T. 6

$$\xi \sim p(x,\theta) = \begin{cases} \frac{\theta-1}{x^\theta}, & x \geq 1 \\ 0, & x < 1 \end{cases}, \text{ где } \theta > 1$$

$\exists \vec{x}_n$

a) $\boxed{\text{О.М.П.}}$

функция правдоподобия:

$$L(\theta) = \prod_{i=1}^{n} p(x_i, \theta) = \frac{(\theta-1)^n}{\left(\prod_{i=1}^{n} x_i\right)^\theta} \left\{\min_{i=\overline{1,n}} x_i \geq 1\right\}$$

$$(\ln L)'_\theta = \left(n \ln(\theta-1) - \theta \left(\sum_{i=1}^{n} \ln x_i\right)\right)'_\theta = \frac{n}{\theta-1} - \sum_{i=1}^{n} \ln x_i$$

$$\frac{n}{\theta-1} - \sum_{i=1}^{n} \ln x_i = 0 \Rightarrow \tilde{\theta} = 1 + \frac{n}{\sum_{i=1}^{n} \ln x_i} = \underline{1 + (\overline{\ln x})^{-1}}$$

О.М.П.

$$(\ln L)''_{\theta\theta} = \frac{-n}{(\theta-1)^2} < 0 \Rightarrow \text{это максимум}$$

b) $\boxed{\text{О.М.П.}}$

- докажем, что ПВМ – сильно регул.: (чтобы воспользоваться теоремой)

1) явл регулярной

- $p(x,\theta)$ – непр. дифф. по θ на $(1;+\infty)$
- $\int_1^{+\infty} \frac{\partial}{\partial\theta} p(x,\theta)\,dx \overset{?}{=} 0$

$$p'_\theta = \left(\frac{\theta-1}{x^\theta}\right)'_\theta = \frac{\ln x - \theta \ln x + 1}{x^\theta}$$

на $(1;+\infty)$

Тогда $\int_1^{+\infty} x^{-\theta}(\ln x(1-\theta)+1)\,dx = \frac{x^{-\theta+1}}{1-\theta}\Big|_1^{+\infty} +$

$$+\int_1^{+\infty} \ln x \cdot d(x^{1-\theta}) = \frac{1}{\theta-1} - \int_1^{+\infty} x^{-\theta}dx = \frac{1}{\theta-1} - \frac{1}{\theta-1} = 0$$

• $$I(\theta) = \int_1^{+\infty} \frac{\theta-1}{x^\theta} \cdot \left(\left(\ln(\theta-1) - \theta \ln x\right)'_\theta\right)^2 dx =$$

$$= \int_1^{+\infty} \frac{\theta-1}{x^\theta}\left(\frac{1}{\theta-1} - \ln x\right)^2 dx = \int_1^{+\infty}\left(\frac{1}{x^\theta(\theta-1)} - 2\cdot\frac{\ln x}{x^\theta} + (\theta-1) \times\right.$$

$$\left.\times \frac{\ln^2 x}{x^\theta}\right)dx = \frac{1}{(\theta-1)^2} - 2\int_1^{+\infty} \ln x \, d\left(-\frac{1}{(\theta-1)x^{\theta-1}}\right) +$$

$$+\int_1^{+\infty} \frac{(\theta-1)\ln^2 x}{x^\theta}dx = \frac{1}{(\theta-1)^2} - 2\tilde{I} + \tilde{\tilde{I}}$$

$$\tilde{I} = \int_1^{+\infty} \ln x \, d\left(-\frac{1}{(\theta-1)x^{\theta-1}}\right) = \underbrace{\ln x \cdot \frac{x^{1-\theta}}{1-\theta}\Bigg|_1^{+\infty}}_{\text{т.к. } \ln 1 = 0} - \int_1^{+\infty} \frac{x^{-\theta}dx}{1-\theta} =$$

$$= -\frac{1}{(\theta-1)^2}$$

$$\tilde{\tilde{I}} = \int_1^{+\infty} \frac{(\theta-1)\ln^2 x}{x^\theta}dx = \int_1^{+\infty} \ln^2 x \, d(-x^{1-\theta}) = -\left(\underbrace{\ln^2 x \cdot x^{1-\theta}\Big|_1^{+\infty}}_{0} - \right.$$

