$$D[\tilde{\Theta}] = \frac{\Theta(1-2\Theta)}{2n} \leq \frac{\Theta(1-2\Theta)}{2n} \Rightarrow$$ эффективная

т.о. О.М.М. — не эффективна.

Т.5 $\xi \sim R[\Theta, 2\Theta]$, $\exists \vec{x}_n \Rightarrow \xi \sim \rho(x,\Theta) = \frac{1}{\Theta}\{(\Theta, 2\Theta)\}$

a, b Найдём $\tilde{\Theta}$:

О. М. М.

Возьмём α_1:

$$\alpha_1 = \int_{-\infty}^{+\infty} x\rho dx = \int_{\Theta}^{2\Theta} \frac{x}{\Theta} dx = \frac{3}{2}\Theta$$

$$\alpha_1 = \hat{\alpha}_1 = \overline{x} \Rightarrow \frac{3}{2}\Theta = \overline{x} \Rightarrow \underline{\tilde{\Theta} = \frac{2}{3}\overline{x}} - О.М.М.$$

- проверим несмещ.:

$$M[\tilde{\Theta}] = \frac{2}{3}M[\overline{x}] = \frac{2}{3}M[\xi] = \Theta \Rightarrow \tilde{\Theta} - \text{несмещ.}$$

- проверим состоятельность:

$$D[\tilde{\Theta}] = \frac{4}{9}D[\overline{x}] = \frac{4}{9n}\underbrace{D[\xi]}_{\mu_2} \Subset$$

$$\alpha_2 = \int_{\Theta}^{2\Theta} \frac{x^2}{\Theta} dx = \frac{8t^3}{3t} - \frac{t^3}{3t} = \frac{7}{3}t^2 \Rightarrow \mu_2 = \alpha_2 - \alpha_1^2 = \frac{\Theta^2}{12}$$

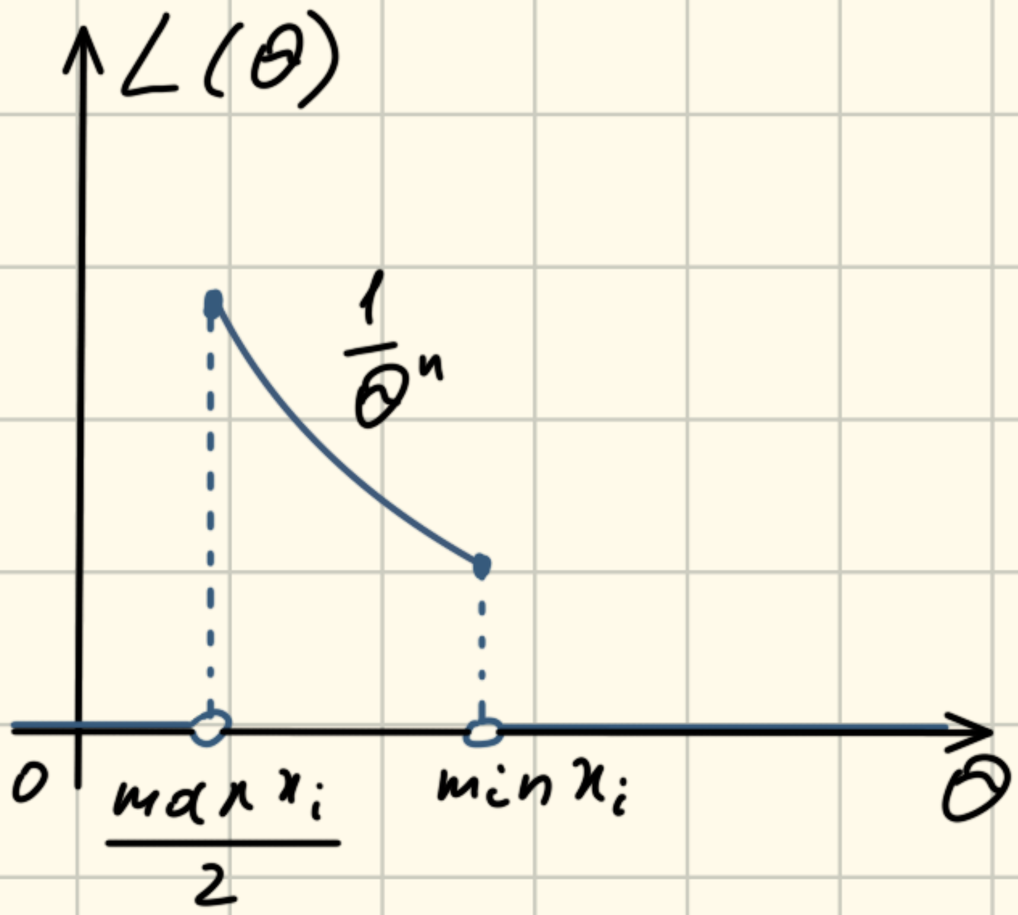
$$\Subset \frac{1}{9n} \cdot \frac{\Theta^2}{12} = \frac{\Theta^2}{27n} \xrightarrow[n\to+\infty]{} 0 \Rightarrow \tilde{\Theta} - \text{состоят.}$$

