Exercises in Digital Signal Processing

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February 2, 2021

Contents

1 The Discrete Fourier Transform

1.1 Compute the DFT of the 2-point signal by hand (without a calculator or computer).

$$x = [20, \quad 5]$$

1.2 Compute the DFT of the 4-point signal by hand.

$$x = [3, \quad 2, \quad 5, \quad 1]$$

1.3 The even samples of the DFT of a 9-point real signal $x(n)$ are given by

$$\begin{aligned} X(0) &= 3.1, \\ X(2) &= 2.5 + 4.6\,\mathrm{j}, \\ X(4) &= -1.7 + 5.2\,\mathrm{j}, \\ X(6) &= 9.3 + 6.3\,\mathrm{j}, \\ X(8) &= 5.5 - 8.0\,\mathrm{j}, \end{aligned}$$

Determine the missing odd samples of the DFT. Use the properties of the DFT to solve this problem.

1.4 The DFT of a 5-point signal $x(n)$, $0 \le n \le 4$ is

$$X(k) = [5, \quad 6, \quad 1, \quad 2, \quad 9], \qquad 0 \le k \le 4.$$

A new signal $g(n)$ is defined by

$$g(n) := W_5^{-2\,n}\, x(n), \qquad 0 \le n \le 4.$$

What are the DFT coefficients $G(k)$ of the signal $g(n)$, for $0 \le k \le 4$?

1.5 Compute by hand the circular convolution of the following two 4-point signals (do not use MATLAB, etc.)

$$g = [1, \ 2, \ 1, \ -1]$$

$$h = [0, \ 1/3, \ -1/3, \ 1/3]$$

1.6 What is the circular convolution of the following two sequences?

```
x = [1 2 3 0 0 0 0];
h = [1 2 3 0 0 0 0];
```

1.7 What is the circular convolution of the following two sequences?

```
x = [1 2 3 0 0 0 0];
h = [0 0 0 0 1 2 3];
```

1.8 What is the circular convolution of the following two sequences?

```
h = [1 2 3 0 0 0 0];
x = [1 1 1 1 1 1 1];
```

1.9 What is y after running the following MATLAB commands? Do not use MATLAB to get your answer and do not explicitly compute the DFT; instead use the properties of the DFT.

```
clear
x = [1 2 0 -1];
g = [1 0 0 2 1];
X = fft([x 0 0 0]);
G = fft([g 0 0]);
Y = X.*G;
y = ifft(Y);
```

1.10 What is y after running the following MATLAB commands?

```
clear
x = [1 2 3 4];
g = [5 6 7 8];
X = fft([x 0 0 0]);
G = fft([g 0 0]);
Y = X.*G;
y = ifft(Y);
```

1.11 Suppose you run the following DFT example in MATLAB.

```
x1 = [1 2 3 4 3 2 1]';
x2 = [1 2 3 4 4 3 2]';
x3 = [1 2 3 4 -4 -3 -2]';
x4 = [0 2 3 4 -4 -3 -2]';

X1 = fft(x1)
X2 = fft(x2)
X3 = fft(x3)
X4 = fft(x4)
```

This vectors X1, X2, X3, X4 are shown below out of order. Using properties of the DFT, match them to vectors A, B, C, D, by completing the table. Explain how you get your answers. Do not use MATLAB for this problem and do not explicitly compute the DFT; instead use the properties of the DFT.

Signal	DFT
x1	
x2	
x3	
x4	

```
A:
  16.0000
  -4.5489 - 2.1906i
   0.1920 + 0.2408i
  -0.1431 - 0.6270i
  -0.1431 + 0.6270i
   0.1920 - 0.2408i
  -4.5489 + 2.1906i

C:
  19.0000
  -5.0489
  -0.3080
  -0.6431
  -0.6431
  -0.3080
  -5.0489

B:
        0
        0 -12.4480i
        0 + 4.9582i
        0 - 4.8440i
        0 + 4.8440i
        0 - 4.9582i
        0 +12.4480i

D:
   1.0000
   1.0000 -12.4480i
   1.0000 + 4.9582i
   1.0000 - 4.8440i
   1.0000 + 4.8440i
   1.0000 - 4.9582i
   1.0000 +12.4480i
```

1.12 Let $x(n)$, for $0 \le n \le N-1$, be a *real* N-point signal. The DFT coefficients are $X(k)$, for $0 \le k \le N-1$.

(a) Show that $X(0)$ is a real number.

(b) Assume N is an even number. Is $X(N/2)$ real, imaginary, or a generic complex number? Show your explanation.

1.13 Consider the following 8-point signals, $0 \le n \le 7$.

(a) $[1, 1, 1, 0, 0, 0, 1, 1]$

(b) $[1, 1, 0, 0, 0, 0, -1, -1]$

(c) $[0, 1, 1, 0, 0, 0, -1, -1]$

(d) $[0, 1, 1, 0, 0, 0, 1, 1]$

Which of these signals have a real-valued 8-point DFT? Which of these signals have a imaginary-valued 8-point DFT? Do not use MATLAB or any computer to solve this problem and do not explicitly compute the DFT; instead use the properties of the DFT.

1.14 Consider the following 9-point signals, $0 \le n \le 8$.

(a) [3, 2, 1, 0, 0, 0, 0, 2, 1]
(b) [3, 2, 1, 0, 0, 0, 0, −2, −1]
(c) [3, 2, 1, 0, 0, 0, 0, −2, −1]
(d) [0, 2, 1, 0, 0, 0, 0, −2, −1]
(e) [0, 2, 1, 0, 0, 0, 0, 2, 1]
(f) [3, 2, 1, 0, 0, 0, 0, 1, 2]
(g) [3, 2, 1, 0, 0, 0, 0, −1, −2]
(h) [0, 2, 1, 0, 0, 0, 0, −1, −2]
(i) [0, 2, 1, 0, 0, 0, 0, 1, 2]

Which of these signals have a real-valued 9-point DFT? Which of these signals have an imaginary-valued 9-point DFT? Do not use MATLAB or any computer to solve this problem and do not explicitly compute the DFT; instead use the properties of the DFT.

1.15 Consider the N-point sequence $x(n)$ defined for $n = 0, \dots, N-1$. Show that if $x(n)$ is real, then the N-point DFT $X(k)$ satisfies

$$X(N-k) = \overline{X(k)}, \quad k = 0, \dots, N-1, \tag{1}$$

where the overline notation denotes the complex conjugate.

1.16 **DFT and circular convolution.** Verify the circular convolution property of the DFT in Matlab. Write two Matlab functions to compute the circular convolution of two sequences of equal length. One function should use the DFT (`fft` in Matlab), the other function should compute the circular convolution directly not using the DFT. Verify that both Matlab functions give the same results.

Hand in a hard copy of both functions, and an example verifying they give the same results (you might use the `diary` command).

1.17 **DFT and linear convolution.** Write a Matlab function that uses the DFT (`fft`) to compute the linear convolution of two sequences that are not necessarily of the same length. (Use zero-padding.) Verify that it works correctly by comparing the results of your function with the Matlab command `conv`.

1.18 The 13-point DFT of a 13-point signal $x(n)$ is given by

$$X(k) = [0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0], \qquad k = 0, \dots, 12$$

Give a formula for $x(n)$ in terms of trigonometric functions. You answer should not contain j.

1.19 What is the inverse DFT of the following N-point discrete-time sequence $X(k)$?

$$X(k) = [0,\ 1,\ \underbrace{0,\ 0,\ \dots,\ 0}_{N-3 \text{ terms}},\ 1], \qquad k = 0, 1, \dots, N-1.$$

Your answer should be a formula for $x(n)$ in terms of n and N. Is $x(n)$ real-valued? If so, simplify your answer for $x(n)$ so that it does not contain j.

1.20 What is the inverse DFT of the following N-point discrete-time sequence $X(k)$?

$$X(k) = [0,\ \mathrm{j},\ \underbrace{0,\ 0,\ \dots,\ 0}_{N-3 \text{ terms}},\ -\mathrm{j}], \qquad k = 0, 1, \dots, N-1.$$

Your answer should be a formula for $x(n)$ in terms of n and N. Is $x(n)$ real-valued? If so, simplify your answer for $x(n)$ so that it does not contain j.

1.21 The DFT of an N-point signal $x(n)$ is given by

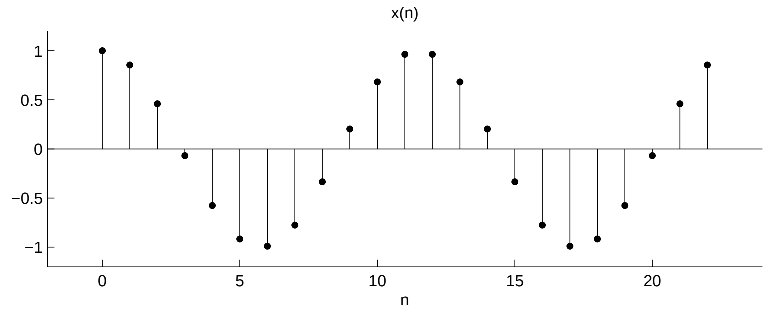
$$X(k) = [0,\ \mathrm{e}^{\mathrm{j}\theta},\ \underbrace{0,\ 0,\ \dots,\ 0}_{N-3 \text{ values}},\ \mathrm{e}^{-\mathrm{j}\theta}], \quad 0 \le k \le N-1.$$

Find the N-point signal $x(n)$. Simplify your answer. Is $x(n)$ real? If so, express your answer without using j.

1.22 The 23-point signal $x(n)$ is two cycles of a cosine signal,

$$\cos\left(2\pi \frac{2}{N} n\right), \qquad 0 \le n \le N-1, \quad \text{with } N = 23.$$

The signal $x(n)$ is illustrated in the figure.



(a) The 23-point DFT of $x(n)$ is computed. The DFT coefficients are denoted $X(k)$. Accurately sketch $|X(k)|$ for $0 \le k \le N-1$.

(b) A 23-point signal $y(n)$ is obtained by circularly shifting $x(n)$ by 3 samples to the right. The 23 DFT coefficients of $y(n)$ are denoted $Y(k)$. Accurately sketch $|Y(k)|$ for $0 \le k \le N-1$.

1.23 Find the DFT of the N-point discrete-time signal,

$$x(n) = \cos\Big(\frac{2\pi}{N} n + \theta\Big), \qquad n = 0, 1, \dots, N-1.$$

1.24 Sketch the signal

```
x = sin(2*pi*[0:7]/8);
```

and find the DFT of x. Do not use direct computation of the DFT.

1.25 The 20-point signal $x(n)$ is given by

$$x(n) = \sin\left(\frac{2\,\pi}{N}\,2\,n\right), \qquad 0 \le n \le N-1$$

where $N = 20$.

(a) Roughly sketch the signal for $0 \le n \le 19$. Do not explicitly calculate the values.

(b) Find the DFT coefficients of the signal. That means, find $X(k)$ for $0 \le k \le 19$. Show the derivation of your answer. You should not use direct computation of the DFT.

1.26 (a) Find the DFT of the N-point signal,

$$x(n) = \cos\left(\frac{4\pi}{N} n\right) \qquad n = 0, \dots, N-1.$$

Also, sketch $x(n)$ and $X(k)$.

(b) Find the DFT of the N-point signal,

$$x(n) = \cos\left(\frac{4\pi}{N} n - \frac{\pi}{4}\right) \qquad n = 0, \dots, N-1$$

1.27 (a) Find the DFT of the N-point signal,

$$x(n) = \sin\left(\frac{4\pi}{N} n\right) \qquad n = 0, \dots, N-1.$$

Also, sketch $x(n)$ and $X(k)$.

(b) Find the DFT of the N-point signal,

$$x(n) = \sin\left(\frac{4\pi}{N} n - \theta\right) \qquad n = 0, \dots, N-1$$

1.28 **Sinusoids.**

(a) What is the 30-point signal $x(n)$ that has the following DFT? (Provide a mathematical formula.)

$$X(k) = \begin{cases} 1, & k = 4 \\ 0, & k \neq 4 \end{cases} \qquad \text{for } 0 \le k < 30$$

(b) What is the DFT of the N-point signal,

$$x(n) = \cos\left(\frac{2\,\pi}{N}\,P\,n\right), \qquad 0 \le n \le N-1$$

where P is an integer between 0 and N? Hint: using the previous part. Show the derivation of your answer.

1.29 Find the N-point DFT of the N-point sequence:

$$x = [\underbrace{1, \quad 0, \quad -1, \quad 0}_{\text{repeats}}, \quad \dots, \quad 1, \quad 0, \quad -1, \quad 0]$$

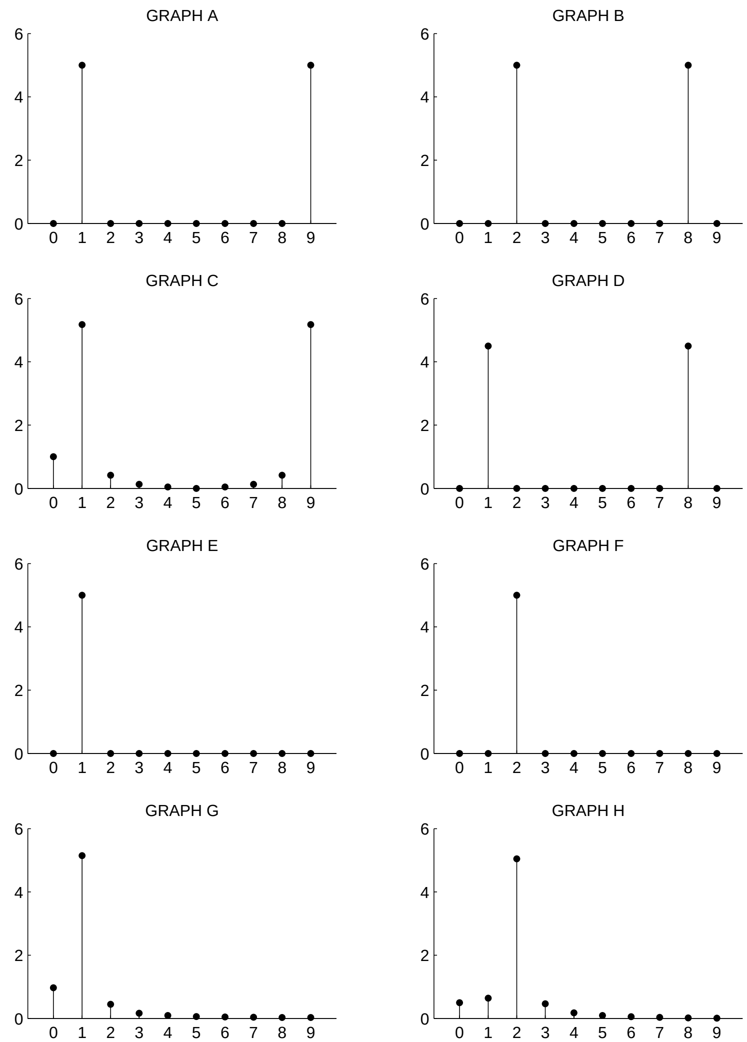
The sequence is made of K periods of the 4-point sequence (1, 0, -1, 0). The length of the sequence is $N = 4K$. Simplify your answer.

1.30 The following MATLAB commands define two ten-point signals and the DFT of each.

```
x1 = cos([0:9]/9*2*pi);
x2 = cos([0:9]/10*2*pi);
X1 = fft(x1);
X2 = fft(x2);
```

(a) Roughly sketch each of the two signals, highlighting the distinction between them.

(b) Which of the following four graphs illustrates the DFT $|X_1(k)|$? Explain your answer. Which graph illustrates the DFT $|X_2(k)|$?



Give your answer by completing the table and provide an explanation for your answer. Note that there are more graphs shown than needed. Your answer will use only two of the eight graphs. Provide a brief explanation for your answer. You should be able to do this exercise without using MATLAB (this was an exam question).

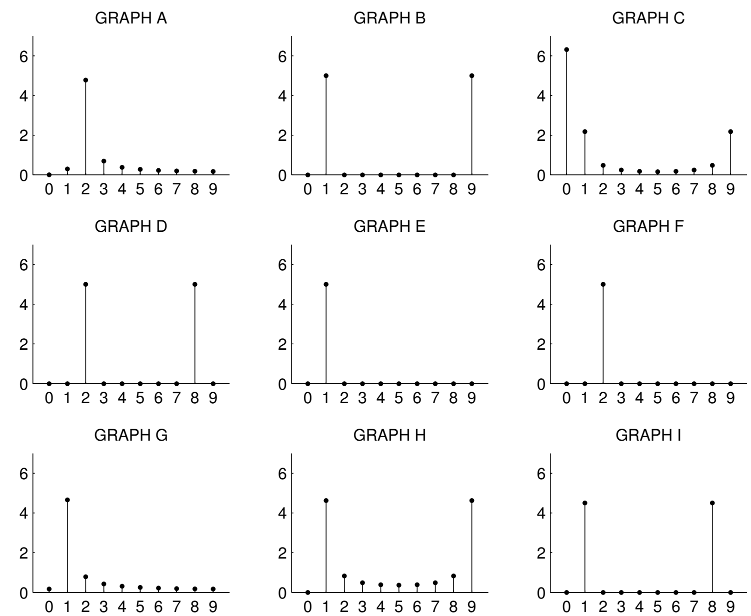
DFT	Graph
X1	
X2	

1.31 The following MATLAB commands define three ten-point signals and the DFT of each.

```
x1 = sin([0:9]/10*pi);
x2 = sin([0:9]/9*2*pi);
x3 = sin([0:9]/10*2*pi);
X1 = fft(x1);
X2 = fft(x2);
X3 = fft(x3);
```

(a) Roughly sketch each of the three signals, highlighting the values of the first and last values of each signal

(b) Which of the following graphs illustrates the DFT $|X(k)|$ of each signal?



Give your answer by completing the table and provide an explanation for your answer. Note that there are more graphs shown than needed. Your answer will use only three of the graphs. Provide a brief explanation for your answer.

DFT	Graph
X1	
X2	
X3	

1.32 What is the result of the following MATLAB commands? Explain your answer.

```
>> x = ones(1,10);
>> X = fft(x)
```

1.33 Let N be an even integer. Let the N-point signal $x(n)$ be defined for $0 \le n \le N-1$ as

$$x(n) = \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd.} \end{cases}$$

Find the DFT of the N-point discrete-time signal $x(n)$. Your answer should be simplified. If $X(k)$ is real, write it without using j.

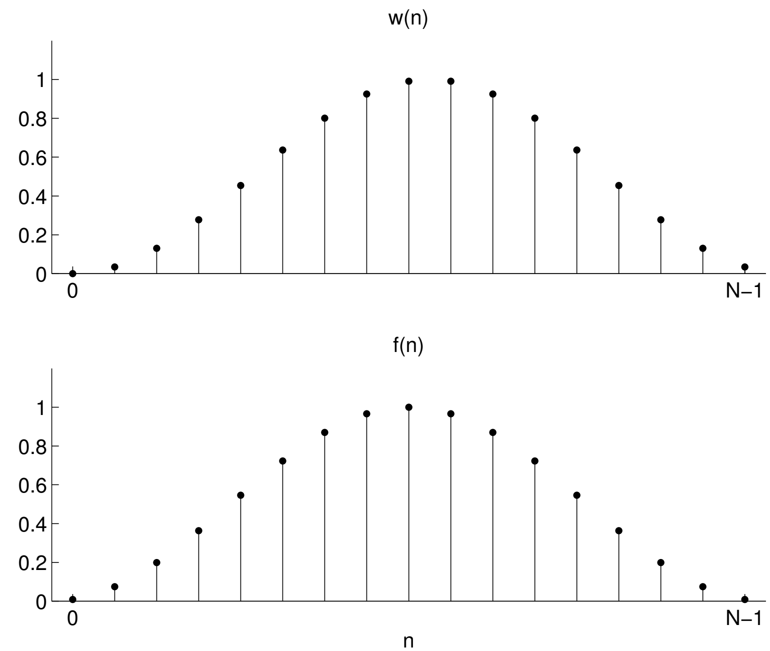
1.34 What is the result of the following MATLAB command? Do not use direct computation of the DFT. Show your derivation or explain your answer.

```
>> fft([1  0  -1  0  1  0  -1  0  1  0  -1  0])
```

1.35 Two N-point sequences are created in Matlab using the commands:

```
n = 0:N-1;
w = 0.5 - 0.5 * cos(2*pi*n/N);
f = 0.5 - 0.5 * cos(2*pi*(n+0.5)/N);
```

The sequences are illustrated in the figure:



(a) Find the N-point DFT $W(k)$ of the sequence $w(n)$. Also, sketch $W(k)$.

(b) Find the N-point DFT $F(k)$ of the sequence $f(n)$. Also, sketch $|F(k)|$.

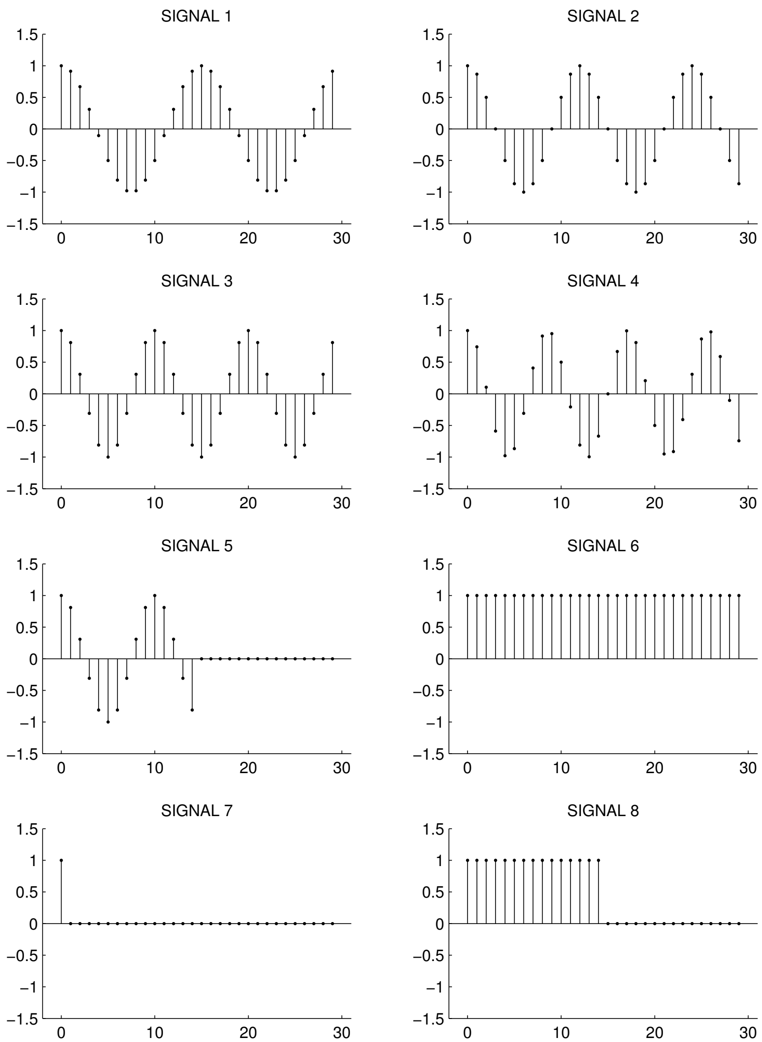
(c) The signal $x(t) = \cos(2\pi t)$ is sampled at interval $T = 0.01$ second for $N = 100$ samples. The frequency of the signal is then measured using a windowed DFT (with no zero padding). The window $w(n)$ is used. Find the resulting N-point DFT.

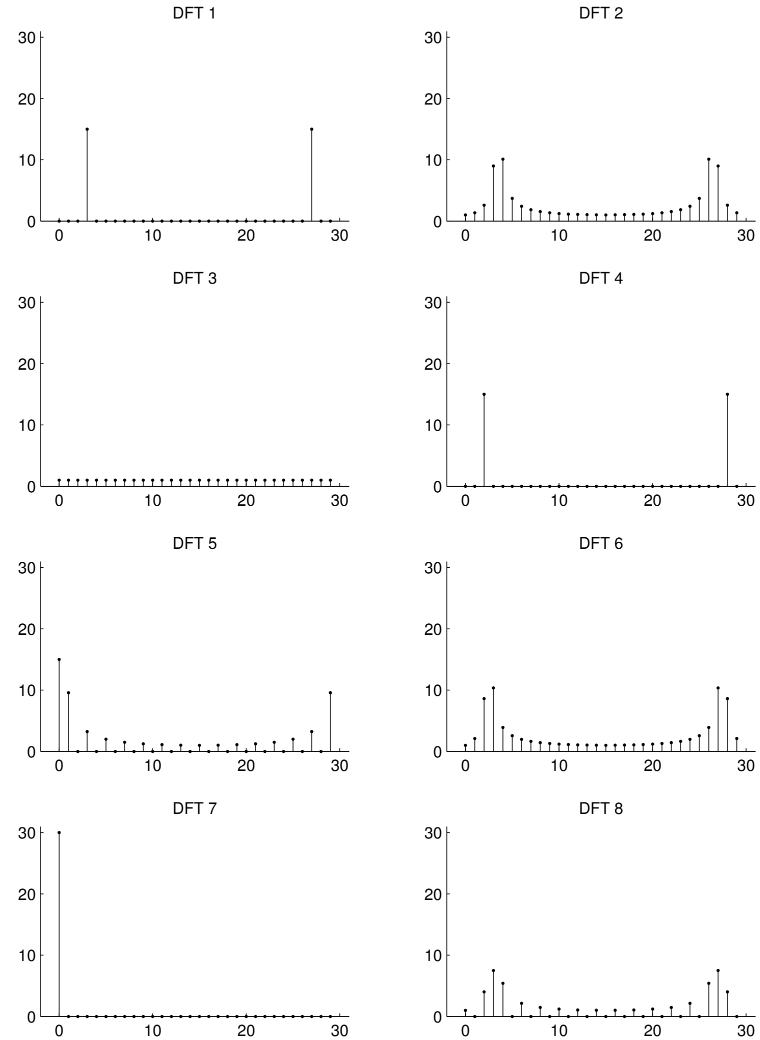
1.36 Let N be an even integer. What is the DFT of the following N-point discrete-time signal x?

$$x = [\underbrace{0,\ 0,\ \ldots,\ 0}_{\frac{N}{2}\text{ terms}},\ 1,\ \underbrace{0,\ 0,\ \ldots,\ 0}_{\frac{N}{2}-1\text{ terms}}]$$

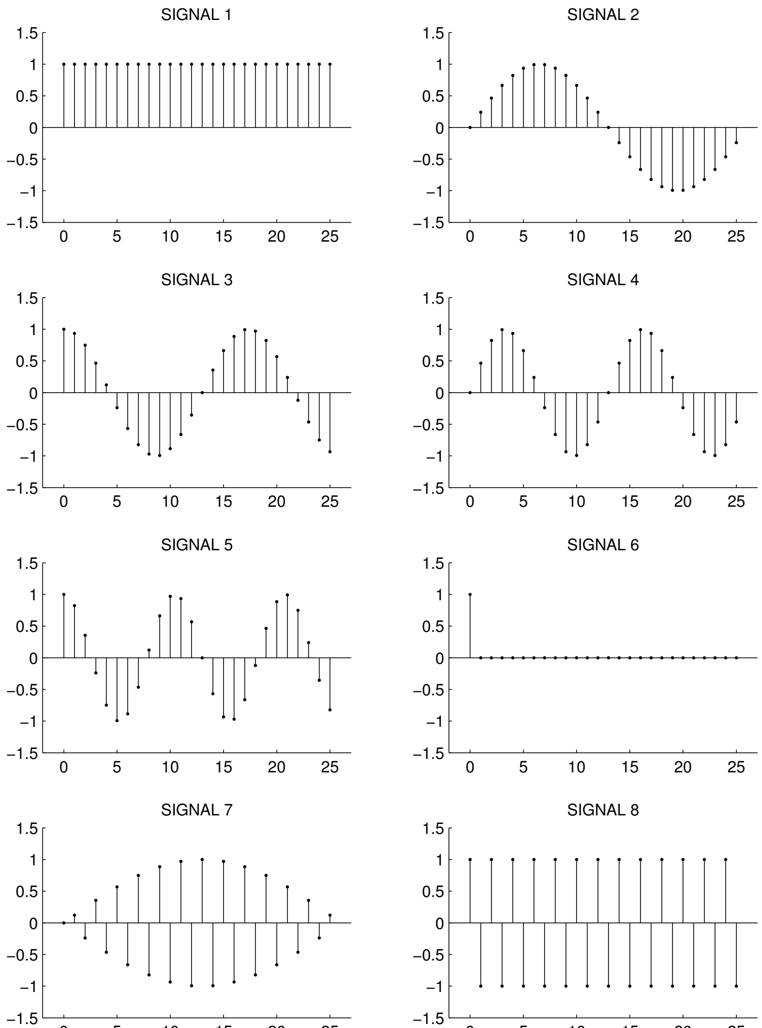
Give the DFT when $N = 10$.

