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Math 261: Midterm Fall 2020

Academic Integrity Commitment: I agree that during the exam:

I will not seek help on the exam from others (students, professors, tutors) by way of text, email, or some other form of communication.

I will not use the internet (except for accessing the exam, the notes, and an online matrix calculator when stated in the exam).

I understand that all work submitted on this exam reflects my own individual work and not that of anyone else. I also understand that any evidence that I have violated this agreement will result in receiving a 0% for this exam, and a referral to the Dean.

Please sign or type your full name if you agree to these conditions.

Signature: Shijie Ren

Date: 10/15/2020

(If you are not printing this exam, sign your name under your printed name on the top of your exam.)

No calculators allowed unless specified. Explain all answers. Good Luck!

1. (15pts) Consider the system of equations:

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$$-5x_1 + 7x_2 + 9x_3 = 7$$
$$x_1 - 2x_2 + 6x_3 = -1$$

(a) Describe what each equation represents geometrically.

Each equation is a plane in $\mathbb{R}^?$?

(b) Can this system have a unique solution? Explain.

Since this system has 3 variables, but only 2 equations, it won't have a single unique solution

(c) Write the augmented matrix that represents this system.

$$\left[\begin{array}{ccc|c} -5 & 7 & 9 & 7 \\ 1 & -2 & 6 & -1 \end{array}\right]$$

(d) Row reduce the matrix showing all steps.

$$\left[\begin{array}{ccc|c} -5 & 7 & 9 & 7 \\ 1 & -2 & 6 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} -5 & 7 & 9 & 7 \\ 5 & -10 & 30 & -5 \end{array}\right] \sim \left[\begin{array}{ccc|c} -5 & 7 & 9 & 7 \\ 0 & -3 & 39 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} -5 & 7 & 9 & 7 \\ 0 & -1 & 13 & \frac{2}{3} \end{array}\right] \sim \left[\begin{array}{ccc|c} -5 & 7 & 9 & 7 \\ 0 & -7 & 91 & \frac{14}{3} \end{array}\right] \sim \left[\begin{array}{ccc|c} -5 & 0 & 100 & \frac{35}{3} \\ 0 & -1 & 13 & \frac{2}{3} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} -1 & 0 & 20 & \frac{7}{3} \\ 0 & -1 & 13 & \frac{2}{3} \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -20 & -\frac{7}{3} \\ 0 & 1 & -13 & -\frac{2}{3} \end{array}\right]$$

(e) Write the general solution to the system. If there are infinitely many, parameterize them and also write the solution in vector form.

$$\begin{cases} x_1 - 20x_3 = -\frac{7}{3} \\ x_2 - 13x_3 = -\frac{2}{3} \end{cases} \Rightarrow \begin{cases} x_1 = 20x_3 - \frac{7}{3} \\ x_2 = 13x_3 - \frac{2}{3} \end{cases} \leftarrow x_3 = x_3$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{7}{3} + 20x_3 \\ -\frac{2}{3} + 13x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{7}{3} \\ -\frac{2}{3} \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 20 \\ 13 \\ 1 \end{bmatrix}$$

(f) Describe the solution geometrically.

It is a line passes through a vector $\vec{u} = \begin{bmatrix} -\frac{7}{3} \\ -\frac{2}{3} \\ 0 \end{bmatrix}$ in the direction of vector $\vec{v} = \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$?

2. (10 points) Let A be the matrix

$$A = \begin{bmatrix} 1 & 0 & -20 \\ 0 & 1 & -13 \end{bmatrix}$$

(a) Find the set of vector(s) that span the solution to $A\vec{x} = \vec{0}$.

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$$\begin{cases} x_1 - 20x_3 = 0 \\ x_2 - 13x_3 = 0 \end{cases} \therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 20x_3 \\ 13x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 20 \\ 13 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1 = 20x_3 \\ x_2 = 13x_3 \end{cases}$$

$x_3 = x_3$

(b) Describe the solution geometrically.

It is a line passes through the origin in the direction of $\begin{bmatrix} 20 \\ 13 \\ 1 \end{bmatrix}$

(c) Compare and contrast the solution of $A\vec{x} = \vec{0}$ to the solution to the system in problem 1. Describe the similarities and differences both algebraically and geometrically.

The solution of $A\vec{x}=\vec{0}$ is homogenous. and for the same $A\vec{x}$, $A\vec{x}=0$ and $A\vec{x}=b$ are parallel to each other, if $\vec{b} \neq \vec{0}$, passes through different points.

3. (5 points) Compute the products. If a product is undefined, explain why.

(a) $\begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = 3\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} + (-2)\begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} -12 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -4 \\ -12 \\ -2 \end{bmatrix} = \begin{bmatrix} -16 \\ -9 \\ -2 \end{bmatrix}$ ✓

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(b) $\begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Undefined. the columns of $\begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix}$ is the the same numbers of rows of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ✓

4. (5 points) Determining consistency/uniqueness:
Given an augmented matrix representing the equation $A\vec{x} = \vec{b}$, What would you have to know about the pivots in the matrix A in order to know that the linear system is consistent and has a unique solution for any vector $\vec{b}$? Feel free to give an example if you'd like, but your answer should be in general terms.

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If an augmented matrix is consistent and has a unique solution, there must be no pivot in the last column of the augmented matrix, but a pivot position in each column of the coefficient matrix, and there must be ~~more or~~ equal rows than columns

5. (10 points) Given that the matrix A:

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$$A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 0 \\ 2 & -5 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 4 \\ 2 & -5 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) Do the column vectors span $\mathbb{R}^4$? Explain.

It is not possible for columns vectors span $\mathbb{R}^4$, as there are 3 variables at most only.

