Stat4DS Homework 02

Solutions

Old Topics

3.16 Define the following events:

- S = {You are sick},
  S^c = {You are healthy},
- $+$ = {A diagnostic test is positive, i.e. it says your are sick},
  $- = +^c$ = {A diagnostic test is negative, i.e. it says your are healthy}.

The test is not perfect and it is known that:

- $\mathsf{P}(+|S) = 0.98$,
- $\mathsf{P}(-|S^c) = 0.95$.

In addition the disease we are after is quite rare so that $\Pr(S) = 0.01$, that is, if we pick a random person in out population he/she will we sick only 1% of the times.

a. Here we are asked to find $\Pr(S|+)$. From the Bayes' Rule (see page 32) we get

$$\mathsf{P}(S|+) = \frac{\mathsf{P}(S \cap +)}{\mathsf{P}(+)} = \frac{\mathsf{P}(+|S) \cdot \mathsf{P}(S)}{\mathsf{P}(+|S) \cdot \mathsf{P}(S) + \mathsf{P}(+|S^c) \cdot \mathsf{P}(S^c)} = \frac{0.98 \cdot 0.01}{0.98 \cdot 0.01 + (1 - 0.95)(1 - 0.01)} = 0.165.$$

b. The Bayes' Rule is quite useful because it allows for sequential updates of some initial (probabilistic) belief. In the previous point we started from the *prior* probability $\Pr(S) = 0.01$ to get the *posterior* (i.e. after we have seen a positive test) probability $\mathsf{P}(S|+)$. Here, since we have already gathered this information, we can substitute $\mathsf{P}(S|+)$ to $\mathsf{P}(S)$ in the Bayes'formula to get the probability of the event of interest. In other words:

$$\begin{aligned}\mathsf{P}(S| +_2 \cap +_1) &= \frac{\mathsf{P}(+_2|S) \cdot \mathsf{P}(S|+_1)}{\mathsf{P}(+_2|S) \cdot \mathsf{P}(S|+_1) + \mathsf{P}(+_2|S^c) \cdot (1 - \mathsf{P}(S|+_1))} = \frac{0.98 \cdot 0.165}{0.98 \cdot 0.165 + (1 - 0.95) \cdot (1 - 0.165)} = \\ &= \frac{0.1617}{0.1617 + 0.0417} = 0.7948 \quad \text{(very high!)}\end{aligned}$$

To convince you, let's work out the math! Define the following events:

- S = {You are sick},
  S^c = {You are healthy},
- $+_1$ = {A diagnostic test is positive the first time you use it},
  $-_1 = +_1^c$ = {A diagnostic test is negative the first time you use it},
  $+_2$ = {A diagnostic test is positive the second time you use it},
  $-_2 = +_2^c$ = {A diagnostic test is negative the second time you use it}.

So we want to find:

$$\mathsf{P}(S| +_2 \cap +_1) = \frac{\mathsf{P}(S \cap +_2 \cap +_1)}{\mathsf{P}(+_2 \cap +_1)} = \frac{\mathsf{P}(+_2|S \cap +_1)\mathsf{P}(+_1|S)\mathsf{P}(S)}{\mathsf{P}(+_2 \cap +_1)}.$$

Now, remember that the results of sequential tests are independent by assumption, so we can pull the event $+_1$ out of the conditioning in the numerator, obtainig: $\mathsf{P}(+_2|S \cap +_1) = \mathsf{P}(+_2|S)$. Consequently, expanding the denominator by the LAW OF TOTAL PROBABILITY, we have

$$\mathsf{P}(S| +_2 \cap +_1) = \frac{\mathsf{P}(+_2|S) \cdot \left[\mathsf{P}(+_1|S)\mathsf{P}(S)\right]}{\mathsf{P}(+_2 \cap +_1)} = \frac{\mathsf{P}(+_2|S) \cdot \left[\mathsf{P}(+_1|S)\mathsf{P}(S)\right]}{\mathsf{P}(+_2|S) \cdot \left[\mathsf{P}(+_1|S)\mathsf{P}(S)\right] + \mathsf{P}(+_2|S^c) \cdot \left[\mathsf{P}(+_1|S^c)\mathsf{P}(S^c)\right]}.$$

Finally, if we multiply and divide the last quantity by $\frac{1}{\mathsf{P}(+_1)}$, we get

$$\mathsf{P}(S|+_2 \cap +_1) = \frac{\mathsf{P}(+_2|S) \cdot \left[\frac{\mathsf{P}(+_1|S)\mathsf{P}(S)}{\mathsf{P}(+_1)}\right]}{\mathsf{P}(+_2|S) \cdot \left[\frac{\mathsf{P}(+_1|S)\mathsf{P}(S)}{\mathsf{P}(+_1)}\right] + \mathsf{P}(+_2|S^c) \cdot \left[\frac{\mathsf{P}(+_1|S^c)\mathsf{P}(S^c)}{\mathsf{P}(+_1)}\right]} = \frac{\mathsf{P}(+_2|S) \cdot \mathsf{P}(S|+_1)}{\mathsf{P}(+_2|S) \cdot \mathsf{P}(S|+_1) + \mathsf{P}(+_2|S^c) \cdot (1 - \mathsf{P}(S|+_1))},$$