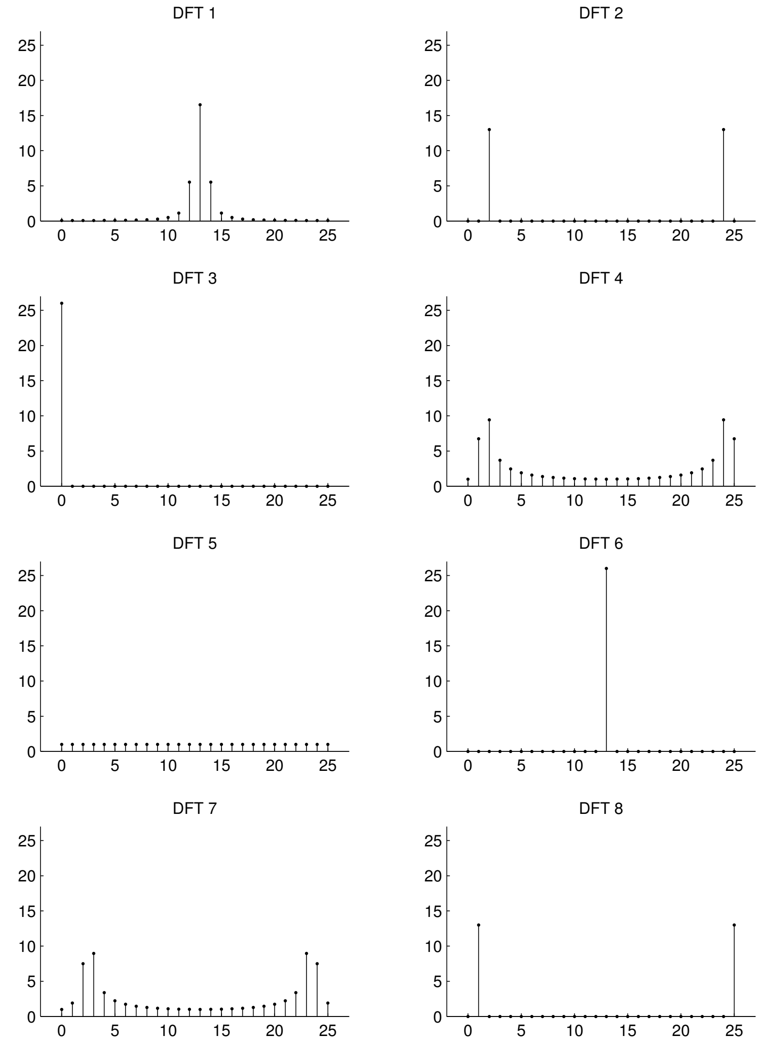
1.37 **DFT Matching.** Match each discrete-time signal with its DFT. You should be able to do this problem with out using a computer.



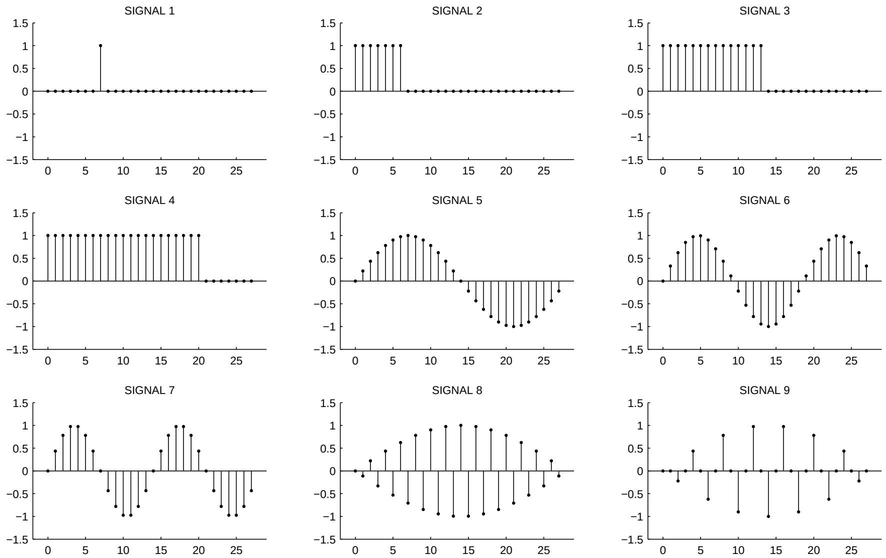


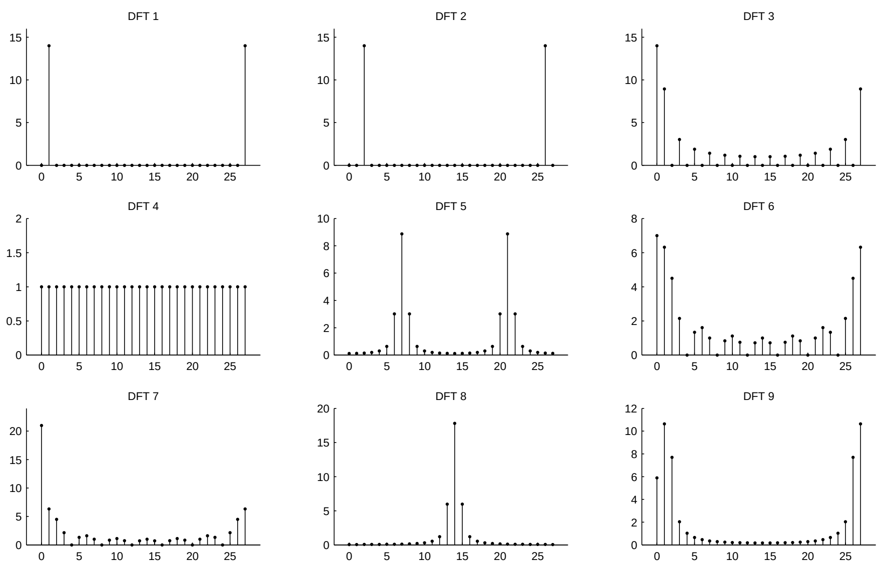
1.38 **DFT Matching.** Match each discrete-time signal with its DFT by filling out the following table. You should be able to do this problem with out using a computer.



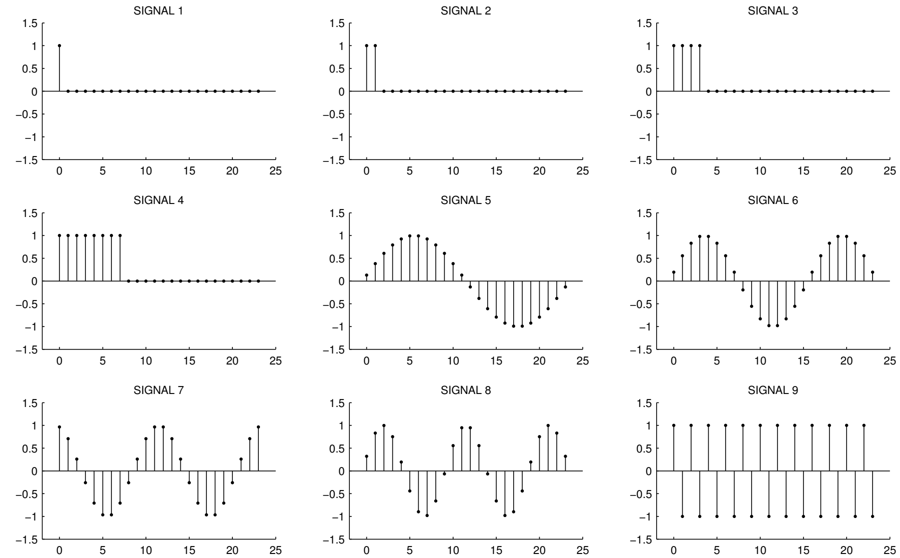


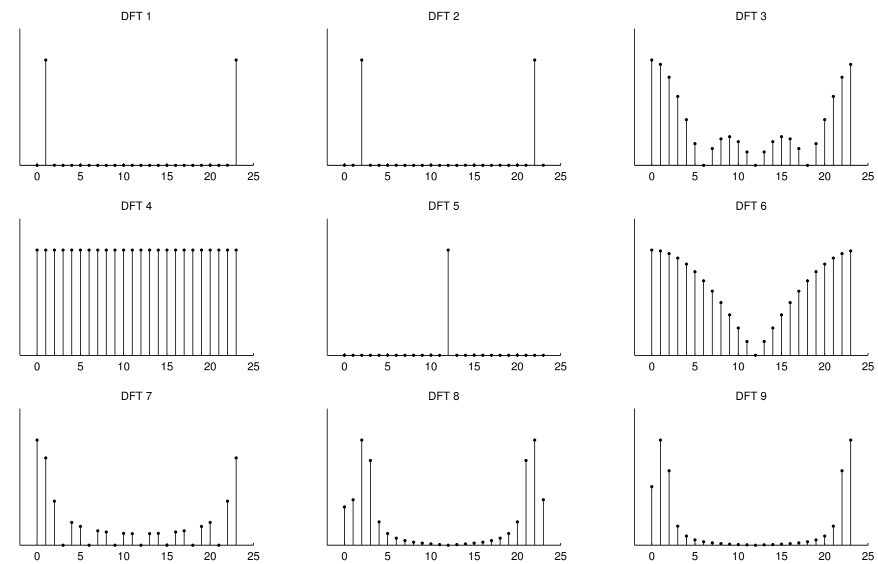
1.39 **DFT Matching.** Match each of the following 28-point discrete-time signals with its DFT by completing a table. (The DFT plots show the magnitude of the complex DFT coefficients.)



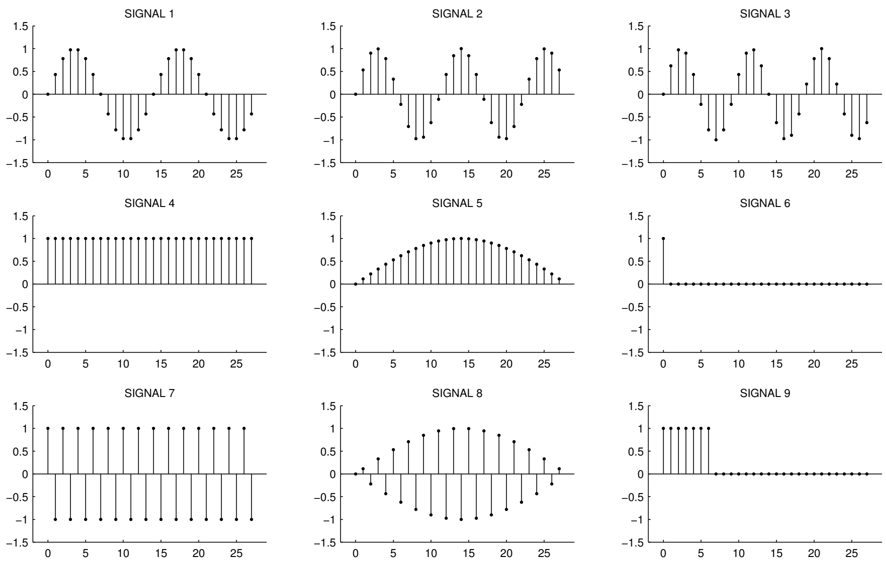


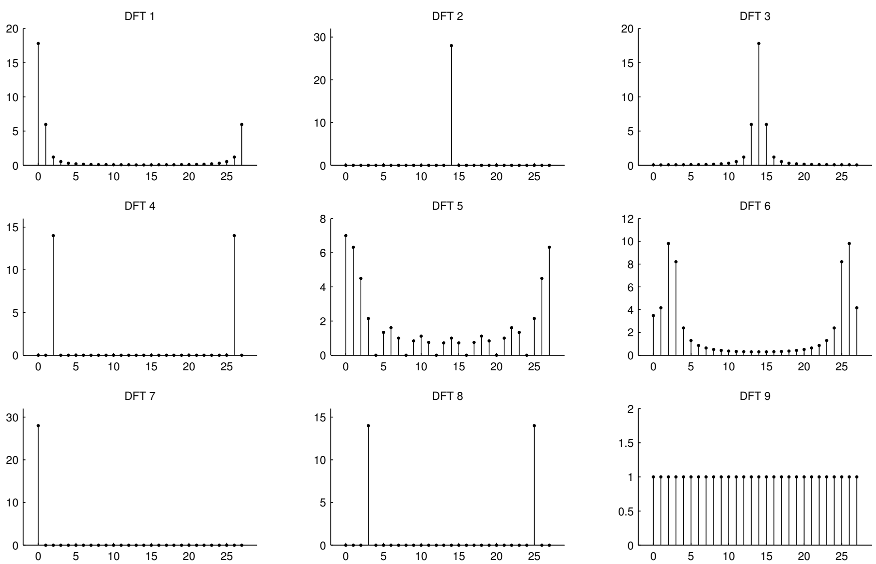
1.40 **DFT Matching.** Match each of the following 28-point discrete-time signals with its DFT by completing a table. (The DFT plots show the magnitude of the complex DFT coefficients.)



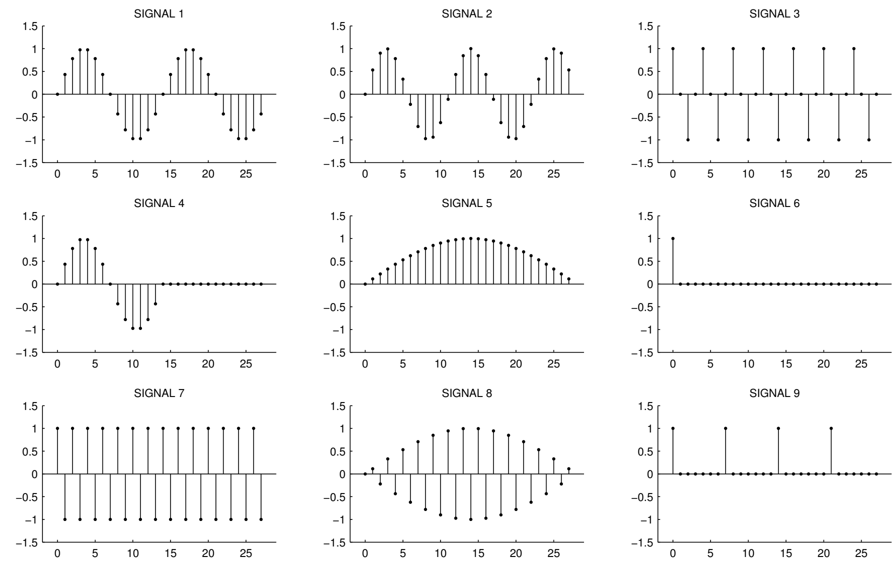


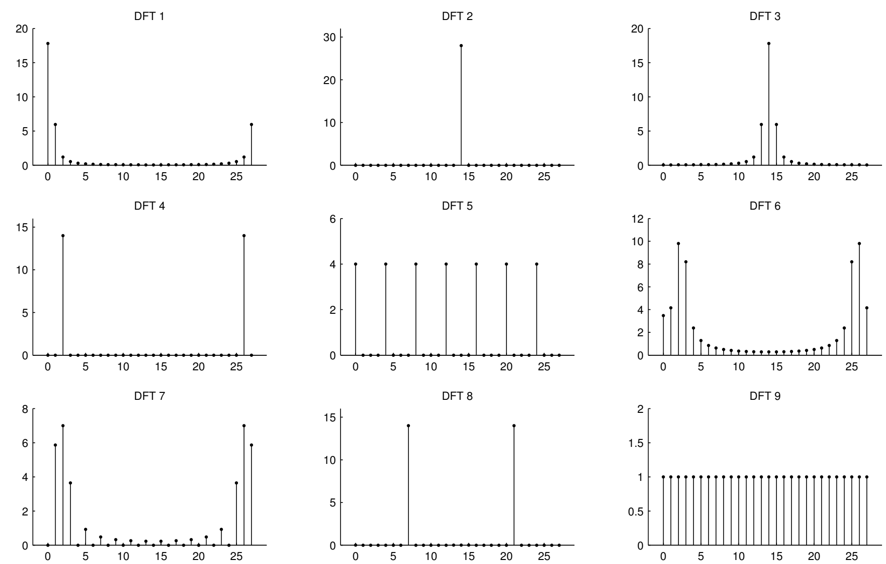
1.41 **DFT Matching.** Match each of the following 28-point discrete-time signals with its DFT by filling out a table. (The DFT plots show the magnitude of the complex DFT coefficients.)





1.42 **DFT Matching.** Match each of the following 28-point discrete-time signals with its DFT by completing a table. (The DFT plots show the magnitude of the complex DFT coefficients.)





1.43 What is the result of the following MATLAB command? Explain your answer.

```
>> fft([0 1 1 1 1 1])
```

Do not use direct computation of the DFT. Show your use of the appropriate DFT properties.

1.44 (a) The sequence $d(k)$ is defined as

$$d(k) = \sum_{n=0}^{N-1} \exp\left(\mathrm{j}\frac{2\pi}{N}nk\right), \quad k \in \mathbb{Z}.$$

Find a simplified expression for $d(k)$. Also, sketch $d(k)$ when $N = 5$. Note that $d(k)$ is an infinitely long sequence in both directions.

(b) Let $N = KM$ where K and M are positive integers. Consider the N-point sequence, $x(n)$, defined for $0 \le n \le N-1$, as

$$x(n) = \delta(\langle n \rangle_K) = \begin{cases} 1 & n \text{ is a multiple of } K \\ 0 & \text{otherwise.} \end{cases}$$

For $K = 4$, $M = 3$, sketch the 12-point sequence $x(n)$.
For $K = 3$, $M = 4$, sketch the 12-point sequence $x(n)$.
For general N, find the N-point DFT $X(k)$, $0 \le k \le N-1$, in simplified form. You might use part (a) in your work.

(c) Illustrate your answer to part (b) using a sketch of $X(k)$.

1.45 What is the result of the following MATLAB commands? Explain your answer.

```
>> x = [6 4 3 2 1];
>> X = fft(x);
>> G = conj(X);
>> g = ifft(G)
```

Do not use direct computation of the DFT. Show your use of the appropriate DFT properties.

1.46 What is the result of the following MATLAB commands? Explain your answer.

```
>> x = [6 5 4 3 2 1];
>> X = fft(x);
>> G = [X(1) X(end:-1:2)];
>> g = ifft(G)
```

Do not use direct computation of the DFT. Show your use of the appropriate DFT properties.

1.47 What is the result of the following MATLAB commands? Explain your answer.

```
>> x = [8 7 6 5 4 3 2 1];
>> X = fft(x);
>> G = [X(5:8) X(1:4)];
>> g = ifft(G)
```

Do not use direct computation of the DFT. Show your use of the appropriate DFT properties.

1.48 What is the result of the following MATLAB commands? Explain your answer.

```
>> x = [6 5 4 3 2 1];
>> X = fft(x);
>> G = X .* exp(-j*[0:5]*2*pi/6);
>> g = ifft(G)
```

Do not use direct computation of the DFT. Show your use of the appropriate DFT properties.

1.49 What is the result of the following MATLAB commands? Explain your answer.

```
>> x = [6 5 4 3 2 1];
>> X = fft(x);
>> G = X .* (-1).^[0:5];
>> g = ifft(G)
```

Do not use direct computation of the DFT nor MATLAB. Show your use of the appropriate DFT properties.

1.50 The DFT of the 10-point signal

$$x = [6, \quad 5, \quad 4, \quad 3, \quad 2, \quad 1, \quad 2, \quad 3, \quad 4, \quad 5]$$

is

$$X = [35, \quad 10.4721, \quad 0, \quad 1.5279, \quad 0, \quad 1, \quad 0, \quad 1.5279, \quad 0, \quad 10.4721]$$

Using properties of the DFT, what is the DFT of the following signal g?

$$g = [1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 5, \quad 4, \quad 3, \quad 2]$$

Do not use direct computation of the DFT. Show your use of the appropriate DFT properties.

1.51 A 6-point signal $x(n)$ has the DFT:

$$X(k) = [4 \quad 1 \quad 2 \quad 6 \quad 3 \quad 5], \qquad k = 0, \ldots, 5$$

Find the DFT $G(k)$ of the 6-point signal $g(n)$ defined as $g(n) = x(n)\,(-1)^n$.

Do not use direct computation of the DFT. Show your use of the appropriate DFT properties.

1.52 **DFT and deconvolution.** (Porat 4.26) We are given the two sequences

$$x = [1, \ -3, \ 1, \ 5], \qquad y = [7, \ -7, \ -9, \ -3].$$

Does there exist a sequence h such that y is the circular convolution of x and h,

$$y = h \circledast x \, ?$$

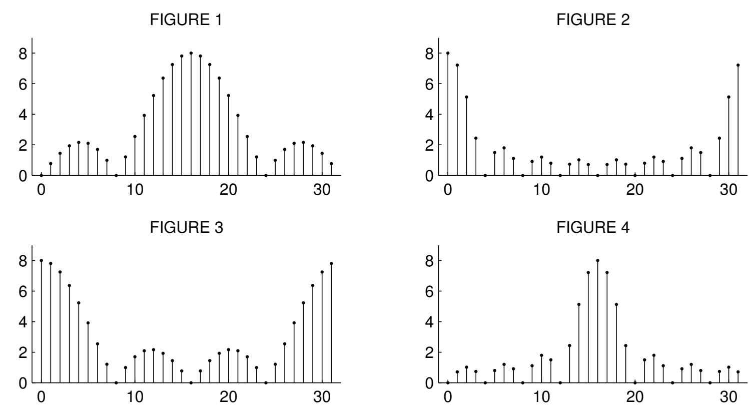
Here, $h \circledast x$ denotes circular convolution of h and x. If so, find h; if not, prove that such h does not exist. (Use the DFT command in Matlab, `fft`.)

1.53 **DFT and deconvolution.** (Porat 4.27) Given $x = [1, \ 3, \ 2, \ 1]$, $y = [1, \ 5, \ 10, \ 12, \ 9, \ 4, \ 1]$, use the DFT to find h such that $y = h * x$ (such that $y(n)$ is the linear convolution of $h(n)$ and $x(n)$). You may use Matlab.

