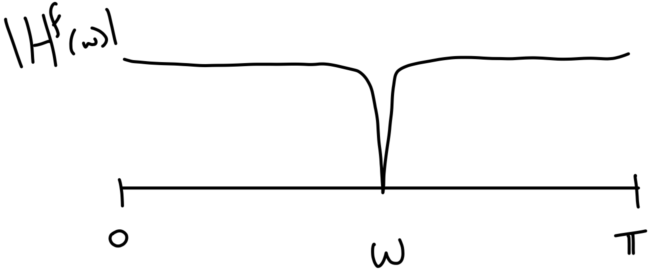
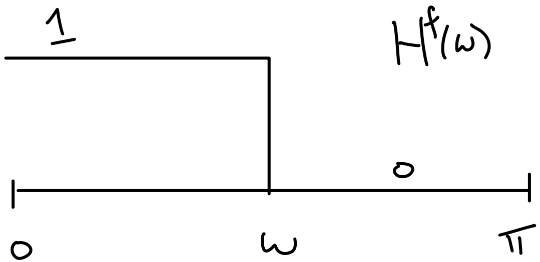


notch filter

we can implement this using a difference eq.



Ideal low-pass filter can not be implemented by any difference equation ☹

Low-pass filter has the effect of smoothing data.

In practice we use Low-pass filters that are not ideal.

Simple low-pass filter:

$$y(n) = \frac{1}{K}\left[x(n) + (n-1) + \cdots + x(n-K) \right]$$

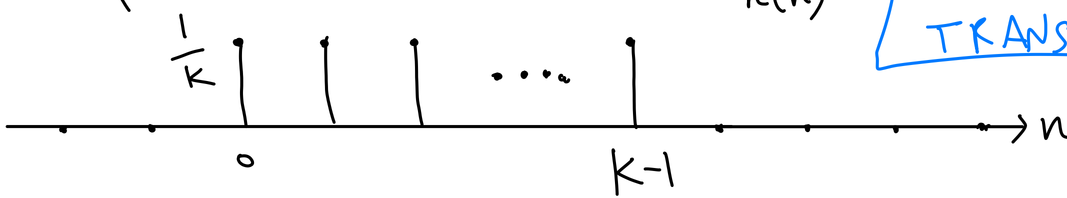
average of the last K input signal values

"K-point moving average" filter.

(FIR filter)

impulse response.

$h(n)$



$$H(z) = \frac{1}{K} + \frac{1}{K}z^{-1} + \frac{1}{K}z^{-2} \cdots + \frac{1}{K}z^{-(K-1)}$$

TRANSFER FUNCTION

DC gain? $\sum_n h(n) = \underbrace{\frac{1}{K} + \cdots \frac{1}{K}}_{K} = 1$ unit dc gain

$H^f(0)$ = also dc gain $\quad H^f(\omega) = \sum_n h(n) e^{-j\omega n}$

$$H^f(0) = \sum_n h(n) e^0 = \sum_n h(n) \checkmark$$

$H(1)$ = also dc gain

$$H(z) = \sum_n h(n) z^{-n}, \quad H(1) = \sum_n h(n) \checkmark$$

Lets sketch the pole-zero diagram...

$$H(z) = \frac{1}{K}\left[1 + z^{-1} + z^{-2} + \cdots + z^{-(K-1)}\right]$$

$$= \frac{1}{K}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \cdots + \frac{1}{z^{K-1}}\right]$$

$$H(z) = \frac{1}{K} \frac{z^{k-1} + z^{k-2} + z^{k-3} + \cdots + 1}{z^{k-1}}$$

$z^{k-1} = 0$, $\rightarrow z = 0$ multiplity $k-1$

poles are all at $z=0$.

To find the zeros of $H(z)$,..

$(z-1)\left[z^{k-1} + z^{k-2} + \cdots + 1 = 0 \right]$ solve for z.

