Unit III

Linear Transformation

Definition: Linear Transformation (L T)

$\boxed{Q}$ Q Define $\boxed{L.T.}$ with an example. Mention one application.

Sol^n Let U & V be two vector spaces over the field F. A **Linear Transformation** from U into V is function T from U into V such that

$$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$$

for all $\alpha, \beta \in U$ and all $a, b \in F$

Examples Prove $T: V(R^3) \longrightarrow V(R^2)$ defined by $\boxed{T(a, b, c) = (a\ b)}$ ★ $\forall\ a, b, c \in R$ (real number) is a LT.

Let $\alpha = (a_1, b_1, c_1)$

$\beta = (a_2, b_2, c_2)$

$\alpha, \beta \in R^3$

★★
$$\boxed{\begin{aligned} T(\alpha) &= T(a_1, b_1, c_1) = (a_1, b_1) \\ T(\beta) &= T(a_2, b_2, c_2) = (a_2, b_2) \end{aligned}}$$

To prove ★ is LT Consider

$$T(a\alpha + b\beta) = T\left(a(a_1, b_1, c_1) + b(a_2, b_2, c_2)\right)$$
$$= T(aa_1 + ba_2, ab_1 + bb_2, ac_1 + bc_2)$$
$$= (aa_1 + ba_2, ab_1 + bb_2)$$
$$= a(a_1, b_1) + b(a_2, b_2)$$

By ★★ $= aT(\alpha) + bT(\beta)$

$\therefore T$ is LT

Example Verify if T given by

$$T : R^3 \to R^3, \quad T(x, y, z) = (x, y, 1)$$

is LT.

let $\alpha = (x_1, y_1, z_1) \Rightarrow T(\alpha) = T(x_1, y_1, z_1) = (x_1, y_1, 1)$

$\beta = (x_2, y_2, z_2) \Rightarrow T(\beta) = T(x_2, y_2, z_2) = (x_2, y_2, 1)$

Consider LHS $T(a\alpha + b\beta)$

$$= T[a(x_1, y_1, z_1) + b(x_2, y_2, z_2)]$$

$$= T[(ax_1 + bx_2,\ ay_1 + by_2,\ az_1 + bz_2)]$$

$$= (ax_1 + bx_2,\ ay_1 + by_2,\ 1) \to *$$

RHS $=$ $aT(\alpha) = aT(x_1, y_1, z_1) = a(x_1, y_1, 1)$

$bT(\beta) = aT(x_2, y_2, z_2) = b(x_2, y_2, 1)$

$aT(\alpha) + bT(\beta) = (ax_1 + bx_2,\ ay_1 + by_2,\ a + b)$ $**$

From $*$, $**$ $T(a\alpha + b\beta) \neq aT(\alpha) + bT(\beta)$

$\therefore$ T is NOT LT

if $(x_1, y_1, z_1) = (1\ 2\ 3)$ $\quad (x_2, y_2, z_2) = (4\ 5\ 6)$

$a = 1 \quad b = 1$

$*$ LHS $= (5\ 7\ 1) \neq (5\ 7\ 2) \equiv$ RHS $**$

Matrix of Linear Transformation

Q: Find the Matrix of LT for the following wrt standard Basis

U → V, Co-Domain

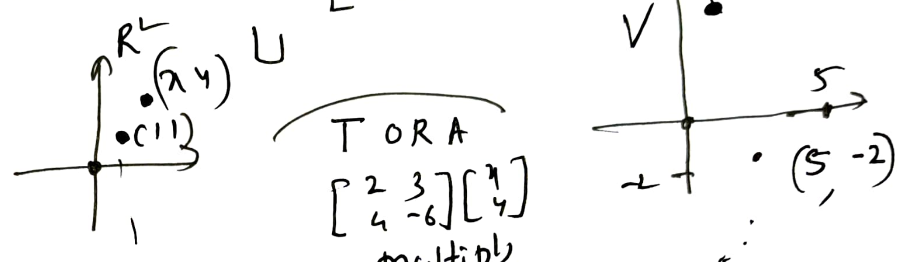
① $T: R^2 \to R^2$, $T(x, y) = (2x+3y,\ 4x-6y)$

Domain

$B = \{(1,0)(0\ 1)\} = \{e_1, e_2\} \equiv$ standard Basis

Matrix of LT is $\begin{bmatrix}2 & 3\\ 4 & -6\end{bmatrix} = A$

Observe



$T(0\ 0) = (0\ 0)$

ZERO of U should map on to ZERO of V

② $T: R^2 \to R^2$ $T(x, y) = (x, y) = (1x+0y,\ 0x+1y)$

Matrix of L.T. is $A = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \equiv$ Identity matrix

∴ $T(x, y) = (x, y)$ is known as Identity operator

"Operator" ~~is~~ term is used for L.T. when ~~the~~ LT is from same vector space V to V (OR R^2 to R^2 or R^3 to R^3 etc)

Matrix of L.T. examples contd wrt std Basis.