1.54 Match each MATLAB code fragment to the figure it produces by completing the table.

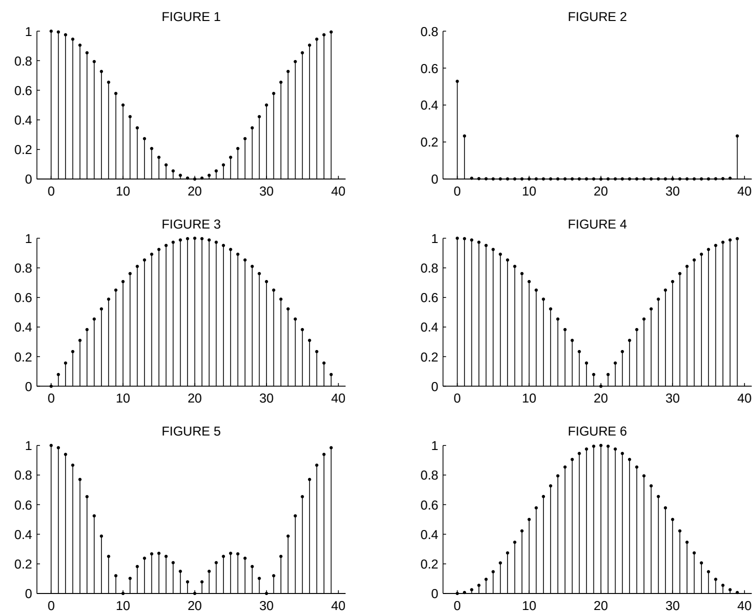
Code	Figure
1	
2	
3	
4	

```
% CODE 1
h = [1 1 1 1 1 1 1 1];
H = fft(h,32);
stem(0:31,abs(H),'.')

% CODE 2
h = [1 1 1 1 1 1 1 1];
H = fft(h,32);
stem(0:31,abs(fftshift(H)),'.')

% CODE 3
h = [2 2 2 2 0 0 0 0];
H = fft(h,32);
stem(0:31,abs(H),'.')

% CODE 4
h = [2 2 2 2 0 0 0 0];
H = fft(h,32);
stem(0:31,abs(fftshift(H)),'.')
```



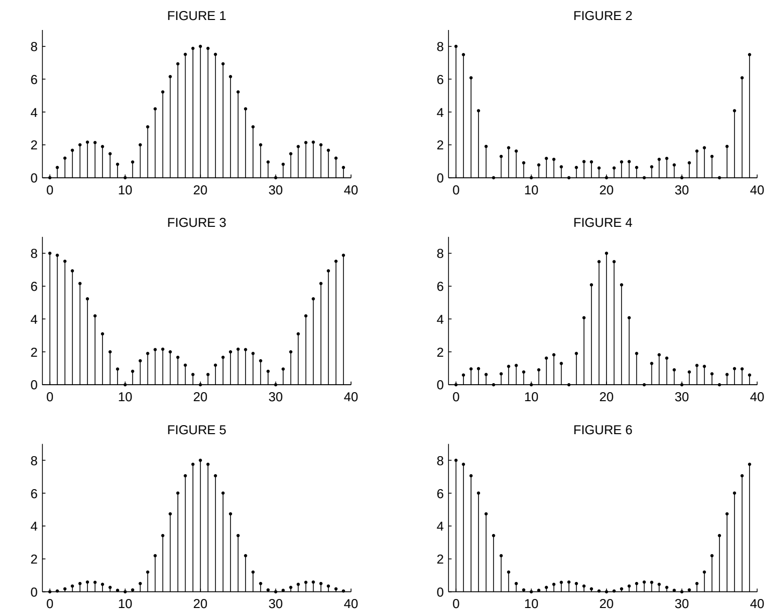
1.55 Match each MATLAB code fragment to the figure it produces. Provide a brief explanation for your answers.

```
N = 40;
n = 0:N-1;

% CODE 1
h = hamming(N)/N;
H = fft(h,N);
stem(n,abs(H),'.')

% CODE 2
h = [0.5 0.5];
H = fft(h,N);
stem(n,abs(H),'.')

% CODE 3
h = [0.5 -0.5];
H = fft(h,N);
stem(n,abs(H),'.')

% CODE 4
h = [0.25 0.5 0.25];
H = fft(h,N);
stem(n,abs(H),'.')

% CODE 5
h = [-0.25 0.5 -0.25];
H = fft(h,N);
stem(n,abs(H),'.')

% CODE 6
h = ones(1,4)/4;
H = fft(h,N);
stem(n,abs(H),'.')
```



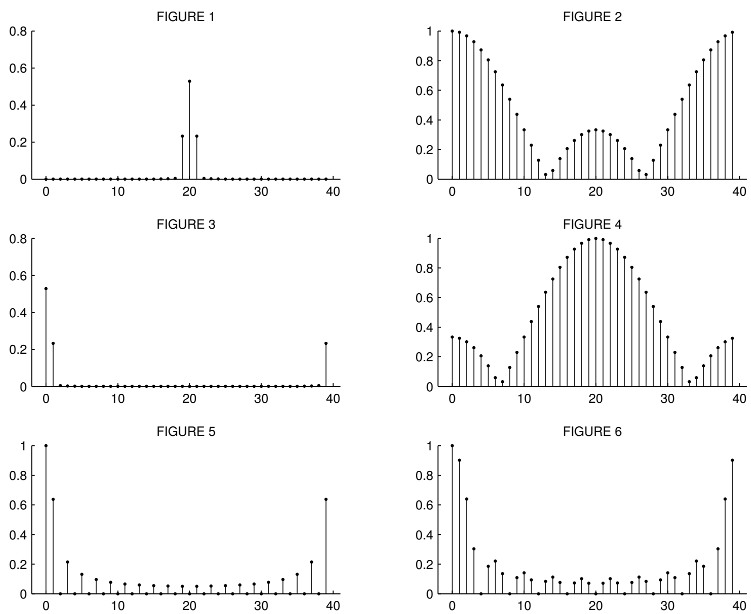
1.56 Match each MATLAB code fragment to the figure it produces. Provide a brief explanation for your answers.

```
% CODE 1
h = [1 1 1 1 1 1 1 1];
H = fft(h,40);
stem(0:39,abs(H),'.')

% CODE 2
h = [1 1 1 1 1 1 1 1];
H = fft(h,40);
stem(0:39,abs(fftshift(H)),'.')

% CODE 3
h = [2 2 2 2];
H = fft(h,40);
stem(0:39,abs(H),'.')

% CODE 4
h = [2 2 2 2];
H = fft(h,40);
stem(0:39,abs(fftshift(H)),'.')

% CODE 5
h = conv([1 1 1 1],[1 1 1 1])/2;
H = fft(h,40);
stem(0:39,abs(H),'.')

% CODE 6
h = conv([1 1 1 1],[1 1 1 1])/2;
H = fft(h,40);
stem(0:39,abs(fftshift(H)),'.')
```



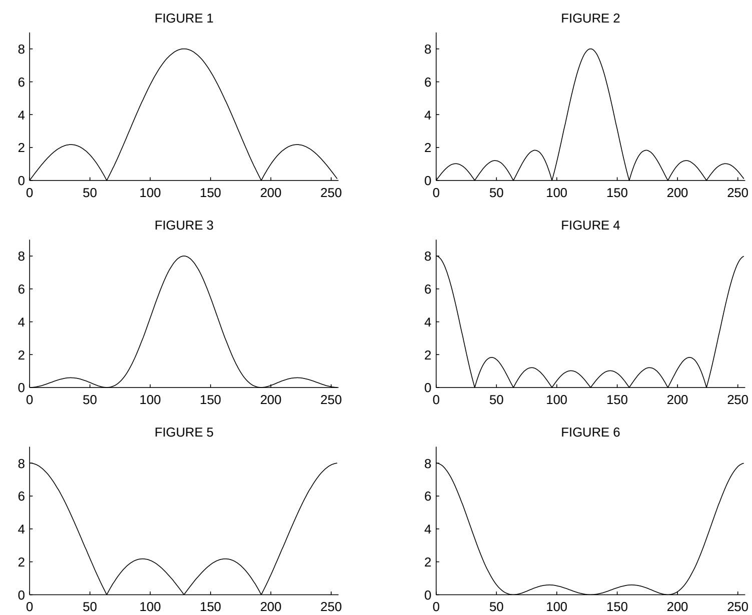
1.57 Match each MATLAB code fragment to the figure it produces. Provide a brief explanation for your answers.

```
N = 40;
n = 0:N-1;

% CODE 1
h = [1/3 1/3 1/3];
H = fft(h,N);
stem(n,abs(H),'.')

% CODE 2
h = [1/3 -1/3 1/3];
H = fft(h,N);
stem(n,abs(H),'.')

% CODE 3
h = hamming(N)/N;
H = fft(h,N);
stem(n,abs(H),'.')

% CODE 4
h = hamming(N)/N .* (-1).^n';
H = fft(h,N);
stem(n,abs(H),'.')

% CODE 5
h = ones(1,N/4)/(N/4);
H = fft(h,N);
stem(n,abs(H),'.')

% CODE 6
h = ones(1,N/2)/(N/2);
H = fft(h,N);
stem(n,abs(H),'.')
```



1.58 In this problem, simple sequences are zero-padded and their FFTs are computed. Some are displayed using `fftshift`. Match each MATLAB code fragment to the figure it produces by completing a table.

```
% CODE 1
h = [1 1 1 1 1 1 1 1];
L = 2^8;
H = fft(h,L);
plot(0:L-1,abs(H))

% CODE 2
h = [1 1 1 1 1 1 1 1];
H = fft(h,L);
plot(0:L-1,abs(fftshift(H)))

% CODE 3
h = [2 2 2 2];
H = fft(h,L);
plot(0:L-1,abs(H))

% CODE 4
h = [2 2 2 2];
H = fft(h,L);
plot(0:L-1,abs(fftshift(H)))

% CODE 5
h = conv([1 1 1 1],[1 1 1 1])/2;
H = fft(h,L);
plot(0:L-1,abs(H))

% CODE 6
h = conv([1 1 1 1],[1 1 1 1])/2;
H = fft(h,L);
plot(0:L-1,abs(fftshift(H)))
```



1.59 When using both windowing and zero padding for an N-point sequence $x(n)$, which is the proper procedure: (1) zero-pad first to length M, then use a window of length M or (2) use a window of length N on $x(n)$ and then zero-pad to length M? Give reasons.

1.60 **DFT and samples of the DTFT.** If a discrete-time signal $x(n)$ is finite in length, then samples of its DTFT can be computed using the DFT. Let $x(n)$ be given by

$$x(n) = (1, 2, 3, \underline{2}, 3, 4, 2)$$

for $n = -3, \ldots, 3$. $x(n)$ is zero outside this range. $x(0)$ is represented by $\underline{2}$. Please explain how the DFT can be used to find the samples of the DTFT $X(2\pi k/12)$ for $k = 0, \ldots, 11$. Write a Matlab mfile to implement it, with comments relating each command to the math. Include a print-out of the 12 DTFT samples in your write-up.

Repeat the same problem with the following sequence

$$g(n) = (1, 2, 4, \underline{8}, 4, 2, 1)$$

for $n = -3, \ldots, 3$. $g(n)$ is zero outside this range. Note that $g(-n) = g(n)$. What effect does this have on your result?

1.61 An analog signal is sampled at 8192 Hz and 600 samples are collected. These 600 samples are available on the course webpage as the file `signal1.txt`. Plot the signal versus time in seconds; and using the DFT, plot the spectrum of the signal versus *physical frequency* in hertz with the DC component in the center of the plot. If your computer has sound capability, use the `soundsc` command to listen to the signal.

1.62 (Mitra 11.1) A bandlimited analog signal is sampled at 7500 Hz (sufficient to ensure no aliasing), and N samples are collected.

(a) What is the frequency resolution of the DFT in Hz if $N = 1250$?

(b) To attain a frequency resolution of 4.5 Hz, what should N be?

1.63 (Mitra 11.3b) A bandlimited analog signal is sampled at 14 kHz (sufficient to ensure no aliasing) and $N = 1010$ samples are collected. The DFT of these 1010 samples is computed. Determine the physical frequencies corresponding to DFT indices $k = 195$, 339, and 917.

1.64 (Porat 4.23) A bandlimited analog signal is sampled (with no aliasing) at 500 Hz and 980 samples are collected. The DFT of these 980 samples is computed. We wish to compute the value of the spectrum of the sampled signal at 120 Hz.

(a) Which DFT index k is nearest to 120 Hz, and what is its physical frequency in hertz?

(b) What is the minimum number of zeros we must pad onto the 980 samples to obtain a DFT value at 120 Hz exactly? What is the DFT index k then corresponding to 120 Hz?

1.65 An analog signal $x(t)$, bandlimited to 50 Hz, is sampled for 2 seconds at 80 Hz. The array of collected samples is zero-padded to length 256. A 256-point DFT of the (zero-padded) data is computed.

(a) Which DFT coefficients are free of aliasing?

(b) Which DFT coefficient corresponds approximately to 20 Hz? What is the true frequency corresponding to this DFT coefficient?

(c) Suppose the data were zero-padded to 512 (instead of to 256) and a 512-point DFT were computed. How is the 256-point DFT and this 512-point DFT related? Be specific.

Explain your answers with appropriate figures.

1.66 The analog signal $x(t)$ is bandlimited to 30 Hz.

$$X(f) = 0 \quad \text{for} \quad |f| > 30 \text{ Hz}.$$

The signal $x(t)$ is sampled with a sampling rate of 50 Hz and 1000 samples are collected. You then take the DFT of these 1000 samples. Which DFT coefficients $X^d(k)$ are free of aliasing?

1.67 The analog signal $x(t)$ is band-limited to 40 Hz. Suppose the signal is sampled at the rate of 100 samples per second and that at this rate 200 samples are collected. Then 200 zeros are appended to the 200 samples to form a 400-point vector. Then the 400-point DFT of this vector is computed to get $X(k)$ for $0 \le k \le 399$.

(a) Which DFT coefficients are free of aliasing?

(b) The DFT coefficient $X(50)$ represents the spectrum of the analog signal at what frequency f? (Give your answer in Hz).

1.68 An analog signal $s(t)$ only contains frequencies between 30 Hz and 40 Hz. (It is a bandpass signal). You want to know the value of the spectrum at 35 Hz.

The signal is sampled at a rate of 60 samples per second and 200 samples are collected. Call them $x(n)$ for $n = 0, \ldots, 199$. You then take a 200-point DFT of these 200 samples. Call the DFT values $X(k)$, for $k = 0, \ldots, 199$.

Which DFT coefficient corresponds most closely to 35 Hz of the *analog* signal $s(t)$? Show your work and show diagrams.

1.69 An analog signal $x_a(t)$, band-limited to 30 Hz, is sampled 100 times per second for 10 seconds. The DFT of the collected data is computed and 1000 DFT coefficients, $X(k)$, are obtained.

(a) What frequency (in Hz) does the DFT coefficient $X(100)$ correspond to?

(b) The individual performing this operation expects that the DFT coefficients $X(k)$ corresponding to the frequency range 30 Hz $< f <$ 50 Hz will be zero, because the analog signal $x_a(t)$ was band-limited to 30 Hz and the signal is over-sampled (no aliasing takes place). However, the DFT coefficients in that frequency range are not exactly zero. Explain clearly why those DFT coefficients are not exactly zero.

1.70 **DFT.** A 3 Hz continuous-time cosine signal with amplitude 1 is sampled at a rate of 10 samples per second to produce a discrete-time signal $x(n)$, $n \in \mathbb{Z}$.

(a) Sketch $|X(e^{j\omega})|$ for $|\omega| \le \pi$ where $X(e^{j\omega})$ is the discrete-time Fourier transform (DTFT) of $x(n)$.

(b) A block of 10 contiguous samples is taken from $x(n)$ and defined as the 10 point vector $g(n)$, $n = 0, \dots, 9$. The 10 point DFT of $g(n)$ is computed to obtain 10 DFT coefficients $G(k)$, $k = 0, \dots, 9$.
Sketch $|G(k)|$ for $k = 0, \dots, 9$.

(c) The 10 point vector $g(n)$ in part (b) can be extended with zeros outside the range $0 \le n \le 9$ to define the signal $v(n)$, $n \in \mathbb{Z}$,

$$v(n) = \begin{cases} g(n) & 0 \le n \le 9 \\ 0 & \text{otherwise} \end{cases}$$

Sketch $|V(e^{j\omega})|$ for $|\omega| \le \pi$ where $V(e^{j\omega})$ is the DTFT of $v(n)$.

2 The Fast Fourier Transform

2.1 Which is likely to be faster?

(a) A 96-point FFT,

(b) or an 87-point FFT?

Explain your answer.

2.2 Which can be computed faster?

(a) A 509-point DFT,

(b) or a 512-point DFT?

Explain your answer.

2.3 If a 512-point radix-2 FFT takes about 50 microseconds on a particular computer, how long would you expect a 2048-point radix-2 FFT to take on the same computer? Select the best answer below, and explain your choice.

(a) Slightly less than twice as long.

(b) Exactly twice as long.

(c) Slightly more that twice as long.

(d) Slightly less than four-times as long.

(e) Exactly four-times as long.

(f) Slightly more that four-times as long.

(g) Slightly less than eight-times as long.

(h) Exactly eight-times as long.

(i) Slightly more that eight-times as long.

2.4 The following MATLAB code gives four methods to compute the linear convolution of $x(n)$ and $g(n)$. Identify which of the four methods are *wrong*! Of the remaining methods, which one do you think requires the most additions and multiplications (the least efficient) and which method requires the fewest (the most efficient). Explain!

```
x = rand(1,126);
g = rand(1,126);

% FOUR METHODS TO COMPUTE THE
% LINEAR CONVOLUTION OF x AND g

% METHOD 1
y = conv(x,g);

% METHOD 2
X = fft([x zeros(1,120)]);
G = fft([g zeros(1,120)]);
Y = X.*G;
y = ifft(Y);

% METHOD 3
X = fft([x zeros(1,125)]);
G = fft([g zeros(1,125)]);
Y = X.*G;
y = ifft(Y);

% METHOD 4
X = fft([x zeros(1,130)]);
G = fft([g zeros(1,130)]);
Y = X.*G;
y = ifft(Y);
```

2.5 **Flops of the MATLAB FFT.** Measure the number of flops (floating point operations) of the MATLAB `fft` command for lengths $2 \leq N \leq 512$. Make these measurements by placing the flop measurement of a single FFT in a `for` loop which steps through the lengths `n = 2:512`. Use `help flops` to see how to measure flops. Plot the number of flops versus the length N using `plot(n,f,'.')` to prevent MATLAB from connecting the dots.

Comment on your results. For which values N is the MATLAB FFT algorithm *most* and *least* efficient?

On recent versions of MATLAB, the `flops` command is no longer available. In this case just time the `fft` function with the MATLAB command `etime` or with the commands `tic` and `toc`.

2.6 **FFT-based Fast convolution.** Modify the DFT-based linear convolution MATLAB function you wrote in HW 1 so that it always uses a radix-2 FFT. (Zero-pad up to the nearest power of 2.)

In MATLAB you can get the next highest power of 2 using

```
2^(ceil(log2(n)))
```

In the past, we asked that you compare the efficiency of your new MATLAB function with the `conv` function by plotting the flops of each verses N, where N is the length of both $x(n)$ and $h(n)$. However, the `flops` command that measures the number of flops is not available in recent versions of MATLAB.

2.7 (a) Represent circular convolution as a matrix vector multiplications. Let $y(n)$ be the circular convolution of two five-point sequences $a(n)$ and $x(n)$ $(n = 0, \ldots, 4)$, then write

$$y = Ax$$

where y and x are 5-point column vectors and A is a 5 by 5 matrix. Write out explicitly what the matrix A is - what structure does it have?

(b) As part of an algorithm you need to compute the matrix vector product $g = A\,x$ where A is the following kind of matrix,

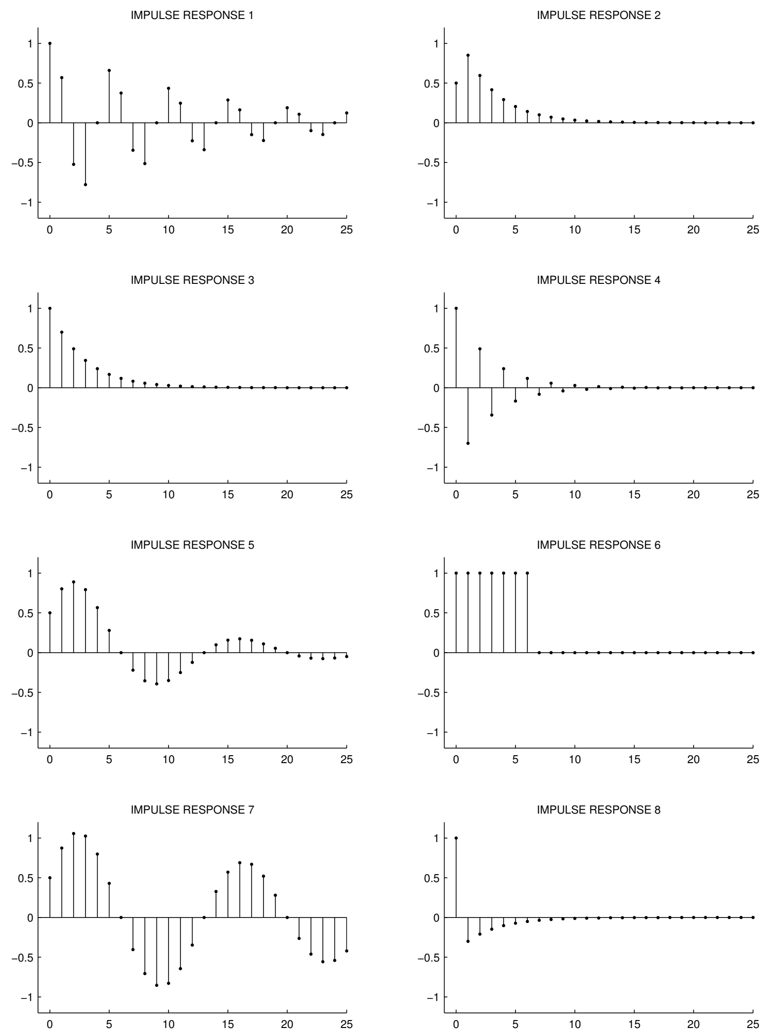
$$A = \begin{bmatrix} a(0) & a(1) & a(2) & a(3) & a(4) \\ b(1) & a(0) & a(1) & a(2) & a(3) \\ b(2) & b(1) & a(0) & a(1) & a(2) \\ b(3) & b(2) & b(1) & a(0) & a(1) \\ b(4) & b(3) & b(2) & b(1) & a(0) \end{bmatrix}$$

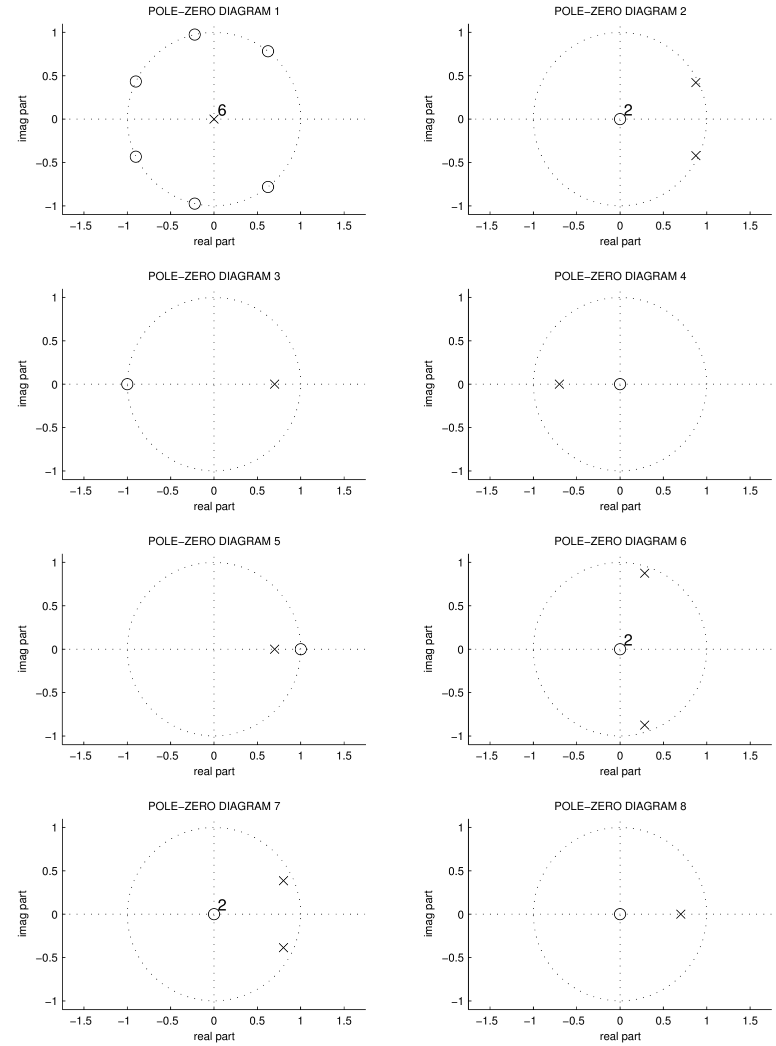
By extending A appropriately, you can compute the vector g by using circular convolution and the FFT. Describe how to extend A and x and how to use the FFT to efficiently compute the matrix-vector product $A\,x$.

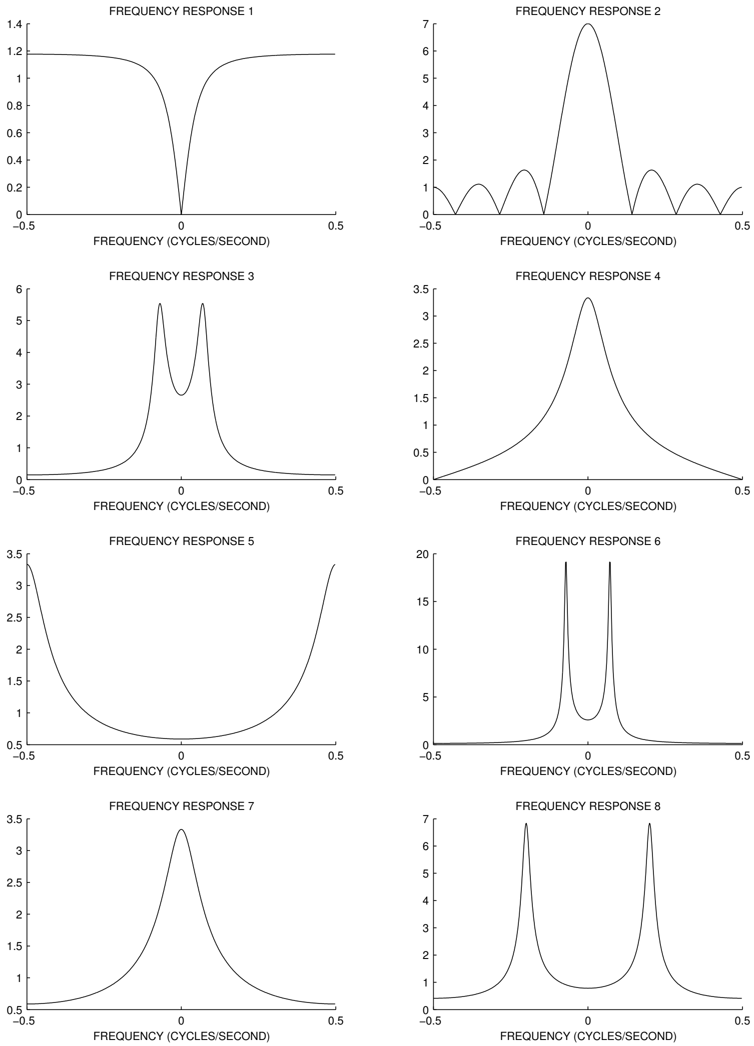
(The FFT method is more efficient for large N than direct matrix vector multiplication, because it takes $O(N \log(N))$ instead of $O(N^2)$.)

3 Filters

3.1 **Matching.** The diagrams on the following three pages show the impulse responses, pole-zero diagrams, and frequency responses magnitudes of 8 discrete-time causal LTI systems. But the diagrams are out of order. Match each diagram by filling out the following table.



