Inner Product Spaces

Contents: Inner product, Inner product spaces
Orthogonal Sets and orthogonal projections
Gram-Schmidts process, QR- Factorization

Q: Define | Explain Inner product and Inner product spaces, with examples
OR
OR State the conditions for a vector space to be normed space with example
Give examples of vector spaces with norms [OR Normed vector space]

Ans: If $V \equiv V(F)$ is a vector space over a field (F reals) then the inner product is a function from $V \times V$ into F $[f : V \times V \to F]$ which assign ~~such that~~ for each pair $\alpha, \beta \in V$ $\exists$ a scalar in F ~~such that~~ denoted by $\langle \alpha, \beta \rangle \equiv (\alpha, \beta)$ in such a way that

(i) $(\alpha, \beta) = (\beta, \alpha)$

(ii) $(a\alpha + b\beta, \gamma) = a(\alpha, \gamma) + b(\beta, \gamma)$

(iii) $(\alpha, \alpha) \geq 0$ and $(\alpha, \alpha) = 0 \Rightarrow \alpha = 0$

$\boxed{\langle \alpha, \beta \rangle \text{ is also known as normed space.}}$

Vector space in which Inner product (or norm) is defined is Inner product OR normed space

Q State and prove Triangle Inequality

OR

If α, β are vectors in an inner product space V ~~such~~ that prove

$$\|\alpha+\beta\| \leq \|\alpha\| + \|\beta\|$$

Note: If $\alpha \in V$ if V is a inner product space then the norm or length of vector α is written as $\|\alpha\|$

Also $\|\alpha\|^2 = (\alpha, \alpha) = \langle \alpha, \alpha \rangle$

Ans to Q

$$\|\alpha+\beta\|^2 = \langle \alpha+\beta, \alpha+\beta \rangle$$

$$= \langle \alpha \; \alpha+\beta \rangle + \langle \beta \; \alpha+\beta \rangle$$

$$= \langle \alpha, \alpha \rangle + \langle \alpha, \beta \rangle + \langle \beta \; \alpha \rangle + \langle \beta \; \beta \rangle$$

$$= \|\alpha\|^2 + 2\langle \alpha, \beta \rangle + \|\beta\|^2$$

$$\leq \|\alpha\|^2 + 2\|\alpha\| \, \|\beta\| + \|\beta\|^2$$

$$\|\alpha+\beta\|^2 \leq [\|\alpha\| + \|\beta\|]^2$$

$$\text{i.e.} \quad \|\alpha+\beta\| \leq \|\alpha\| + \|\beta\|$$

Example 5 & If α and β are vectors in a real innerproduct space and $\alpha+\beta$ is orthogonal to $\alpha-\beta$ then prove that $\|\alpha\| = \|\beta\|$

OR

If two vectors are such that their sum vector and difference vector are orthogonal then prove that the norms of these two vectors should be same

Soln Let α, β ~~&~~ vectors in a real inner product space (given)

also $\alpha+\beta$ orthogonal to $\alpha-\beta$

$\Rightarrow \langle \alpha - \beta, \alpha+\beta \rangle = 0$

$\Rightarrow \langle \alpha\ \alpha+\beta \rangle - \langle \beta\ \alpha+\beta \rangle = 0$

$\Rightarrow \langle \alpha, \alpha \rangle + \langle \alpha\ \beta \rangle - \langle \beta, \alpha \rangle - \langle \beta\ \beta \rangle = 0$

$\Rightarrow \langle \alpha, \alpha \rangle - \langle \beta, \beta \rangle = 0$

$\Rightarrow \|\alpha\|^2 - \|\beta\|^2 = 0$

$\Rightarrow \|\alpha\|^2 = \|\beta\|^2$

$\Rightarrow \|\alpha\| = \|\beta\|$ //

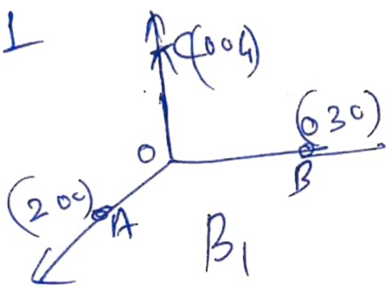
Q1 Define (1) orthogonal, orthonormal set of vectors (2) orthogonal orthonormal Basis

