Linear Functional

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LA
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Q Define Linear functional.
Demonstrate with two examples
OR
Differentiate between Linear Transformation
Linear Operator and Linear Functional

OR

Q Prove that (Establish / Illustrate / Demonstrate)

(i) the "trace" function is a linear functional from the space of all $n \times n$ matrices over a field of $F = \mathbb{R}$ (real numbers)

(ii) The "function f_i" which assigns to each vector α in vector space V the i^{th} co-ordinate of α relative to the ordered basis β is a linear functional over $F = \mathbb{R}$ (set of reals)

where ~~for~~

$$\boxed{f_i(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n) = a_i}$$

PTO

Linear Functional

Sol^n Let $M_{n\times n} = V$ the set of all $n \times n$ matrices for a +ve integer n be the vector space over a field $F = R =$ real numbers

The the "trace" as L.T.

$$\boxed{L(A) = tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{12} + \dots + a_{1n}}$$

for $A \in M_{n\times n}$, $A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$

If $A \in M_{3\times3}$

Eg $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$trace(A) = tr(A) = 1+5+9 = 15$
$= a_{11} + a_{22} + a_{33}$

Trace of matrix A = sum of diagonal element (It is a real number)

Def^n Linear Functional is Linear Transformation from a vector space V to Field F (say R)

LT → $T: V \to F$

Linear Functional is LT (or function) $\boxed{\text{"Scalar Valued"}}$

Now To prove that $L(A)$ is Linear functional we need to prove its L.T.

Eg let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \alpha$ $\quad \beta = \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 9 & 10 & 11 \end{bmatrix} = \beta$

$a = 10$, $b = 20$

& $a, b \in R$ = real

$Tr(a\alpha + b\beta) = Tr(10A + 20B) = Tr\Big($

$= Tr \begin{bmatrix} 10+80 & 20+100 & 30+120 \\ 40+140 & 50+160 & 60+180 \\ 70+180 & 80+200 & 90+220 \end{bmatrix} = \begin{matrix} 10+80 \\ +50+160 \\ +90+220 \end{matrix} = 610$

Consider

$a\,T(\alpha) + b\,T(\beta)$

$= a\,T(A) + b\,T(B)$

$= 10\ tr(A) + 20\ tra(B)$

$= 10\ tr\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\right) + 20\ tr\begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 9 & 10 & 11 \end{bmatrix}$

$= 10(1+5+9) + 20(4+8+11)$

$= 10 \times 15 + 20(23)$

$= 150 + 460 \implies tr(a\alpha + b\beta) = a\,tr(\alpha) + b\,tr(\beta)$

$= 610$

$\therefore\ tr(A)$ is LT

In general for $a, b \in F$ $A_{n\times n} = [a_{ij}]_{n\times n}$, $B_{n\times n} = [b_{ij}]_{n\times n}$

$= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{22} & \\ a_{n1} & & a_{nn} \end{bmatrix}$, $\begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}$

$tra(aA + bB) = tr\left([a a_{ij}]_{n\times n} + [b b_{ij}]_{n\times n}\right)$

$= \sum_{i=1}^{n} a a_{ii} + b b_{ii})$

$= a \sum_{i=1}^{n} a_{ii} + b \sum b_{ii}$

$tra(aA + bB) = a\ tra A + b\ tra B$

$\therefore\ tr(A)$ is Linear Functional

(ii) $T : V \to F \equiv (R)$

Let $B = \{\alpha_1, \alpha_2 \dots \alpha_n\}$ be an ordered basis. Let $a_1, a_2 \dots a_n \in F$

then given Let $\alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n \in V$

(Then Given) $\Rightarrow f_i = T(\alpha) = T(a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n)$

$f_i = T(\alpha) \Rightarrow = a_i$

Eg $T : R^3 \longrightarrow R$ (say)

