Symmetric Matrices and Quadratic Forms

→ Diagonalization (1) → Constrained Optimization

→ Quadratic Forms → Singular Value Decomposition (SVD)

~~Gauss Jordan method~~

(1) **Q** Orthogonally diagonalize matrix

$$A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$$ ~~whose characteristic~~

Sol^n characteristic equation

$$\det(A - \lambda I) = \begin{vmatrix} 6-\lambda & -2 & -1 \\ -2 & 6-\lambda & -1 \\ -1 & -1 & 5-\lambda \end{vmatrix}$$

$$= \lambda^3 - (\text{sum of diagonals})\lambda^2 + \begin{bmatrix}\text{sum of minors} \\ \text{of order 2}\end{bmatrix}\lambda - (\det A) = 0$$

$$= \lambda^3 - 17\lambda^2 + 90\lambda - 144 = 0$$

Roots are $\lambda = \underset{\lambda_1}{3}, \underset{\lambda_2}{6}, \underset{\lambda_3}{8}$

For Eigenvectors

For V_1 $\begin{bmatrix} 6-3 & -2 & -1 \\ -2 & 6-3 & -1 \\ -1 & -1 & 5-3 \end{bmatrix}$

$V_1 = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} \simeq \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

V_2 $\begin{bmatrix} 0 & -2 & -1 \\ -2 & 0 & -1 \\ -1 & -1 & -1 \end{bmatrix}$

$V_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

V_3 $\begin{bmatrix} -2 & -2 & -1 \\ -2 & -2 & -1 \\ -1 & -1 & - \end{bmatrix}$

$V_3 = \begin{bmatrix} 5 \\ -5 \\ 0 \end{bmatrix} \simeq \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

Transition matrix

$$P = [V_1 \ V_2 \ V_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & 0 \end{bmatrix}$$

Diagonalization

$$\bar{P}^{1} \equiv \begin{bmatrix} & & \\ & & \end{bmatrix}$$

$$D = \bar{P}^{1} A P = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 8\,0 \end{bmatrix}$$

Ex Compute matrix P which diagonalises

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}.$$

Sol^n Char. Eq^n $\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$ $\quad \lambda^3 - 7\lambda^2 + 36 = 0$

$$\lambda = \underset{\lambda_1}{-2}, \underset{\lambda_2}{3}, \underset{\lambda_3}{6}$$

For V_1 $\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix}$ $V_1 = \begin{bmatrix} +20 \\ 0 \\ -20 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ +1 \end{bmatrix}$

For V_2 $\begin{vmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{vmatrix}$ $V_2 = \begin{bmatrix} -5 \\ +5 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

For V_3 $\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix}$ $V_3 = \begin{bmatrix} 4 \\ -8 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ +2 \\ 1 \end{bmatrix}$

$$P = [V_1 \ V_2 \ V_3] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & +2 \\ +1 & 1 & 1 \end{bmatrix}$$

$$D = \bar{P}^{1} A P = \begin{bmatrix} -2 & & \\ & 3 & \\ & & 6 \end{bmatrix}$$

Constrained Optimizal

Scientific necessity always emphasises to find maximum or minimum value of Quadratic form $Q(X)$ for condition put on X. This constrained optimization has interesting solution and elegant approach in connection with eigenvalues

Consider $\|X\| = 1$, $\|X\|^2 = 1 \cdot X^T X = 1$

OR $x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2 = 1$

For $Q(X)$ without any cross product terms $(x_1 x_2, x_2 x_3, x_3 x_1)$;

Example: Find the maximum and minimum values of $Q(X) = 9x_1^2 + 4x_2^2 + 3x_3^2$ subject to the constraint $X^T X = 1$

Sol^n Since $x_2^2, x_3^2 \geq 0$, $4x_2^2 \leq 9x_2^2$, $3x_3^2 \leq 9x_3^2$

$$\therefore Q(X) = 9x_1^2 + 4x_2^2 + 3x_3^2$$
$$\leq 9x_1^2 + 9x_2^2 + 9x_3^2$$
$$= 9(x_1^2 + x_2^2 + x_3^2)$$

For max $\boxed{Q(X)}_{max} = 9$ as $X^T X = 1$

For min
$9x_1^2 \geq 3x_1^2$
$4x_2^2 \geq 3x_2^2$

$$Q(X) \geq 3x_1^2 + 3x_2^2 + 3x_3^2$$
$$= 3(x_1^2 + x_2^2 + x_3^2)$$
$$\boxed{Q(X)}_{min} = 3$$

Eg If matrix of $Q(X)$ is

$$A = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$ find $Q(X)$ and

hence compute maximum and minimum values of $Q(X)$, $XX^T = 1$

Soln $Q(X)$ of A is $Q(X) = 9x_1^2 + 4x_2^2 + 3x_3^2$

Rest as in previous example

Eg Find Max and Min $Q(X) = 9x_1^2 + 4x_2^2 + 3x_3^2$

subject to $XX^T = 1$ and $X^T u_1 = 0$ $u_1 = (1\ 0\ 0)$

where u_1 is unit eigenvector corresponding to greatest eigenvalue $\lambda_1 = 9$

