}A^{\mu}\phi^{*}\phi - m^{2}\phi^{*}\phi\end{aligned} \tag{8}$$

Extracting the terms concerning A^{μ} out and we obtain:

$$\mathcal{L}_{\text{Interaction}} \equiv iQ\left(\partial_{\mu}\phi^{*}\right)\phi A^{\mu} - iQA_{\mu}\phi^{*}\partial^{\mu}\phi + Q^{2}A_{\mu}A^{\mu}\phi^{*}\phi \equiv \widehat{J}^{\mu}_{\mathrm{KG}}A_{\mu} + Q^{2}A_{\mu}A^{\mu}\phi^{*}\phi \tag{9}$$

This interation term shares the same form of the interations of charged particles with e.m. fields(We shall discuss later)

To summary, when e.m. fields and particles are coupled, the phycial observable has U(1) symmetry.To help build the bridge of local U(1) symmetry at different time and positions and connect them, we must constrain the e.m. field to have gauge symmetry.