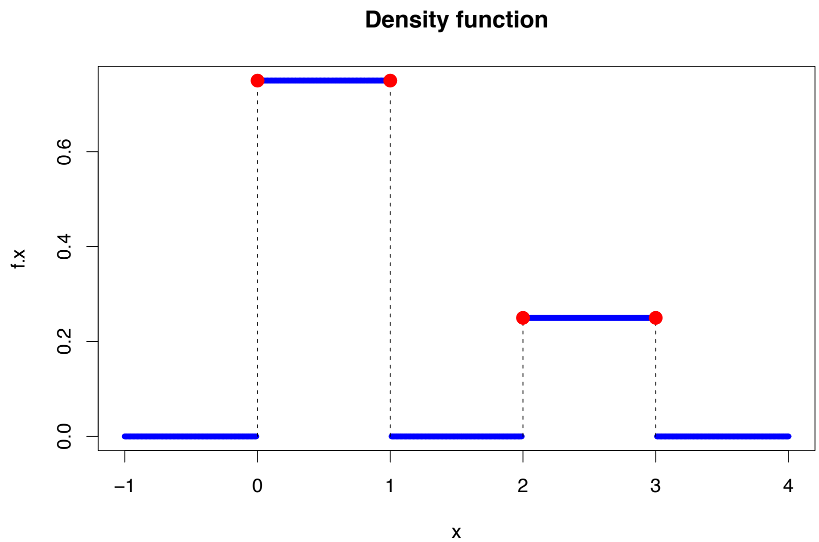
as expected.

Chapter 5

5.1 Let X be a continuous r.v. with density function

$$f(x) = \begin{cases} \frac{3}{4} & 0 \leq x \leq 1 \\ \frac{1}{4} & 2 \leq x \leq 3 \\ 0 & elsewhere \end{cases}$$

a. Here is a graphical representation.



R code:

```
x = seq(-1,4,.01)
f.x = 3/4*((x>=0)&(x<=1)) + 1/4*((x>=2)&(x<=3))
plot(x,f.x,lwd=2,col=4,cex=.5,main="Density function")
segments(0,0,0,3/4,lty=2)
segments(1,0,1,3/4,lty=2)
segments(2,0,2,1/4,lty=2)
segments(3,0,3,1/4,lty=2)
points(0,3/4,col=2,pch=19,cex=1.5)
points(1,3/4,col=2,pch=19,cex=1.5)
points(2,1/4,col=2,pch=19,cex=1.5)
points(3,1/4,col=2,pch=19,cex=1.5)
```

b. Deriving the c.d.f. is straightforward just looking at the previous figure. We can proceed in this way:

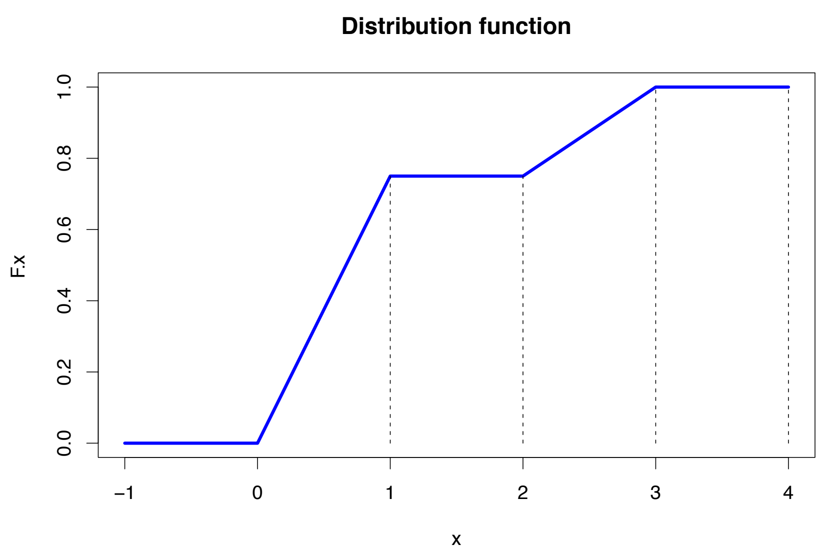
- If $x < 0$ then $F(x) = 0$
- If $0 \leq x \leq 1$ then $F(x)$ will be the area of a rectangle of height $\frac{3}{4}$ and base x, for $0 \leq x \leq 1$, i.e. $F(x) = \frac{3}{4}x$.
  More formally $F(x) = \int_0^x \frac{3}{4} u du = \left[\frac{3}{4}u\right]_0^x = \frac{3}{4}x$

- If $1 < x < 2$ then $F(x) = 3/4$.
- If $2 \leq x \leq 3$ then $F(x)$ will be the sum of the areas of two rectangles: the former has dimensions $\frac{3}{4}$ and 1, the latter has height $\frac{1}{4}$ and base $x-2$, for $2 \leq x \leq 3$, i.e. $F(x) = \frac{3}{4} + \frac{1}{4}(x-2) = \frac{x+1}{4}$
  More formally $F(x) = \int_0^1 \frac{3}{4} u du + \int_2^x \frac{1}{4} u du = \frac{3}{4} + \left[\frac{1}{4}u\right]_2^x = \frac{3}{4} + \frac{x}{4} - \frac{2}{4} = \frac{x+1}{4}$
- Finally for $x > 1$, $F(x) = 1$ that is the sum of the areas of two rectangles: the former has dimensions $\frac{3}{4}$ and 1, the latter has dimensions $\frac{1}{4}$ and 1.