Soln $X^T u_1 = [x_1, x_2, x_3] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0 \Rightarrow x_1 = 0$

$\Rightarrow XX^T = 1 = [x_1^2 + x_2^2 + x_3^2 = 1]$ reduces to $x_2^2 + x_3^2 = 1$

$Q(X) = 9x_1^2 + 4x_2^2 + 3x_3^2 = 0 + 4x_2^2 + 3x_3^2 \leq 4(x_2^2 + x_3^2)$

$Q(X) \leq 4$ max $\boxed{Q(X) = 4}$

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Quadratic Forms

If $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ or $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ then

Quadratic form ~~x~~ is a second degree function in $x_1, x_2 \ldots x_n$

Ex $Q(X) = x_1^2 + 2x_1x_2 + x_2^2$ | $Q(X) = 9x_1^2 + 3x_2^2 + 2x_3^2$

$Q(X) = 3x_1^2 + 9x_2^2$ | $Q(X) = -4x_1^2 - 3x_2^2 - 5x_3^2$

$Q(X) = 5x_1^2$ | $Q(X) = 5x_1^2 - 2x_2^2 + 3x_3^2$

$Q(X) = -6x_2^2$

$$\boxed{Q(X) = \sum_{i=1}^{n} x_i^2 + \prod_{\substack{i \neq j \\ i=j=1}}^{n} x_i x_j}$$

Is $Q(X) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$ positive definite

Sol Matrix of $Q(X)$, $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

Eigen values are $5, 2, -1$.

$Q(X)$ not +ve definite but "Indefinite"

~~Define~~
Explain, positive definite
negative
~~pve~~ semidefinite
—ve "
with examples, ~~and~~ ~~with~~ Draw diagrams

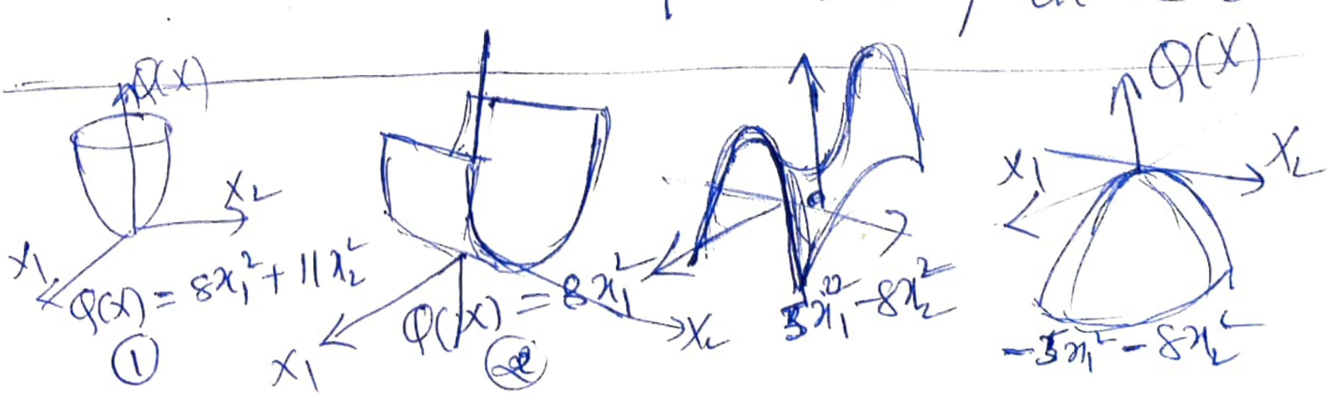
Classify the $Q(x)$

① $Q(x) = 8x_1^2 + 11x_2^2$, ② $Q(x) = 8x_1^2$ ③ $Q(x) = 5x_1^2 - 8x_2^2$

④ $Q(x) = -5x_1^2 - 7x_2^2$

SoM

A quadratic form $Q(X)$	is with matrix of $Q(x)$ as A
① positive definite	all eigen values of A +ve $Q(x)>0$
④ —ve "	" " " of A —ve $Q(x)<0$
② " semidefinite	" " " are ≥ 0 $Q(x) \geq 0$
③ Indefinite	if few eigen values ≤ 0 " " " ≥ 0 $Q(x) \geq 0$ / $Q(x) \leq 0$



Singular value Decomposition

Let A be $m \times n$ matrix with rank r.
Then $\exists$ (1) $m \times n$ matrix Σ with diagonal entries $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$

(2) $m \times m$ orthogonal matrix U

(3) $n \times n$ " " V

such that $\boxed{A = U \Sigma V^T}$.

This is known as SVD or Singular Value Decomposition of A.

