AI1103-Assignment 5

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Download latex-tikz codes from

https://github.com/Umesh−k26/AI−1103/blob/main/Assignment5/assignment5.tex

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Question

Suppose X and Y are independent random variables where Y is symmetric about 0. Let $U = X + Y$ and $V = X - Y$. Then

1) U and V are always independent.
2) U and V have the same distribution.
3) U is always symmetric about 0.
4) V is always symmetric about 0.

Solution

Lemma 0.1. *If a random variable X has a symmetrical distribution about the origin, i.e., $f(-x) = f(x)$, then $\phi_X(t)$ is even function of t.*

Proof.

$$\phi_X(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx \tag{0.0.1}$$

Put $x = -y$

$$\phi_X(t) = \int_{-\infty}^{\infty} e^{-ity} f(-y)\, dy \tag{0.0.2}$$

$$\phi_X(t) = \int_{-\infty}^{\infty} e^{-ity} f(y)\, dy \tag{0.0.3}$$

$$\phi_X(t) = \phi_X(-t) \tag{0.0.4}$$

□

Lemma 0.2. *If X_1 and X_2 are independent random variables, then $\phi_{X_1+X_2}(t) = \phi_{X_1}(t)\,\phi_{X_2}(t)$.*

Proof.

$$\phi_{X_1+X_2}(t) = \mathbb{E}\left(e^{it(X_1+X_2)}\right) \tag{0.0.5}$$

$$= \mathbb{E}\left(e^{itX_1+itX_2}\right) \tag{0.0.6}$$

$$= \mathbb{E}\left(e^{itX_1}e^{itX_2}\right) \tag{0.0.7}$$

$$= \mathbb{E}\left(e^{itX_1}\right)\mathbb{E}\left(e^{itX_2}\right) \tag{0.0.8}$$

$$\phi_{X_1+X_2}(t) = \phi_{X_1}(t)\phi_{X_2}(t) \tag{0.0.9}$$

□

Since Y is symmetric about 0, from **Lemma 0.1** $\phi_Y(t)$ is an even function.

$$\phi_Y(-t) = \phi_Y(t) \tag{0.0.10}$$

Since X and Y are independent random variables, from **Lemma 0.2**,

$$\phi_{X+Y}(t) = \phi_X(t)\,\phi_Y(t) \tag{0.0.11}$$

$$\phi_{X-Y}(t) = \phi_X(t)\,\phi_Y(-t) \tag{0.0.12}$$

$$\phi_{X-Y}(t) = \phi_X(t)\,\phi_Y(t) \qquad \text{(from(0.0.10))} \tag{0.0.13}$$

Let $U = X + Y$ and $V = X - Y$.

$$\phi_U(t) = \phi_X(t)\,\phi_Y(t) \tag{0.0.14}$$

$$\phi_V(t) = \phi_X(t)\,\phi_Y(t) \tag{0.0.15}$$

$$\phi_U(t)\,\phi_V(t) = \phi_X^2(t)\,\phi_Y^2(t) \tag{0.0.16}$$

$$\phi_{U+V}(t) = \phi_{2X}(t) = \phi_X(2t) \tag{0.0.17}$$

Examining each option :

1) If U and V are independent, then

$$\phi_{U+V}(t) = \phi_U(t)\,\phi_V(t)$$

But from (0.0.16) and (0.0.17),

$$\phi_{U+V}(t) \neq \phi_U(t)\,\phi_V(t) \tag{0.0.18}$$

Hence, **Option 1 is incorrect.**

2) From (0.0.14) and (0.0.15),

$$\phi_U(t) = \phi_V(t)$$

$\implies$ U and V have same distribution.
Hence, **Option 2 is correct.**

3)

$$\phi_U(-t) = \phi_X(-t)\,\phi_Y(-t) \quad (0.0.19)$$
$$\phi_U(-t) = \phi_X(-t)\,\phi_Y(t) \quad (0.0.20)$$

$\implies$ $\phi_U(-t) \neq \phi_U(t)$
$\implies$ U is not symmetric about 0.
Hence, **Option 3 is incorrect.**

4) Since U and V have the same distribution, V is also not symmetric about 0.
Hence, **Option 4 is incorrect.**