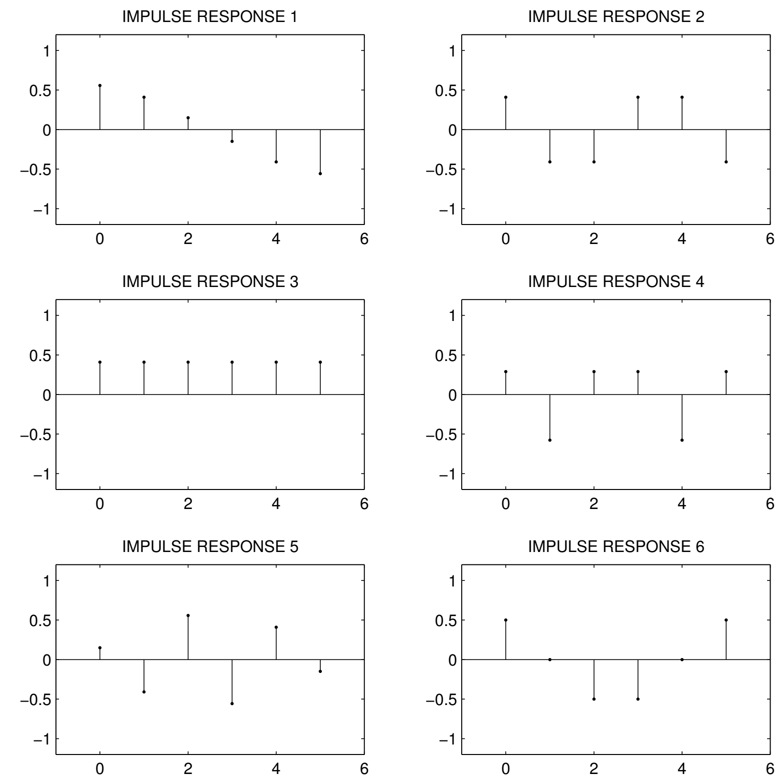


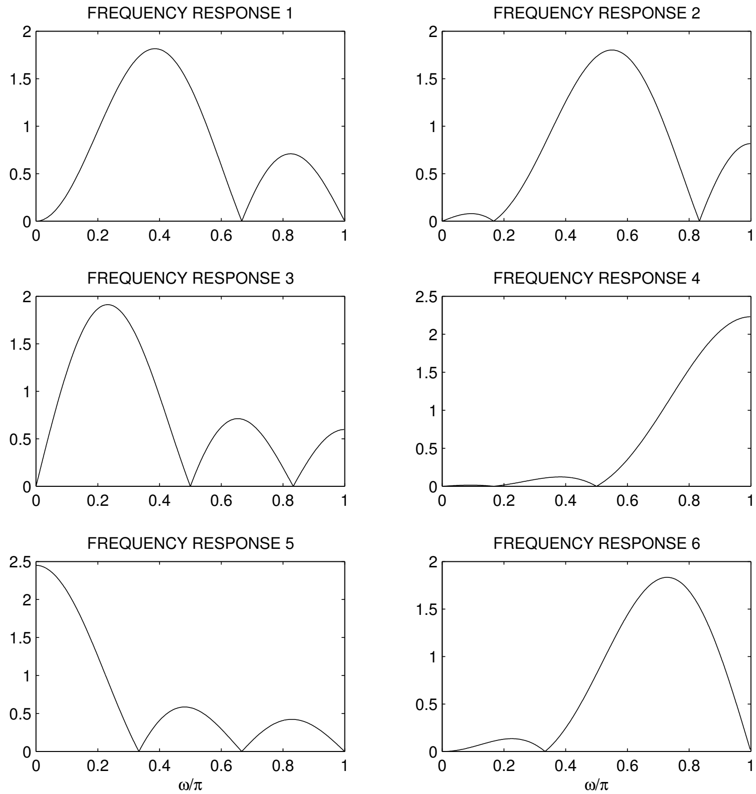


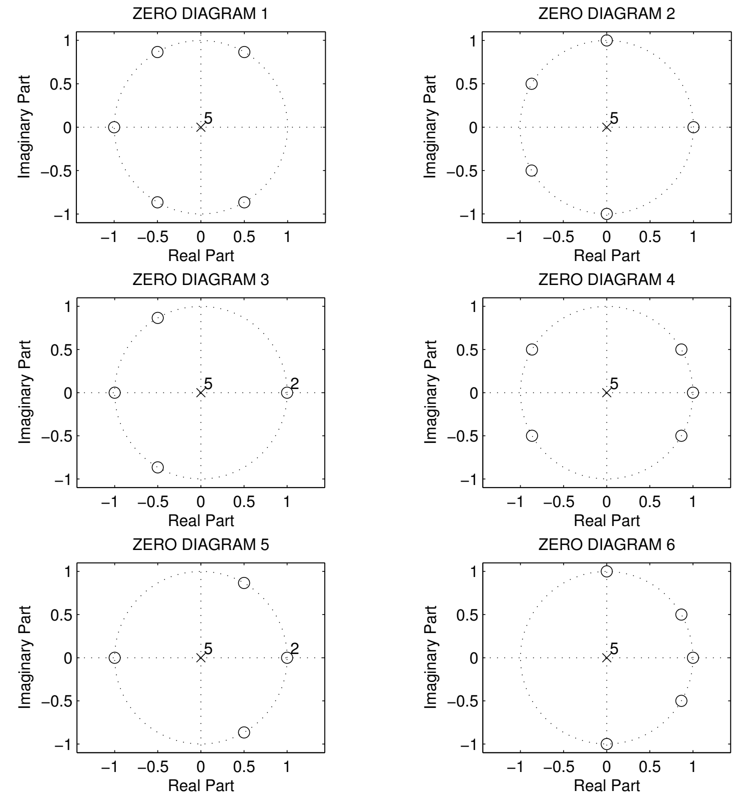
Impulse response	Pole-zero	Frequency response
1		
2		
3		
4		
5		
6		
7		
8		

3.2 **Matching.** The following figures show 6 impulse responses, frequency responses, and zero diagrams. Match each frequency response and zero diagram to the corresponding impulse response.

Impulse Response	Frequency Response	Zero Diagram
1		
2		
3		
4		
5		
6		







3.3 An FIR digital filter has the transfer function

$$H(z) = (1 - z^{-1})^3 \, (1 + z^{-1})^3$$

(a) Sketch the pole-zero diagram of this system.

(b) Sketch $|H^f(\omega)|$.

(c) Would you classify this as a low-pass, high-pass, band-pass, or band-stop filter? Please briefly explain.

3.4 An analog signal $x_a(t)$, band-limited to 5 Hz, is sampled at a rate of 10 samples per second to give a discrete-time signal $x(n) = x_a(0.1\,n)$. The discrete-time signal $x(n)$ is then filtered with an digital FIR filter, h.

Find the shortest impulse response $h(n)$ so that

(a) The filter completely rejects the frequencies 5 Hz and 2.5 Hz.

(b) The DC gain of the filter is 1.

(c) The filter has linear-phase.

Sketch the frequency response magnitude of the filter.

3.5 Design a simple causal real discrete-time LTI system with the properties:

(a) The system should exactly preserve the signal $\cos(0.5\,\pi\,n)$.

(b) The system should annihilate constant signals. That is, the frequency response should have a null at dc.

Hint: It can be done with an impulse response of length 3.

For the system you design:

(a) Find the difference equation to implement the system.

(b) Sketch the impulse response of the system.

(c) Sketch the poles and zeros of the system.

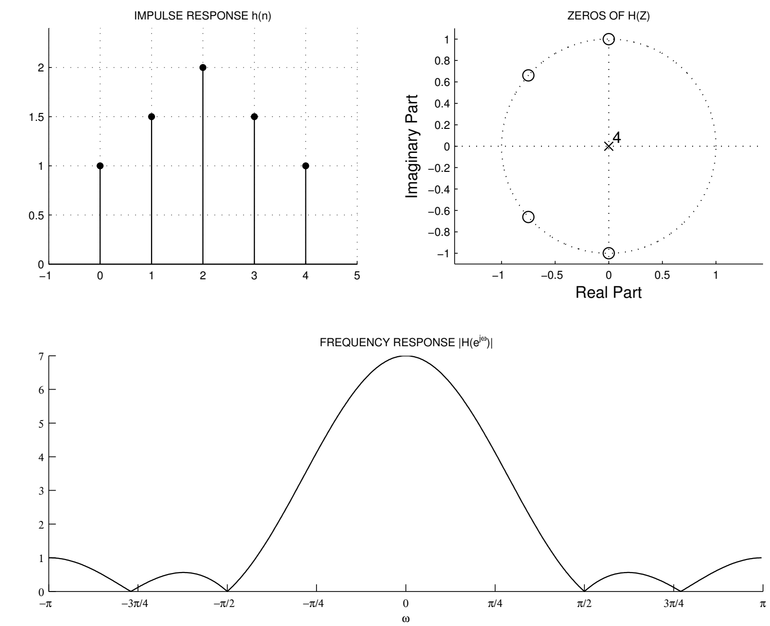
(d) Find and sketch the frequency response magnitude $|H^f(\omega)|$. Clearly show the nulls of the frequency response.

3.6 **Filter Transformations.** An FIR filter with impulse response $h(n)$ is illustrated below. Define a new FIR filter with impulse response $g(n) = (-1)^n\,h(n)$. Sketch the,

(a) impulse response,

(b) zero-diagram, and

(c) frequency response of the new filter.



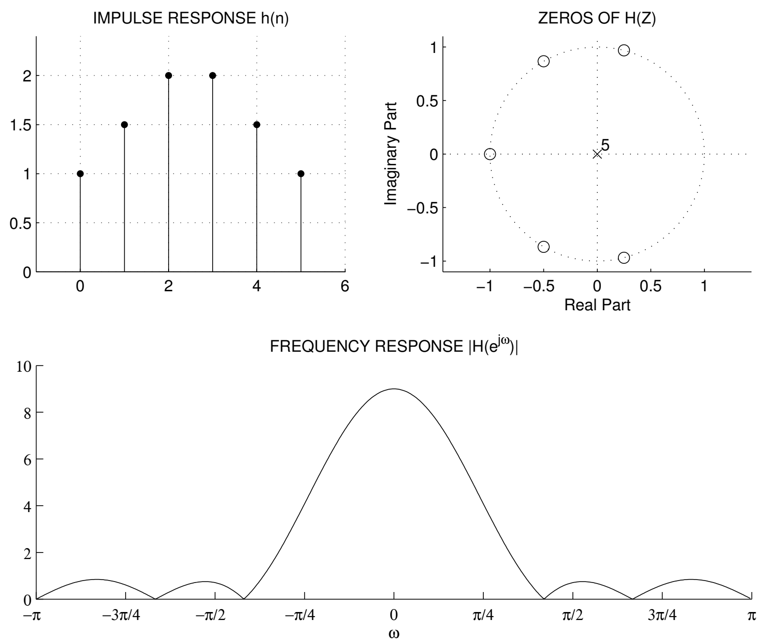
3.7 **Filter Transformations.** An FIR filter with impulse response $h(n)$ is illustrated below. Define a new FIR filter with impulse response $g(n) = (-1)^n\,h(n)$. Sketch the,

(a) impulse response,

(b) zero-diagram, and

(c) frequency response of the new filter.

(d) how would you classify the new filter (low-pass filter, high-pass, band-pass, band-stop, etc)?

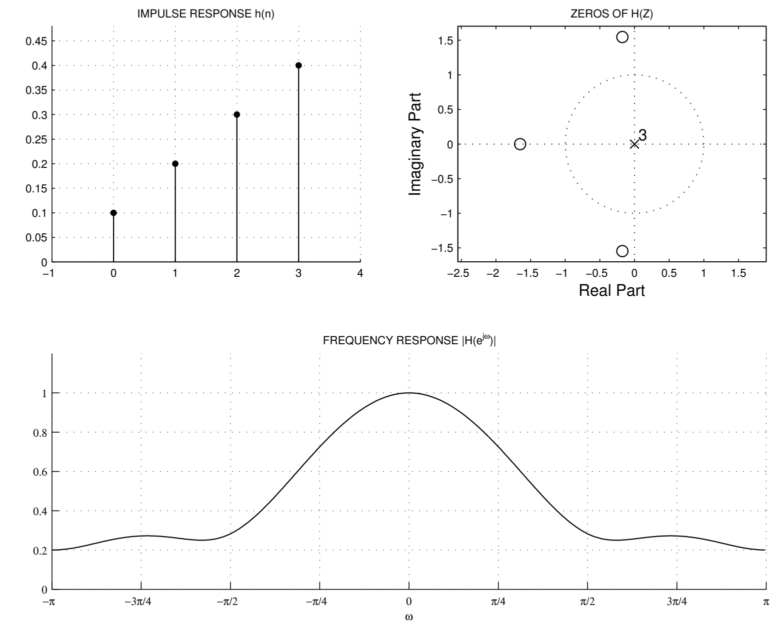


3.8 **Filter Transformations.** An N-point FIR filter with impulse response $h(n)$ is illustrated below. Define a new FIR filter with impulse response $g(n) = h(6-n)$.

(a) Sketch the impulse response of the filter, $g(n)$.

(b) What is the relationship between the zeros of $G(z)$ and the zeros of $H(z)$?
Based on the zero-diagram of $H(z)$ shown below, sketch the zero-diagram of $G(z)$.

(c) Find the frequency response $G^f(\omega)$ in terms of the frequency response $H^f(\omega)$.
Sketch $|G^f(\omega)|$.



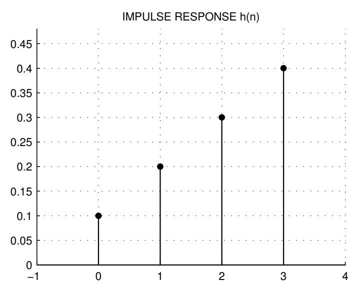
3.9 **Filter Transformations.** An N-point FIR filter with impulse response $h(n)$ is illustrated below. Define a new FIR filter with impulse response $g(n) = (-1)^n\, h(N-1-n)$.

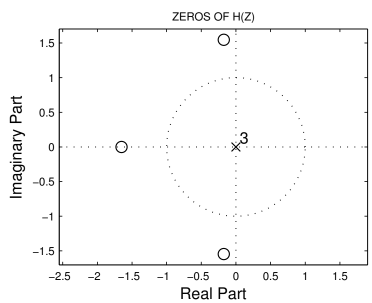
(a) Sketch the impulse response of the new filter, $g(n)$.

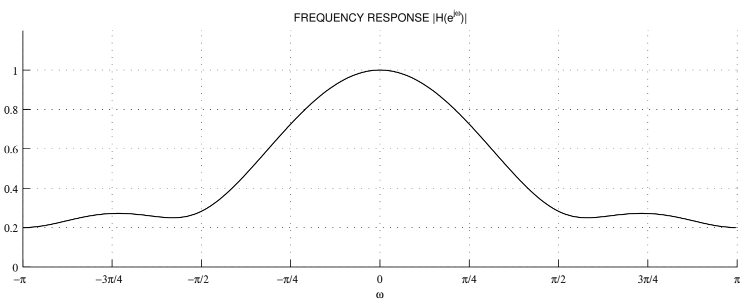
(b) What is the dc gain of the filter $g(n)$?

(c) What is the relationship between the zeros of $G(z)$ and the zeros of $H(z)$? Based on the zero-diagram of $H(z)$ shown below, sketch the zero-diagram of $G(z)$.

(d) Find an expression for the frequency response $G^f(\omega)$ in terms of the frequency response $H^f(\omega)$. Sketch $|G^f(\omega)|$.







3.10 **Filter Transformations.** An FIR filter with impulse response $h(n)$ is illustrated below. Define a new FIR filter with impulse response $g(n) = (-1)^n\, h(N-1-n)$.

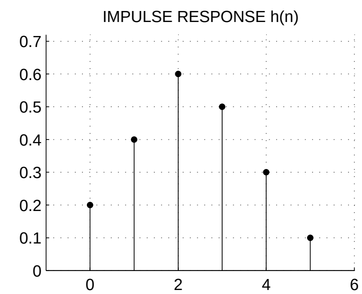
(a) Sketch the impulse response of the new filter, $g(n)$.

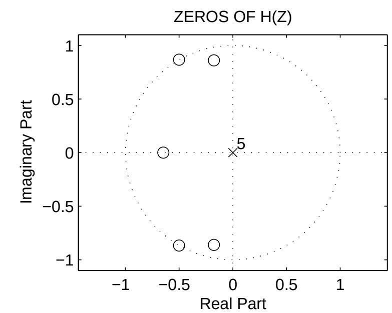
(b) What is the dc gain of the filter $g(n)$?

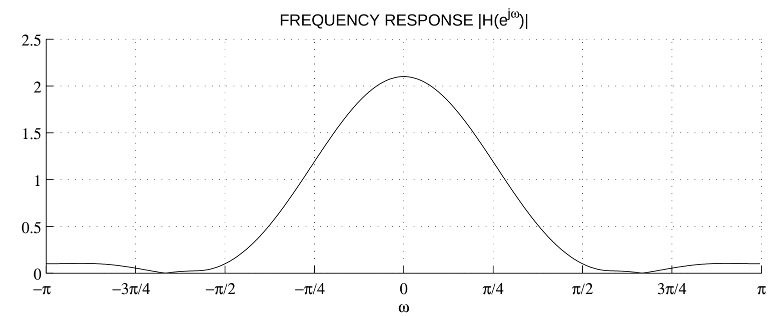
(c) Find an expression for the frequency response $G^f(\omega)$ of the new filter $g(n)$ in terms of the frequency response $H^f(\omega)$ of the filter $h(n)$. Sketch the frequency response magnitude $|G^f(\omega)|$ of the new filter.

(d) What is the relationship between the zeros of $G(z)$ and the zeros of $H(z)$? Based on the zero-diagram of $H(z)$ shown below, sketch the zero-diagram of $G(z)$.

(e) How would you classify the new filter (low-pass filter, high-pass, band-pass, band-stop, etc)?







3.11 Very simple low-pass digital filtering can be performed by a running average of N consecutive samples,

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} x(n-k)$$

where $x(n)$ is the signal being smoothed by the filter. In the following, assume $N = 8$.

(a) Sketch the impulse response of this low-pass filter.

(b) What is the dc gain of the filter? That means, find $H^f(0)$.

(c) Find the zeros of the transfer function and sketch the pole/zero diagram.

(d) Based on the zero diagram in (c), sketch the frequency response amplitude $A(\omega)$ and the frequency response magnitude $|H(\omega)|$.

3.12 Very simple low-pass digital filtering can be performed by a running average of N consecutive samples,

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} x(n-k)$$

where $x(n)$ is the signal being smoothed by the filter. In the following, assume $N = 7$.

(a) Sketch the impulse response of this low-pass filter.

(b) What is the dc gain of the filter? That means, find $H^f(0)$.

(c) Find the zeros of the transfer function and sketch the pole/zero diagram.

(d) Based on the zero diagram in (c), sketch the frequency response ampltude $A(\omega)$. (This is a linear-phase FIR filter.)

(e) A simple filter to remove the dc component of a signal can be implemented using the equation

$$y(n) = x(n-M) - \frac{1}{N} \sum_{k=0}^{N-1} x(n-k)$$

where $M = (N-1)/2$. (N needs to be odd.) Sketch the impulse response and frequency response amplitude of this filter using $N = 7$.

3.13 For the discrete-time LTI system having the impulse response $h(n)$

$$h(n) = \begin{cases} 0.5 & 0 \le n \le 5, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Accurately sketch the pole-zero diagram,

(b) Sketch the frequency response magnitude $|H(e^{j\,\omega})|$. Accurately indicate the values at $\omega = 0$ and $\omega = \pi$. Also accurately indicate the nulls of the frequency response.

(c) Sketch the phase of the frequency response $\angle H(e^{j\,\omega})$.

(d) Sketch the step response of the system.

3.14 Consider a filter implemented using the difference equation,

$$y(n) = \frac{1}{5} y(n-1) + \frac{1}{N} \sum_{k=0}^{N-1} x(n-k)$$

where $x(n)$ is the signal being filtered. In the following, assume $N = 8$.

(a) What is the dc gain of the filter? That means, find $H^f(0)$.

(b) Find the zeros of the transfer function and sketch the pole/zero diagram.

(c) Based on the zero diagram in (c), roughly sketch the frequency response ampltude $A(\omega)$. Explicitly indicate any frequency response nulls if there are any.

3.15 Consider a filter implemented using the difference equation,

$$y(n) = 0.04\, y(n-2) + \frac{1}{N} \sum_{k=0}^{N-1} x(n-k)$$

where $x(n)$ is the signal being filtered. In the following, assume $N = 8$.

(a) What is the dc gain of the filter? That means, find $H^f(0)$.

(b) Find the zeros of the transfer function and sketch the pole/zero diagram.

(c) Based on the zero diagram, roughly sketch the frequency response ampltude $A(\omega)$. Explicitly indicate any frequency response nulls if there are any.

3.16 Very simple high-pass digital filtering can be performed by the difference equation:

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} (-1)^k \, x(n-k)$$

where $x(n)$ is the signal being filtered. In the following, assume $N = 8$.

(a) Sketch the impulse response of this high-pass filter.

(b) What is the dc gain of the filter? That means, find $H^f(0)$.

(c) Find the zeros of the transfer function and sketch the pole/zero diagram.

(d) Based on the zero diagram in (c), sketch the frequency response amplitude $A(\omega)$ (this is a linear-phase FIR filter) and the frequency response magnitude $|H^f(\omega)|$.

3.17 Very simple signal smoothing can be performed by a moving average of N consecutive samples,

$$v(n) = \frac{1}{N} \sum_{k=0}^{N-1} x(n-k)$$

where $x(n)$ is the input signal. In the following, assume $N = 6$. Consider a filter implemented by running a signal **twice** through the N-point moving average. That is, the filter consists of two N-point moving average systems in cascade.

(a) Sketch the impulse response $h(n)$ of the total filter.

(b) What is the dc gain of the total filter? That means, find $H^f(0)$.

(c) Find the zeros of the transfer function and accurately sketch the pole/zero diagram.

(d) Based on the zero diagram in (c), sketch the frequency response magnitude $|H(\omega)|$.

3.18 Consider a filter implemented using the difference equation,

$$y(n) = x(n) - x(n-N) + y(n-1)$$

where $x(n)$ is the signal being filtered.

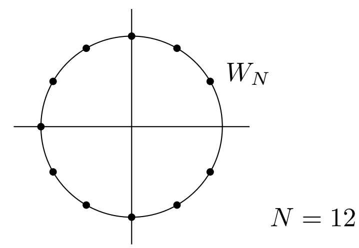
(a) Find the zeros of the transfer function and sketch the pole/zero diagram.

(b) Based on the pole-zero diagram, roughly sketch the frequency response magnitude $|H(\omega)|$. Explicitly indicate any frequency response nulls if there are any.

(c) Sketch the impulse response of the filter.

(d) What is the dc gain of the filter?

3.19 The transfer function $H(z)$ of an FIR filter has zeros in the z-plane as illustrated.



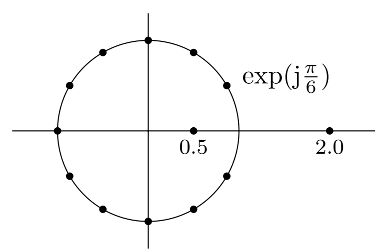
The zeros on the unit circle are at powers of W_{12}. The dc-gain of the filter is 2. All the poles are at $z = 0$.

(a) Accurately sketch the impulse response of the filter.

(b) Sketch the frequency response magnitude $|H(\mathrm{e}^{\mathrm{j}\,\omega})|$. Accurately indicate the nulls of the frequency response.

(c) Sketch the phase of the frequency response $\angle H(\mathrm{e}^{\mathrm{j}\,\omega})$.

3.20 The transfer function $H(z)$ of an FIR filter has 13 zeros in the z-plane as illustrated.



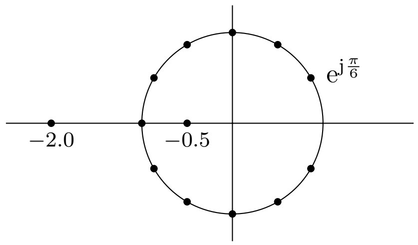
The zeros on the unit circle are at powers of W_{12}. The dc-gain of the filter is unity. All the poles are at $z = 0$.

(a) Accurately sketch the impulse response of the filter.

(b) Sketch the frequency response magnitude $|H(\mathrm{e}^{\mathrm{j}\,\omega})|$. Accurately indicate the nulls of the frequency response.

(c) Sketch the phase of the frequency response $\angle H(\mathrm{e}^{\mathrm{j}\,\omega})$.

3.21 The transfer function $H(z)$ of an FIR filter has 13 zeros in the z-plane as illustrated.



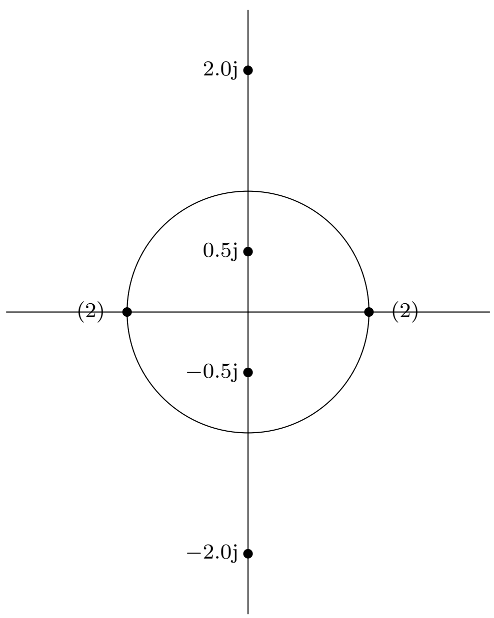
The zeros on the unit circle are at powers of W_{12}. The dc-gain of the filter is unity. All the poles are at $z = 0$.

(a) Accurately sketch the impulse response of the filter.

(b) Sketch the frequency response magnitude $|H(e^{j\omega})|$. Accurately indicate the nulls of the frequency response.

(c) Sketch the phase of the frequency response $\angle H(e^{j\omega})$.

3.22 **Transfer functions.** The transfer function $H(z)$ of a causal FIR filter has zeros in the z-plane as illustrated.



The roots on the unit circle are double zeros.

(a) What is the dc gain of the filter?

(b) Accurately sketch the impulse response of the filter (as far as it can be determined).

(c) Sketch the frequency response magnitude $|H(e^{j\omega})|$. Accurately indicate the nulls of the frequency response.

(d) Accurately sketch the phase of the frequency response $\angle H(e^{j\omega})$.

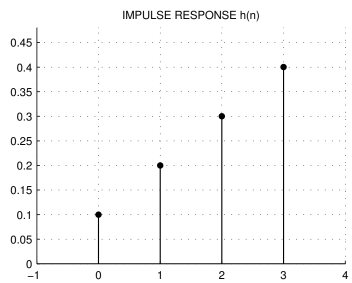
(e) Find a spectral factorization of $H(z)$. Accurately sketch the impulse response of the spectral factor and its zero-diagram.

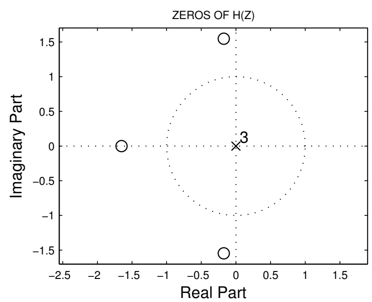
3.23 **Filter Transformations.** An N-point FIR filter with impulse response $h(n)$ is illustrated below. Define a new FIR filter with complex-valued impulse response $g(n) = (\mathrm{j})^n \, h(n)$.