О.М.П.

функция правдоподобия:

$$L(\Theta) = \prod_{i=1}^{n} \rho(x_i, \Theta) = \frac{1}{\Theta^n}\left\{\forall i \hookrightarrow \begin{matrix} \min\limits_{i=\overline{1,n}} x_i \geq \Theta, \\ \max\limits_{i=\overline{1,n}} x_i \leq 2\Theta \end{matrix}\right\} =$$

$$= \frac{1}{\Theta^n}\left\{\frac{\max x_i}{2} \leq \Theta \leq \min x_i\right\}$$



$$\Rightarrow \max: \quad \widehat{\theta} = \frac{\max x_i}{2}$$

О.М.П.

- проверим несмещ.:

$$M[\widehat{\theta}] = \frac{1}{2} M[\max x_i] = \frac{1}{2} \int_{\theta}^{2\theta} t \cdot \frac{n}{\theta} \left(\frac{t}{\theta} - 1\right)^{n-1} dt =$$

$$= \left\{ \begin{array}{l} u = \frac{t}{\theta} \\ du = \frac{1}{\theta} dt \\ t = u\theta \\ dt = \theta\, du \end{array} \right\} = \frac{\theta}{2} \int_1^2 n u (u-1)^{n-1} du = \frac{\theta}{2} \cdot \widetilde{I} \quad (=)$$

$\int u\, dv = uv - \int v\, du$

$$\widetilde{I} = \int_1^2 u \cdot \underbrace{n(u-1)^{n-1} du}_{d((u-1)^n)} = u(u-1)^n \Big|_1^2 - \int_1^2 (u-1)^n du =$$

$$= \left(u(u-1)^n - \frac{(u-1)^{n+1}}{n+1} \right) \Big|_1^2 = 2 + \frac{1^{n+1}}{n+1}$$

$$(=) \quad \theta - \frac{\theta}{2(n+1)} = \frac{\theta(2n+1)}{2n+2} \Rightarrow \widehat{\theta} - \text{смещенная}$$

Тогда

$$\widehat{\theta}' = \frac{2n+2}{2n+1} \cdot \frac{\max x_i}{2} = \frac{n+1}{2n+1} \max x_i - \text{несмещ}$$

- проверим оптимальность:

$$D[\widehat{\theta}] = M[\widehat{\theta}^2] - M^2[\widehat{\theta}]$$

$$M[\widehat{\theta}^2] = \frac{1}{4} \int_{\theta}^{2\theta} t^2 \cdot \frac{n}{\theta} \left(\frac{t}{\theta} - 1\right)^{n-1} dt = \left\{ u = \frac{t}{\theta} \right\} =$$

$$= \frac{\theta^2}{4} \int_1^2 n \cdot u^2 (u-1)^{n-1} du = \frac{\theta^2}{4} \cdot \widetilde{\widetilde{I}}$$

$$\approx I = \int_1^2 n \cdot u^2 (u-1)^{n-1} du = u^2 (u-1)^n \Big|_1^2 - 2\int_1^2 u(u-1)^n du =$$

$$= 4 - \frac{2}{n+1} \int_1^2 (n+1)\, u (u-1)^n du \ominus$$

$$\approx I = \int_1^2 (n+1)\, u (u-1)^n du = u(u-1)^{n+1} \Big|_1^2 - \frac{(u-1)^{n+2}}{n+2} \Big|_1^2 =$$

$$= 2 - \frac{1^{n+2}}{n+2} = 2 - \frac{1}{n+2}$$

$$\ominus \; 4 - \frac{2}{n+1}\left(2 - \frac{1}{n+2}\right) = \frac{2(2n^2+4n+1)}{(n+2)(n+1)}$$

$$\Rightarrow M[\hat{\theta}^2] = \frac{\theta^2}{\cancel{4}_2} \cdot \frac{\cancel{2}^1 (2n^2+4n+1)}{(n+2)(n+1)} = \frac{2n^2+4n+1}{2(n+2)(n+1)} \cdot \theta^2$$