$k-1$ solutions (including potential multiplicities)

Let's multiply both sides by $(z-1)$

$$z^k + 0 \cdot z^{k-1} + 0 \cdot z^{k-2} + \cdots - 1 = 0$$

$$z^k - 1 = 0$$

$z=1$ will be a solution $(1)^k - 1 = 0$ ✓

K solutions, one solution is $z=1$

the <u>other</u> $K-1$ solutions will solutions of

$$z^{K-1} + z^{K-2} + \cdots + 1 = 0.$$

How to solve $z^K - 1 = 0$?

$$z^K = 1$$

$$e^{j2\pi} = 1 \quad \text{also} \quad e^{j4\pi} = 1 \quad \text{also} \quad e^{j6\pi} = 1$$

$$e^{j2\pi m} = 1 \text{ for any integer } m.$$

$$m \in \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

We want to solve $z^k = 1$

Lets write $z^k = e^{j2\pi m}$

$$z = \left(e^{j2\pi m}\right)^{1/k}$$

$$z = e^{jm\frac{2\pi}{k}}, \quad m \in \mathbb{Z}$$

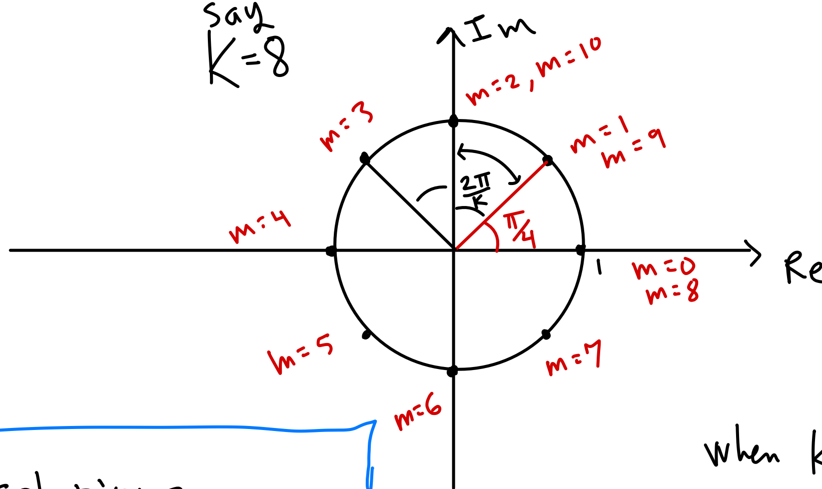
$$z^k = \left(e^{j\frac{2\pi}{k}m}\right)^k = e^{j2\pi m} = 1 \checkmark \qquad m \in \mathbb{Z}$$

$$\boxed{z = e^{jm\frac{2\pi}{k}} \text{ solves } z^k - 1 = 0}$$

how many solutions are there? $m \in \mathbb{Z}$

where are the complex numbers $e^{j\frac{2\pi}{k}m}$ in the complex plane?

Say $K=8$



when $m=0$

$$e^{j\frac{2\pi}{K}m} = e^{0} = 1$$

$$\angle e^{j\frac{2\pi}{K}m} = \frac{2\pi}{K}m$$

angles are integer multiples of $\frac{2\pi}{K}$

when $K=8$, $\frac{2\pi}{K} = \frac{2\pi}{8} = \frac{\pi}{4}$

when $m=1$, $\frac{\pi}{4}$ is the $\angle$

when $m=2$, $\frac{2\pi}{K}m$ is $\frac{\pi}{2}$

K solutions to $z^K - 1 = 0$ given by

$$z = e^{j\frac{2\pi}{K}m}, \quad m = 0, 1, 2, \ldots, K-1$$

"roots of unity"