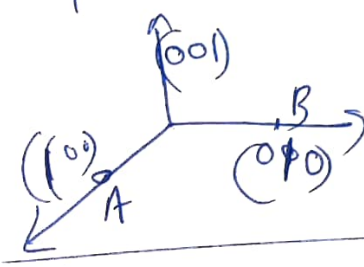
Orthogonal & Orthonormal Basis

Consider $B_1 = \{(2\ 0\ 0), (0, 3, 0), (0, 0, 4)\}$
$= \{u_1, u_2, u_3\}$

$B_2 = \{\underset{e_1}{(1\ 0\ 0)}, \underset{e_2}{(0, 1, 0)}, \underset{e_3}{(0, 0, 1)}\}$

Eg 1





$B_1 = \{p_1, p_2, p_3\}$

Eg 2 $B_1 = \{2 + 0x + 0x^2, \ 0 + 3x + 0x^2, \ 0 + 0 + 4x^2\}$

$B_2 = \{\underset{p_1'}{1 + 0x + 0x^2}, \ \underset{p_2'}{0 + 1x + 0x^2}, \ \underset{p_3'}{0 + 0 + 1x^2}\}$

In example 1 $(u_1 \cdot u_2) = \langle u_1, u_2 \rangle = 2\times 0 + 0\times 3 + 0\times 0 = 0$

||ly $(u_1 \cdot u_3) = \langle u_1, u_3 \rangle = 0$

$(u_2 \cdot u_3) = 0$

that is all products $\underset{=0}{u_1 \cdot u_2}$, $\underset{=0}{u_2 \cdot u_3}$, $\underset{=0}{u_3 \cdot u_1}$ are ZEROs

Just like $i \cdot j = 0$ $j \cdot k = 0$ $k \cdot i = 0$

$B_1 = \{u_1, u_2, u_3\}$ is a basis of R^3 in which all vectors are mutually orthogonal. ∴ B_1 is orthogonal set or Basis

But $\|u_1\| = 2 \neq 1$ $\|u_2\| \neq 1$ $u_3 \neq 1$ Not orth. NORMAL

whereas $B_2 = \{e_1, e_2, e_3\}$ is Orthonormal. $\boxed{\text{Similarly Eg 2}}$

Orthogonal (sets) Basis : ~~A~~ Basis

~~Set~~ ~~Sequence~~ ~~of~~ ~~vecto~~

Set of vectors $B = \{u_1, u_2 \ldots u_k\}$ is ~~case~~

$\boxed{\text{Orthogonal set}}$ If $u_i \cdot u_j = 0$ $\forall\, i = j = 1, 2, k$

If set $B = \{u_1, u_2, \ldots u_k\}$ is a basis of a vector space V then B is Orthogonal Basis

But $\|u_i\|$ need not be 1 = unit

If each of $\boxed{\|u_i\| = 1}$ in the above case then B is known as set of ~~orthogo~~ orthonormal vectors and if Basis of V Orthonormal Basis

Also in this case $u_i \cdot u_i = 1$

3rd example $\boxed{V = M_{2\times 3}} = \left\{ \begin{bmatrix} a & b & c \\ e & f & g \end{bmatrix} \middle/ a, b, c\, e f g \in R \right\}$

Then $B_1 = \left\{ \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 5 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 6 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix} \right\}$

$= \{u_1\ u_2\ u_3\ u_4\ u_5\ u_6\}$

is a $\boxed{\text{orthogonal set}}$ as $u_1 \cdot u_2 = 0$ etc $u_i \cdot u_j = 0$ $\forall\, i j\ i \neq j$