In summary, this is the c.d.f.

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{3}{4}x & 0 \leq x \leq 1 \\ \frac{3}{4} & 1 < x < 2 \\ \frac{x+1}{4} & 2 \leq x \leq 3 \\ 1 & x > 3 \end{cases}$$

and this is how it looks like:



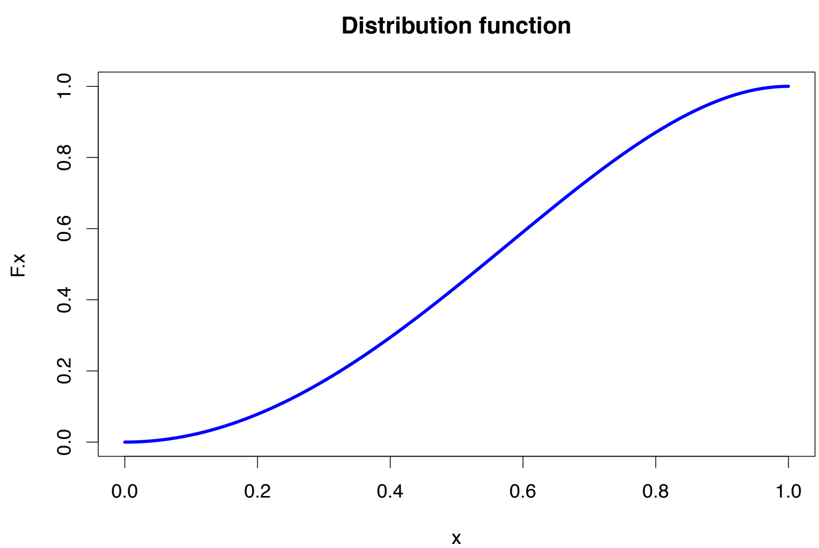
R code:

```
x = seq(-1,4,.01)
F.x = 3/4*x*((x>=0)&(x<=1)) + 3/4*((x>1)&(x<2)) + (1/4+ 1/4*x)*((x>=2)&(x<=3)) + (x>3)
plot(x,F.x,lwd=4,col=4,cex=.5,ylim=c(0,1),xlim=c(-1,4),type="l",main="Distribution function"
segments(1,0,1,3/4,lty=2)
segments(2,0,2,3/4,lty=2)
segments(3,0,3,1,lty=2)
segments(4,0,4,1,lty=2)
```

5.3 Let X be a r.v. with c.d.f. $F(x) = 2x^2 - x^4$.

R code:

```
x = seq(0,1,.01)
F.x = 2*x^2 - x^4
plot(x,F.x,ylim=c(0,1),type="l",main="Distribution function",col=4,lwd=4)
```



a. We are interested in $P\left(\frac{1}{4} \leq X \leq \frac{3}{4}\right)$, that can be expressed in terms of the c.d.f.:

$$P\left(\frac{1}{4} \leq X \leq \frac{3}{4}\right) = F\left(\frac{3}{4}\right) - F\left(\frac{1}{4}\right) = 2\left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^4 - 2\left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^4 = \frac{176}{256} = \frac{11}{16}$$

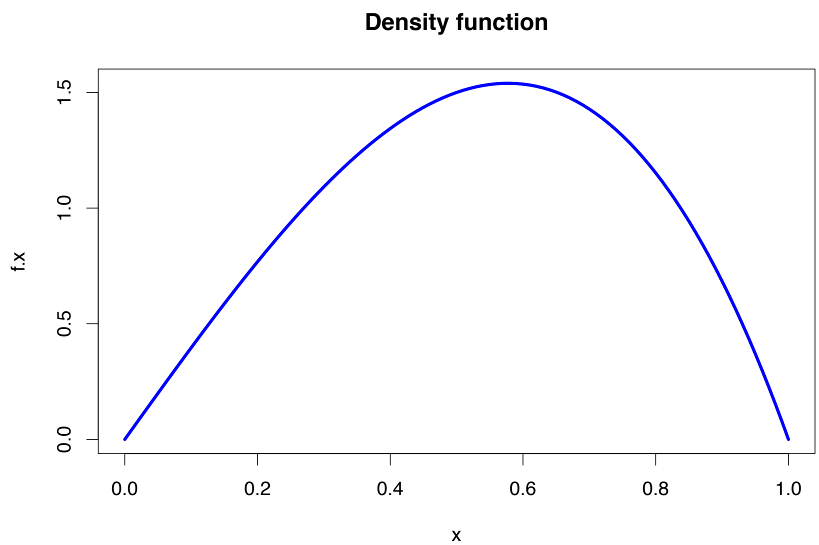
b. In order to obtain the p.d.f. we only need to derive the c.d.f., that is:

$$f(x) = F'(x) = 2 \cdot 2x - 4x^3 = 4x(1 - x^2)$$

And here it is:

