AI1103-Assignment 5

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Download latex-tikz codes from

https://github.com/Umesh−k26/AI−1103/blob/main/Assignment5/assignment5.tex

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Question

Suppose X and Y are independent random variables where Y is symmetric about 0. Let $U = X + Y$ and $V = X - Y$. Then

1) U and V are always independent.
2) U and V have the same distribution.
3) U is always symmetric about 0.
4) V is always symmetric about 0.

Solution

Y is symmetric about 0,

$$\implies f_Y(-y) = f_Y(y) \tag{0.0.1}$$

$$F_U(u) = \Pr(U \le u) \tag{0.0.2}$$
$$= \Pr(X + Y \le u) \tag{0.0.3}$$
$$= \Pr(X \le u - Y) \tag{0.0.4}$$
$$F_U(u) = \int_{-\infty}^{\infty} f_Y(y) \int_{-\infty}^{u-y} f_X(x)\,dx\,dy \tag{0.0.5}$$

differentiating equation (0.0.5) gives,

$$f_U(u) = \int_{-\infty}^{\infty} f_Y(y) f_X(u - y)\,dy \tag{0.0.6}$$
$$f_U(u) = \int_{-\infty}^{\infty} f_Y(-y) f_X(u + y)\,dy$$
$$\left(\because \int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx\right) \tag{0.0.7}$$

from (0.0.1),

$$f_U(u) = \int_{-\infty}^{\infty} f_Y(y) f_X(u + y)\,dy \tag{0.0.8}$$

$$F_V(v) = \Pr(V \le v) \tag{0.0.9}$$
$$= \Pr(X - Y \le v) \tag{0.0.10}$$
$$= \Pr(X \le v + Y) \tag{0.0.11}$$
$$F_V(v) = \int_{-\infty}^{\infty} f_Y(y) \int_{-\infty}^{v+y} f_X(x)\,dx\,dy \tag{0.0.12}$$

differentiating equation (0.0.12) gives,

$$f_V(v) = \int_{-\infty}^{\infty} f_Y(y) f_X(v + y)\,dy \tag{0.0.13}$$

Option 2

From (0.0.8) and (0.0.13), U and V have same distribution.
$\therefore$ Option 2 is **correct**.

Examining other options :
To disprove, it is enough to examine one case.

Let $X \sim U(0, 1)$ and $Y \sim U(0, 1)$. $R_U \in [0, 2]$ and $R_V \in [-1, 1]$

$$f_U(u) = \begin{cases} u & 0 < u < 1 \\ 2 - u & 1 < u < 2 \end{cases} \tag{0.0.14}$$
$$f_V(v) = \begin{cases} 1 - v & 0 < v < 1 \\ 1 + v & -1 < v < 0 \end{cases} \tag{0.0.15}$$
$$f_{U,V}(u, v) = \frac{1}{2} \qquad (u, v) \in R_{U,V} \tag{0.0.16}$$

(U, V) is uniformly distributed on T.

Option 1

Here, $f_{U,V}(u, v) = \frac{1}{2}$,
And

$$f_U(u) f_V(v) = \begin{cases} u(1 - v) & 0 < u < 1, 0 < v < 1 \\ u(1 + v) & 0 < u < 1, -1 < v < 0 \\ (2 - u)(1 - v) & 1 < u < 2, 0 < v < 1 \\ (2 - u)(1 + v) & 1 < u < 2, -1 < v < 0 \end{cases}$$

$\implies f_{U,V}(u,v) \neq f_U(u) f_V(v)$
$\therefore$ U and V are not always independent.
Hence, Option 1 is **incorrect**.

Option 3

From (0.0.14),

$$f_U(-u) = \begin{cases} -u & 0 < u < 1 \\ 2+u & 1 < u < 2 \end{cases} \tag{0.0.17}$$

But,

$$f_U(-u) \neq f_U(u) \tag{0.0.18}$$

$\therefore$ U is not always symmetric about 0.
Hence, Option 3 is **incorrect**.

Option 4

From (0.0.15),

$$f_V(-v) = \begin{cases} 1+v & 0 < v < 1 \\ 1-v & -1 < v < 0 \end{cases} \tag{0.0.19}$$

But,

$$f_V(-v) \neq f_V(v) \tag{0.0.20}$$

$\therefore$ V is not always symmetric about 0.
Hence, Option 4 is **incorrect**.