$\alpha_1 = (5\ 6\ 7)$ $\quad a_1 = 27$ $\quad$ Some assumption

$\alpha_2 = (10\ 20\ 30)$ $\quad a_2 = 31$

$\alpha_3 = (1, 2, 3)$ $\quad a_3 = 43$

$\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$

$\alpha = \boxed{27(5, 6, 7) + 31(10, 20, 30) + 43(1, 2, 3)}$

$\boxed{f_i(\alpha) = a_i}$

$f_1(\alpha) = a_1 = 27$

$f_2(\alpha) = a_2 = 31$

$f_3(\alpha) = a_3 = 43$

If $\beta = b_1\beta_1 + b_2\beta_2 + b_3\beta_3$
$= \boxed{277(1, 5, 6) + 567(2, 3, 4) + 721(5, 9, 15)}$

$f_2(k_1\alpha + k_2\beta) = f_2(100\alpha + 200\beta)$

$k_1 = 100 \quad k_2 = 200$ say

$= f_2(100\ \boxed{} + 200\ \boxed{})$

$= 100 \times 31 + 200 \times 43$

$= 100\ f_2(\alpha) + 200\ f_2(\beta)$

$= k_1(\alpha) + k_2(\beta)$

In general

For $T: V \longrightarrow R$ V n-dimensional

α for $\alpha = (\alpha_1 \cdots \alpha_n)$ $a_1 \cdots a_n$
$\beta = (\beta_1 \cdots \beta_n)$ $b_1 \cdots b_n \in F$

$$f_i(k_1 a\alpha + k_2 b\beta) = f_i(k_1 a_1\alpha_1 + k_2 b_1\beta_1, \cdots k_1 a_i\alpha_i + k_2 b_i\beta_i \cdots k_1 a_n\alpha_n + k_2 b_n\beta_n)$$

$$= k_1 a_i + k_2 b_i$$

$$= k_1 f_i(\alpha) + k_2 f_i(\beta)$$

Also define Linear Functional

Differentiation LT, Lin operator, Lin functional

L T	Lin Operator	Lin functional
$L: V \rightarrow U$ V & U any two vector space R^n R^3 R^2 R $M_{m\times n}$ polynomial	$L: V \rightarrow U$ $V \equiv U$ any vector space Same	$L: V \rightarrow F$ V any vector space F should be field say Real nos

But all are Linear Transformations

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Linear Transformation

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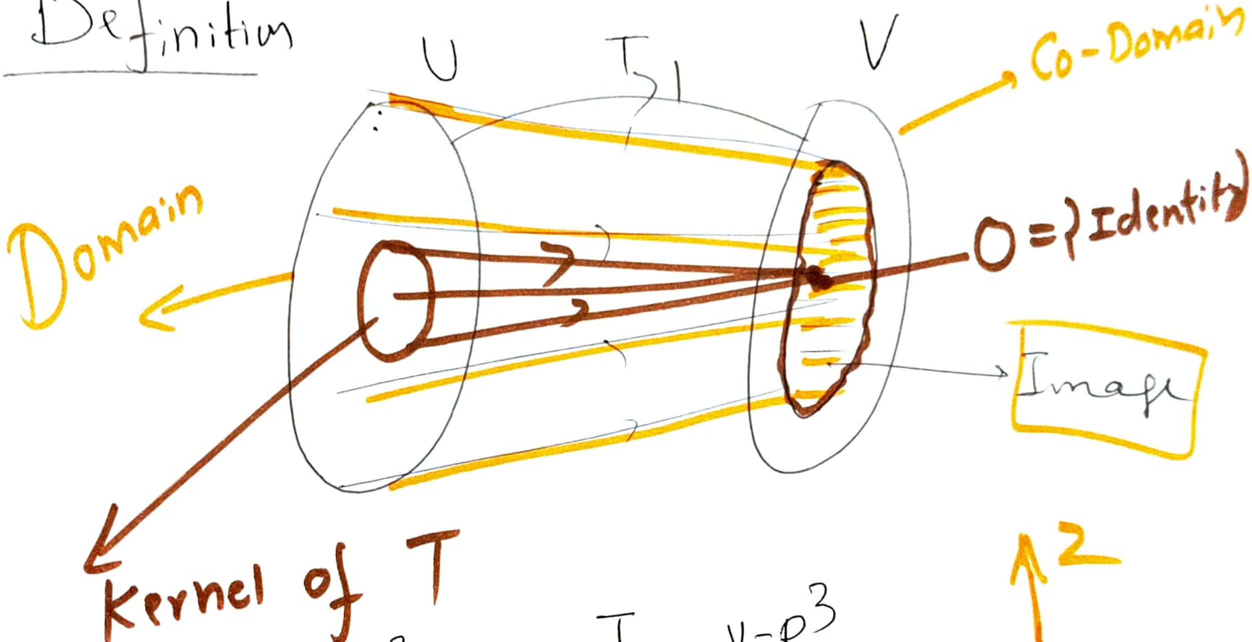
Lin. Algebra