R code:

```
x = seq(0,1,.01)
f.x = 4*x*(1- x^2)
plot(x,f.x,type="l",main="Density function",col=4,lwd=4)
```



5.7 Let us consider the following r.v.:

S = "score of a student"

X = "the student pass the exam"

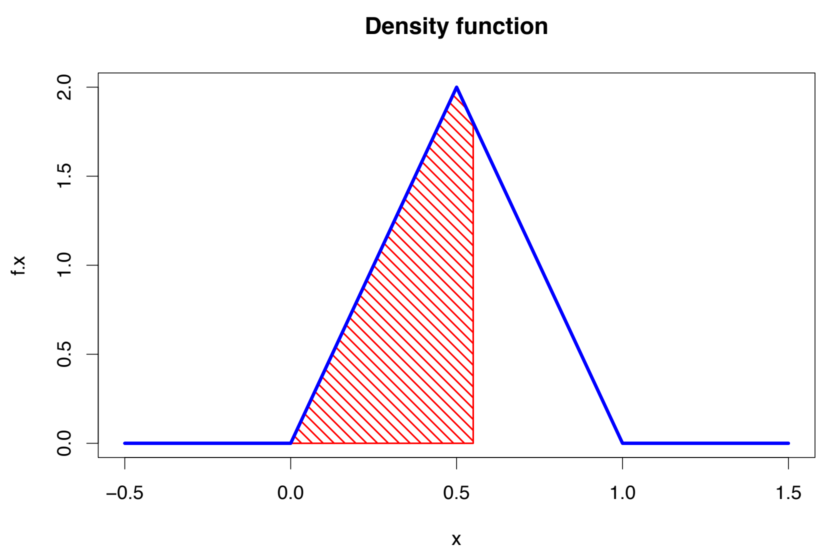
The probability density function associated to S is

$$f_S(x) = \begin{cases} 4x & 0 \leq x \leq 1/2 \\ 4 - 4x & 1/2 \leq x \leq 1 \\ 0 & \textit{elsewhere} \end{cases}$$

We also know that the probability that a student passes the exam is $P(X = 1) = P(S \geq 0.55)$.

a. Conversely, the probability that the student fails the exam is

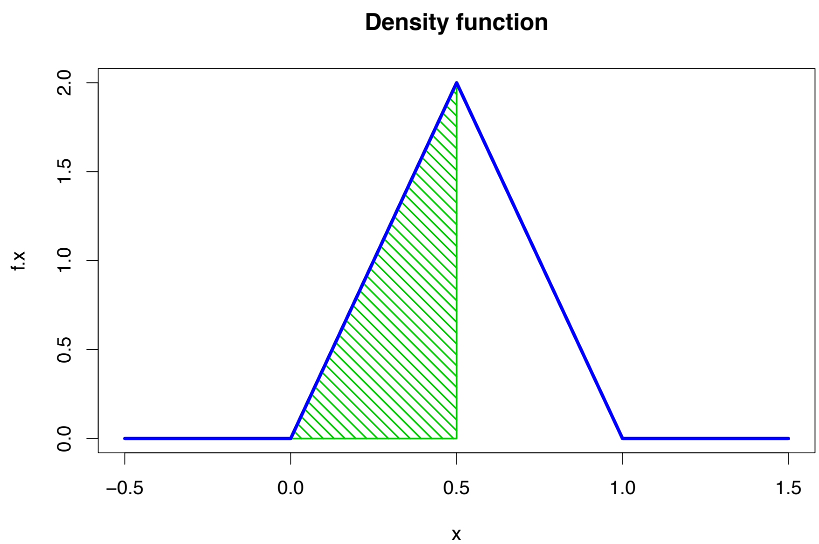
$$\begin{aligned}
P(X = 0) &= P(S < 0.55) = F_S(0.55) = \int_0^{0.55} f_S(x)dx = \\
&= \int_0^{0.5} f_S(x)dx + \int_{0.5}^{0.55} f_S(x)dx = \\
&= \left[4\frac{x^2}{2}\right]_0^{0.5} + \left[4x - 4\frac{x^2}{2}\right]_{0.5}^{0.55} = \\
&= 2 \cdot (0.5)^2 + 4 \cdot 0.55 - 2(0.55)^2 - 4 \cdot 0.5 + 2(0.5)^2 = 0.595
\end{aligned}$$



R code:

```
par(cex=1.5)
x = seq(-.5,1.5,.01)
f.x = 4*x*((x>=0)&(x<=1/2)) + (4-4*x)*((x>1/2)&(x<=1))
plot(x,f.x,type="l",main="Density function",col=4,lwd=4)
polygon(c(x[x<=0.55],rev(x[x<=0.55])),c(f.x[x<=0.55],rep(0,length(x[x<=.55]))),
col=2,density=c(10,20),angle=c(-45,45),lwd=2)
```

b. The 50th percentile of the distribution of S can be easily derived noting that the area under the density curve can be splitted in two triangles of area 1/2 (as shown in the following picture).