(b) Do the column vectors span $\mathbb{R}^3$? Explain.

Yes, ~~the~~ the three vectors are all linearly independ, so they span $\mathbb{R}^3$

(c) Do the columns of A form a linearly independent set? Explain.

Yes they form a linear independent set, since the system has only the trivial solution where $x_1 = x_2 = x_3 = 0$

6. (3 points) Write the system of equations that is equivalent to the vector equation.

$$\begin{bmatrix} 6 & 1 \\ 3 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ -8 \end{bmatrix}.$$

$\because$ $x_1 \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ -8 \end{bmatrix}$

$\therefore$ the system of equation is $\begin{cases} 6x_1 + x_2 = 9 \\ 3x_1 + 2x_2 = 0 \\ 2x_1 + 4x_2 = -8 \end{cases}$

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7. (10 points) Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 9 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 3 \\ -1 \\ -9 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 2 \\ -10 \\ -54 \end{bmatrix}$. Show that $\vec{w}$ is in the span of $\{\vec{u}, \vec{v}\}$. Show all relevant work and make sure your conclusion is clear. You may use an RREF calculator for this problem.

if $\vec{w}$ is in the span of $\{\vec{u}, \vec{v}\}$, there would be an a, b make that $\vec{w} = a \cdot \vec{u} + b \cdot \vec{v}$

we can get this augmented matrix

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 2 & -1 & -10 \\ 9 & -9 & -54 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 9 & -9 & -54 \\ 0 & 4 & 8 \\ 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

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we get $a = -4$, $b = 2$. plug in $\vec{u}$ and $\vec{v}$

$$a \begin{bmatrix} 1 \\ 2 \\ 9 \end{bmatrix} + b \begin{bmatrix} 3 \\ -1 \\ -9 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ -8 \\ -36 \end{bmatrix} + \begin{bmatrix} 6 \\ -2 \\ -18 \end{bmatrix} = \begin{bmatrix} 2 \\ -10 \\ -54 \end{bmatrix} = w$$

$\therefore$ $\vec{w}$ is indeed in the span of $\{\vec{u}, \vec{v}\}$

8. (15 points) Choose one of the following two problems. For this problem you should use an RREF calculator. But be sure to show all relevant work and note when you have used a calculator.

√ **Problem A:** Suppose an economy has four sectors, Agriculture (A), Energy (E), Manufacturing (M), and Transportation (T). Sector A sells 10% of its output to E and 25% to M, and retains the rest. Sector E sells 30% of its output to A, 35% to M, and 25%to T and retains the rest. Sector M sells 30% of its output to A, 15% to E, and 40% to T and retains the rest. Sector T sells 20% of its output to A, 10% to E, and 30% to M and retains the rest.

(a) Construct the exchange table for this economy.

(b) Find the set of equilibrium prices if the output of Sector A is 5 billion dollars.

Problem B: The following is census data for Mathia (a fictitious town):

Year	Population in Hundred Thousands
1980	1.79
1990	2.03
2000	2.27
2010	2.50

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(a) Use linear algebra to find a 3rd degree polynomial to fit the the data.

(b) Use the polynomial to estimate the population in 2020.

(a)

	A	E	M	T
A	65%	10%	25%	
E	30%	10%	35%	25%
M	30%	15%	15%	40%
T	20%	10%	30%	40%

$10\% \cdot A + 25\% A = 30\% E + 30\% M + 20\% T$

$(30\% + 35\% + 25\%) E = 10\% A + 15\% M + 10\% T$

$(30\% + 15\% + 40\%) M = 25\% A + 35\% E + 30\% T$

$(20\% + 10\% + 30\%) T = 25\% E + 40\% M$

$$\begin{array}{cccc} A & E & M & T \end{array}$$
$$\left[\begin{array}{cccc|c} -0.35 & 0.3 & 0.3 & 0.2 & 0 \\ 0.1 & -0.9 & 0.15 & 0.1 & 0 \\ 0.25 & 0.35 & -0.85 & 0.3 & 0 \\ 0 & 0.25 & 0.4 & -0.6 & 0 \end{array}\right] \Rightarrow \begin{cases} A = 2.02785T \\ E = 0.53110T \\ M = 1.16805T \\ T = T \end{cases}$$

$\therefore 0.35A = 5 \times 10^9$, $A = 1.43 \times 10^{10}$

$T = 7.04 \times 10^9$

$M = 8.23 \times 10^9$

$E = 3.74 \times 10^9$

$\therefore$ The set of equilibrium prices are \$ $\begin{bmatrix} 1.43 \times 10^{10} \\ 3.74 \times 10^9 \\ 8.23 \times 10^9 \\ 7.04 \times 10^9 \end{bmatrix} \begin{array}{c} A \\ E \\ M \\ T \end{array}$

9. (2 points) What is something about the class structure that is working for you this quarter? Write a couple of sentences for full credit.

I particularly like the part where we can get credits for uploading class notes/ group assignment. It is not only useful for when we learn, it is also very helpful before we do homework and exams to review and brush up.

10. **BONUS (5 points)** At the start of each chapter in our textbook, there is an "Introductory Example" (found the page before the first section). Look through the book and find one that fits your major or sounds interesting. Read it and write a small paragraph about what you learned.

+5

Chapter: 8. The Geometry of Vector Spaces

Introductory Example: The Platonic Solids

What did you learn?

I started painting since I was very little, infact, I used to struggle very hard with basic math, until I met a teacher who helped me understand concepts more with Geometry. I think visualing vectors as geometric subjects will certainly help me better linear alegebra as a whole. It seems like there are a lot of such application in computer graphics and I wish to really grasp and learn more.