Examples

$T(x, y, z) = (x - y + z,\ 2x + 3y + 4z,\ -2x + 6y - 7z)$

Standard Basis $= \beta = \{(1,0,0)\ (0,1,0)\ (0,0,1)\}$

Matrix of LT, $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 4 \\ -2 & 6 & -7 \end{bmatrix}$

same

Note $L\left([x, y, z]_{3\times1}\right) = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 4 \\ -2 & 6 & -7 \end{bmatrix}_{3\times3} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times1} = \begin{bmatrix} x - y + z \\ 2x + 3y + 4z \\ -2x + 6y - 7z \end{bmatrix}$

$\boxed{L([x, y, z]) = A[x, y, z]}$

NOTE

$\Rightarrow$ L and A are same OR Interchangable this is very Important

Instead of L, A can be used in many situations this means a lot

Eg $T(x\ y\ z) = (x\ y\ z) = (x + 0y + 0z,\ 0x + y + 0z,\ 0x + 0y + z)$

$\rightarrow A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Identity operator Special case of LT

Eg $T(x, y, z) = (0\ 0\ 0)$ $A = \begin{bmatrix} & & \end{bmatrix}$

$T(x\ y\ z) = (x + 2y + 3z,\ 2x + 3y + 4z,\ 3x + 4y + 5z)$

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ Observe this is Symmetric matrix

Q. Find the matrix of LT

$T: R^3 \longrightarrow R^3$

$T(a, b, c) = (2b+c, a-4b, 3a)$ wrt

Basis

(i) $B = \{(1,0,0)\ (0,1,0)\ (0,0,1)\}$ standard Basis

(ii) $B' = \{(1,1,1)\ (1,1,0)\ (1,0,0)\}$

Soln Given $T: R^3 \longrightarrow R^3$

$T(a, b, c) = (2b+c, a-4b, 3a)$

$\equiv (0a+2b+c,\ a-4b+0c,\ 3a+0b+0c)$

Matrix of LT T wrt standard Base T is

$\cancel{A_B} \neq\!\!/\ A_B = [T]_B = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -4 & 0 \\ 3 & 0 & 0 \end{bmatrix}$

Think A^{1} and Hence T^{1} ☺

(ii) Given $\boxed{T(a\ b\ c) = (2b+c,\ a-4b,\ 3a)\ *}$

Basis $= B' = \{(1,1,1)\ (1,1,0)\ (1,0,0)\}$ with f_1, f_2, f_3

in $*$ put $(a\ b\ c) = (1,1,1)$

__Step 1__ Find $T(f_1) = T(1\ 1\ 1)$ $T(f_2) = T(1\ 1\ 0)$

$T(f_3) = T(1\ 0\ 0)$ using $*$

$$T(1\ 1\ 1) = (3\ \ -3\ \ 3)$$

$$|||^{ly}\quad T(1\ 1\ 0) = (2\ \ -3\ \ 3)$$

$$T(1\ 0\ 0) = (0\ \ 1\ \ 3)$$

__Step 2__ Express each of $T(1,1,1)$ $T(1,1,0)$ $T(1,0,0)$ in terms of f_1 f_2 f_3 as

$$x(1,1,1) + y(1,1,0) + z(1,0,0) = (3,-3,+3)$$

$$'' \qquad = (2,-3,3)$$

$$= (0,1,3)$$

$$\Rightarrow \quad \begin{array}{l|c|c} x+y+z = 3 & \text{OR} & \text{OR} \\ x+y = -3 & 2 & 0 \\ x = +3 & -3 & 1 \\ & 3 & 3 \end{array}$$

1st

$x+y+z = 3$

$x+y = -3$

$x = +3$

$\Rightarrow$ $x = +3$

$y = -6$

$z = 6$

Similarly

$x+y+z = 2$

$x+y = -3$

$x = 3$

$\Rightarrow$ $x = 3$

$y = -6$

$z = 5$

$x+y+z = 0$

$x+y = 1$

$x = 3$

$\Rightarrow$ $x = 3$

$y = -2$

$z = -1$

Matrix of LT with Basis B' $= [T]_{B'} = [A]_{B'} = \begin{bmatrix} 3 & -6 & 6 \\ 3 & -6 & 5 \\ 3 & -2 & -1 \end{bmatrix}$

Unit III
LA

Q Prove that T is invertible and find $\cancel{A}$ T^{-1} given

$$\boxed{T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)} \quad 1$$

Sol^n wrt Std Base $= \{(1\,0\,0)\ (0\,1\,0), (0\,0\,1)\}$

Matrix of $\boxed{}$ 1

is $A = \begin{bmatrix} 3 & 0 & 1 \\ -2 & 1 & 0 \\ -1 & 2 & 4 \end{bmatrix}$

$|A| = 9 \neq 0$

$\Rightarrow A^{-1}$ exists

$\Rightarrow A$ is invertible

$\Rightarrow T$ is invertible

$$A^{-1} = \frac{1}{9}\begin{bmatrix} 4 & 2 & -1 \\ 8 & 13 & -2 \\ -3 & -6 & 3 \end{bmatrix}$$

$$T^{-1}(x, y, z) = A^{-1}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 4 & 2 & -1 \\ 8 & 13 & -2 \\ -3 & 6 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$T^{-1}(x, y, z) = \left(\frac{4x + 2y - z}{9}, \frac{8x + 13y - 2z}{9}, \frac{-3x + 6y + 3z}{9}\right)$$