

AI1103-Assignment 4

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Download latex-tikz codes from

<https://github.com/Umesh-k26/AI-1103/blob/main/Assignment4/assignment4.tex>

and python codes from

<https://github.com/Umesh-k26/AI-1103/tree/main/Assignment4/codes>

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Let X_1, X_2, X_3, X_4, X_5 be *i.i.d.* random variables having a continuous distribution function. Then

$\Pr(X_1 > X_2 > X_3 > X_4 > X_5 | X_1 = \max(X_1, X_2, X_3, X_4, X_5))$

equals _____.

- (A) $\frac{1}{4}$
- (B) $\frac{1}{5}$
- (C) $\frac{1}{4!}$
- (D) $\frac{1}{5!}$

SOLUTION

Required probability

$$= \Pr(X_1 > X_2 > X_3 > X_4 > X_5 | X_1 = \max(X_1, X_2, X_3, X_4, X_5)) \quad (0.0.1)$$

$$= \Pr(X_2 > X_3 > X_4 > X_5) \quad (0.0.2)$$

$$= \Pr(X_2 = \max(X_2, X_3, X_4, X_5)) \times \Pr(X_3 = \max(X_3, X_4, X_5)) \times \Pr(X_4 = \max(X_4, X_5)) \quad (0.0.3)$$

Since X_1, X_2, X_3, X_4 and X_5 are identical and independently distributed random variables, required probability is

$$= \int_{-\infty}^{\infty} \Pr(X_3, X_4, X_5 < x | X_2 = x) \times \int_{-\infty}^{\infty} \Pr(X_4, X_5 < x | X_3 = x) \quad (0.0.4)$$

$$\times \int_{-\infty}^{\infty} \Pr(X_5 < x | X_4 = x) = \int_{-\infty}^{\infty} F_{X_3}(x) F_{X_4}(x) F_{X_5}(x) f_{X_2}(x) dx \times \int_{-\infty}^{\infty} F_{X_4}(x) F_{X_5}(x) f_{X_3}(x) dx \times \int_{-\infty}^{\infty} F_{X_5}(x) f_{X_4}(x) dx \quad (0.0.5)$$

$$= \int_{-\infty}^{\infty} (\Phi(x))^3 \phi(x) dx \times \int_{-\infty}^{\infty} (\Phi(x))^2 \phi(x) dx \times \int_{-\infty}^{\infty} \Phi(x) \phi(x) dx \quad (0.0.6)$$

Substituting

$$u = \Phi(x) \quad (0.0.7)$$

$$du = \phi(x) dx \quad (0.0.8)$$

Hence, the required probability is

$$= \int_0^1 u^3 du \times \int_0^1 u^2 du \times \int_0^1 u du \quad (0.0.9)$$

$$= \left. \frac{u^4}{4} \right|_0^1 \times \left. \frac{u^3}{3} \right|_0^1 \times \left. \frac{u^2}{2} \right|_0^1 \quad (0.0.10)$$

$$= \frac{1}{4} \times \frac{1}{3} \times \frac{1}{2} \quad (0.0.11)$$

$$= \frac{1}{4!} \quad (0.0.12)$$

Here $\phi(x)$ and $\Phi(x)$ represent the pdf and cdf of respectively of random variables X_1, X_2, X_3, X_4, X_5 .

The correct option is **Option (C)**.