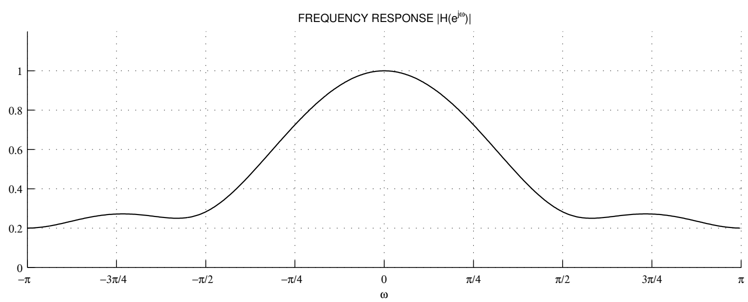
(a) Sketch the real and imaginary parts of the impulse response of the new filter, $g(n)$.

(b) What is the relationship between the zeros of $G(z)$ and the zeros of $H(z)$? Based on the zero-diagram of $H(z)$ shown below, sketch the zero-diagram of $G(z)$.

(c) Find an expression for the frequency response $G^f(\omega)$ in terms of the frequency response $H^f(\omega)$. Sketch $|G^f(\omega)|$.

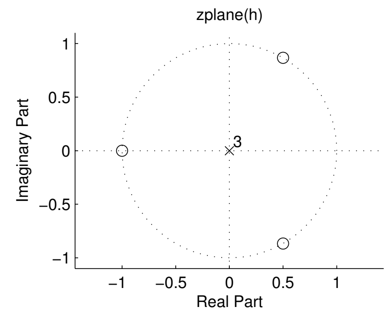






3.24 **Filter Transformations.**

(a) Here is the zero diagram of a linear-phase FIR filter with transfer function $H(z)$. Sketch the frequency response magnitude of the filter, $|H(e^{j\omega})|$.



(b) Find and sketch the impulse response $h(n)$. (The zeros of $H(z)$ are evenly spaced on the unit circle.)

(c) Define $F(z) = H(-z)$. Sketch the zero-diagram of $F(z)$ and the impulse response $f(n)$. Sketch the frequency response magnitude of the filter, $|F(e^{j\omega})|$.

(d) Define $S(z) = H(z^2)$. Sketch the zero-diagram of $S(z)$ and the impulse response $s(n)$. Sketch the frequency response magnitude of the filter, $|S(e^{j\omega})|$.

(e) Define $G(z) := H(z)\, H(1/z)$. Sketch the zero-diagram of $G(z)$ and the impulse response $G(n)$. Sketch the frequency response magnitude of the filter, $|G(e^{j\omega})|$.

3.25 **Filter specifications.** An analog signal, bandlimited to 20 Hz is corrupted by high-frequency noise. The spectrum of the noise is from 25 Hz to 35 Hz. The noisy analog signal is sampled at 60 Hz. A digital lowpass filter is to be designed so as to remove the noise from the signal. What would you choose for the desired frequency response $D(\omega)$ and weighting function $W(\omega)$? Sketch those functions for $0 \le \omega \le \pi$ where ω is normalized frequency (radians/sample).

3.26 **Filter Specifications.** The spectrum of an analog signal, $x(t)$, is contained between 5 Hz and 30 Hz. The signal is contaminated by additive noise in the frequency band 50 Hz - 60 Hz. The contaminated signal is then sampled 100 times per second. A linear-phase FIR digital filter is to be designed so as to remove the noise from the signal. Consider the design of the filter using either the weighted least-square filter design method or the Remez algorithm. These filter design methods entail choosing a desired frequency response $D(\omega)$ and weighting function $W(\omega)$.

What would you choose for the desired frequency response $D(\omega)$ and weighting function $W(\omega)$? Sketch those functions for $0 \leq \omega \leq \pi$ where ω is normalized frequency (radians/sample).

3.27 **Filter specifications.** An analog signal, bandlimited to 10 Hz is corrupted by high-frequency noise. The spectrum of the noise is from 20 Hz to 40 Hz. The noise between 30 Hz to 40 Hz is twice as strong as the noise between 20 Hz and 30 Hz. The noisy analog signal is sampled at 80 samples/second. A digital lowpass filter is to be designed so as to remove the noise from the signal. What would you choose for the desired frequency response $D(\omega)$ and weighting function $W(\omega)$? Sketch those functions for $0 \leq \omega \leq \pi$ where ω is in units of radians/sample.

3.28 **Filter specifications.** An analog signal, bandlimited to 15 Hz is corrupted by high-frequency noise. The spectrum of the noise is from 20 Hz to 30 Hz. The noisy analog signal is sampled at 50 Hz. A digital lowpass filter is to be designed so as to remove the noise from the signal. What would you choose for the desired frequency response $D(\omega)$ and weighting function $W(\omega)$? Sketch those functions for $0 \leq \omega \leq \pi$ where ω is normalized frequency (radians/sample).

3.29 **Filter specifications.** An analog signal, bandlimited to 10 Hz is corrupted by high-frequency noise. The spectrum of the noise is from 40 Hz to 70 Hz. The noisy analog signal is sampled at 100 samples/second. A digital lowpass filter is to be designed so as to remove the noise from the signal. What would you choose for the desired frequency response $D(\omega)$ and weighting function $W(\omega)$? Sketch those functions for $0 \leq \omega \leq \pi$ where ω is normalized frequency (radians/sample).

3.30 **Filter specifications.** The spectrum of an analog bandpass signal is contained between 50 Hz and 80 Hz. The signal is corrupted by low-frequency noise; the noise is bandlimited to 30 Hz. The noisy analog signal is sampled at 200 samples/second. A digital filter is to be designed so as to remove the noise from the signal. What would you choose for the desired frequency response $D(\omega)$ and weighting function $W(\omega)$? Sketch those functions for $0 \leq \omega \leq \pi$ where ω is normalized frequency (radians/sample).

3.31 **Filter Specifications.** An analog signal, bandlimited to 30 Hz is corrupted by high-frequency noise. The spectrum of the noise is from 50 Hz to 80 Hz. The noisy analog signal is sampled at 200 Hz. A digital lowpass filter is to be designed so as to remove the noise from the signal. What would you choose for the desired frequency response $D(\omega)$ and weighting function $W(\omega)$? Sketch those functions for $0 \leq \omega \leq \pi$ where ω is normalized frequency (radians/sample).

3.32 **Filter Specifications.** The spectrum of an analog signal, $x(t)$, is contained between 5 Hz and 30 Hz. The signal is contaminated by a 60 Hz hum. The contaminated signal is then sampled at 100 Hz. A linear-phase FIR digital filter is to be designed so as to remove the 60 Hz noise from the signal. To ensure that the frequency response has a null at 60 Hz the constrained weighted least-square filter design method will be used to insert a null (zero) at 60 Hz. That filter design method entails choosing a desired frequency response $D(\omega)$ and weighting function $W(\omega)$.

What would you choose for the desired frequency response $D(\omega)$ and weighting function $W(\omega)$? Sketch those functions for $0 \leq \omega \leq \pi$ where ω is normalized frequency (radians/sample). In normalized frequency, at what frequency ω_o should the null be located? ($0 \leq \omega_o \leq \pi$). Provide an explanation for your answer.

3.33 **Filter specifications.** An analog signal, bandlimited to 10 Hz is corrupted by high-frequency noise. The spectrum of the noise is from 30 Hz to 50 Hz. The noisy analog signal is sampled at 70 Hz. A digital lowpass filter is to be designed so as to remove the noise from the signal. For the filter design problem, what would you choose for the desired frequency response $D(\omega)$ and error weighting function $W(\omega)$? Sketch those functions for $0 \leq \omega \leq \pi$ where ω is normalized frequency (radians/sample). Clearly explain using appropriate spectral diagrams.

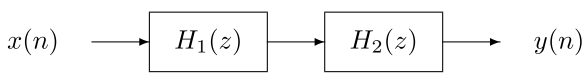
4 Linear-Phase FIR Digital Filters

4.1 (Porat 9.4) The impulse response of a causal linear-phase FIR filter starts with the values

$$h(0) = 1, \quad h(1) = 3, \quad h(2) = -2.$$

Find the shortest FIR impulse response $h(n)$ for each of the four types. ($h(3) =?$, $h(4) =?$, etc.)

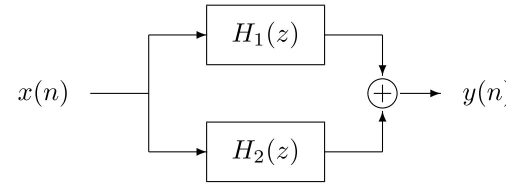
4.2 (Based on Porat 9.7) When two linear-phase FIR filters $h_1(n)$ and $h_2(n)$ are connected in cascade as shown the total filter linear-phase.



What is the filter type of the total filter as a function of the types of $h_1(n)$ and $h_2(n)$? (Fill in the table.)

TYPE (I, II, III, or IV)		
$h_1(n)$	$h_2(n)$	$h_1(n) * h_2(n)$
I	I	?
I	II	?
I	III	?
I	IV	?
II	II	?
II	III	?
II	IV	?
III	III	?
III	IV	?
IV	IV	?

If two linear-phase FIR filters $h_1(n)$ and $h_2(n)$ are connected in parallel as shown, is the total filter linear-phase?



4.3 (Porat 9.5) Let $H(z)$ be the transfer function of a linear-phase FIR filter with real coefficients. The filter is known to have zeros in the following locations:

$$z = \left\{1, \quad 0.5\,e^{j\pi/3}, \quad -5, \quad j\right\}$$

(a) Because $H(z)$ is a linear-phase FIR filter it must have other zeros as well, because the zeros of a linear-phase FIR must exist in particular configurations. What are the other zeros of $H(z)$? Sketch the zero diagram of this filter.
(b) What is the minimal length of the filter?
(c) What is the filter type of the minimal length filter? (I,II,III, or IV)

4.4 Consider a Type-I linear-phase FIR filter $H(z)$ (with real-valued impulse response). You are told that $H(z)$ has zeros at $z = 1$ and $z = 2 + 2j$.

(a) What other zeros must $H(z)$ have? Sketch the zero-diagram.
(b) Find the impulse response, $h(n)$.

4.5 For the transfer function

$$H(z) = z^{-2} - z^{-6}$$

of an FIR linear-phase filter,

(a) sketch the impulse response
(b) what is the type of the filter (I, II, III, or IV)?
(c) sketch the frequency response magnitude $|H^f(\omega)|$.
(d) sketch the zero diagram

4.6 For the transfer function

$$H(z) = z^{-1} + z^{-6}$$

of an FIR linear-phase filter,

(a) sketch the impulse response
(b) what is the type of the filter (I, II, III, or IV)?
(c) sketch the frequency response magnitude $|H^f(\omega)|$.
(d) sketch the zero diagram

4.7 (a) List all of the four types of linear-phase FIR filters can be used for the implementation of a *bandpass* filter.

(b) List all of the four types of linear-phase FIR filters can be used for the implementation of a *bandstop* filter.

4.8 **The real-valued amplitude response.** The frequency response of a linear-phase FIR filter can be written as

$$H(\omega) = A(\omega)\, e^{j\theta(\omega)}$$

where amplitude response $A(\omega)$ is continuous and real-valued. Write a Matlab function that uses the FFT to compute $A(\omega)$ from the impulse response $h(n)$ at the frequency points

$$\omega_k = \frac{2\pi}{L} k \qquad 0 \le k \le L-1.$$

The input to your Matlab function should be a vector `h` containing the values of the impulse response, the filter Type (1,2,3,4), and the number of samples L. You may assume that L is greater than the length of the filter. The Matlab function should provide vectors `A` and `w` such that `plot(w,A)` gives a plot of $A(\omega)$.

An example Matlab header:

```
function [A,w] = firamp(h,type,L)
% [A,w] = firamp(h,type,L)
% Amplitude response of a linear-phase FIR filter
% A : amplitude response at the frequencies w
% w : [0:L-1]*(2*pi/L);
% h : impulse response
% type = [1,2,3,4]
```

Verify your Matlab function works correctly for each of the 4 filter types by using it to plot $A(\omega)$ for the following filters.

- Test for Type I using the following impulse response.

```
N = 29;
n = 0:N-1;
wo = 0.34*pi;
h1 = (wo/pi)*sinc((wo/pi)*(n-(N-1)/2));
```

- Test for Type II using the truncated sinc function again.

```
N = 28;
n = 0:N-1;
wo = 0.34*pi;
h2 = (wo/pi)*sinc((wo/pi)*(n-(N-1)/2));
```

- Test for Type IV using a differentiator with length 16

```
N = 16;
n = 0:N-1;
h4 = (-1).^(n-(N-2)/2)./(pi*(n-(N-1)/2).^2);
```

- Test for Type III using a *lowpass* differentiator — obtained by convolving the impulse response of a Type II lowpass filter and Type IV differentiator.

```
h3 = conv(h2,h4);
```

4.9 **Filter transformations.** Let $h(n)$ be the Type I linear-phase lowpass FIR filter obtained in the previous problem. Consider the three transformations:

A) $G(z) = H(-z)$

B) $G(z) = z^{-M} - H(z)$

C) $G(z) = z^{-M} - H(-z)$

Lets denote the amplitude response of $h(n)$ by $A_h(\omega)$ and the amplitude response of $g(n)$ by $A_g(\omega)$. For each of the filter transformations,

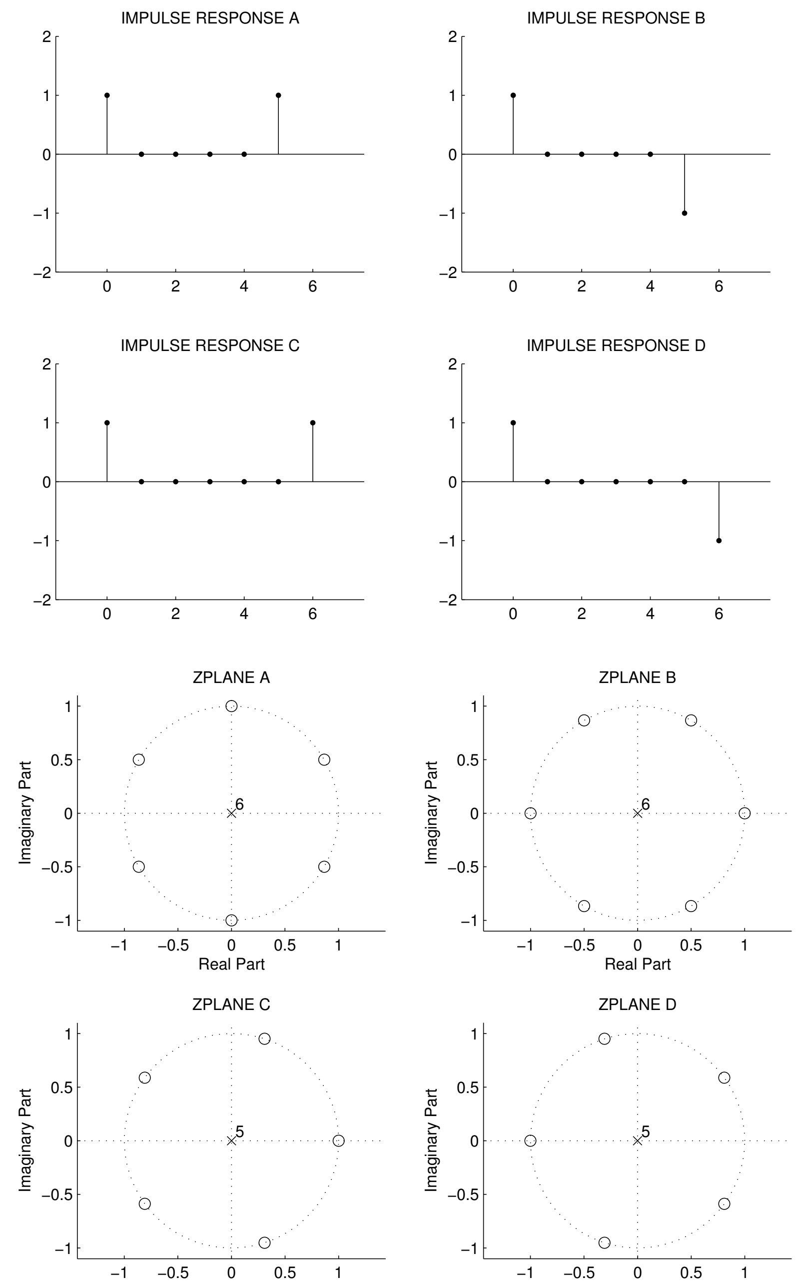
(a) Find an expression for $g(n)$ in terms of $h(n)$.

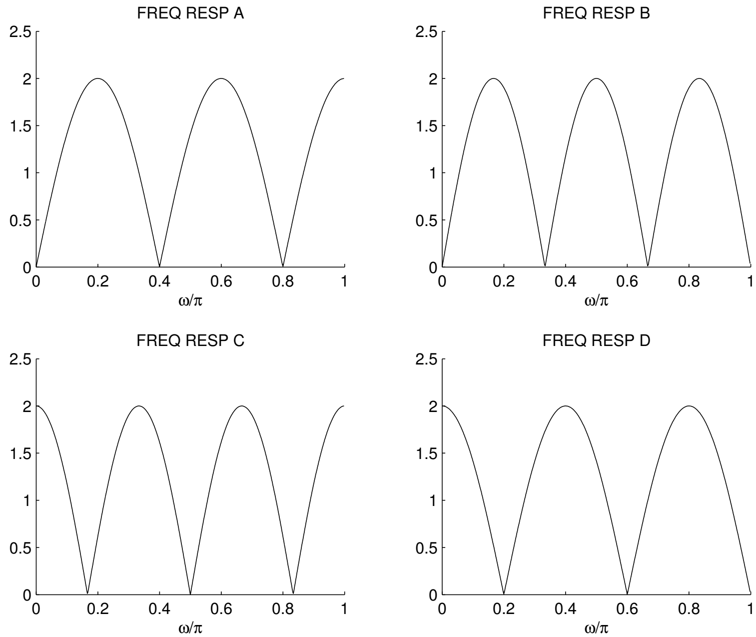
(b) Find an expression for $A_g(\omega)$ in terms of $A_h(\omega)$.

(c) Plot $g(n)$, $A_g(\omega)$ and use `zplane` to plot the zeros of $G(z)$. (In the command `zplane(g)`, the impulse response `g` should be a row vector. If it is a column vector you can type `zplane(g')`.) Use the Matlab command `subplot` to save paper. Also use `orient tall` so that when printed out, the plots use the whole page.

(d) If $H(z)$ is a low-pass filter with cutoff frequency ω_o, what kind of filter is $G(z)$ and what is its cutoff frequency?

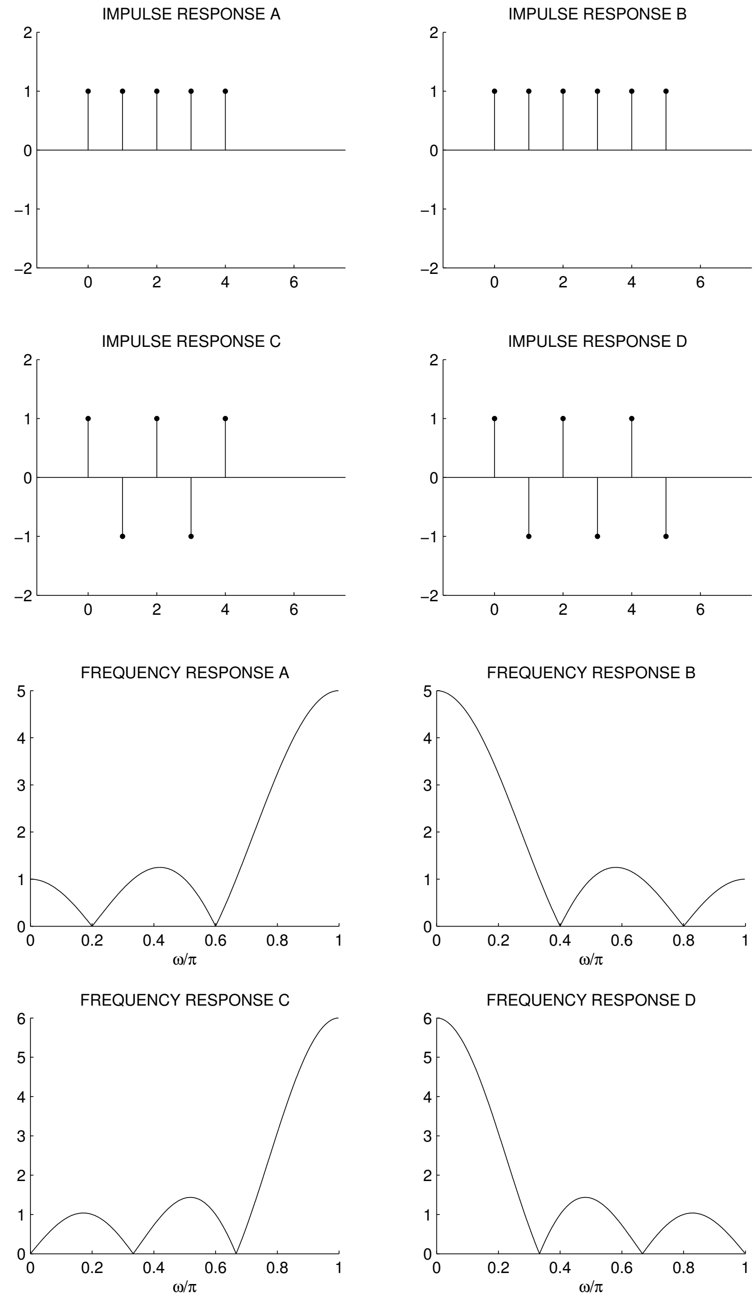
4.10 **Matching.** Match each impulse response with its frequency response and zero diagram. You should do this problem with out using a computer.

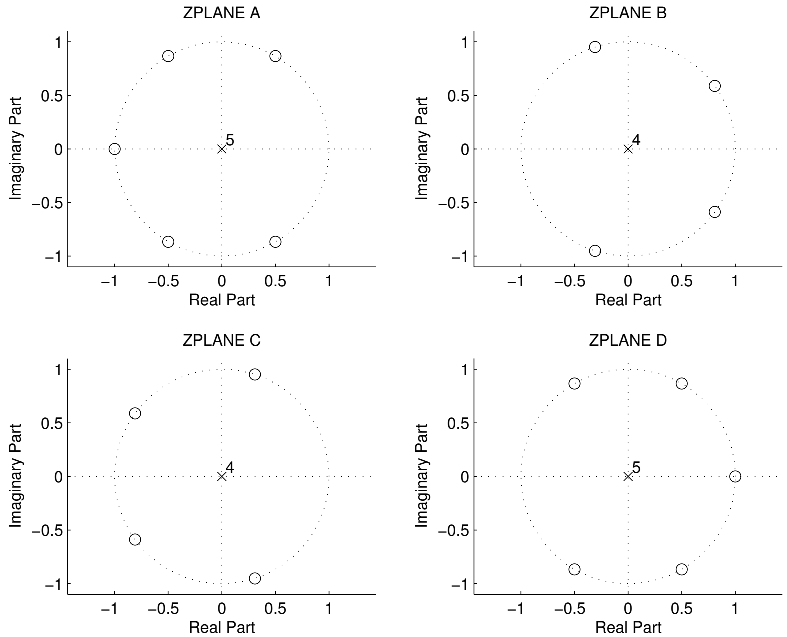




Impulse Response	Zero Diagram	Frequency Response
A		
B		
C		
D		

4.11 **Matching.** Match each impulse response with its frequency response and zero diagram by filling out the following table. You should do this problem with out using a computer.

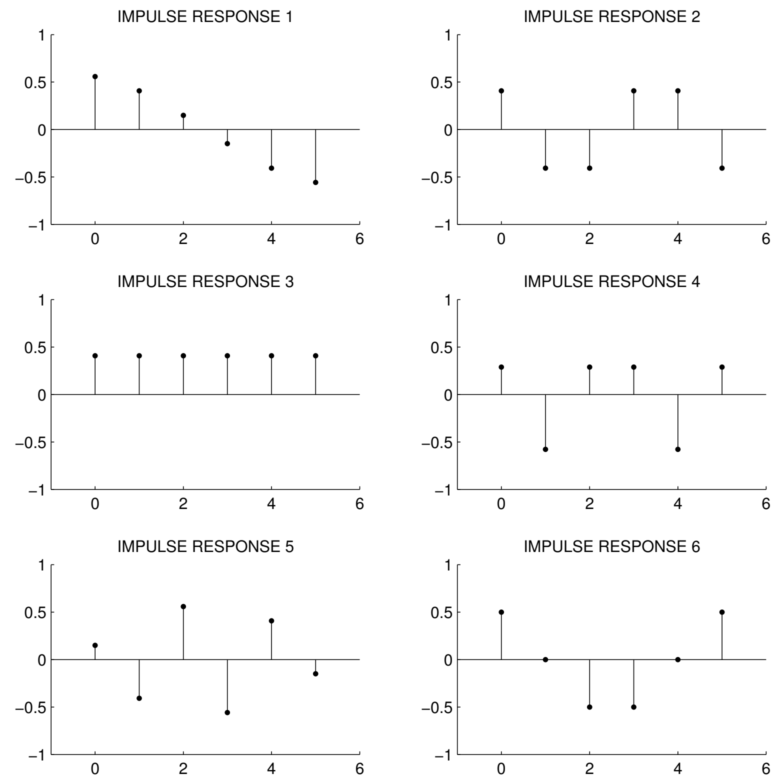


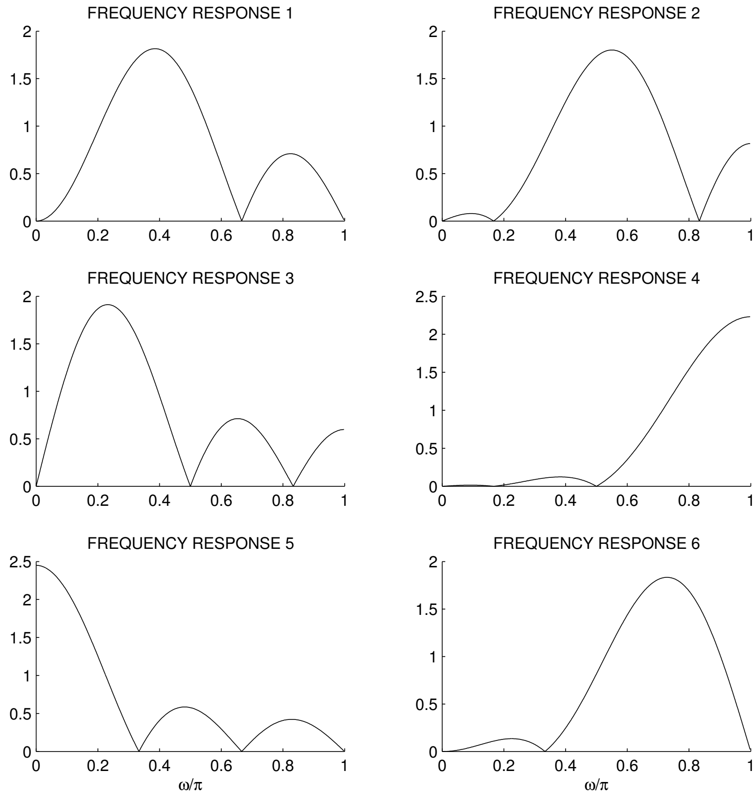


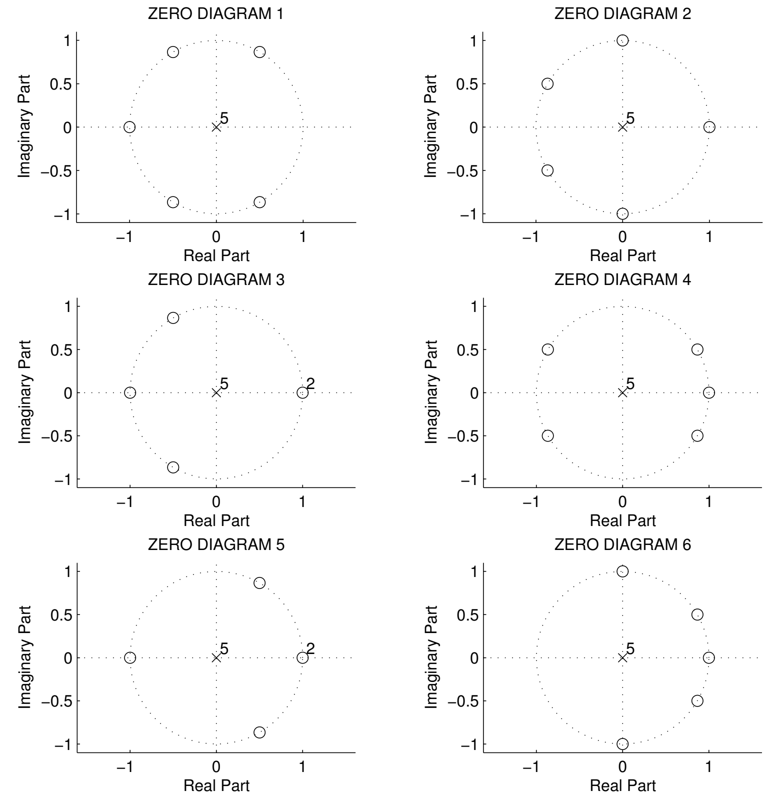
Impulse Response	Zero Diagram	Frequency Response
A		
B		
C		
D		

4.12 **Matching.** The following figures show 6 impulse responses, frequency responses, and zero diagrams. Match each frequency response and zero diagram to the corresponding impulse response.