Formally we need to find

$$q_{50}: \quad P(S < q_{50}) = 0.50$$

and we can check that for $0 \leq y \leq 1/2$

$$P(S < y) = \int_0^y 4x dx = \left[4\frac{x^2}{2}\right]_0^y = 2y^2 = 0.50 \Longleftrightarrow y^2 = \frac{1}{4} \Longleftrightarrow y = \frac{1}{2}$$

Hence we can conclude that $q_{50} = 0.5$

5.11 Recalling the definition the **median** $q_{0.5}$ is the value such that

$$F(q_{0.5}) = P(X \leq q_{0.5}) = 0.5$$

and, if the c.d.f is invertible we can directly compute:

$$q_{0.5} = F^{INV}(0.5)$$

When we consider $X \sim Exp(\lambda)$ we have

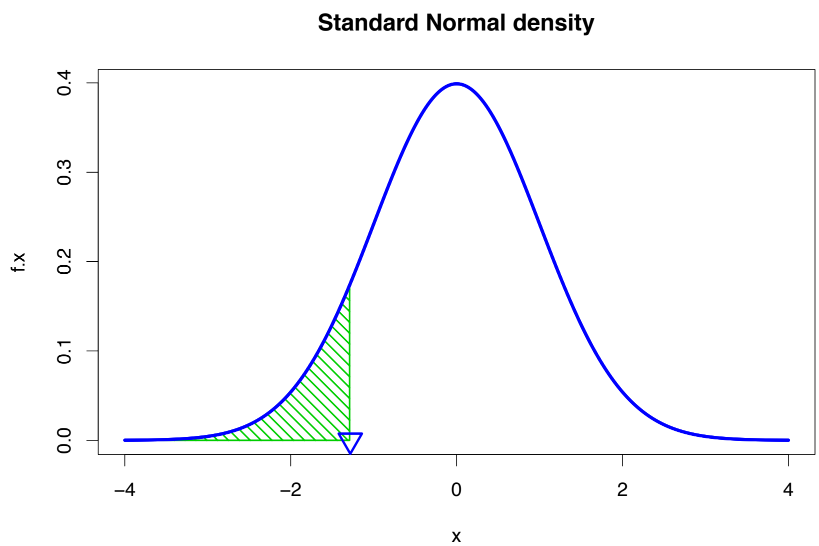
$$0.5 = 1 - e^{-\lambda q_{0.5}} \Rightarrow q_{0.5} = \frac{ln(0.5)}{-\lambda}$$

5.14 In order to compute the 10th percentile of a standard normal distribution we could use the table (Table B.1 in the book), but...we don't want to!!

Using R we easily get

```
> qnorm(0.1)
[1] -1.281552
```

Note that we do not need to specify the parameters because the default values already correspond to the standard normal (mean=0, sd=1).



R code:

```
par(cex=1.5)
x = seq(-4,4,.01)
f.x = dnorm(x)
plot(x,f.x,type="l",main="Standard Normal density",col=4,lwd=4)
polygon(c(x[x<=qnorm(0.1)],rev(x[x<=qnorm(0.1)])),
c(f.x[x<=qnorm(0.1)],rep(0,length(x[x<=qnorm(0.1)]))),
col=3,density=c(10,20),angle=c(-45,45),lwd=2)
points(qnorm(.1),0,col=4,lwd=3,pch=6,cex=2)
```

7.3 From the ALTERNATIVE EXPRESSION FOR THE VARIANCE (see page 97) we know that:

$$\mathbb{E}\left(X^2\right) = \mathbb{V}ar(X) + [\mathbb{E}(X)]^2$$

so, since $\mathbb{E}(X) = 2$ and $\mathbb{V}ar(X) = 3$, we get

$$\mathbb{E}\left(X^2\right) = 3 + 2^2 = 7.$$

7.4 Let X be a random variable with $\mathbb{E}(X) = 2$ and $\mathbb{V}ar(X) = 4$ and let $Y = -2X + 3 = g(X)$ where $g(\cdot)$ is just a linear transformation/change of units. By using the basic properties of expectation and variance (see page 98), we get:

- $\mathbb{E}(Y) = \mathbb{E}(-2X + 3) = -2 \cdot \mathbb{E}(X) + 3 = -2 \cdot 2 + 3 = -1$;
- $\mathbb{V}ar(Y) = \mathbb{V}ar(-2X + 3) = (-2)^2 \cdot \mathbb{V}ar(X) = 4 \cdot 4 = 16$.

7.7 Let X be a continuous random variable having the following density

$$f_X(x) = \begin{cases} 4x - 4x^3 & x \in [0, 1]; \\ 0 & \text{otherwise}; \end{cases}$$

and let $Y = 2 \cdot X + 3 = g(X)$ where $g(\cdot)$ is, once again, a linear transformation. By using the basic properties of expectation and variance (see page 98) we know that:

- $\mathbb{E}(Y) = \mathbb{E}(2X + 3) = 2 \cdot \mathbb{E}(X) + 3$;
- $\mathbb{V}ar(Y) = \mathbb{V}ar(2X + 3) = (2)^2 \cdot \mathbb{V}ar(X)$;

so, to complete the job, we need to calculate $\mathbb{E}(X)$ and $\mathbb{V}ar(X)$.

