Reinforcement Learning Formula Sheet

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Multi-Armed Bandit Problem

Expected reward of action a: $q_*(a) \equiv \mathbb{E}[R_t \mid A_t = a]$

Estimate of $q_*(a)$ at time t: $Q_t(a) \equiv \dfrac{\sum_{i=1}^{t-1} R_i \cdot \mathbb{1}_{A_i=a}}{\sum_{i=1}^{t-1} \mathbb{1}_{A_i=a}}$

Optimization: $Q_{n+1} = Q_n + \dfrac{1}{n}(R_n - Q_n)$

$\lim_{t\to\infty} Q_t(a) = q_*(a)$ by LLN

Greedy action selection: $A_t = \operatorname{argmax}_a Q_t(a)$

ϵ-greedy selection: greedy most of time but selects random action w/ small probability ϵ

Nonstationary problems: constant step-size parameter

$$Q_{n+1} \equiv Q_n + \alpha(R_n - Q_n), \quad \alpha \in [0,1)$$

$$Q_{n+1} = (1-\alpha)^n Q_1 + \sum_{i=1}^{n} \alpha(1-\alpha)^{n-i} R_i$$

Notice exponentially decaying past rewards.

1: A simple bandit algorithm

Initialize, for $a = 1$ to k:
  $Q(a) \leftarrow 0$
  $N(a) \leftarrow 0$
Loop:

$$A \leftarrow \begin{cases} \operatorname{argmax}_a Q(a), & \text{with probability 1 - } \epsilon \\ \text{random action}, & \text{with probability } \epsilon \end{cases}$$

  $R \leftarrow bandit(A)$
  $N(A) \leftarrow N(A) + 1$
  $Q(A) \leftarrow Q(A) + \dfrac{1}{N(A)}[R - Q(A)]$

Upper Confidence Bound (UCB) Action Selection

"Optimism in the face of uncertainty"

Same as greedy except initialize $Q_t(a)$ to a high value, select value that optimizes an action A_t, and updates the upper bound to $Q_t(a)$.

$$A_t \equiv \operatorname{argmax} \left[Q_t(a) + c\sqrt{\frac{\ln t}{N_t(a)}} \right]$$

Finite Markov Decision Processes

State: $S_t \in \mathcal{S}$, Action: $A_t \in \mathcal{A}(s)$, Reward: $R_{t+1} \in \mathcal{R} \subset \mathbb{R}$

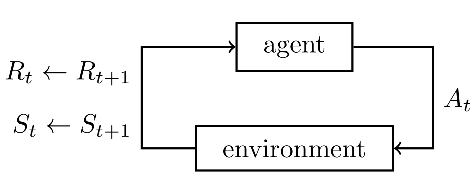
Transition dynamics fn (joint PMF):

Joint prob of next state s' and reward r given state s and action a.

$$p(s', r \mid s, a) \equiv \Pr\{S_t = s', R_t = r \mid S_{t-1} = s, A_{t-1} = a\}$$

$$p : \mathcal{S} \times \mathcal{R} \times \mathcal{S} \times \mathcal{A} \to [0,1]$$

$$\sum_{s' \in \mathcal{S}} \sum_{r \in \mathcal{R}} p(s', r \mid s, a) = 1, \quad \forall s \in \mathcal{S}, \forall a \in \mathcal{A}(s)$$



State-Transition Probabilities (Alternative Forms)

$$p(s', r \mid s, a) \equiv \Pr\{S_t = s', R_t = r \mid S_{t-1} = s, A_{t-1} = a\}$$

$$p(s' \mid s, a) = \Pr\{S_t = s' \mid S_{t-1} = s, A_{t-1} = a\} = \sum_{r \in \mathcal{R}} p(s', r \mid s, a)$$

$$r(s, a) \equiv \mathbb{E}[R_t \mid S_{t-1} = s, A_{t-1} = a] = \sum_{r \in \mathcal{R}} r \sum_{s' \in \mathcal{S}} p(s', r \mid s, a)$$

$$r(s, a, s') \equiv \mathbb{E}[R_t \mid S_{t-1} = s, A_{t-1} = a, S_t = s'] = \sum_{r \in \mathcal{R}} r \frac{p(s', r \mid s, a)}{p(s' \mid s, a)}$$

Markov property: future states of Markov process depend only on present state and not on past events.

