$$D[\tilde{\theta}] = \frac{\theta(1-2\theta)}{2n} \equiv \frac{\theta(1-2\theta)}{2n}$$

$\Rightarrow$ эффективн

т.о. О.М.М. — не эффектив.

Т. 5 $\xi \sim R[\theta, 2\theta]$, $\exists \vec{x}_n$

$\Rightarrow \xi \sim \rho(x,\theta) = \frac{1}{\theta}\{(\theta, 2\theta)\}$

a, b Найдём $\tilde{\theta}$:

$\boxed{\text{О.М.М.}}$ Возьмём α_1:

$$\alpha_1 = \int_{-\infty}^{+\infty} x \rho dx = \int_{\theta}^{2\theta} \frac{x}{\theta} dx = \frac{3}{2}\theta$$

$$\alpha_1 = \tilde{\alpha}_1 = \bar{x} \Rightarrow \frac{3}{2}\theta = \bar{x} \Rightarrow \underline{\tilde{\theta} = \frac{2}{3}\bar{x}} \text{ — О.М.М.}$$

- проверим несмещ.:

$$M[\tilde{\theta}] = \frac{2}{3} M[\bar{x}] = \frac{2}{3} M[\xi] = \theta \Rightarrow \tilde{\theta} \text{ — несмещ.}$$

- проверим состоятельность:

$$D[\tilde{\theta}] = \frac{4}{9} D[\bar{x}] = \frac{4}{9n} \underbrace{D[\xi]}_{\mu_2} \ominus$$

$$\alpha_2 = \int_{\theta}^{2\theta} \frac{x^2}{\theta} dx = \frac{8t^3}{3t} - \frac{t^3}{3t} = \frac{7}{3}t^2 \Rightarrow \mu_2 = \alpha_2 - \alpha_1^2 = \frac{\theta^2}{12}$$

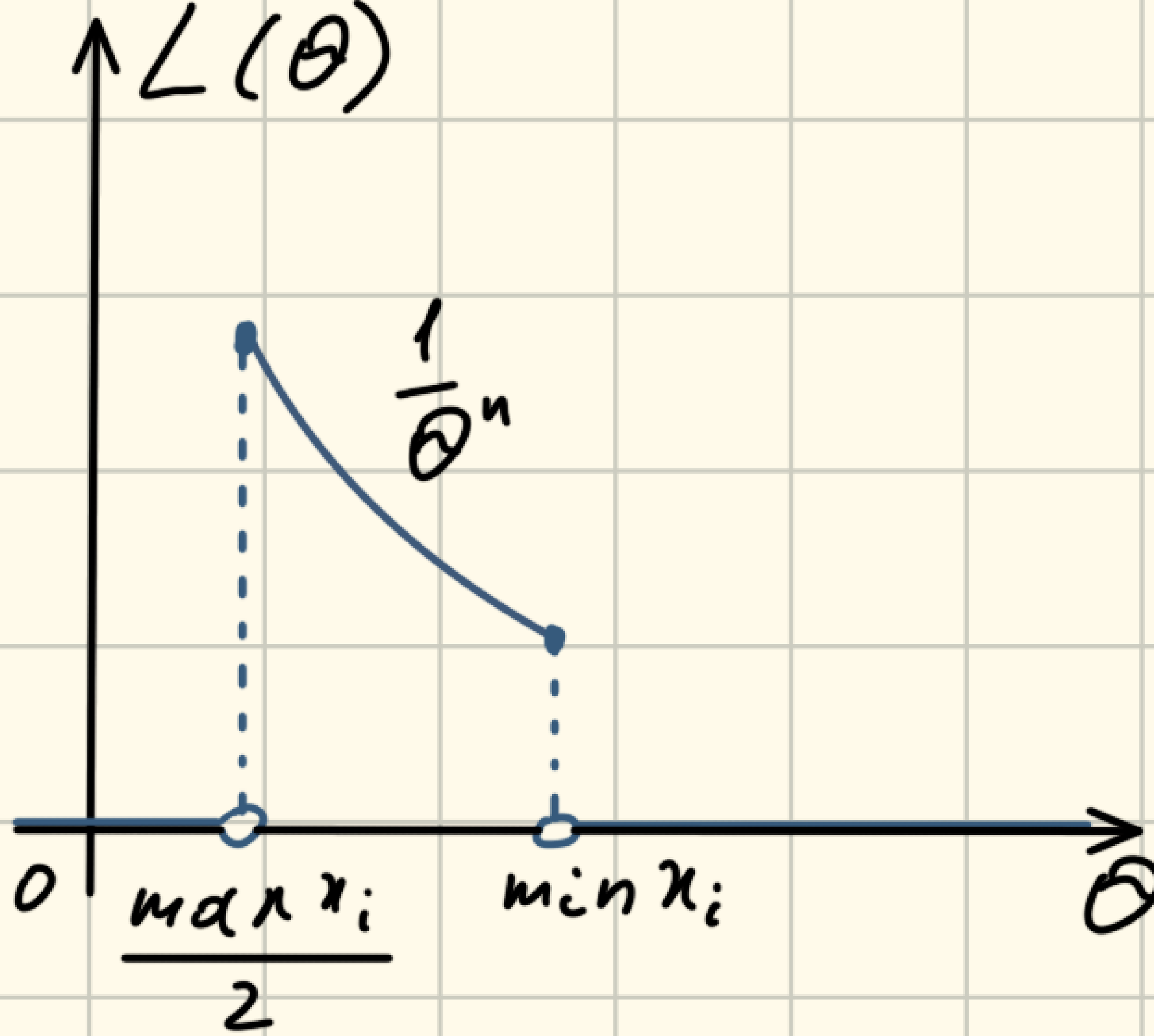
$$\ominus \frac{1\cancel{4}}{9n} \cdot \frac{\theta^2}{\cancel{12}_{3}} = \frac{\theta^2}{27n} \xrightarrow[n \to +\infty]{} 0 \Rightarrow \tilde{\theta} \text{ — состоят.}$$