- By definition of expectation for continuous r.v. (see page 91), and noticing that $f_X(\cdot)$ is 0 outside the interval $[0, 1]$, we get:

$$\mathbb{E}(X) = \int x f_X(x) \mathrm{d}x = \int_0^1 x(4x - 4x^3) \mathrm{d}x = 4 \cdot \left[\frac{x^3}{3}\Big|_0^1\right] - 4 \cdot \left[\frac{x^5}{5}\Big|_0^1\right] = 4 \cdot \frac{1}{3} - 4 \cdot \frac{1}{5} = \frac{4}{3} - \frac{4}{5} = \frac{8}{15}.$$

- From the ALTERNATIVE EXPRESSION FOR THE VARIANCE (see page 97) we know that:

$$\mathbb{V}ar(X) = \mathbb{E}\left(X^2\right) - [\mathbb{E}(X)]^2,$$

so we need to find $\mathbb{E}\left(X^2\right)$. By applying what we called the Law of the lazy Statistician (a.k.a. the Chance of Variable Formula, see page 96), we get that:

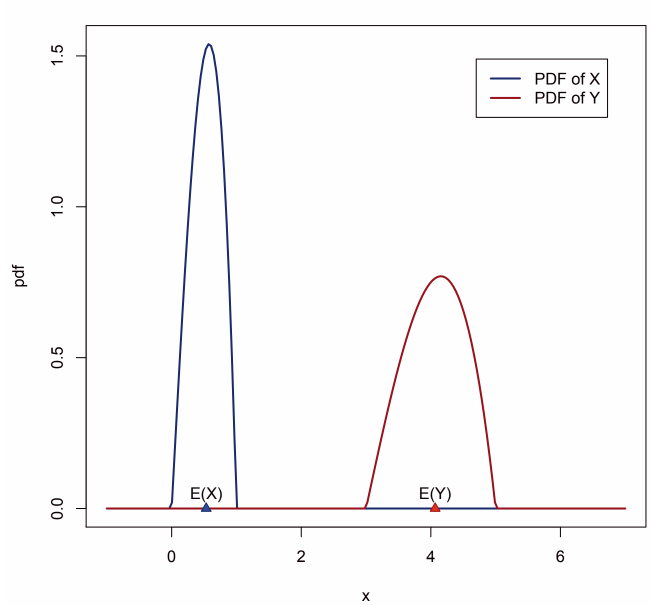
$$\mathbb{E}\left(X^2\right) = \int x^2 f_X(x)\mathrm{d}x = \int_0^1 x^2(4x - 4x^3)\mathrm{d}x = \left[\frac{x^4}{4}\Big|_0^1\right] - 4 \cdot \left[\frac{x^6}{6}\Big|_0^1\right] = 4 \cdot \frac{1}{4} - 4 \cdot \frac{1}{6} = 1 - \frac{2}{3} = \frac{1}{3}.$$

Hence

$$\mathbb{V}ar(X) = \mathbb{E}\left(X^2\right) - [\mathbb{E}(X)]^2 = \frac{1}{3} - \left(\frac{8}{15}\right)^2 = \frac{11}{225}.$$

So, finally

- $\mathbb{E}(Y) = \mathbb{E}(2X + 3) = 2 \cdot \mathbb{E}(X) + 3 = 2 \cdot \frac{8}{15} + 3 = \frac{61}{15}$;
- $\mathbb{V}ar(Y) = \mathbb{V}ar(2X + 3) = (2)^2 \cdot \mathbb{V}ar(X) = 4 \cdot \frac{11}{225} = \frac{44}{225}$.



R code:

```
# Define the PDF of the r.v. X
x.pdf = function(x){(4*x - 4*(x^3))*(x>=0)*(x<=1)}

# Define the PDF of the r.v. Y = 2*X + 3 = g(X)
```

```
# by using the usual formula with g^-1(y) = (y - 3)/2
y.pdf = function(y){ x.pdf((y - 3)/2)*abs(1/2)}

# Plot the pdf of X
curve(x.pdf(x), from = -1, to = 7, n = 200, lwd = 2, col = "navy", xlab ="x", ylab = "pdf")

# Add the pdf of Y
curve(y.pdf(x), n = 200, lwd = 2, col = "darkred", add = TRUE)

# Add a legend (you have to click on the plot to place it!)
legend(locator(1), c("PDF of X", "PDF of Y"), col = c("navy","darkred"), lwd = 2)

# Add a dot for E(X) and some text
points(8/15,0, pch = 24, col = "navy", bg = "blue")
text(8/15,0, "E(X)", pos = 3)

# Add a dot for E(Y) and some text
points(61/15,0, pch = 24, col = "darkred", bg = "red")
text(61/15,0, "E(Y)", pos = 3)
```