Agent-envir interactions: episode $\to$ terminal state $\to$ reset

Goal of agent: maximize expected return, G_t

Episodic tasks: $G_t \equiv R_{t+1} + R_{t+2} + \cdots + R_T$

Continuing Tasks (no terminal state)

$$G_t \equiv R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots + \gamma^{k-1} R_{t+k} + \cdots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

$$G_t = R_{t+1} + \gamma G_{t+1}, \qquad \sum_{k=0}^{\infty} \gamma^k = \frac{1}{1-\gamma}, \qquad \gamma \in [0,1) \text{ is discount rate}$$

$$G_t \equiv \sum_{k=t+1}^{T} \gamma^{k-t-1} R_k \qquad T = \infty \text{ or } \gamma = 1 \text{ (but not both)}$$

Notice that future rewards are discounted more.

$\gamma = 0$: agent only cares about immediate reward (greedy).

$\gamma \to 1$: future rewards contribute more.

Policies

Law of total expectation: $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid Y]]$

Partition formula: $\mathbb{E}[X] = \sum_i \mathbb{E}[X \mid A_i] P(A_i)$

Policy: mapping from states to probs of selecting each possible action.

$$\pi(a|s) = p(a \mid s) = \Pr\{A_t = a \mid S_t = s\}$$

Expectation of R_{t+1} in terms of π and p:

$$\mathbb{E}[R_{t+1} \mid S_t = s] = \sum_a \pi(a \mid S_t) \sum_{s', r} p(s', r \mid s, a) r$$

Value Functions

Value fns give expected return G_t when starting in state s and following policy π thereafter.

State-value fn: $v_\pi(s) \equiv \mathbb{E}_\pi[G_t \mid S_t = s]$ $\qquad G_t = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$

Value of terminal state is always 0.

Action-value fn: $q_\pi(s, a) \equiv \mathbb{E}_\pi[G_t \mid S_t = s, A_t = a]$

v_π in terms of q_π and π: $v_\pi(s) = \sum_a \pi(a \mid S_t) q_\pi(s, a)$

q_π in terms of v_π and p: $q_\pi(s, a) = \sum_{r, s'} p(s', r \mid s, a)[r + \gamma v_\pi(s')]$

Bellman Equations

$$v_\pi(s) = \sum_a \pi(a \mid s) \sum_{s',r} p(s', r \mid s, a)[r + \gamma v_\pi(s')]$$

$$q_\pi(s,a) = \sum_{s',r} p(s', r \mid s, a) \left[r + \gamma \sum_{a'} \pi(a' \mid s') q(s', a') \right]$$

Optimal value fns: $\pi_1 \geq \pi_2 \iff v_{\pi_1}(s) \geq v_{\pi_2}(s), \quad \forall s \in \mathcal{S}$

$$v_*(s) = \max_\pi v_\pi(s) = \max_a \sum_{s',r} p(s', r \mid s, a)[r + \gamma v_*(s')], \quad \forall s \in \mathcal{S}$$

$$q_*(s,a) = \max_\pi q_\pi(s,a) = \sum_{s',r} p(s', r \mid s, a)[r + \gamma \max_{a'} q_*(s', a')]$$

$\forall s \in \mathcal{S}, \forall a \in \mathcal{A}$

Policy Evaluation

$$\pi_* = \operatorname*{argmax}_a \sum_{s',r} p(s', r \mid s, a)[r + \gamma v_*(s')]$$

2: Iterative Policy Evaluation

Input π, the policy to be evaluated
$\vec{V} \leftarrow \vec{0}, \vec{V}' \leftarrow \vec{0}$
loop:
  $\Delta \leftarrow 0$
  loop for each $s \in \mathcal{S}$:
    $V'(s) \leftarrow \sum_a \pi(a \mid s) \sum_{s',r} p(s', r \mid s, a)[r + \gamma V(s')]$
    $\Delta \leftarrow \max(\Delta, |V'(s) - V(s)|)$
  $V \leftarrow V'$
until $\Delta < \theta$ (small positive number)
return $V \approx v_\pi$