Тогда

$$D[\hat{\theta}] = \theta^2 \left(\frac{2n^2+4n+1}{2(n+2)(n+1)} - \left(\frac{2n+1}{2n+2}\right)^2 \right) = \theta^2 \cdot \frac{n}{4(n+1)^2(n+2)}$$

(если смотреть состоят. θ, то по определению)

$$D[\hat{\theta}^1] = \cancel{4} \cdot \frac{\cancel{(n+1)^2}}{(2n+1)^2} \cdot \theta^2 \cdot \frac{n}{\cancel{4}\cancel{(n+1)^2}(n+2)} \xrightarrow[n \to \infty]{} 0$$

$\Rightarrow \hat{\theta}^1$ – состоят.

(c) $$D[\hat{\theta}]_{\text{ОММ}} = \frac{\theta^2}{27n} \geq \frac{n\theta^2}{(n+2)(2n+1)^2} = D[\hat{\theta}^1]_{\text{ОМП}}$$

(при $n \geq 3$)

$\Rightarrow$ О.М.П. (исправл.) эффективнее О.М.М.

(d) Пусть $f(\vec{x_n}, \theta) = \dfrac{\max\limits_{i=\overline{1,n}} x_i}{\theta} - 1$

Тогда $P(f < t) = P(\max x_i < \theta t + \theta) =$

$= (F(\Theta t + \Theta))^n$, где $F(x) = \begin{cases} 0, & x < \Theta \\ \frac{x}{\Theta} - 1, & \Theta \le x \le 2\Theta \\ 1, & x > 2\Theta \end{cases}$

т.о. $P(\xi < t) = \begin{cases} 0, & t < 0 \\ t^n, & 0 \le t \le 1 \\ 1, & t > 1 \end{cases}$

$t_1 : F(t_1) = \frac{1-\beta}{2}$

$t_2 : F(t_2) = \frac{1+\beta}{2}$

$\Rightarrow t_{1,2} = \sqrt[n]{\frac{1 \mp \beta}{2}}$

Тогда $P\left(t_1 < \frac{\max x_i}{\Theta} - 1 < t_2\right) = \beta$

т.е. $t_1 + 1 < \frac{\max x_i}{\Theta} < t_2 + 1$

$$\Rightarrow P\left(\frac{\max x_i}{1 + \sqrt[n]{\frac{1+\beta}{2}}} < \Theta < \frac{\max x_i}{1 + \sqrt[n]{\frac{1-\beta}{2}}}\right) = \beta$$

ТОЧНЫЙ ДОВЕРИТЕЛЬНЫЙ ИНТЕРВАЛ

ⓒ а) О.М.М.: $\hat{\Theta} = \frac{2}{3}\overline{x} = g(\hat{\alpha}_1)$

$g(\alpha_1) = \Theta$; $\vec{\nabla} g = g' = \frac{2}{3}$; $K_{11} = \alpha_2 - \alpha_1^2$

$$\frac{\sqrt{n}(\hat{\Theta} - \Theta)}{\sqrt{\left(\frac{2}{3}\right)^2 (\alpha_2 - \alpha_1^2)}} = \frac{3n(\hat{\Theta} - \Theta)}{2\sqrt{s^2(n-1)}} \rightsquigarrow N(0,1),$$ где $S^2 = \frac{n}{n-1} \cdot \hat{\mu}_2$

т.к. норм. распр.: $t_{1,2} = \mp 1{,}96$

$$P\left(t_1 < \frac{3n(\hat{\Theta} - \Theta)}{2\sqrt{s^2(n-1)}} < t_2\right) = \beta$$

тогда $\frac{2t_1\sqrt{s^2(n-1)}}{3n} < \hat{\Theta} - \Theta < \frac{2t_2\sqrt{s^2(n-1)}}{3n}$

$$\Rightarrow P\left(\frac{2}{3}\overline{x} - \frac{3{,}92\sqrt{s^2(n-1)}}{3n} < \Theta < \frac{2}{3}\overline{x} + \frac{3{,}92\sqrt{s^2(n-1)}}{3n}\right) = \beta$$

АСИМПТОТИЧЕСКИЙ ДОВЕРИТЕЛЬНЫЙ ИНТЕРВАЛ О.М.М.