Impulse Response	Frequency Response	Zero Diagram
1		
2		
3		
4		
5		
6		







4.13 **Double zeros at $z = 1$ and $= -1$** This problem deals with linear-phase FIR filters with zeros at $z = 1$ and $z = -1$ of multiplicity two.

For each of the following linear-phase filters: sketch the zero diagram, impulse response, amplitude response, and classify each filter as type I, II, III, IV. Comment on your observations, especially with regard to what you know about the zero locations of linear-phase filters.

(a)

$$H(z) = (1 - z^{-1})(1 + z^{-1})$$

(b)

$$H(z) = (1 - z^{-1})^2(1 + z^{-1})$$

(c)

$$H(z) = (1 - z^{-1})(1 + z^{-1})^2$$

(d)

$$H(z) = (1 - z^{-1})^2 (1 + z^{-1})^2$$

4.14 Show a derivation for each of your answers. Do not use MATLAB to answer any part of this question.

(a) The transfer function of an FIR digital filter is

$$H_1(z) = (1 + z^{-1})^3$$

Which of the four types of linear-phase FIR filter is this?

(b) The transfer function of a second FIR digital filter is

$$H_2(z) = H_1(z)\,(1 - z^{-1})^2$$

Which of the four types of linear-phase FIR filter is this?

(c) The transfer function of a third FIR digital filter is

$$H_3(z) = H_1(-z)$$

Which of the four types of linear-phase FIR filter is this?

(d) The transfer function of a fourth FIR digital filter is

$$H_4(z) = H_2(-z)$$

Which of the four types of linear-phase FIR filter is this?

(e) Characterize each of these filters as lowpass, highpass, or bandpass.

4.15 **Design by DFT-based interpolation.** Implement the DFT-based interpolation approach for Type II. (Modify the code in the notes for the Type I case.) In particular, consider the design of a length $N = 12$ low-pass Type II FIR filter where the amplitude function $A(\omega)$ interpolates 1 and 0 as follows.

$$A\left(\frac{2\pi}{N}k\right) = \begin{cases} 1 & k = 0, 1, 2 \\ 0 & k = 3, 4, 5, 6 \end{cases}$$

What values should $A(2\pi k/N)$ interpolate for $7 \le k \le N - 1$? (Look at the characteristics of the $A(\omega)$ function in the notes.) Find the Type II filter $h(n)$ that interpolates those values. Make a plot of $A(\omega)$, $h(n)$, and the zeros of $H(z)$ using `zplane`. Check that $A(\omega)$ interpolates the specified points.

Note: the impulse response $h(n)$ should be *real*. If you get complex values for $h(n)$ then you have made a mistake.

Also, use this approach to design a Type II filter of length $N = 32$.

Optional: Implement the DFT-based interpolation approach for the design of Type III and IV FIR filters and design an example of each.

4.16 A student is asked to design a Type I and a Type II low-pass FIR linear-phase filter using DFT-based interpolation. The student turns in the work shown below, which has two problems.

(a) The student does not provide an explanation.

(b) One of the solutions is correct, the other solution is fake (the student just *made up* the vector `h`, it does not follow from the MATLAB commands).

Identify which solution is correct and provide an explanation for it. Why does the fake solution not work? You should be able to do this problem with out using MATLAB.

```
TYPE I FIR IMPULSE RESPONSE:

>> H = [1 1 1 1 1 0 0 0 0 0 0 1 1 1 1]';
>> v = ifft(H);
>> h = [v(9:15); v(1:8)]

h =

    0.0394
   -0.0667
         0
    0.0853
   -0.0667
   -0.0963
    0.3050
    0.6000
    0.3050
   -0.0963
   -0.0667
    0.0853
         0
   -0.0667
    0.0394
```

```
TYPE II FIR IMPULSE RESPONSE:

>> H = [1 1 1 1 1 0 0 0 0 0 1 1 1 1]';
>> v = ifft(H);
>> h = [v(9:14); v(1:8)]

h =

   -0.0318
   -0.0494
    0.0891
   -0.0255
   -0.1287
    0.2892
    0.6429
    0.6429
    0.2892
   -0.1287
   -0.0255
    0.0891
   -0.0494
   -0.0318
```

4.17 *Optional*: Mitra 4.19. (But use the frequencies 0.2π, 0.4π, 0.9π)

4.18 Let $h(n)$ be a (real-valued) FIR impulse response, not necessarily linear-phase. Then the impulse response defined by convolving $h(n)$ with its time-reversed version is always a linear-phase FIR filter. In other words, $p(n) := h(n) * h(-n)$, is a linear-phase FIR impulse response. The question is: which of the four linear-phase FIR filter types can $p(n)$ be?

4.19 (a) A digital filter is implemented using the difference equation

$$y(n) = x(n) + x(n-1) - 0.2y(n-1).$$

Does this system have the linear-phase property? Explain.

(b) Why is it desirable for a filter to have linear phase?

4.20 A digital filter is implemented using the difference equation

$$y(n) = 3\,x(n) + 2\,x(n-1) + x(n-2).$$

Does this system have the linear-phase property?

4.21 The frequency response of a linear-phase discrete-time LTI filter can be written as

$$H(\omega) = A(\omega)\,\mathrm{e}^{\mathrm{j}(a\omega+b)}, \qquad |\omega| < \pi \tag{2}$$

where $A(\omega)$ is a real-valued function called the 'amplitude response', and $a,\, b \in \mathbb{R}$.

(a) Which of the four basic types of FIR filter (I, II, III, IV) have $A(\omega) = A(-\omega)$? Which types have $A(\omega) = -A(-\omega)$? Show a clear derivation.

(b) The linear-phase filter (2) is cascaded with an ideal delay system $D(\omega)$ with delay τ samples, $\tau \in \mathbb{R}$.

$$x(n) \longrightarrow \boxed{H(\omega)} \longrightarrow \boxed{D(\omega)} \longrightarrow y(n)$$

Does the total system have linear-phase? If so, what is the amplitude response of the total system in terms of $A(\omega)$? Show a clear derivation.

(c) Consider an ideal band-pass Hilbert transformer with frequency response

$$H(\omega) = \begin{cases} -\mathrm{j}, & 0.2\pi < \omega < 0.8\pi \\ \mathrm{j}, & -0.8\pi < \omega < -0.2\pi \\ 0, & \text{for other } \omega \in [-\pi, \pi] \end{cases} \tag{3}$$

Does it have linear-phase? If so, what is its amplitude response? Show a clear derivation.

For system (3), find the output produced by input signal $x(n) = 2\sin(0.1\pi n) + 3\cos(0.3\pi n)$.

(d) The ideal band-pass Hilbert transformer (3) can not be realized. Consider the design of a real linear-phase FIR filter to approximate it. Which of the four basic filters types are appropriate? Explain.

(e) Use the 'impulse response truncation' method to design a real linear-phase FIR filter approximating the band-pass Hilbert transform (3). Provide an explicit formula for the impulse response $h(n)$. Simplify so that j does not appear. Which of the four types is your filter?

4.22 (a) Show that the transfer function $H(z)$ of a Type I FIR filter will always have a zero of *even* multiplicity at $z = -1$. One way to do this is by contradiction. Assume the zero if of odd multiplicity: write $H(z)$ as

$$H(z) = (z^{-1} + 1)^{2K+1}\, G(z)$$

where $G(z = -1) \neq 0$. Also use the fact that

$$h(n) = h(N - 1 - n)$$

can be written as

$$H(z) = H(1/z)\, z^{-(N-1)}.$$

Then derive a contradiction.

(b) Suppose $h(n)$ is a Type II FIR filter. We learned that the transfer function $H(z)$ must have a zero at $z = -1$. That is: $H(-1) = 0$.

Classify the following statement as *true or false*. $H(z)$ always has an *odd* number of zeros at $z = -1$.

If true, give a derivation. If false, give an example of a Type II FIR filter with an even number of zeros at $z = -1$.

4.23 An FIR filter with a complex-valued impulse response can have linear phase. Find the frequency response of each of the two FIR filters with the impulse responses:

```
h1 = [1 j 2 j 1];
h2 = [1 j 2 -j 1];
```

One of these two FIR filters has linear phase. Which one? What is its the phase response?

4.24 **Complex-valued Linear-phase FIR Filters.** An FIR filter with a complex-valued impulse response can have linear phase. Which of the following FIR filters have linear phase?

```
h1 = [1+j  2-j  2+2j  2-2j  2+j  1-j];

h2 = [1+j  2-j  2+2j  2+2j  2-j  1+j];

h3 = [1+j  2-j  2+2j  5  2-2j  2+j  1-j];

h4 = [1+j  2-j  2+2j  5 2+2j  2-j  1+j];

h5 = [1+j  2-j  2+2j  5j  2-2j  2+j  1-j];
```

For those filters that you claim have linear phase, can you show why they have linear phase?

4.25 Consider a Type I FIR filter of length N. The impulse response $h(n)$ is non-zero for $0 \le n \le N-1$. We have learned how to compute the real-valued amplitude response $A(\omega)$ on a uniform grid. Hans has suggested another way. He defines the length L signal $g(n)$

$$g(n) = \begin{cases} h(n+M) & 0 \le n \le M \\ 0 & M+1 \le n \le L-1 \end{cases}$$

where $M = (N-1)/2$ as usual. (This is simply the second half of $h(n)$ including the midpoint.) Then he computes:

$$C(k) = 2\,\mathrm{Real}\{\mathrm{DFT}_L\{g(n)\}\} - h(M)$$

for $0 \le k \le L-1$. Hans claims that

$$C(k) = A\left(\frac{2\pi}{L}k\right).$$

Is he correct? If so, derive the correctness of his formula. If not, show why not.

To make it clear, suppose

```
h = [1, 2, 4, 2, 1]
```

Then in Matlab notation, Hans claims that

```
C = 2*real(fft([4 2 1 zeros(1,L-3)])) - 4;
```

will compute L samples of $A(\omega)$ for $\omega = \frac{2\pi}{L}k$, $0 \le k \le L-1$.

5 Windows

5.1 **Linear-phase FIR filter design using windows** (Based on Mitra M7.18, M7.19)

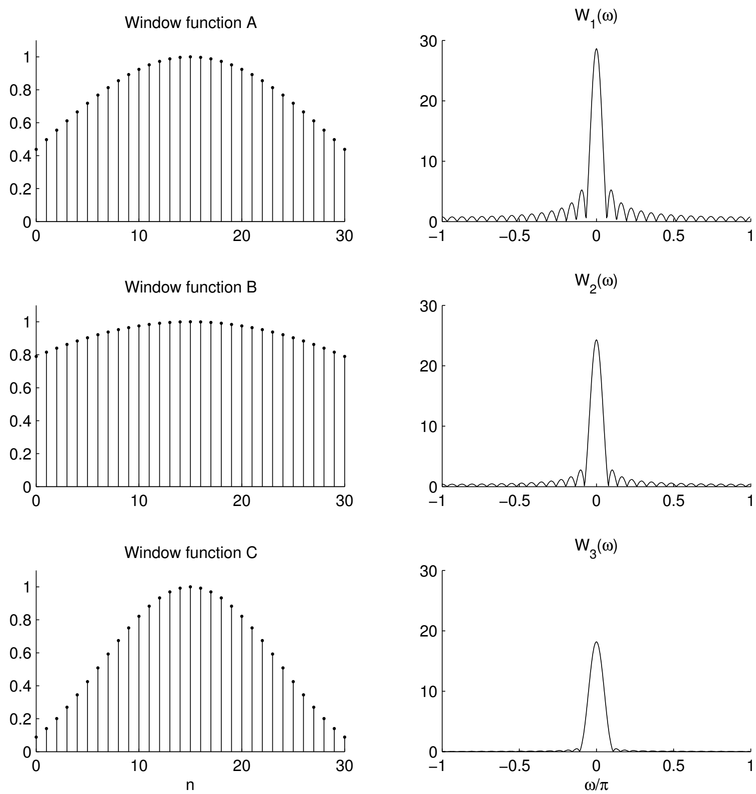
Use each of the Hamming, Hann, Blackman, and Kaiser windows to design four linear-phase FIR digital lowpass filters. Each of the four filters should be of the same length. It is desired that the filters meet the following specifications: passband edge f_p at 2 Hz, stopband edge f_s at 4 Hz, maximum passband attenuation of 0.1 dB, minimum stopband attenuation of 40 dB. The filter is to operate at a sampling frequency F_s of 20 Hz.

First, determine the length of the filter using the MATLAB command `kaiserord` and the specifications listed above. Use the length provided by `kaiserord` for each of the four filters. You should find that the Kaiser window leads to a filter that meets the specifications, but that the other windows lead to filters that do not quite meet the specification.

Display the amplitude response and magnitude response in dB. You will use the commands `sinc`, `hamming`, `hanning`, `blackman`, and `kaiser`. (Don't use the `fir1` command, although you may use it to check your work.)

Comment on your observations

5.2 **Windows.** Below are shown three different window functions, and the discrete-time Fourier transform of each. Match each window function with its Fourier transform.

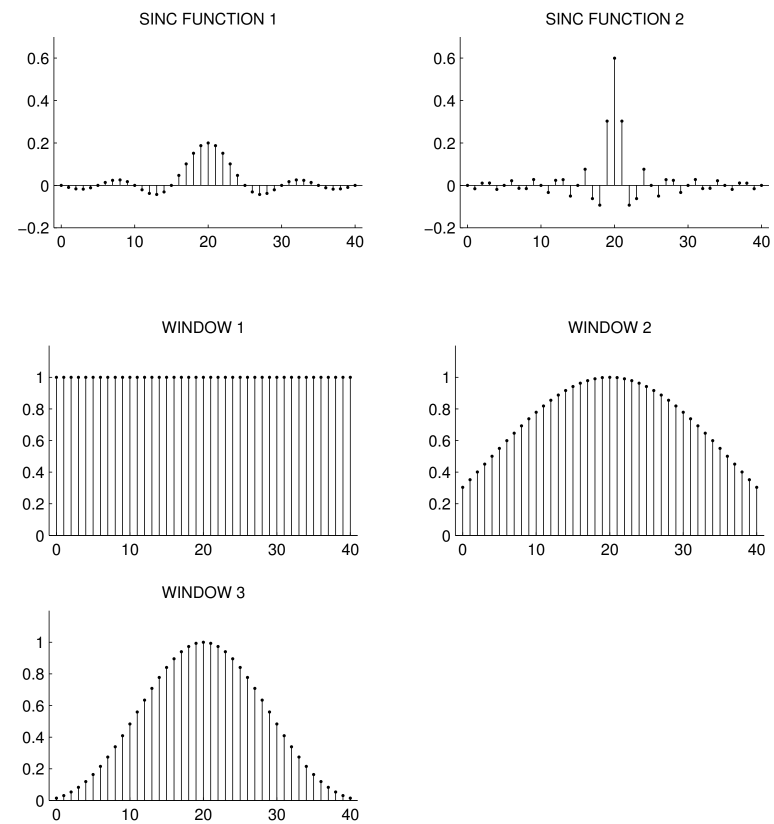


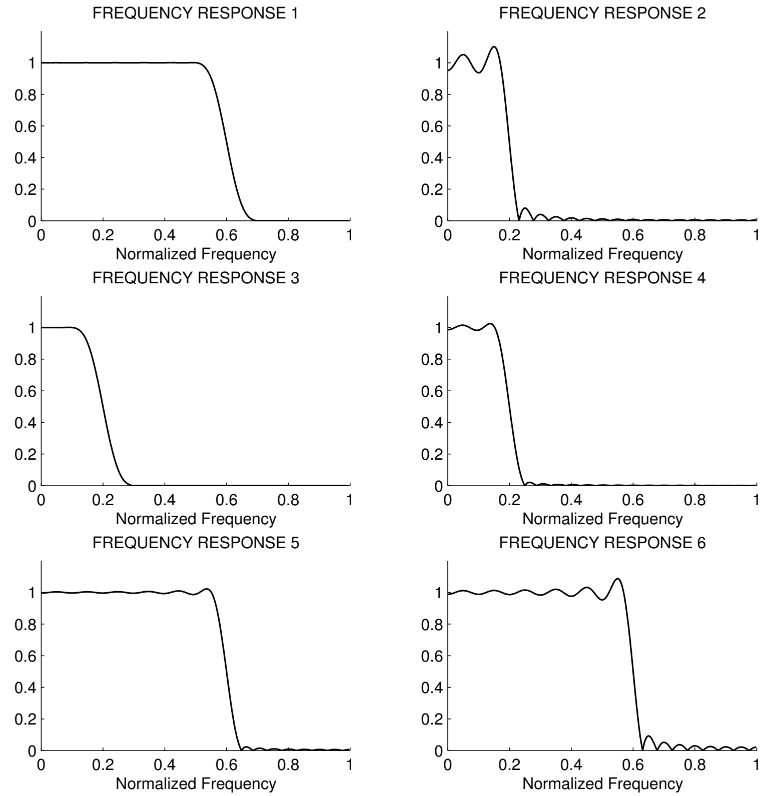
5.3 **Low-pass FIR Filter Design by Windows.** The design of a low-pass FIR filter using a window provides an impulse response of the form

$$h(n) = d(n) \cdot w(n)$$

where $d(n)$ is the inverse DTFT of the ideal low-pass frequency response. The figures below illustrate two different sinc functions $d(n)$ and 3 different window functions $w(n)$. In each case identify the frequency response of the resulting filter $h(n)$ by filling the table below.

Sinc Function	Window	Frequency Response
1	1	
1	2	
1	3	
2	1	
2	2	
2	3	



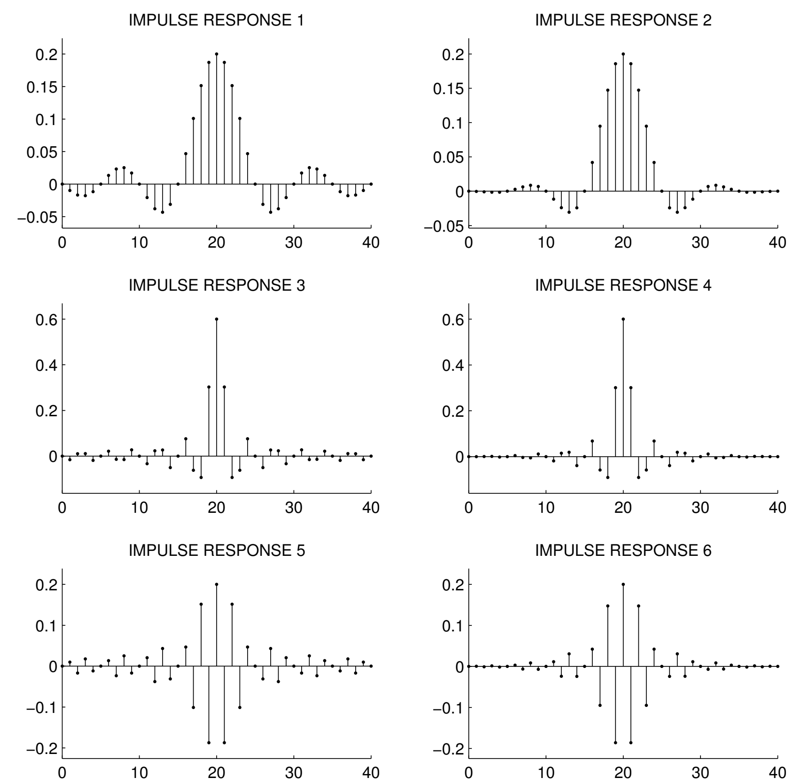


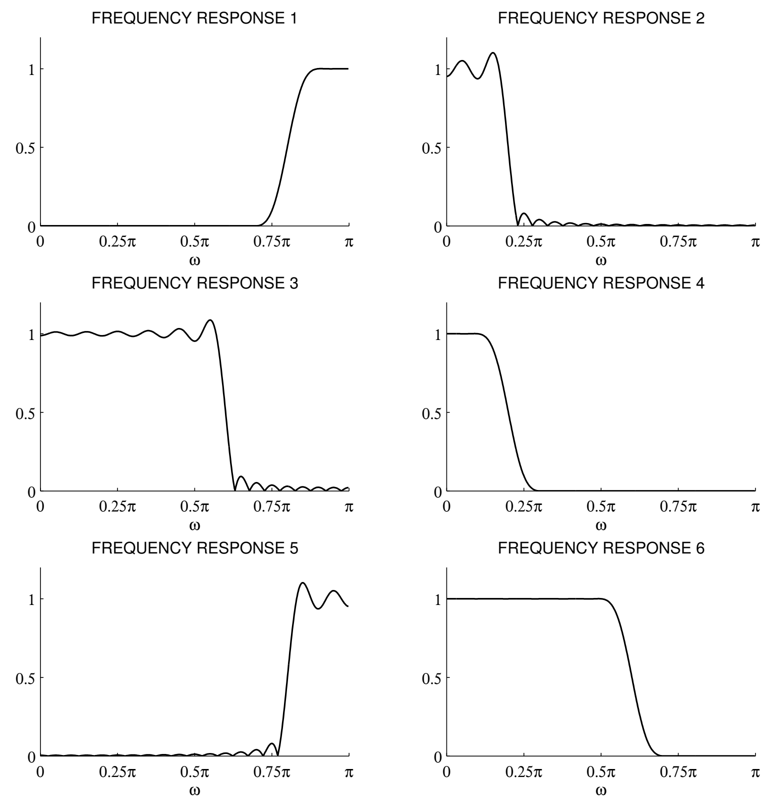
5.4 **Linear-Phase FIR Filter Design by Windows.** The design of a linear-phase FIR filter using the window method gives an impulse response of the form

$$h(n) = w(n) \cdot d(n)$$

where $w(n)$ is the window function and $d(n)$ is the inverse DTFT of the ideal frequency response. The figures below show the impulses responses and frequency responses of six filters designed using the window method. But they are out of order. Match each frequency response to its impulse response by completing the table.

Impulse Response	Frequency Response
1	
2	
3	
4	
5	
6	





5.5 (Porat 6.1) We saw that the height of the largest side-lobe of the rectangular window is about -13.5 dB relative to the main-lobe. What is the relative height of the smallest side-lobe? You may assume the window length N is odd. (Hint: At what frequency will it be located?).

5.6 (Porat 6.7) We are given 128 samples of the signal

$$x(n) = \sin\left(2\pi \frac{6.3}{128} n\right) + 0.001 \sin\left(2\pi \frac{56}{128} n\right).$$

(a) Explain why a rectangular window is not adequate for detecting the second component. (Hint: the previous problem is relevant.)

(b) Of the Hann and Hamming windows, guess which one is better in this case for detecting the second component, then test your guess on a computer.

5.7 **Frequency measurement.** Consider a signal consisting of three sinusoidal components,

$$x(n) = A_1 \sin(2\pi\, f_1\, n) + A_2 \sin(2\pi\, f_2\, n) + A_3 \sin(2\pi\, f_3\, n)$$

where the frequencies f_i are unknown. On the class webpage, a 100 point signal (`signal2.txt`) of this form is available. Using appropriate windows, estimate the three frequencies f_i.

5.8 **Frequency measurement.** Consider a 50-point signal consisting of three sinusoidal components,

$$x(n) = A_1 \sin(\omega_1\, n) + A_2 \sin(\omega_2\, n) + A_3 \sin(\omega_3\, n).$$

Suppose you want to use windowing and the FFT to estimate the frequencies of the sinusoidal components from the data (you do not know the frequencies). For which of the following two signals is the estimation of the frequencies *more* difficult? (MATLAB is not required for this problem.)

A) $x(n) = 10 \sin(0.31\pi\, n) + 11 \sin(0.32\pi\, n) + 9 \sin(0.33\pi\, n)$.

B) $x(n) = 0.01 \sin(0.2\pi\, n) + 0.011 \sin(0.5\pi\, n) + 0.012 \sin(0.8\pi\, n)$.

Clearly explain your answer.

5.9 **Frequency Measurement.** Consider a 50-point signal consisting of two sinusoids,

$$x(n) = A_1 \sin(\omega_1\, n) + A_2 \sin(\omega_2\, n).$$

Assume the signals are observed without noise. You plan to use the FFT to estimate the two frequencies present in the data. For which of the following signals would the estimation of both frequencies be *easiest*?

Clearly explain your answer, using diagrams as appropriate.