Not Orthonormal

Also $B_1 = \{u_1\ u_2\ u_3, u_4, u_5\ u_6\}$ is a $\boxed{\text{orthogonal Basis}}$ of V

whereas $B_2 = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$

$= \{e_1, e_2\ e_3, e_4, e_5\ e_6\}$

is Orthonormal set as well as Orthonormal Basis of $M_{2\times 3}$ as $e_i \cdot e_j = 0$ $i \neq j$, $e_i \cdot e_j = 1$

Gram-Schmidts process helps to transform Set of Linearly independent vectors (not orthogonal also) to orthogonal/orthonormal vectors

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Q Apply Gram-Schmidts process to transform $S = \{u_1, u_2, u_3\}$, $u_1 = (1, 1, 1)$, $u_2 = (-1, 0, -1)$, $u_3 = (-1, 2, 3)$ to an orthonormal basis of $\mathbb{R}^3$.

Soln Step 1: Let $V_1 = u_1 = (1, 1, 1)$

Step 2: compute V_2 V_3

$$V_2 = u_2 - \left(\frac{u_2 \cdot V_1}{V_1 \cdot V_1}\right) V_1$$

$$= (-1, 0, -1) - \left[\frac{(-1, 0, -1) \cdot (1, 1, 1)}{(1, 1, 1) \cdot (1, 1, 1)}\right](1, 1, 1)$$

$$= (-1, 0, -1) - \left[-\frac{2}{3}(1\ 1\ 1)\right] = (-1\ 0\ -1) + \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$V_2 = \left(-1 + \tfrac{2}{3}, 0 + \tfrac{2}{3}, -1 + \tfrac{2}{3}\right)$$

$$V_2 = \left(-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right)$$ Multiply by 3 just to clear fraction $\boxed{V_2 = (-1, 2, -1)}$

$$V_3 = u_3 - \left(\frac{u_3 \cdot V_1}{V_1 \cdot V_1}\right) V_1 - \left[\frac{u_3 \cdot V_2}{V_2 \cdot V_2}\right] V_2$$

$$= (-1, 2, 3) - \left[\frac{(-1, 2, 3) \cdot (1, 1, 1)}{(1\ 1\ 1) \cdot (1, 1, 1)}\right](1, 1, 1) - \left[\frac{(-1, 2, 3) \cdot (-1, 2, -1)}{(-1, 2, -1) \cdot (-1, 2, -1)}\right](-1, 2, -1)$$

$$= (-1, 2, 3) - \left[\frac{4}{3}(1\ 1\ 1)\right] - \left[\frac{2}{6}(-1, 2, -1)\right]$$

$$= (-1\ 2, 3) - \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right) - \left(-\frac{2}{6}\ \frac{4}{6}, -\frac{2}{6}\right)$$

$$= \left(-1 - \frac{4}{3} + \frac{2}{6},\ 2 - \frac{4}{3} - \frac{4}{6},\ 3 - \frac{4}{3} + \frac{2}{6}\right)$$

$$V_3 = (-2, 0, 2)$$

Now

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$\boxed{T = \{v_1, v_2, v_3\}} = \{(1,1,1), (-1,2,-1), (-2,0,2)\}$

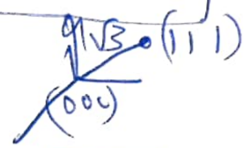
is an orthogonal Basis of $\mathbb{R}^3$

$\boxed{\text{As } v_1\cdot v_2 = 0 \quad v_2\cdot v_3 = 0 \quad v_3\cdot v_1 = 0}$

Step 3 def $\boxed{w_1} = \frac{1}{\|v_1\|} v_1 = \boxed{\frac{(1,1,1)}{\sqrt{3}}}$