Policy improvement thm: $q_\pi(s, \pi'(s)) \geq v_\pi(s), \quad \forall s \in \mathcal{S}$

$$\pi'(s) \equiv \operatorname*{argmax}_a q_\pi(s,a) = \operatorname*{argmax}_a \sum_{s',r} p(s', r \mid s, a)[r + \gamma v_\pi(s')]$$

3: Policy Iteration

1. Initialization
$V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily $\forall s \in \mathcal{S}$

2. Policy Evaluation
Loop:
  $\Delta \leftarrow 0$
  Loop for each $s \in \mathcal{S}$:
    $v \leftarrow V(s)$
    $V(s) \leftarrow \sum_{s',r} p(s', r \mid s, \pi(s))[r + \gamma V(s')]$
    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$

3. Policy Improvement
policy-stable $\leftarrow$ true
For each $s \in \mathcal{S}$:
  old-action $\leftarrow \pi(s)$
  $\pi(s) \leftarrow \operatorname*{argmax}_a \sum_{s',r} p(s', r \mid s, a)[r + \gamma V(s')]$
  If *old-action* $\neq \pi(s)$, then *policy-stable* $\leftarrow$ false
If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2.

Monte Carlo Methods

- Reqs only sample sequences of states, actions, rewards from interactions w/ envir. Works in RL by averaging sample returns.
- MC only for episodic tasks b/c only upon completion of episode are value estimates and policies changed.

4: First-visit MC prediction for estimating $V \approx v_\pi$

Input: policy π to be evaluated

Initialize:
  $V(s) \in \mathbb{R}$, arbitrarily $\forall s \in \mathcal{S}$
  Returns$(s) \leftarrow$ empty list $\forall s \in \mathcal{S}$

Loop (for each episode):
  Generate episode following π
  $G \leftarrow 0$
  Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:
    $G \leftarrow \gamma G + R_{t+1}$
    Unless S_t appears in $S_0, S_1, \ldots, S_{t-1}$:
      Append G to *Returns*(S_t)
      $V(S_t) \leftarrow$ average(*Returns*(S_t))

MC Estimation of Action Values

$\pi(s) \equiv \operatorname*{argmax}_a q(s,a), \qquad q_{\pi_k}(s, \pi_{k+1}(s)) \geq q_{\pi_k}(s, \pi_k(s)) \geq v_{\pi_k}(s)$

5: First-visit MC prediction for estimating $V \approx v_\pi$

Input: policy π to be evaluated

Initialize:
  $\pi(s) \in \mathcal{A}(s)$, arbitrarily $\forall s \in \mathcal{S}$
  $Q(s,a) \in \mathbb{R}$, arbitrarily $\forall s \in \mathcal{S}, \forall a \in \mathcal{A}$
  Returns$(s) \leftarrow$ empty list $\forall s \in \mathcal{S}, \forall a \in \mathcal{A}$

Loop (for each episode):
  Choose $S_0 \in \mathcal{S}, A_0 \in \mathcal{A}(S_0)$ randomly st all pairs have probabilities greater than 0
  Generate episode from S_0, A_0 following π
  $G \leftarrow 0$
  Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:
    $G \leftarrow \gamma G + R_{t+1}$
    Unless S_t appears in $S_0, A_0, \ldots, S_{t-1}, A_{t-1}$:
      Append G to *Returns*(S_t)
      $Q(S_t, A_t) \leftarrow$ average(*Returns*(S_t, A_t))
      $\pi(S_t) \leftarrow \operatorname*{argmax}_a Q(S_t, a)$

Note the last three lines can be made more efficient:

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \frac{1}{n}(G - Q(S_t, A_t))$$

$$\pi(S_t) \leftarrow \operatorname*{argmax}_a Q(S_t, a))$$