$\boxed{\text{О.М.П.}}$

функция правдоподобия:

$$L(\theta) = \prod_{i=1}^{n} \rho(x_i, \theta) = \frac{1}{\theta^n} \left\{ \forall i \Leftrightarrow \begin{array}{l} \min\limits_{i=\overline{1,n}} x_i \geq \theta, \\ \max\limits_{i=\overline{1,n}} x_i \leq 2\theta \end{array} \right\} =$$

$$= \frac{1}{\theta^n} \left\{ \frac{\max x_i}{2} \leq \theta \leq \min x_i \right\}$$



$$\Rightarrow \max: \ \hat{\theta} = \frac{\max x_i}{2}$$

О.М.П.

• проверим несмещ.:

$$M[\hat{\theta}] = \frac{1}{2} M[\max x_i] = \frac{1}{2}\int_{\theta}^{2\theta} t \cdot \frac{n}{\theta}\left(\frac{t}{\theta} - 1\right)^{n-1} dt =$$

$$= \left\{ \begin{array}{l} u = \frac{t}{\theta} \\ du = \frac{1}{\theta} dt \\ t = u\theta \\ dt = \theta du \end{array} \right\} = \frac{\theta}{2}\int_1^2 n u (u-1)^{n-1} du = \frac{\theta}{2} \cdot \tilde{I} \ominus$$

$$\int u\,dv = uv - \int v\,du$$

$$\tilde{I} = \int_1^2 u \cdot \underbrace{n(u-1)^{n-1} du}_{d((u-1)^n)} = u(u-1)^n \Big|_1^2 - \int_1^2 (u-1)^n du =$$

$$= \left(u(u-1)^n - \frac{(u-1)^{n+1}}{n+1}\right)\Bigg|_1^2 = 2 + \frac{1^{n+1}}{n+1}$$

$$\ominus \ \theta - \frac{\theta}{2(n+1)} = \frac{\theta(2n+1)}{2n+2} \Rightarrow \hat{\theta} \text{ – смещенная}$$