~~th~~ $\sqrt{\lambda_1}\ \sqrt{\lambda_2} \cdots \sqrt{\lambda_r}$ non-zero eigenvalue of A
$= \sigma_1 = \sigma_2 \cdots \sigma_r$

Find the Singular value Decomposition of $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$

Sol^n step 1 $A^T = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix}$, $A^TA = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$

Eigen values of A^TA, $\lambda_1 = 360$ $\lambda_2 = 90$, $\lambda_3 = 0$

Respective unit eigen vecter $v_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$ $v_2 = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$ $v_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$

Singular values are roots of eigen values

$\sqrt{\lambda_1} = \sqrt{360} = 6\sqrt{10}$ $\sqrt{\lambda_2} = \sqrt{90} = 3\sqrt{10}$ $\sigma_3 = 0$

$\Rightarrow D = \begin{bmatrix} 6\sqrt{10} & 0 \\ 0 & 3\sqrt{10} \end{bmatrix}$, $\Sigma = \begin{bmatrix} D & 0 \end{bmatrix} = \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}$

using non-zero singular values

Now $u_1 = \frac{1}{\sigma_1} Av_1 = \frac{1}{6\sqrt{10}} \underset{2\times3}{\begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}} =$

$u_1 = \frac{1}{6\sqrt{10}} \underset{2\times1}{\begin{bmatrix} 18 \\ 6 \end{bmatrix}} = \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$

similarly $u_2 = \frac{1}{\sigma_2} Av_2 = \frac{1}{3\sqrt{10}} \begin{bmatrix} 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$

$U = [u_1 \ u_2] = \begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & -3/\sqrt{10} \end{bmatrix}$

$A = U \Sigma V = \begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & -3/\sqrt{10} \end{bmatrix} \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix} \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ 1/3 & -1/3 & -2/3 \\ 1/3 & 2/3 & 1/3 \end{bmatrix}$

Find SVD of

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$$

Sol^n $A^T = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}$ $\quad AA^T = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$

Eigen values of A^TA are $\lambda_1 = 18$ and $\lambda_2 = 0$

& respective e.vectors $v_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$\therefore V = [v_1 \; v_2] = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

Singular values $\sigma_1 = \sqrt{\lambda_1} = 3\sqrt{2}$ $\quad \sigma_2 = 0$

same order of A $\quad \Sigma = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{3\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$, $\boxed{Av_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$

Now $\boxed{x_1 - 2x_2 + 2x_3 = 0}$

For w_1 let $x_3 = 0$, $x_2 = 1$ $\quad w_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

For w_2 $x_2 = 0$ $x_3 = 1$ $x_1 = -2$ $\quad , w_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

$u_2 = \hat{w_1} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$ $\quad u_3 = \hat{w_3} = \begin{bmatrix} -2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix}$

$U = [u_1 \; u_2 \; u_3] = \begin{bmatrix} 1/3 & 2/\sqrt{5} & -2/\sqrt{5} \\ -2/3 & 1/\sqrt{5} & 0 \\ 2/3 & 0 & 1/\sqrt{5} \end{bmatrix}$

$\boxed{A = U\Sigma V} = [\quad][\quad][\quad]$

Q Find all possible Jordan cannonical form of matrix with eigen values 8, 7 with 3 and 2 as multiplicities

Sol^n char eq^n $(\lambda-8)^3(\lambda-7)^2$. matrix order 5

$5 = 3+2 = 2+1+2 = 1+1+1+2 = [2+1]+1+1$
$= [2]+[1]+1+1 = 1+1+1+1+1$

$$\begin{pmatrix} 2 & 1 & & & \\ & 2 & 1 & & \\ & & 2 & & \\ & & & 5 & 1 \\ & & & & 5 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 & & & \\ & 2 & & & \\ & & 2 & & \\ & & & 5 & 1 \\ & & & & 5 \end{pmatrix} \quad \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & 5 & 1 \\ & & & & 5 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & & & \\ & 2 & 1 & & \\ & & 2 & & \\ & & & 5 & \\ & & & & 5 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 & & & \\ & 2 & & & \\ & & 2 & & \\ & & & 5 & \\ & & & & 5 \end{pmatrix}$$

$$\begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & 5 & \\ & & & & 5 \end{pmatrix}$$

$\mathbb{R}^n$, the vector $(\mathbf{u}_j\mathbf{u}_j^T)\mathbf{x}$ is the orthogonal projection of $\mathbf{x}$ onto the subspace spanned by $\mathbf{u}_j$. (See Exercise 35.)

EXAMPLE 4 Construct a spectral decomposition of the matrix A that has the orthogonal diagonalization

$$A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

SOLUTION Denote the columns of P by $\mathbf{u}_1$ and $\mathbf{u}_2$. Then

$$A = 8\mathbf{u}_1\mathbf{u}_1^T + 3\mathbf{u}_2\mathbf{u}_2^T$$

To verify this decomposition of A, compute

$$\mathbf{u}_1\mathbf{u}_1^T = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{bmatrix}$$

$$\mathbf{u}_2\mathbf{u}_2^T = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1/5 & -2/5 \\ -2/5 & 4/5 \end{bmatrix}$$

and

$$8\mathbf{u}_1\mathbf{u}_1^T + 3\mathbf{u}_2\mathbf{u}_2^T = \begin{bmatrix} 32/5 & 16/5 \\ 16/5 & 8/5 \end{bmatrix} + \begin{bmatrix} 3/5 & -6/5 \\ -6/5 & 12/5 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} = A \quad \blacksquare$$