Kernel and Image of Linear Transformation

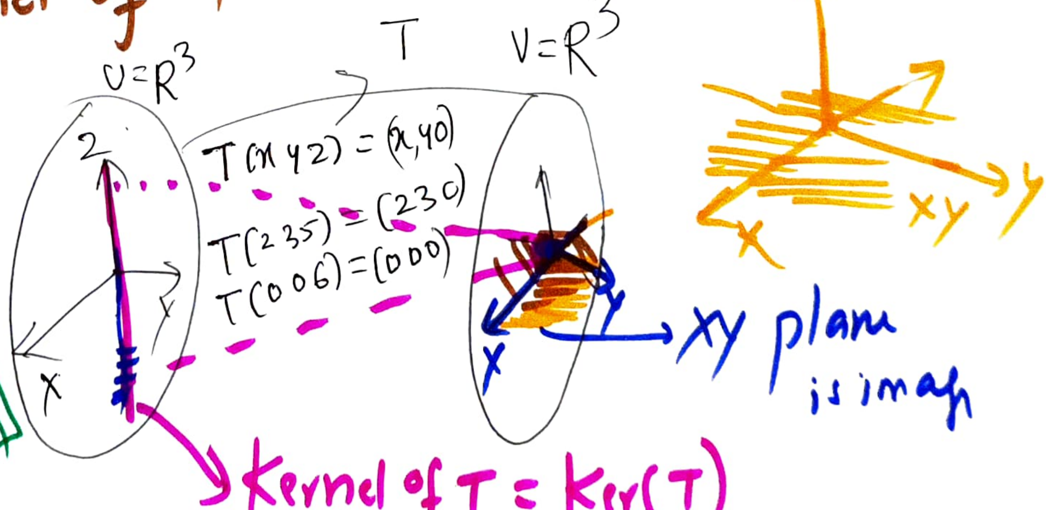
Definition

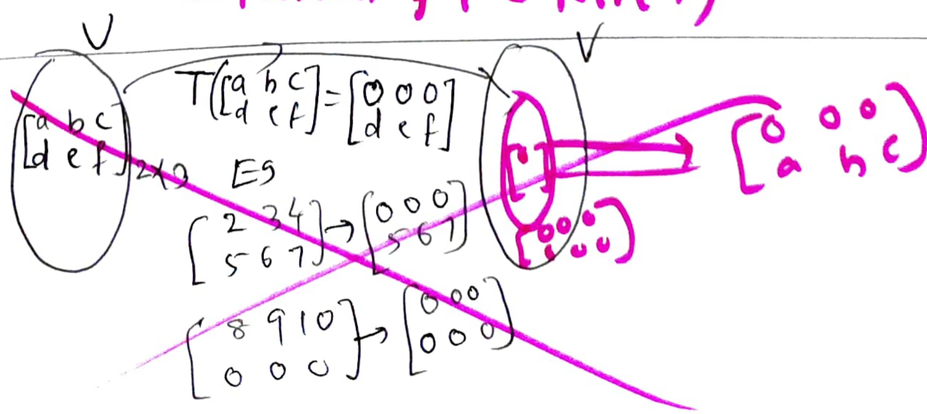


Example

$\dim U = 3$ ✓
$\dim V = 3$ ✓
$\dim \operatorname{Imag} T = 2$ ✓
$\dim \operatorname{Ker} T = 1$ ✓