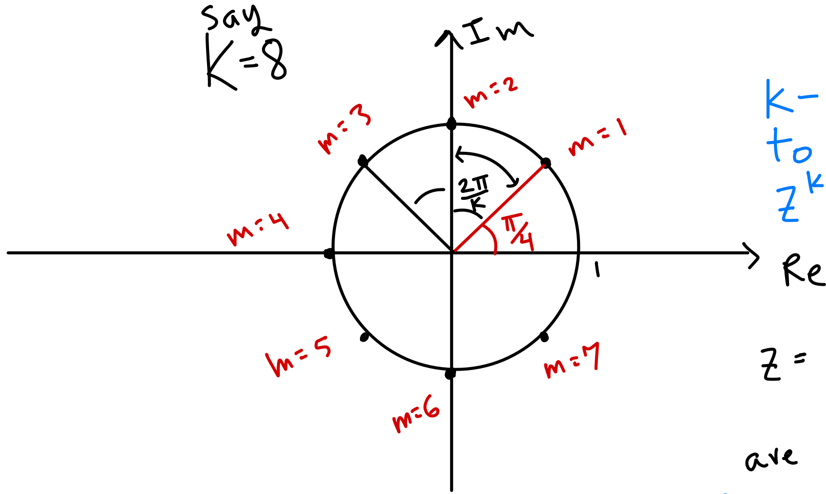
Note

$$e^{j\frac{2\pi}{K}(m+K)} = e^{j\frac{2\pi}{K}m + j2\pi}$$

$$= e^{j\frac{2\pi}{K}m} e^{j2\pi}$$

$$= e^{j\frac{2\pi}{K}m}$$

$$\boxed{e^{j\frac{2\pi}{K}(m+K)} = e^{j\frac{2\pi}{K}m}}$$



K-1 solutions to

$$z^{K-1} + z^{K-2} + \cdots + 1 = 0$$

$$z = e^{j\frac{2\pi}{K}m} \quad m = 1, 2, \ldots, K-1$$

are the solutions to

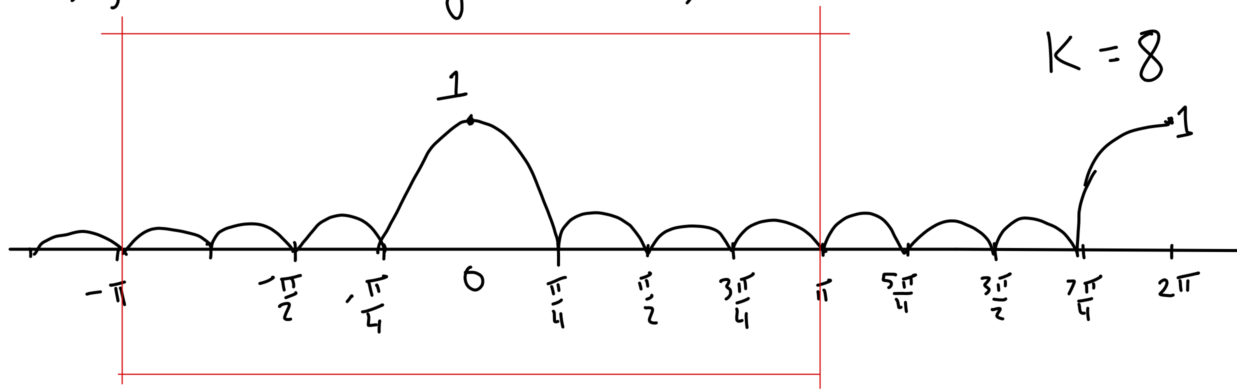
These are the zeros of the K-Point moving average filter.