Chapter 9

9.1 From the table of the joint probabilities $P(X = a, Y = b)$ we have to determine the marginal probability distribution of X and Y, i.e. $P(X = a)$ and $P(Y = b)$.

		a 1	2	3	4	$P(Y = b)$
	1	16/136	3/136	2/136	13/136	34/136
b	2	5/136	10/136	11/136	8/136	34/136
	3	9/136	6/136	7/136	12/136	34/136
	4	4/136	15/136	14/136	1/136	34/136
	$P(X = a)$	34/136	34/136	34/136	34/136	1

9.3 Given the above distribution for X and Y we can compute some probabilities of interest.

a. $P(X = Y)$

		a 1	2	3	4	$P(Y = b)$
	1	16/136	3/136	2/136	13/136	34/136
b	2	5/136	10/136	11/136	8/136	34/136
	3	9/136	6/136	7/136	12/136	34/136
	4	4/136	15/136	14/136	1/136	34/136
	$P(X = a)$	34/136	34/136	34/136	34/136	1

$$\begin{aligned} P(X = Y) &= P(X = 1, Y = 1) + P(X = 2, Y = 2) + P(X = 3, Y = 3) + P(X = 4, Y = 4) = \\ &= \frac{16}{136} + \frac{10}{136} + \frac{7}{136} + \frac{1}{136} = \frac{34}{136} \end{aligned}$$

b. $P(X + Y = 5)$

		a 1	2	3	4	$P(Y = b)$
	1	16/136	3/136	2/136	13/136	34/136
b	2	5/136	10/136	11/136	8/136	34/136
	3	9/136	6/136	7/136	12/136	34/136
	4	4/136	15/136	14/136	1/136	34/136
	$P(X = a)$	34/136	34/136	34/136	34/136	1

$$
\begin{aligned}
P(X+Y=5) &= P(X=1,Y=4)+P(X=2,Y=3)+P(X=3,Y=2)+P(X=4,Y=1) = \\
&= \frac{4}{136}+\frac{6}{136}+\frac{11}{136}+\frac{13}{136}=\frac{34}{136}
\end{aligned}
$$

c. $P(1 < X \leq 3, 1 < Y \leq 3)$

		a 1	2	3	4	$P(Y=b)$
	1	16/136	3/136	2/136	13/136	34/136
b	2	5/136	10/136	11/136	8/136	34/136
	3	9/136	6/136	7/136	12/136	34/136
	4	4/136	15/136	14/136	1/136	34/136
	$P(X=a)$	34/136	34/136	34/136	34/136	1

$$
\begin{aligned}
P(1 < X \leq 3, 1 < Y \leq 3) &= P(X=2,Y=2)+P(X=2,Y=3)+P(X=3,Y=2)+P(X=3,Y=3) = \\
&= \frac{10}{136}+\frac{6}{136}+\frac{11}{136}+\frac{7}{136}=\frac{34}{136}
\end{aligned}
$$

d. $P((X,Y) \in \{1,4\} \times \{1,4\})$

		a 1	2	3	4	$P(Y=b)$
	1	16/136	3/136	2/136	13/136	34/136
b	2	5/136	10/136	11/136	8/136	34/136
	3	9/136	6/136	7/136	12/136	34/136
	4	4/136	15/136	14/136	1/136	34/136
	$P(X=a)$	34/136	34/136	34/136	34/136	1

$$
\begin{aligned}
P((X,Y) \in \{1,4\} \times \{1,4\}) &= P(X=1,Y=1)+P(X=1,Y=4)+P(X=4,Y=1)+P(X=4,Y=4) = \\
&= \frac{16}{136}+\frac{4}{136}+\frac{13}{136}+\frac{1}{136}=\frac{34}{136}
\end{aligned}
$$

9.7 Data on hair color and eye color are summarized in the following table:

	hair color		
eye color	*fair*	*medium*	*dark*
light	1168	825	305
dark	573	1312	1200

Dividing each number by the total 5383 the table is turned in the joint probability distribution of X and Y.

a. Determine the joint and marginal probability distributions of X and Y.

	hair color			
eye color	*fair*	*medium*	*dark*	$P(Y = b)$
light	0.22	0.15	0.06	0.43
dark	0.11	0.24	0.22	0.57
$P(X = a)$	0.33	0.39	0.28	1

a. In order to check whether X and Y are independent we should verify that $P(X = a, Y = b) = P(X = a)P(Y = b)$ is satisfied for all a and b. In this case, for instance:

$$P(X = 1, Y = 1) = 0.22 \neq P(X = 1) \cdot P(Y = 1) = 0.33 \cdot 0.45 = 0.14,$$

then we can conclude that X and Y are not independent.