$\boxed{\dim V = \dim(\operatorname{Ker} T) + \dim(\operatorname{Im} T)}$





Examples

$U = M_{2\times 3} \xrightarrow{T_2} V = M_{2\times 3}$ LA

$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 \\ d & e & f \end{bmatrix}$ → Image

Exam $\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 \\ 5 & 6 & 7 \end{bmatrix}$

$\begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 8 & 9 & 10 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} x & y & z \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ → ZERO of $V = M_{2\times 3}$

$\dim U = 6$

$\dim V = 6$

$\dim(\mathrm{Im}\,T) = 3$

$\dim(\mathrm{Ker}\,T) = 3$

$\dim V = \dim(\mathrm{Ker}\,T) + \dim(\mathrm{Im}\,T)$

Kernel T
$= \mathrm{Ker}(T)$
= Set of all 2x3 matrices with second row zero

$U = P_3(x) \xrightarrow{T_3} V = P_3(x)$

$a_0 + a_1 x + a_2 x^2 + a_3 x^3 \to a_0 + a_1 x^3$ → Image

$2 + 3x + 4x^2 + 5x^3 \to 2 + 0x + 0x^2 + 5x^3$

$0 + 5x + 6x^2 + 0x^3$, $0 - 3x + 8x^2 + 0x^3 \to 0 + 0x + 0x^2 + 0x^3$ → ZERO of V

Kernel

$\dim U = 4$

$\dim V = 4$

$\dim(\mathrm{Im}\,T) = 2$

$\dim(\mathrm{Ker}\,T) = 2$

$\dim V = \dim(\mathrm{Ker}\,T) + \dim \mathrm{Im}\,T$

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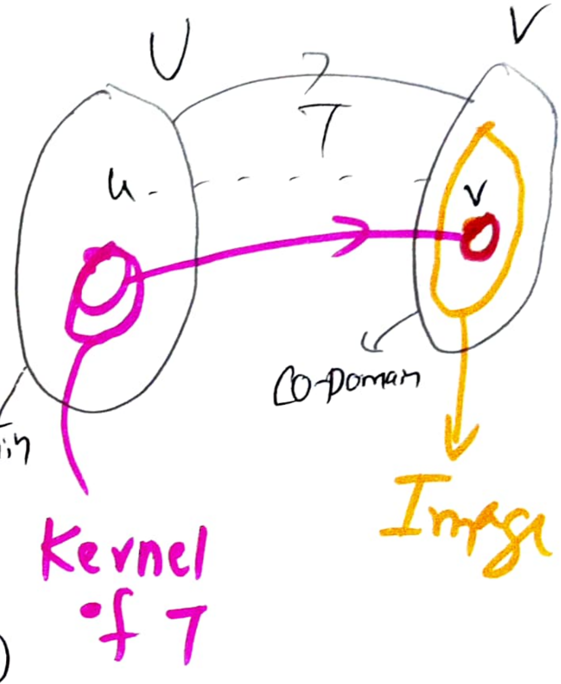
Definition Let $T: U \to V$ be a Linear Transformation

Image: The Image of T is the set of all $v \in V$ such that $f(u) = v$ for some $u \in V$.



Kernel of T $\equiv \ker T$ is the subset of U and is $\{u \in U \mid f(u) = 0 \in V\}$
set of all $u \in U$ which map onto ZERO (Identity) of V.

Q Define Image, Kenel of T.
Give Three Examples and explain with $\dim U$, $\dim V$, $\dim$ Image(T) $\dim \ker(T)$