A) $x(n) = 10 \sin(0.66\pi\, n) + 1 \sin(0.7\pi\, n)$

B) $x(n) = 0.1 \sin(0.66\pi\, n) + 1 \sin(0.7\pi\, n)$

C) $x(n) = 10 \sin(0.66\pi\, n) + 10 \sin(0.7\pi\, n)$

D) $x(n) = 0.1 \sin(0.3\pi\, n) + 0.1 \sin(0.7\pi\, n)$

E) $x(n) = 10 \sin(0.3\pi\, n) + 0.01 \sin(0.7\pi\, n)$

5.10 **Frequency measurement.** Consider a 100-point signal consisting of three sinusoidal components,

$$x(n) = A_1 \sin(\omega_1 n) + A_2 \sin(\omega_2 n) + A_3 \sin(\omega_3 n).$$

Suppose you want to use windowing and the FFT to estimate the frequencies of the sinusoidal components from the data (you do not know the frequencies). For which of the following two signals is the estimation of the frequencies *more* difficult? (MATLAB is not required for this problem.)

A) $x(n) = 0.9 \sin(0.2\pi n) + 0.001 \sin(0.6\pi n) + 0.002 \sin(0.61\pi n)$.

B) $x(n) = 0.001 \sin(0.2\pi n) + 0.9 \sin(0.6\pi n) + 0.8 \sin(0.61\pi n)$.

Clearly explain your answer.

5.11 **Frequency Measurement.** Consider a 50-point signal consisting of three sinusoidal components,

$$x(n) = A_1 \sin(\omega_1 n) + A_2 \sin(\omega_2 n) + A_3 \sin(\omega_3 n).$$

Assume the signals are observed without noise. Suppose you want to use windowing and the FFT to estimate the frequencies of the sinusoidal components from the data (you do not know the frequencies). For which of the following two signals is the estimation of the frequencies *more* difficult?

A) $x(n) = 1 \sin(0.31\pi n) + 1 \sin(0.33\pi n) + 0.1 \sin(0.7\pi n)$.

B) $x(n) = 10 \sin(0.31\pi n) + 1 \sin(0.33\pi n) + 1 \sin(0.7\pi n)$.

Clearly explain your answer.

5.12 **Frequency Measurement.** Consider a 50-point signal consisting of three sinusoids,

$$x(n) = A_1 \sin(\omega_1 n) + A_2 \sin(\omega_2 n) + A_3 \sin(\omega_3 n).$$

Assume the signals are observed without noise. Suppose you want to use windowing and the FFT to estimate each of the three frequencies of the sinusoidal components from the data (you do not know the frequencies). For which of the following signals is the estimation of all the frequencies *easiest*? Clearly explain your answer.

A) $x(n) = 10 \sin(0.31\pi n) + 1 \sin(0.33\pi n) + 10 \sin(0.7\pi n)$.
B) $x(n) = 1 \sin(0.31\pi n) + 1 \sin(0.33\pi n) + 0.1 \sin(0.7\pi n)$.
C) $x(n) = 1 \sin(0.31\pi n) + 1 \sin(0.33\pi n) + 10 \sin(0.7\pi n)$.
D) $x(n) = 10 \sin(0.31\pi n) + 1 \sin(0.33\pi n) + 1 \sin(0.7\pi n)$.

5.13 **Frequency Measurement.**

(a) Consider the discrete-time signal

$$x(n) = 2\cos(0.4\,\pi\, n).$$

Sketch its discrete-time Fourier transform (DTFT), $X(\omega)$, for $|\omega| \le \pi$.

(b) Consider the discrete-time signal

$$p(n) = \begin{cases} 1 & 0 \le n \le 9 \\ 0 & \text{otherwise} \end{cases}$$

Sketch its DTFT, $|P(\omega)|$ for $|\omega| \le \pi$. Indicate the nulls of $P(\omega)$ on your sketch.

(c) Consider the discrete-time signal

$$f(n) = \begin{cases} 2\cos(0.4\,\pi\, n) & 0 \le n \le 9 \\ 0 & \text{otherwise} \end{cases}$$

which can be written as $f(n) = p(n)\,x(n)$. Sketch the DTFT, $|F(\omega)|$, for $|\omega| \le \pi$. Show your work. Indicate the nulls of $F(\omega)$ on your sketch. At what frequencies (approximately) does $|F(\omega)|$ have its maximum value?

(d) Similarly, consider the discrete-time signal

$$v(n) = \begin{cases} \cos(0.5\,\pi\, n) & 0 \le n \le 9 \\ 0 & \text{otherwise} \end{cases}$$

Roughly sketch the DTFT, $|V(\omega)|$, for $|\omega| \le \pi$. At what frequencies (approximately) does $|V(\omega)|$ have its maximum value?

(e) If 10 samples of the discrete-time signal

$$g(n) = \begin{cases} 2\cos(0.4\,\pi\, n + \theta_2) + \cos(0.5\,\pi\, n + \theta_1) & 0 \le n \le 9 \\ 0 & \text{otherwise} \end{cases}$$

are observed, and the zero-padded FFT is graphed, would you see two distinct peaks or not? Explain.

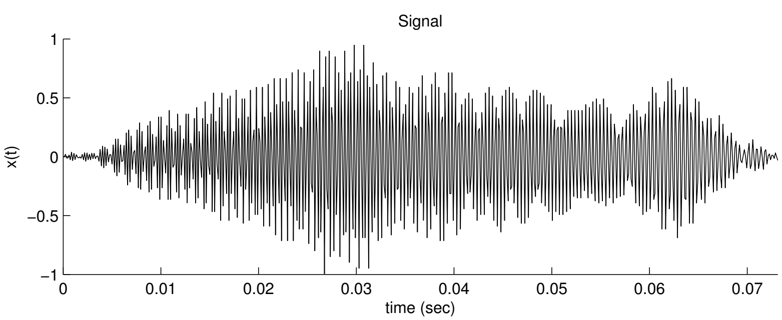
(f) Based on the previous parts of the problem: If 10 samples are observed of a signal that consists of two cosine signals, what should be the minimum difference between their frequencies so that one can easily distinguish them on a graph of the zero-padded FFT?

5.14 **STFT.** (Mitra 11.16) For this problem, it is convenient to use the following definition of the STFT, which is only superficially different from the form in the lecture notes. Show that the sampled STFT defined by

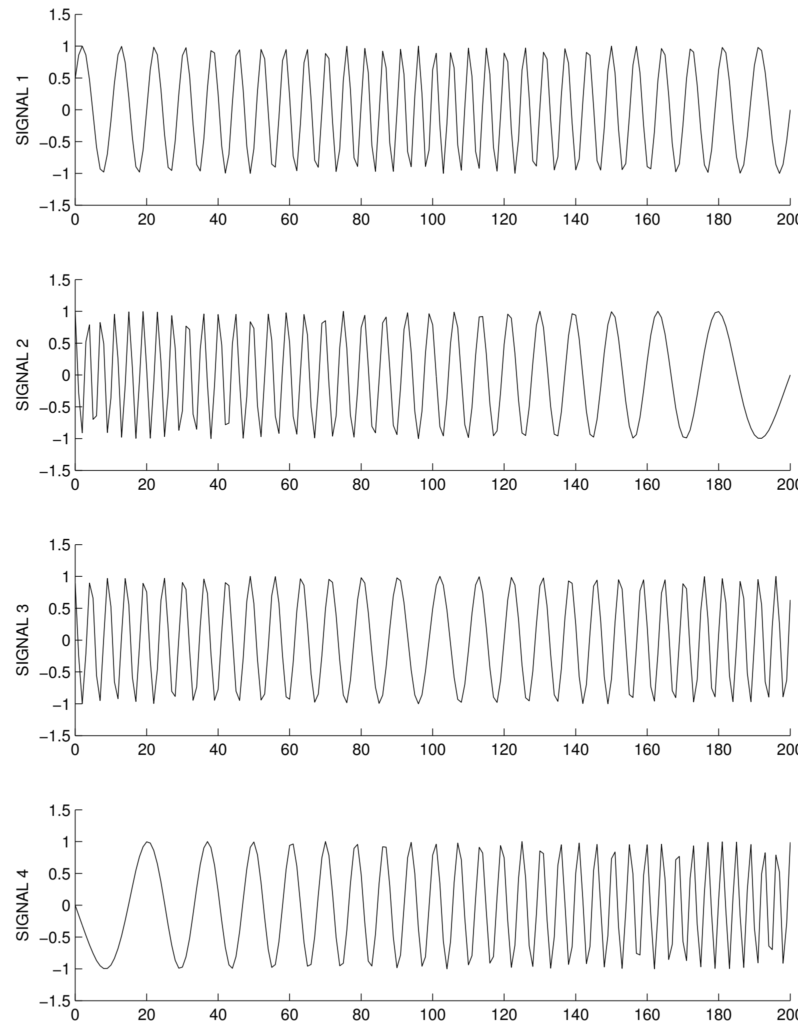
$$X^d(k,n) = \sum_{m=0}^{R-1} x(n-m)\, w(m)\, W_N^{-km}$$

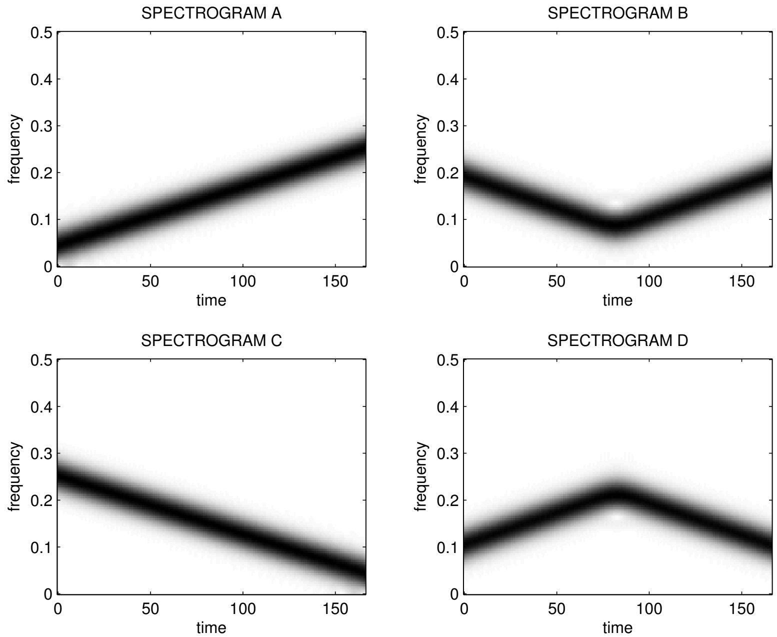
can be interpreted as a bank of N digital filters $h_k(n)$, $0 \le k \le N-1$ whose outputs $y_k(n)$ are the values $X^d(k,n)$. Determine the expression for the impulse response $h_k(n)$ of the N filters.

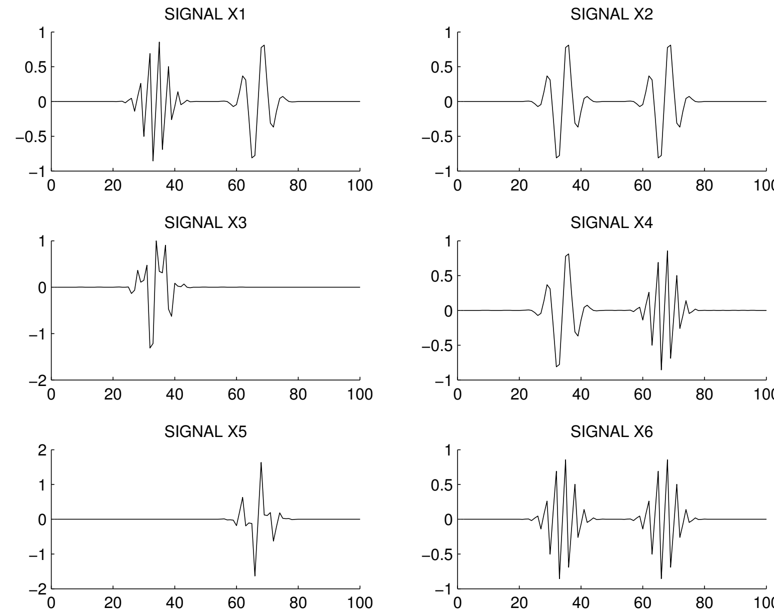
5.15 **Spectrogram.** Look at the Matlab documentation for the `specgram` function (use the `help` command). Make a spectrogram plot of the signal in the data file `signal1.txt` that was used in a previous problem. Try it with several different block lengths, amount of overlap, etc, until you get a spectrogram that best reveals the structure of the signal. Compare and contrast the spectrogram with the spectrum of this signal you got already.



5.16 **STFT.** The following figures show four signals and their spectrograms. Match each signal to its spectrogram.

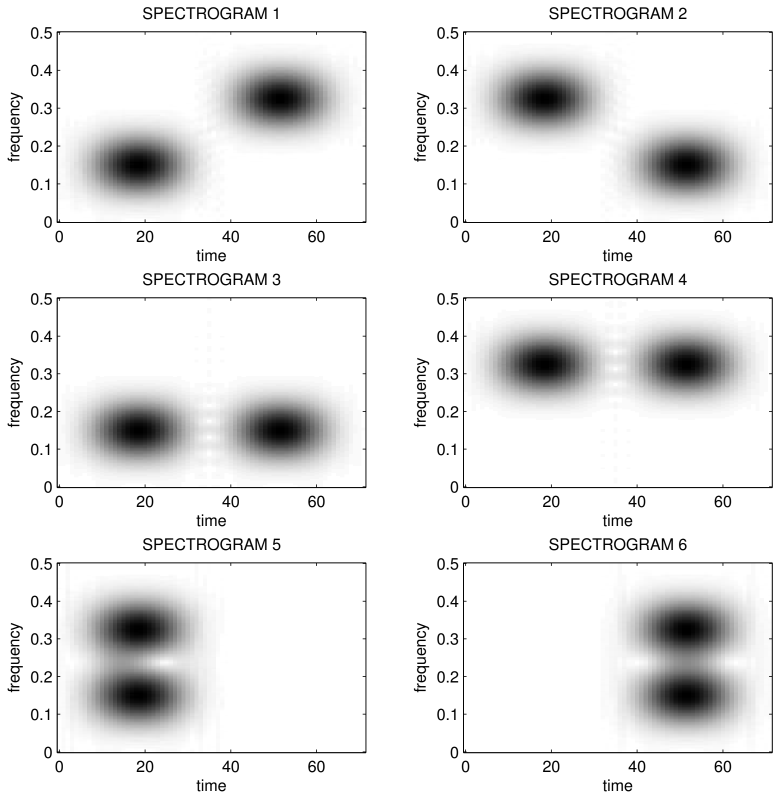






5.17 **The Spectrogram.** Shown below are 6 signals and 6 spectrograms. Match each signal to its spectrogram. Give an explanation of your answer.

Signal	Spectrogram
1	
2	
3	
4	
5	
6	



5.18 **STFT.** Shown below are four spectrograms of the same signal. Each spectrogram is computed using a different set of parameters.

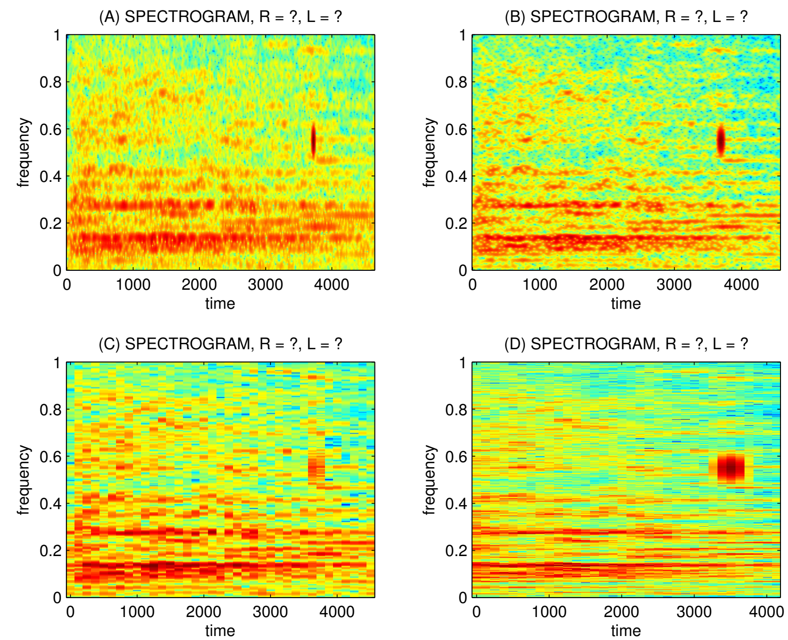
$$R \in \{120,\ 256,\ 1024\}, \qquad L \in \{35,\ 250\}$$

where

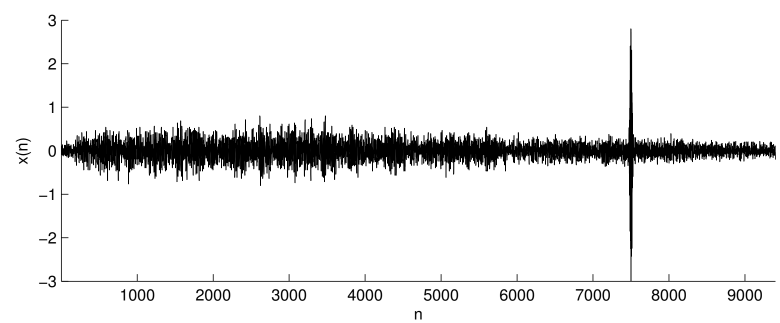
R = block length.

L = time lapse between blocks.

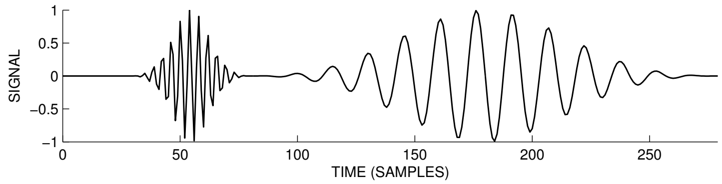
For each of the four spectrograms, indicate what you think R and L are. Briefly explain your choices.



If you like, you may listen to this signal with the `soundsc` command; the data is in the file: `signal3.txt`. Here is a figure of the signal.



5.19 **Spectrograms.** Consider the spectrogram of the following signal.



The spectrograms were computed with parameters:
$R \in \{20, 80\}$, $L \in \{2, 25\}$, $N \in \{20, 256\}$ where

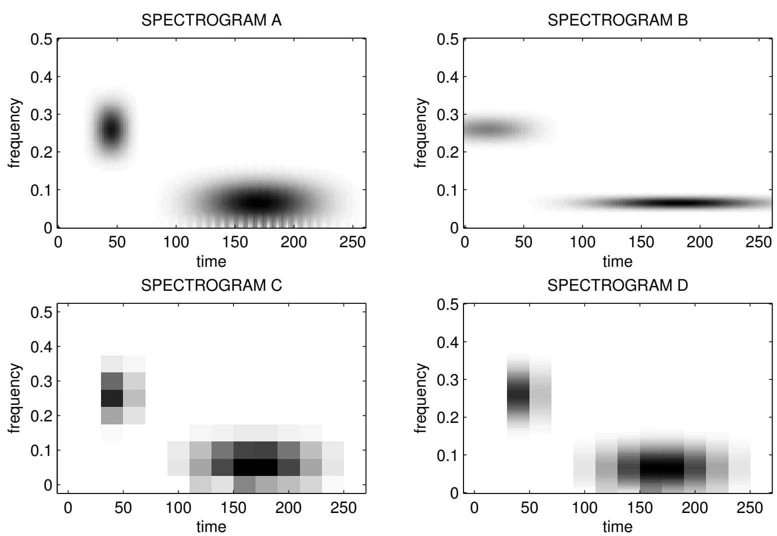
R = block length

L = time lapse between segments.

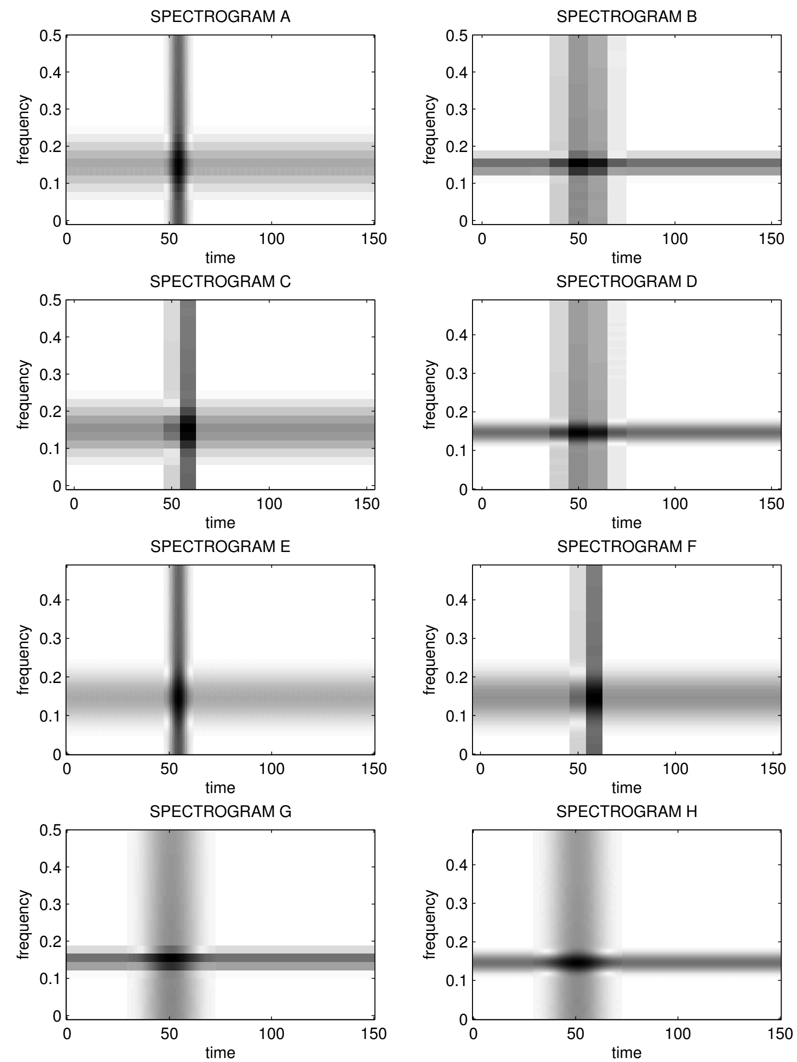
N = FFT length (Each signal block is zero-padded to length N.)

Indicate values R, L, and N by completing the table.

Spectrogram	R	L	N
A			
B			
C			
D			



5.20 Shown below are eight spectrograms of the same signal. (The signal is the sum of a cosine and an impulse.)



Each spectrogram is computed using a different set of parameters.

$$R \in \{18,\ 45\}, \qquad L \in \{1,\ 10\} \qquad N \in \{45,\ 256\}$$

where

R = block length.

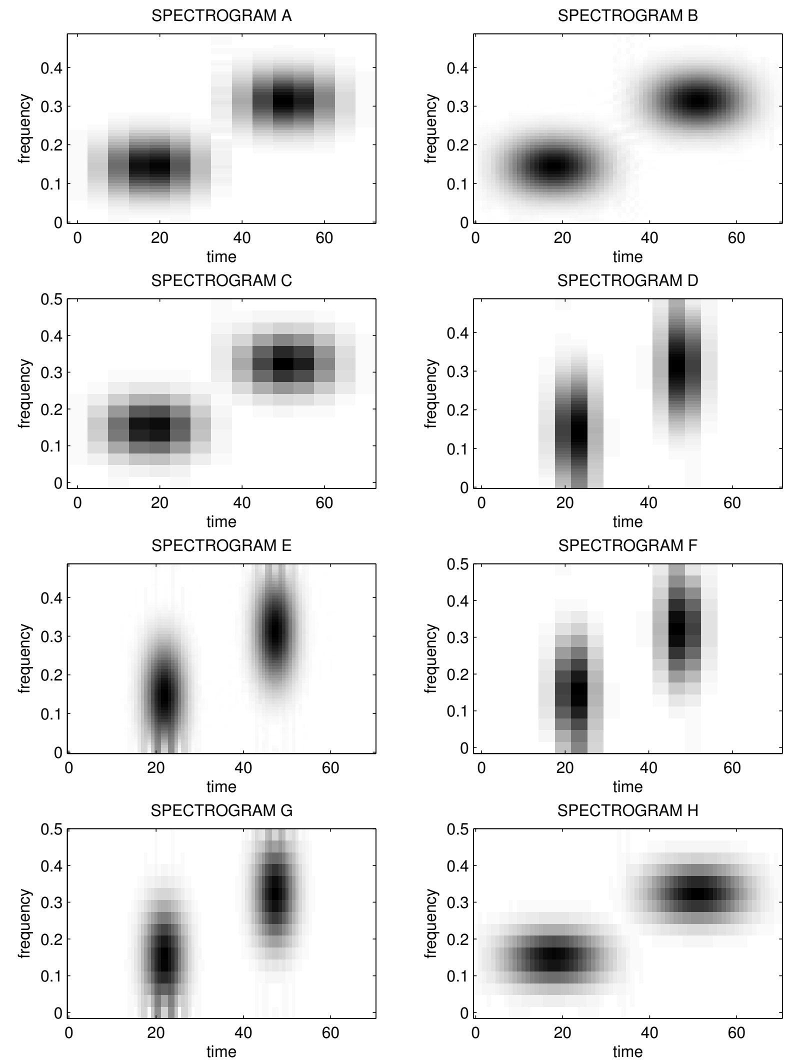
L = time lapse between blocks.

N = FFT length. (Each block is zero-padded to length N.)

For each of the eight spectrograms, indicate what you think R, L, and N are, by filling out the table. Explain!

Spectrogram	R	L	N
A			
B			
C			
D			
E			
F			
G			
H			

5.21 **The Spectrogram.** Shown below are eight spectrograms of the same signal.



Each spectrogram is computed using a different set of parameters.

$$R \in \{10,\ 31\}, \qquad L \in \{1,\ 5\} \qquad N \in \{31,\ 128\}$$

where

R = block length.

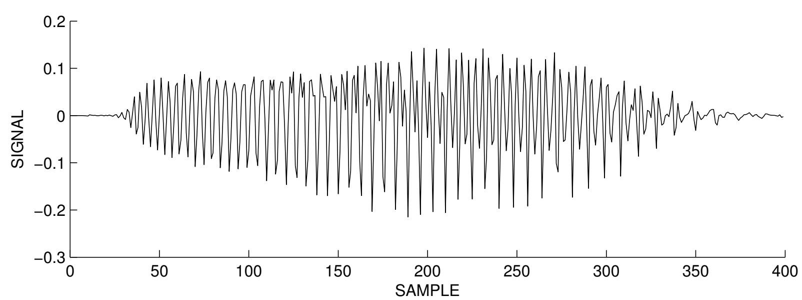
L = time lapse between blocks.