тогда

$$\hat{\theta}' = \frac{2n+2}{2n+1} \cdot \frac{\max x_i}{2} = \frac{n+1}{2n+1} \max x_i \text{ – несмещ}$$

• проверим оптимальность:

$$D[\hat{\theta}] = M[\hat{\theta}^2] - M^2[\hat{\theta}]$$

$$M[\hat{\theta}^2] = \frac{1}{4}\int_{\theta}^{2\theta} t^2 \cdot \frac{n}{\theta}\left(\frac{t}{\theta} - 1\right)^{n-1} dt = \left\{ u = \frac{t}{\theta} \right\} =$$

$$= \frac{\theta^2}{4}\int_1^2 n \cdot u^2 (u-1)^{n-1} du = \frac{\theta^2}{4} \cdot \tilde{\tilde{I}}$$

$$\tilde{\tilde{I}} = \int_1^2 n\cdot u^2 (u-1)^{n-1} du = u^2(u-1)^n \Big|_1^2 - 2\int_1^2 u(u-1)^n du =$$

$$= 4 - \frac{2}{n+1}\int_1^2 (n+1)u(u-1)^n du \ominus$$

$$\tilde{\tilde{\tilde{I}}} = \int_1^2 (n+1)u(u-1)^n du = u(u-1)^{n+1}\Big|_1^2 - \frac{(u-1)^{n+2}}{n+2}\Big|_1^2 =$$

$$= 2 - \frac{1^{n+2}}{n+2} = 2 - \frac{1}{n+2}$$

$$\ominus\ 4 - \frac{2}{n+1}\left(2 - \frac{1}{n+2}\right) = \frac{2(2n^2+4n+1)}{(n+2)(n+1)}$$

$$\Rightarrow M[\tilde{\theta}^2] = \frac{\theta^2}{\cancel{4}_2}\cdot\frac{\cancel{2}^1(2n^2+4n+1)}{(n+2)(n+1)} = \frac{2n^2+4n+1}{2(n+2)(n+1)}\cdot\theta^2$$

Тогда

$$D[\tilde{\theta}] = \theta^2\left(\frac{2n^2+4n+1}{2(n+2)(n+1)} - \left(\frac{2n+1}{2n+2}\right)^2\right) = \theta^2\cdot\frac{n}{4(n+1)^2(n+2)}$$

(если смотреть состоят. $\tilde{\theta}$, то по определению)

$$D[\tilde{\theta}'] = \cancel{4}\cdot\frac{\cancel{(n+1)^2}}{(2n+1)^2}\cdot\theta^2\cdot\frac{n}{\cancel{4(n+1)^2}(n+2)} \xrightarrow[n\to\infty]{} 0$$

$\Rightarrow \tilde{\theta}'$ – состоят.

(c)

$$D[\tilde{\theta}_{\text{ОММ}}] = \frac{\theta^2}{27n} \geq \frac{n\theta^2}{(n+2)(2n+1)^2} = D[\tilde{\theta}'_{\text{ОМП}}]$$

(при $n \geq 3$)

$\Rightarrow$ О.М.П. (исправл.) эффективнее О.М.М.

(d) Пусть $f(\vec{x}_n, \theta) = \dfrac{\max\limits_{i=\overline{1,n}} x_i}{\theta} - 1$

Тогда $P(f < t) = P(\max x_i < \theta t + \theta) =$

$= (F(\theta t+\theta))^n$, где $F(x)=\begin{cases} 0, & x<\theta \\ \frac{x}{\theta}-1, & \theta \le x \le 2\theta \\ 1, & x>2\theta \end{cases}$

т.о. $P(\xi<t)=\begin{cases} 0, & t<0 \\ t^n, & 0\le t\le 1 \\ 1, & t>1 \end{cases}$

$t_1 : F(t_1)=\frac{1-\beta}{2}$

$t_2 : F(t_2)=\frac{1+\beta}{2}$ $\Rightarrow t_{1,2}=\sqrt[n]{\frac{1\mp\beta}{2}}$