Norm

$\|v_1\| = \sqrt{(1\,1\,1)\cdot(1\,1\,1)}$



$\boxed{w_2} = \frac{1}{\|v_2\|}(-1,2,-1) = \boxed{\frac{1}{\sqrt{6}}(-1,2,-1)}$

$w_3 = \frac{1}{\|v_3\|}(-2\,0\,2) = \frac{1}{\sqrt{8}}(-2,0,2)$

$\|v_2\| = \sqrt{(-1)^2+2^2+(-1)^2} = \sqrt{6}$

OR $\boxed{w_3} = \boxed{\left(\frac{-1}{\sqrt{2}} \quad 0 \quad \frac{1}{\sqrt{2}}\right)}$

$\|v_3\| = \sqrt{4+0+4} = \sqrt{8}$

$\boxed{T = \{w_1, w_2, w_3\}}$

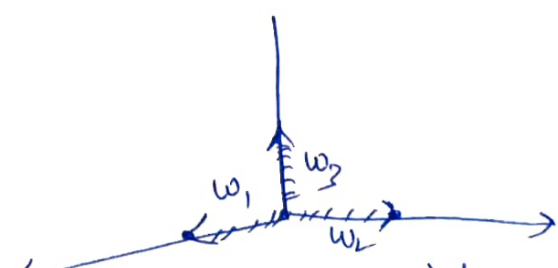
$= \{(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}), (-1/\sqrt{6}, 2/\sqrt{6}, -1/\sqrt{6}), (-1/\sqrt{2}, 0, 1/\sqrt{2})\}$

is an orthonormal of $\mathbb{R}^3$

As $w_1\cdot w_2 = 0 \quad w_1\cdot w_3 = 0 \quad w_2\cdot w_3 = 0$

& $\|w_1\| = 1 \quad \|w_2\| = 1 \quad \|w_3\| = 1$

as w_1, w_2, w_3 unit vector



just like i j k

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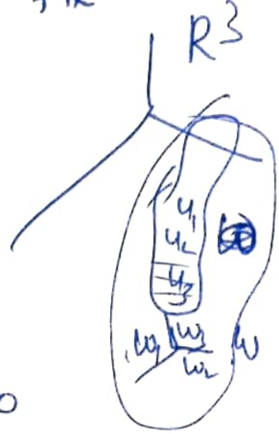

Q. Let $S = \{u_1, u_2, u_3\}$ ~~be the subspace of $\mathbb{R}^3$~~ when

$u_1 = (1, -2, 0, 1)$ $u_2 = (-1, 0, 0, -1)$

and $u_3 = (1, 1, 0, 0)$ be the basis of W, a subspace of $\mathbb{R}^3$

Apply Gram-Schmidt process to to develop an orthonormal basis of W.

Soln. Step 1: Let $u_1 = v_1 = (-1, 2, 0, +1)$

Step 2: Compute V_2, V_3

$$V_2 = u_2 - \left(\frac{u_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1$$

$$= (-1, 0, 0, -1) - \left[\left(\frac{(-1, 0, 0, -1)\cdot(1, -2, 0, 1)}{(1, -2, 0, 1)\cdot(1, -2, 0, 1)}\right)\right](1, -2, 0, 1)$$

$$= (-1, 0, 0, -1) - \left[-\frac{2}{6}(1, -2, 0, 1)\right]$$

$$= (-1, 0, 0, -1) + \frac{2}{6}(1, -2, 0, 1)$$

$$= \left(-1 + \frac{2}{6}, 0 + \frac{4}{6}, 0 + 0, -1 + \frac{2}{6}\right)$$

$$V_2 = \left(-\frac{4}{6}, \frac{4}{6}, 0, -4/6\right)$$

$$V_2 = (-2/3, -2/3, 0, -2/3)$$

Multiply V_2 by 3 just eliminate 3

new $V_2 = (-2, -2, 0, -2)$

$$V_3 = U_3 - \left(\frac{U_3 \cdot V_1}{V_1 \cdot V_1}\right)V_1 - \left(\frac{U_3 \cdot V_2}{V_2 \cdot V_2}\right)V_2$$