9.12 The joint probability density function of (X,Y) is

$$f(x, y) = K(3x^2 + 8xy) \qquad for \quad 0 \leq x \leq 1 \quad and \quad 0 \leq y \leq 2$$

and 0 otherwise.

a. To find K we have to ensure that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)dxdy = 1$.

$$\begin{aligned} 1 &= \int_0^1 \int_0^2 K(3x^2 + 8xy)dxdy = \\ &= \int_0^1 3Kx^2 \left(\int_0^2 dy \right) dx + \int_0^1 8Kx \left(\int_0^2 ydy \right) dx = \\ &= \int_0^1 3Kx^2 \, [y]_0^2 \, dx + \int_0^1 8Kx \left[\frac{y^2}{2} \right]_0^2 dx = \\ &= 6K \left[\frac{x^3}{3} \right]_0^1 + 16K \left[\frac{x^2}{2} \right]_0^1 = 2K + 8K = 10K \Longrightarrow K = 1/10 \end{aligned}$$

b. In order to determine the probability $P(2X \leq Y)$ we need to integrate over all possible values of y, i.e. $0 \leq y \leq 2$, and with respect to those value of x such that $2x < y$, i.e. $0 \leq x \leq y/2$.

$$\begin{aligned}
& \int_0^2 \left(\int_0^{y/2} \frac{1}{10}(3x^2 + 8xy)dx \right) dy = \\
= & \int_0^2 \frac{1}{10} \left[3\frac{x^3}{3} + 8y\frac{x^2}{2} \right]_0^{y/2} dy = \\
= & \frac{1}{10} \int_0^2 \left(\frac{y^3}{8} + 4y\frac{y^2}{4} \right) dy = \\
= & \frac{1}{10} \int_0^2 \frac{9}{8} y^3 dy = \frac{1}{10}\frac{9}{8}\frac{16}{4} = \frac{9}{20}
\end{aligned}$$

1. The CDF $G(y)$ corresponding to the PDF $g(y)$ is, for $0 < y < 2$,

$$G(y) = \int_0^y g(t)\mathrm{d}t = \int_0^y \frac{3}{8}t^2\mathrm{d}t = \frac{1}{8}y^3.$$

By the *Quantile Trasform*, we know that the CDF of the random variable $Y = G^{-1}(X)$ will be $G(\cdot)$. We must therefore determine the inverse function $G^{-1}(\cdot)$. If

$$X = G(Y) = \frac{Y^3}{8},$$

then $Y = G^{-1}(X) = 2X^{1/3}$. It follows that $Y = 2X^{1/3}$.

2. To start with, lets define

- $N = \{\text{die score}\}$,
- $X = \{\text{number of } \texttt{Heads} \text{ in } N \text{ launches}\} = R_1 + R_2 + \cdots + R_N = \sum_{i=1}^{N} R_i$, where

$$R_i = \begin{cases} 1, & \text{if you see } \texttt{Head} \text{ in the } i^{\text{th}} \text{ coin toss;} \\ 0, & \text{otherwise;} \end{cases}$$

are N independent `Bernoulli` random variables with parameter $p = \mathbb{P}(\texttt{Head}) = \frac{1}{2}$.

Clearly we have

- $\mathbb{E}(R_i) = p = \frac{1}{2}$.
- $\mathbb{V}ar(R_i) = p(1-p) = \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{4}$.
- $\mathbb{E}(N) = \frac{1}{6}\Big(1+2+3+4+5+6\Big) = \frac{1}{6} \cdot \left(\frac{6(6+1)}{2}\right) = \frac{7}{2} = 3.5$.
- $\mathbb{V}ar(N) = \mathbb{E}(N^2) - \Big(\mathbb{E}(N)\Big)^2 = \frac{1}{6}\Big(1^2+2^2+3^2+4^2+5^2+6^2\Big) - \left(\frac{7}{2}\right)^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12} \approx 2.917$.

Then, for each fixed $n \in \{1,2,3,4,5,6\}$, we have

- $\mathbb{E}(X|N=n) = \mathbb{E}(R_1 + R_2 + \cdots + R_n) = n \cdot \mathbb{E}(R) = n \cdot p = \frac{n}{2}$.
- $\mathbb{V}ar(X|N=n) = \mathbb{V}ar(R_1 + R_2 + \cdots + R_n) = n \cdot \mathbb{V}ar(R) = n \cdot p(1-p) = \frac{n}{4}$.

So finally, applying the **law of iterated expectations** and the **law of total variance** we saw in class, we get

- $\mathbb{E}(X) = \mathbb{E}_N\Big(\mathbb{E}(X|N)\Big) = \mathbb{E}_N\left(\frac{N}{2}\right) = \frac{1}{2}\mathbb{E}(N) = \frac{7}{4} = 1.75$.

- $\mathbb{V}ar(X) = \mathbb{E}_N\Big(\mathbb{V}ar(X|N)\Big) + \mathbb{V}ar_N\Big(\mathbb{E}(X|N)\Big) = \mathbb{E}_N\left(\frac{N}{4}\right) + \mathbb{V}ar_N\left(\frac{N}{2}\right) = \frac{1}{4}\mathbb{E}\left(N\right) + \frac{1}{4}\mathbb{V}ar\left(N\right) = \frac{1}{4}\left(\frac{7}{2} + \frac{35}{12}\right) = \frac{1}{4} \cdot \frac{77}{12} = \frac{77}{48} \approx 1.604$.
- $\mathtt{SD}(X) = \sqrt{\mathbb{V}ar(X)} = \sqrt{\frac{77}{48}} \approx 1.266$.

THE END