Frequency Response of K-point moving average filt

Because $H(z)$ has zeros on the unit circle at $z = e^{j\frac{2\pi}{K}m}$, $m=1,\ldots,K-1$, the frequency response will have nulls at $\omega = \frac{2\pi}{K}m$, $m = 1,\ldots,K-1$

Also, since dc gain $= 1$, $H^f(0) = 1$

$K = 8$ $\quad \frac{2\pi}{K} = \frac{\pi}{4}$



$$H^f(2\pi) = H^f(0) = 1$$

$$\star \quad H^f(\omega + 2\pi) = H^f(\omega)$$

$\star$ show derivation for HW exercise.

$$H^f(\omega) = \sum_n h(n)\, e^{-jn\omega}$$

$$= H(e^{j\omega})$$

$$H(z) = \frac{1}{K} \frac{z^{k-1} + z^{k-2} + z^{k-3} + \cdots + 1}{z^{k-1}}$$

$$(z-1)(z^{k-1} + z^{k-2} + \cdots + 1) = z^k - 1$$

$$z^{k-1} + z^{k-2} + \cdots + z + 1 = \frac{z^k - 1}{z - 1}$$

$$H(z) = \frac{1}{K} \frac{z^k - 1}{(z-1)\, z^{k-1}} \qquad H^f(\omega) = \frac{1}{K} \frac{(e^{j\omega})^k - 1}{(e^{j\omega} - 1)(e^{j\omega})^{k-1}}$$

$$H^f(\omega) = \frac{1}{K} \frac{e^{jk\omega} - 1}{e^{j\omega} - 1} \cdot e^{-j(k-1)\omega}$$

$$H^f(\omega) = \frac{1}{K} \frac{e^{j\frac{K}{2}\omega} e^{j\frac{K}{2}\omega} - 1}{e^{j\frac{1}{2}\omega} e^{j\frac{1}{2}\omega} - 1} e^{-j(K-1)\omega}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$e^{j\theta} - e^{-j\theta} = 2j\sin\theta$$

$$H^f(\omega) = \frac{1}{K} \frac{e^{j\frac{K}{2}\omega}\left(e^{j\frac{K}{2}\omega} - e^{-j\frac{K}{2}\omega}\right)}{e^{j\frac{1}{2}\omega}\left(e^{j\frac{1}{2}\omega} - e^{-j\frac{1}{2}\omega}\right)} e^{-j(K-1)\omega}$$

$$H^f(\omega) = \frac{1}{K} e^{j\frac{K-1}{2}\omega} \frac{2j\sin(\frac{K}{2}\omega)}{2j\sin(\frac{1}{2}\omega)} e^{-j(K-1)\omega}$$

$$H^f(\omega) = \frac{1}{K} e^{-j\left(\frac{K-1}{2}\right)\omega} \frac{\sin(\frac{K}{2}\omega)}{\sin(\frac{1}{2}\omega)}$$

Digital sinc function

$$\mathrm{dsinc}(\omega; K) = \frac{1}{K}\frac{\sin\left(\frac{K}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$$

where is $\mathrm{dsinc}(\omega) = 0$?

$$\sin\left(\frac{K}{2}\omega\right) = 0$$

$$\frac{K}{2}\omega = \pi l \quad l \in \mathbb{Z}$$

$$\omega = \frac{2\pi}{K}\cdot l \quad l \in \mathbb{Z}$$

$$\sin\left(\tfrac{1}{2}\omega\right) = 0$$

$$\to \frac{\omega}{2} = \pi l, \ l \in \mathbb{Z}$$

$$\omega = 2\pi l, \ l \in \mathbb{Z}.$$

$\mathrm{dsinc}(\omega) = 0$ at $\omega = \frac{2\pi}{K} l$, $l \in \mathbb{Z}$

except at $\omega = 0$ & integer multiples of 2π.

$$H^f(\omega) = e^{-j\frac{K-1}{2}\omega}\,\mathrm{dsinc}(\omega;K)$$

$$|H^f(\omega)| = \underbrace{\left|e^{-j\frac{K-1}{2}\omega}\right|}_{1}\,|\mathrm{dsinc}(\omega;K)|$$

$$|H^f(\omega)| = |\mathrm{dsinc}(\omega;K)|$$

Discrete-Time Fourier Transform (DTFT)