$$V_3 = (1,1,0,0) - \left[\frac{(1,1,0,0)\cdot(1,-2,0,1)}{(1,-2,0,1)\cdot(1,-2,0,1)}\right](1,-2,0,1)$$

$$- \left[\frac{(1,1,0,0)\cdot(-2,-2,0,-2)}{(-2,-2,0,-2)\cdot(-2,-2,0,-2)}\right](-2,-2,0,-2)$$

$$= (1,1,0,0) - \left[\left(-\tfrac{1}{6}\right)(1,-2,0,1)\right]$$

$$- \left[\frac{-4}{12}(-2,-2,0,-2)\right]$$

$$V_3 = \left(\tfrac{1}{2}, 0, 0, -\tfrac{1}{2}\right)$$

Multiply 2, new $V_3 = (1,0,0,-1)$

$T^* = \{V_1, V_2, V_3\} = \{\underset{V_1}{(1,-2,0,1)}, \underset{V_2}{(-2,-2,0,-2)}, \underset{V_3}{(1,0,0,-1)}\}$

is an orthogonal Basis of T^*

as $V_1 \cdot V_2 = 0$ $V_2 \cdot V_3 = 0$ $V_3 \cdot V_1 = 0$

But $\|V_1\|$, $\|V_2\|$ $\|V_3\|$ are NOT 1

NOT unit vectors

Let $W_1 = \frac{1}{\|V_1\|}V_1$ $W_2 = \frac{1}{\|V_2\|}V_2$ $W_3 = \frac{1}{\|V_3\|}V_3$

$\therefore T = \{W_1, W_2, W_3\}$

$= \{\left(\frac{1}{\sqrt6}, -\frac{2}{\sqrt6}, 0, \frac{1}{\sqrt6}\right), \left(-\frac{1}{\sqrt3}, -\frac{1}{\sqrt3}, 0, -\frac{1}{\sqrt3}\right), \left(\frac{1}{\sqrt2}, 0, 0, -\frac{1}{\sqrt2}\right)\}$

is an Orthonormal Basis

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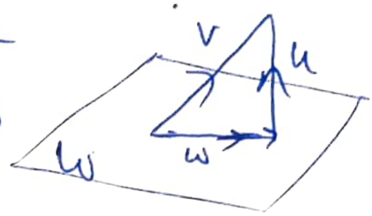
Projections

Q Let W be the two dim subspace of R^3 find the orthogonal projection of $V = (2, 1, 3)$ on W when $\{w_1, w_2\}$ is orthonormal Basis of W with



$$\boxed{\{w_1, w_2\} = \{(2/3, -1/3, -2/3), (1/\sqrt{2}, 0, 1/\sqrt{2})\}}$$

w = proj of V
$u = V - u$
OR $u = V - w$
Or $V = u + w$

Also find vector u orthogonal to every vector in W

Soln $w = \text{proj}_W V = (V \cdot w_1) w_1 + (V \cdot w_2) w_2$

$$w = \{(2\ 1\ 3) \cdot (2/3, -1/3, -2/3)\} (2/3, -1/3, -2/3)$$
$$+ \{(2, 1, 3) \cdot (1/\sqrt{2}, 0, 1/\sqrt{2})\} \{(1/\sqrt{2}, 0, 1/\sqrt{2})\}$$

$$= (-1)(2/3, -1/3, -2/3)$$
$$+ 5/\sqrt{2} (1/\sqrt{2}\ 0\ 1/\sqrt{2})$$

$\frac{5}{2} - \frac{2}{3}$

$\frac{15 - 4}{6}$

$$= (11/6, 1/3, 19/6)$$

Also $u = V - w = (1/6, 2/3, -1/6)$