$$\left.-\int_1^{+\infty} 2\cdot\frac{\ln x}{x}\cdot x^{1-\theta}dx\right) = 2\int_1^{+\infty} \ln x \cdot x^{-\theta}dx =$$

$$= 2\int_1^{+\infty} \ln x \, d\left(\frac{x^{1-\theta}}{1-\theta}\right) = 2\left(\underbrace{\frac{x^{1-\theta}}{1-\theta}\ln x\Bigg|_1^{+\infty}}_{0} - \int_1^{+\infty}\frac{x^{-\theta}dx}{1-\theta}\right) =$$

$$= -\frac{2}{(\theta-1)^2}$$

$$\Rightarrow I(\theta) = \frac{1}{(\theta-1)^2} + \cancel{2\cdot\frac{1}{(\theta-1)^2}} - \cancel{\frac{2}{(\theta-1)^2}} = \frac{1}{(\theta-1)^2}$$

непр. и >0 на $(1,+\infty)$

2) $p(x,\theta)$ — $k \geq 2$ раз непр. дифф по θ на $(1,+\infty)$

(очевидно)

3) $\int\limits_1^{+\infty} \frac{\partial^l}{\partial \theta^l} p(x,\theta)dx \overset{?}{=} 0$ — верно для $l=1$

проверим $l=2$:

$$p''_{\theta\theta} = \ln x \cdot x^{-\theta}(\theta \ln x - \ln x - 2)$$

на $(1;+\infty)$

$$\int\limits_1^{+\infty} (\ln^2 x \cdot x^{-\theta}(\theta-1) - 2\ln x \cdot x^{-\theta})dx = 0$$

везде будет $\ln 1$

$\Rightarrow$ О.М.П. состоят., асимп. несмещ., асимп. эффект и асимп. нормальн

по определению медиана: $F(x_{0,5}) = 0,5$

т.е. $\int\limits_1^{x_{0,5}} \frac{\theta-1}{x^\theta} dx = -\frac{1}{x_{0,5}^{\theta-1}} + 1 = 0,5$

$$\Rightarrow x_{0,5} = 2^{\frac{1}{\theta-1}}$$

Тогда $g(\hat{\theta}) = \sqrt[\hat{\theta}-1]{2}$

Имеем: $\sqrt{n}\frac{g(\hat{\theta}) - g(\theta)}{\sqrt{\vec{\nabla}^T g(\hat{\theta}) I^{-1}(\hat{\theta}) \vec{\nabla} g(\hat{\theta})}} \rightsquigarrow N(0,1)$

$$\vec{\nabla} g(\hat{\theta}) = 2^{\frac{1}{\hat{\theta}-1}} \cdot \ln 2 \cdot \frac{-1}{(\hat{\theta}-1)^2}$$

$$\sqrt{n}\frac{(g(\hat{\theta}) - g(\theta))(\hat{\theta}-1)}{2^{\frac{1}{\hat{\theta}-1}} \cdot \ln 2} \rightsquigarrow N(0,1)$$

т.о.

$$P\left(g(\hat{\theta}) - \frac{1,96 \cdot \ln 2 \cdot 2^{\frac{1}{\hat{\theta}-1}}}{\sqrt{n}(\hat{\theta}-1)} < x_{0,5} < g(\hat{\theta}) + \frac{1,96 \cdot \ln 2 \cdot 2^{\frac{1}{\hat{\theta}-1}}}{\sqrt{n}(\hat{\theta}-1)}\right) = \beta$$

асимптотический доверительный интервал (О.М.П.) для медианы

③ $\boxed{\text{О.М.П.}}$

$$\frac{\sqrt{n}\,(\tilde{\theta}-\theta)}{\tilde{\theta}-1} \rightsquigarrow N(0,1)$$

$$-1{,}96 < \sqrt{n}\cdot \overline{\ln x}\left(1+\frac{1}{\overline{\ln x}}-\theta\right) < 1{,}96$$

$$P\left(1-\frac{1{,}96-\sqrt{n}}{\sqrt{n}\,\overline{\ln x}} < \theta < 1+\frac{1{,}96+\sqrt{n}}{\sqrt{n}\,\overline{\ln x}}\right)=\beta$$

Асимптотич. довер. интервал (О.М.П.)