Тогда $P\left(t_1<\frac{\max x_i}{\theta}-1<t_2\right)=\beta$

т.е. $t_1+1<\frac{\max x_i}{\theta}<t_2+1$

$$\Rightarrow P\left(\frac{\max x_i}{1+\sqrt[n]{\frac{1+\beta}{2}}}<\theta<\frac{\max x_i}{1+\sqrt[n]{\frac{1-\beta}{2}}}\right)=\beta$$

точный доверительный интервал

(e) а) О.М.М.: $\tilde{\theta}=\frac{2}{3}\bar{x}=g(\tilde{\alpha}_1)$

$g(\alpha_1)=\theta$; $\vec{\nabla} g=g'=\frac{2}{3}$; $K_{11}=\alpha_2-\alpha_1^2$

$$\frac{\sqrt{n}(\tilde{\theta}-\theta)}{\sqrt{\left(\frac{2}{3}\right)^2(\tilde{\alpha}_2-\tilde{\alpha}_1^2)}}=\frac{3n(\tilde{\theta}-\theta)}{2\sqrt{s^2(n-1)}}\rightsquigarrow N(0,1),$$ где $s^2=\frac{n}{n-1}\cdot\tilde{\mu}_2$

т.к. норм. распр.: $t_{1,2}=\mp 1{,}96$

$$P\left(t_1<\frac{3n(\tilde{\theta}-\theta)}{2\sqrt{s^2(n-1)}}<t_2\right)=\beta$$

тогда $\frac{2t_1\sqrt{s^2(n-1)}}{3n}<\tilde{\theta}-\theta<\frac{2t_2\sqrt{s^2(n-1)}}{3n}$

$$\Rightarrow P\left(\frac{2}{3}\bar{x}-\frac{3{,}92\sqrt{s^2(n-1)}}{3n}<\theta<\frac{2}{3}\bar{x}+\frac{3{,}92\sqrt{s^2(n-1)}}{3n}\right)=\beta$$

асимптотический доверительный интервал О.М.М.

б) О. М. П. (исправл.) : $\tilde{\theta}' = \frac{n+1}{2n+1} \max\limits_{i=\overline{1,n}} x_i$

$$\frac{\sqrt{n}\,(\tilde{\theta}' - \theta)}{\sqrt{I^{-1}(\theta)}} \rightsquigarrow N(0,1) \;;\; p(x,\theta) = \frac{1}{\theta}\{(\theta, 2\theta)\}$$

$$I(\theta) = \int_{\theta}^{2\theta} p \cdot \left((\ln p)'_{\theta}\right)^2 dx = \int_{\theta}^{2\theta} \frac{1}{\theta} \cdot \left(\left(-\frac{1}{\theta}\right)\right)^2 dx =$$

$$= \int_{\theta}^{2\theta} \frac{1}{\theta^3}\, dx = \frac{x}{\theta^3}\Big|_{\theta}^{2\theta} = \frac{1}{\theta^2}$$

т.е.

$$P\left(t_1 < \underbrace{\frac{\sqrt{n}\left(\frac{n+1}{2n+1} \cdot \max x_i - \theta\right)}{\frac{n+1}{2n+1} \cdot \max x_i}}_{\tilde{\theta}} < t_2\right)$$

$$-\frac{1{,}96\,\hat{\theta}}{\sqrt{n}} < \tilde{\theta} - \theta < \frac{1{,}96\,\hat{\theta}}{\sqrt{n}}$$

$$\Rightarrow P\left(\frac{n+1}{2n+1} \max x_i \left(1 - \frac{1{,}96}{\sqrt{n}}\right) < \theta < \frac{n+1}{2n+1} \max x_i \left(1 + \frac{1{,}96}{\sqrt{n}}\right)\right) = \beta$$

АСИМПТОТИЧЕСКИЙ ДОВЕРИТЕЛЬНЫЙ ИНТЕРВАЛ О. М. П. (исправл.)