N = FFT length. (Each block is zero-padded to length N.)

For each of the eight spectrograms, indicate what you think R, L, and N are, by filling out the table. Explain!

Spectrogram	R	L	N
A			
B			
C			
D			
E			
F			
G			
H			

5.22 The following figures shows a digitized echolocation pulse emitted by the Large Brown Bat, *Eptesicus Fuscus*. There are 400 samples; the sampling period was 7 microseconds.



The spectrogram of the bat pulse was computed with parameter values having a subset of the following values.

$$R \in \{22, 50\}, \qquad L \in \{2, 20\}, \qquad N \in \{50, 256\}$$

where

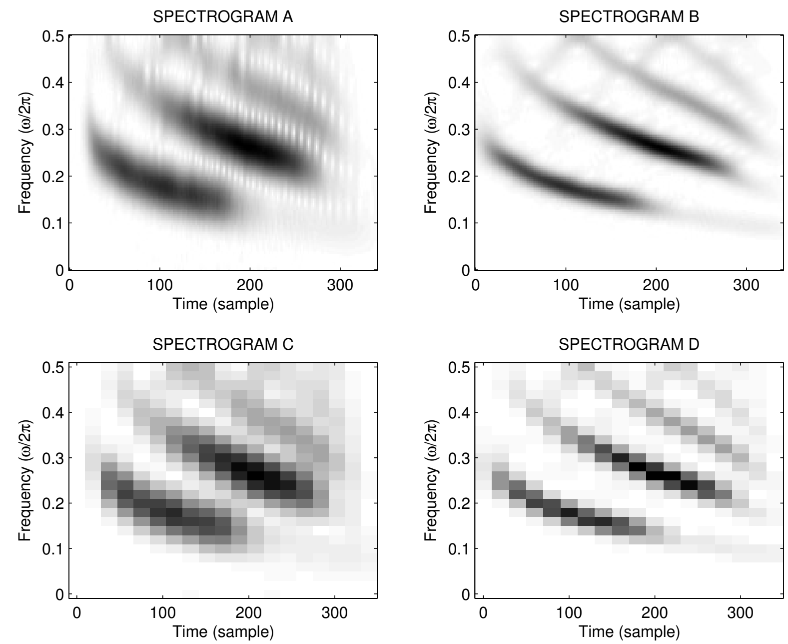
R = block length

L = time lapse between segments.

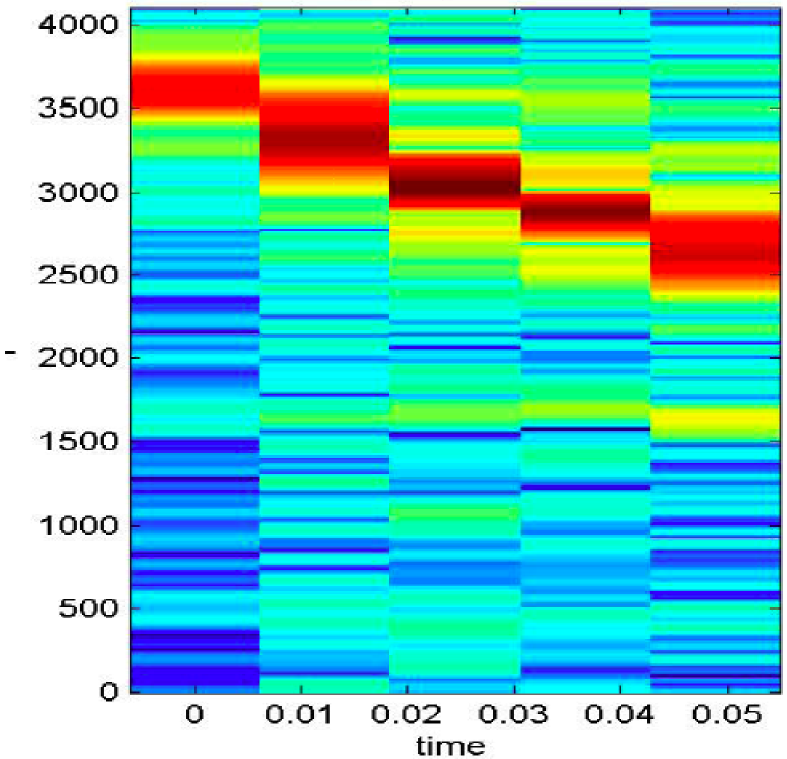
N = FFT length (`nfft` in the MATLAB `specgram` function). (Each block is zero-padded to length N.)

For each of the spectrograms, indicate what you think R, L, and N are by filling out the table, and explain your choices.

Spectrogram	R	L	N
A			
B			
C			
D			



5.23 The following spectrogram is taken from the HW submitted by an EL 713 student.



Which of the following three parameters would you suggest the student modify to best improve the appearance of this spectrogram? What change would you make to that parameter? The three parameters are:

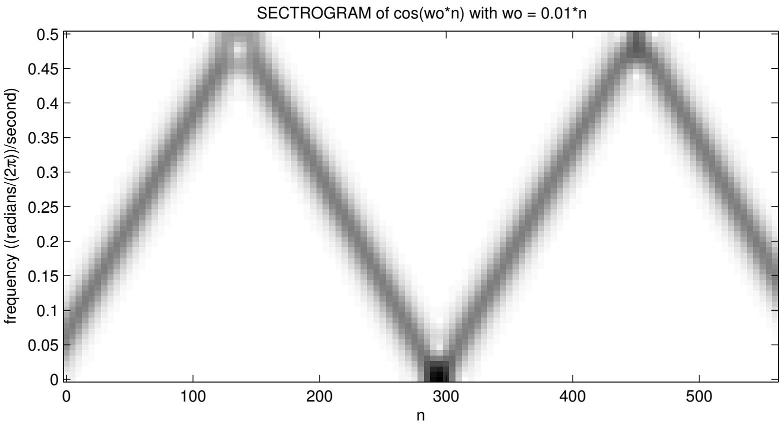
R = block length.

L = time lapse between blocks.

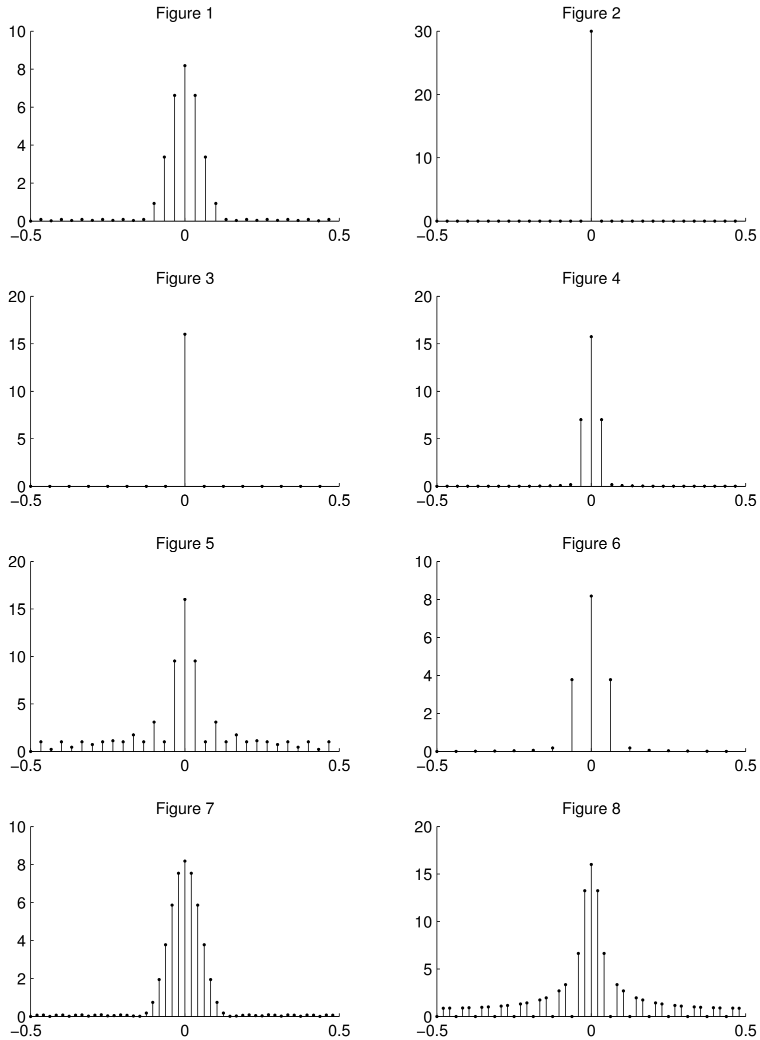
N = FFT length. (Each block is zero-padded to length N.)

Explain your answer.

5.24 **The Spectrogram.** The signal $x(n) = \cos(\omega_o\, n)$ has the frequency ω_o radians/second. If ω_o changes with time, then the signal has a time-varying frequency. Here $\omega_o = 0.01\, n$. It is just *increasing*. But the spectrogram does not look like it is increasing — it shows that the frequency increases, then decreases, then increases, then decreases. Explain why.



5.25 Match each of the following Matlab commands to the figure it produces.




```
>> N1 = 16;
>> N2 = 30;
>> N3 = 48;
>> stem([0:N1-1]/N1-0.5,abs(fftshift(fft(rectwin(N1),N1))),'.')
>> stem([0:N1-1]/N1-0.5,abs(fftshift(fft(hamming(N1),N1))),'.')
>> stem([0:N2-1]/N2-0.5,abs(fftshift(fft(rectwin(N1),N2))),'.')
>> stem([0:N2-1]/N2-0.5,abs(fftshift(fft(hamming(N1),N2))),'.')
>> stem([0:N3-1]/N3-0.5,abs(fftshift(fft(rectwin(N1),N3))),'.')
>> stem([0:N3-1]/N3-0.5,abs(fftshift(fft(hamming(N1),N3))),'.')
>> stem([0:N2-1]/N2-0.5,abs(fftshift(fft(rectwin(N2),N2))),'.')
>> stem([0:N2-1]/N2-0.5,abs(fftshift(fft(hamming(N2),N2))),'.')
```

6 Least Square Filter Design

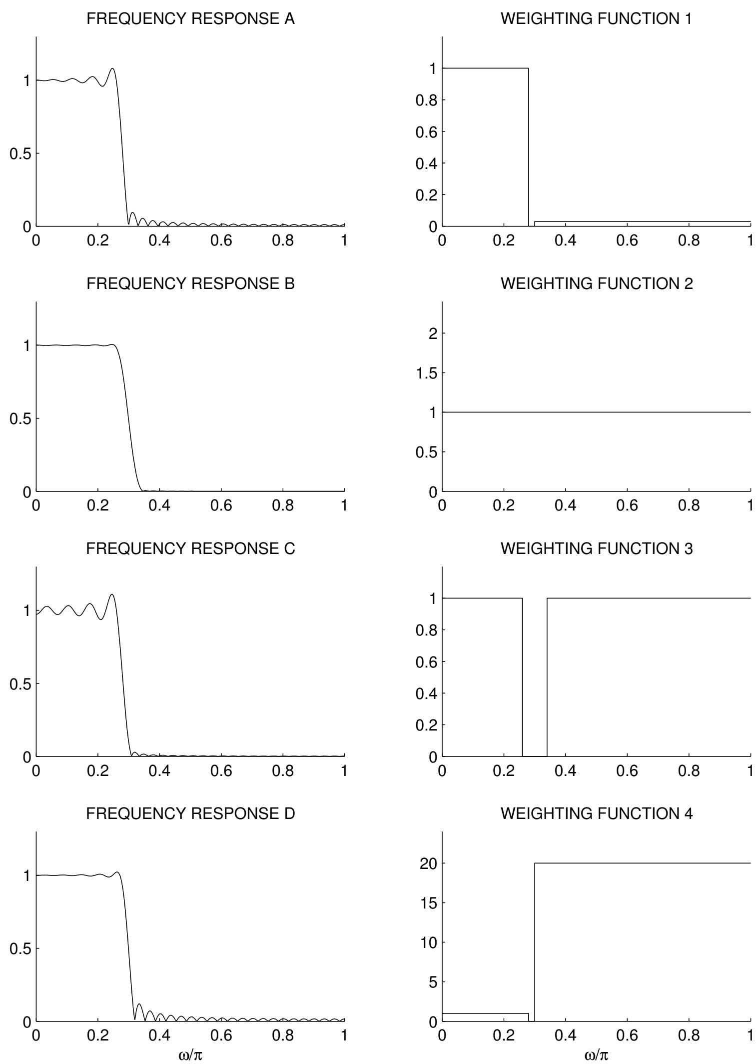
6.1 Redo the example of the weighted low-pass filter from the lecture notes, but increase the stop-band weighting K from 10 to 40.

6.2 In the first design example of the notes we plotted the amplitude response for $N = 31, 61, 121$, but we did not do this for the later design examples. Repeat with longer lengths the design examples of the spline transition low-pass filter and the zero-weighted transition-band low-pass filter. (The spline transition is the straight line connecting the pass-band and stop-bands.) Compare with the 'impulse response truncation' method. Please comment on your observations.

6.3 What is the effect of the *width* of the zero-weighted transition band? As the zero-weighted transition band becomes wider, what happens to the frequency response? Show examples using Matlab to design several filters to illustrate what happens.

6.4 Four FIR lowpass filters of equal length are designed according to different weighting functions $W(\omega)$, as discussed in the lecture notes.

Match each frequency response with the weighting function used for the design.



6.5 Repeat the design example from the lecture notes of a low-pass filter with a specified null, so that the zero is (a) second order, and (b) third order. (Impose the constraint $A'(\omega_1) = 0$ and $A''(\omega_1) = 0$ in addition to the constraint $A(\omega_1) = 0$.) Plot the amplitude response $A(\omega)$, impulse response $h(n)$, and zero-diagram.

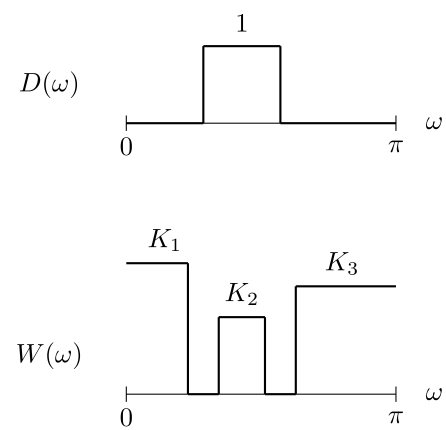
6.6 The weighted square error design of band-pass filters can be done with the following functions $D(\omega)$ and $W(\omega)$,

$$D(\omega) = \begin{cases} 0 & 0 < \omega < \omega_a \quad \text{(stop-band 1)} \\ 1 & \omega_a < \omega < \omega_b \quad \text{(pass-band)} \\ 0 & \omega_b < \omega < \pi \quad \text{(stop-band 2)} \end{cases}$$

and

$$W(\omega) = \begin{cases} K_1 & 0 < \omega < \omega_1 \\ 0 & \omega_1 < \omega < \omega_2 \\ K_2 & \omega_2 < \omega < \omega_3 \\ 0 & \omega_3 < \omega < \omega_4 \\ K_3 & \omega_4 < \omega < \pi \end{cases}$$

where $\omega_1 < \omega_a < \omega_2$ and $\omega_3 < \omega_b < \omega_4$.



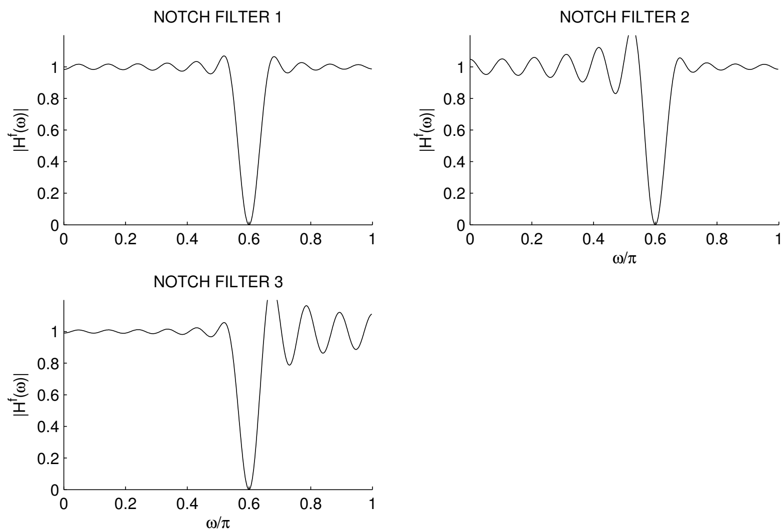
Find the formula for $q(k)$ and $b(k)$. Use the formula to set up the appropriate matrix equation and design band-pass filters of lengths 31 and 61. Use $\omega_1 = 0.28\pi$, $\omega_2 = 0.32\pi$, $\omega_3 = 0.58\pi$, and $\omega_4 = 0.62\pi$ and $K_1 = 1$, $K_2 = 3$, $K_3 = 10$. Plot the amplitude response $A(\omega)$, impulse response $h(n)$, and zero-diagram.

6.7 The design of an FIR digital notch filter is carried out as in the lecture notes using the least-square method. But now, a weighting function is used. The three different weighting functions are used:

$$W_1(\omega) = \begin{cases} 100 & 0 \le \omega \le \omega_1 \\ 0 & \omega_1 < \omega < \omega_2 \\ 1 & \omega_2 \le \omega \le \pi \end{cases} \qquad W_2(\omega) = \begin{cases} 1 & 0 \le \omega \le \omega_1 \\ 0 & \omega_1 < \omega < \omega_2 \\ 100 & \omega_2 \le \omega \le \pi \end{cases}$$

$$W_3(\omega) = \begin{cases} 10 & 0 \le \omega \le \omega_1 \\ 0 & \omega_1 < \omega < \omega_2 \\ 10 & \omega_2 \le \omega \le \pi \end{cases}$$

where $\omega_1 = 0.55\,\pi$ and $\omega_2 = 0.65\,\pi$.



Match each filter to the weighting function used for its design and give an *explanation* for your answer.

Filter	Weighting Function
1	
2	
3	

6.8 Extend the weighted least square error approach to Type II filters. Using a Type II filter instead of Type I filter, repeat the example on pages 18 and 19 of the notes, the 'weighted low-pass example': Design a length 32 symmetric FIR filter with band-edges $\omega_p = 0.26\pi$, $\omega_s = 0.34\pi$ and a stop-band weight of $K = 10$.

Plot the impulse response, frequency response, and zero-diagram of the filter.

6.9 If you try to design a Type II notch filter, similar to the one described in the notes, what problem would arise?

6.10 An analog signal $s(t)$ contains frequencies between 10 Hz and 20 Hz only. However, the signal is corrupted by additive noise $n(t)$. The analog noise signal $n(t)$ contains frequencies between 30 Hz and 40 Hz. The combined signal is $g(t)$,

$$g(t) = s(t) + n(t).$$

You sample $g(t)$ at a sampling rate of 70 Hz to obtain the discrete-time signal $x(n) = g(nT)$ where T is the sampling period.

You want to design an FIR digital filter to remove the noise. If you are using the least-squares design method, what should be the desired frequency response $D(\omega)$ and the weighting function $W(\omega)$? $(-\pi \le \omega \le \pi)$ Make a sketch of both of these functions. Explain!

6.11 *Optional.* We saw that truncating the inverse DTFT of $D(\omega)$ yields a filter that minimizes the unweighted (integral) square error. (With $D(\omega)$ being symmetric, we get a Type I filter.) Similarly, show that truncating the inverse DFT of $D(2\pi k/L)$ yields a Type I filter that minimizes the unweighted *discrete* square error, computed over a uniform grid:

$$\epsilon_2 = \sum_{k=0}^{L-1} (A(\omega_k) - D(\omega_k))^2$$

with

$$\omega_k = \frac{2\pi}{L} k, \qquad 0 \le k \le L - 1.$$

6.12 *Very optional.* Extend the least square error approaches to Type III and IV FIR filters.

6.13 Consider a Type I FIR filter of length N. The impulse response $h(n)$ is non-zero for $0 \le n \le N-1$. We have learned how to compute the real-valued amplitude response $A(\omega)$ on a uniform grid. Hans has suggested another way. He defines the length L signal $g(n)$

$$g(n) = \begin{cases} h(n+M) & 0 \le n \le M \\ 0 & M+1 \le n \le L-1 \end{cases}$$

where $M = (N-1)/2$ as usual. (This is simply the second half of $h(n)$ including the midpoint.) Then he computes:

$$C(k) = 2\,\mathrm{Real}\{\mathrm{DFT}_L\{g(n)\}\} - h(M)$$

for $0 \le k \le L-1$. Hans claims that

$$C(k) = A\left(\frac{2\pi}{L}k\right).$$

Is he correct? If so, derive the correctness of his formula. If not, show why not.

To make it clear, suppose

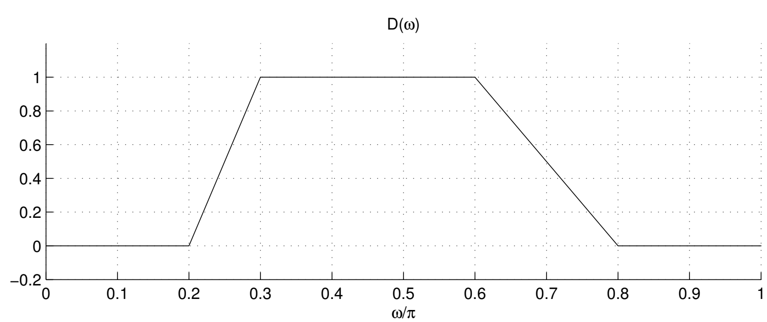
```
h = [1, 2, 4, 2, 1]
```

Then in Matlab notation, Hans claims that

```
C = 2*real(fft([4 2 1 zeros(1,L-3)])) - 4;
```

will compute L samples of $A(\omega)$ for $\omega = \frac{2\pi}{L}k$, $0 \le k \le L-1$.

6.14 Consider the design of a band-pass filter, where the desired amplitude response has the form



The band-edges are

$$\omega_{s1} = 0.2\,\pi, \quad \omega_{p1} = 0.3\,\pi, \quad \omega_{p2} = 0.6\,\pi, \quad \omega_{s2} = 0.8\,\pi.$$

Note that the width of the two transition-bands are not the same. Find a formula for the Type I FIR filter of length N that minimizes the unweighted square error,

$$\mathcal{E}_2 = \int_0^\pi (A(\omega) - D(\omega))^2\, d\omega.$$

6.15 The *Savitsky-Golay* FIR filters (or windows) are a family of lowpass digital filters obtained by minimizing an unweighted square error subject to side constraints.

For example, the following problem leads to a Savitsky-Golay filter when the desired response $D(\omega)$ is set equal to the zero function; $D(\omega) = 0$.

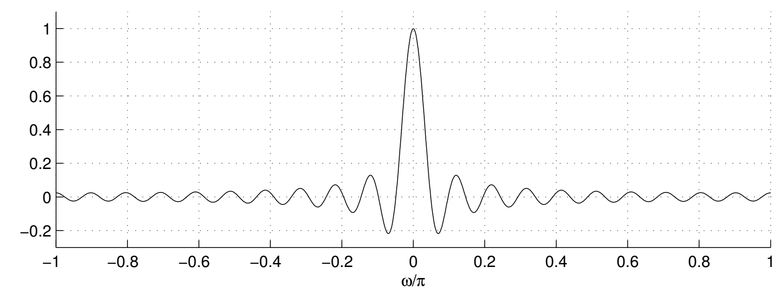
Minimize

$$\int_0^\pi (A(\omega) - D(\omega))^2\, d\omega \tag{4}$$

subject to the side constraint

$$A(0) = 1. \tag{5}$$

The frequency response amplitude of a length-21 Savitsky-Golay filter is shown in the following figure.



(a) How would you solve the constrained minimization problem stated in equations (4) and (5)? (Write down the equations you need to solve.) Solve the equations and determine $h(n)$.

(b) You can see that the stop-band ripples are rather large. By using only an error weighting function how could you reduce the sizes of the stopband ripples without changing the length of the filter? (Explain.) What happens to the other characteristics of the frequency response?

7 Minimax Filter Design

7.1 Consider the design of a type I FIR filter of minimal length satisfying the following specifications:

$$1 - \delta_p \le A(\omega) \le 1 + \delta_p, \qquad 0 \le \omega \le \omega_p$$

$$-\delta_s \le A(\omega) \le \delta_s, \qquad \omega_s \le \omega \le \pi$$

where

$$\delta_p = 0.018, \qquad \delta_s = 0.004,$$

$$\omega_p = 0.31\pi, \qquad \omega_s = 0.38\pi.$$

What should K_p and K_s be? You can estimate what the length of the impulse response should be with the command `remezord`. Use `fircheb` or `remez` to design the filter. For these specifications, does `remezord` correctly indicate what the shortest Type I filter length is? Plot the amplitude response $A(\omega)$ and impulse response $h(n)$.

7.2 Use $N = 81$, with $K_p = 1$, $K_s = 4$, and find the low-pass filter that minimizes the weighted Chebyshev error with band edges:

(a) $\omega_p = 0.27\pi$, $\omega_s = 0.33\pi$.

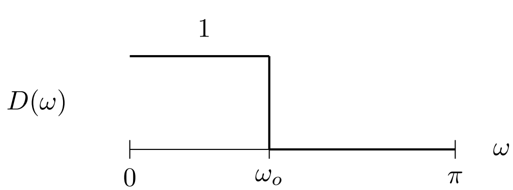
(b) $\omega_p = 0.28\pi$, $\omega_s = 0.32\pi$.

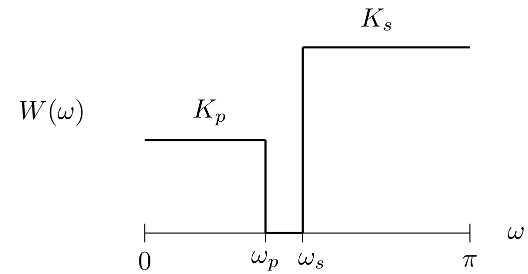
(c) $\omega_p = 0.29\pi$, $\omega_s = 0.31\pi$.

(d) $\omega_p = 0.295\pi$, $\omega_s = 0.305\pi$.

Plot $A(\omega)$ and $h(n)$. What happens to $A(\omega)$ and $h(n)$ as you make the transition band narrower? Why might one not want to use the last filter in practice?

7.3 A low-pass length 51 Type I FIR filter is designed using the Remez (Parks-McClellan) algorithm with the following desired amplitude response $D(\omega)$ and weighting function $W(\omega)$ with $K_p = 1$, $K_s = 10$. The filter has a pass-band ripple of 0.015 (about 0.03 from peak to peak).





(a) What is the size of the stop-band ripple?

(b) If the weighting constants are changed to $K_p = 0.1$, $K_s = 1$ and the filter is redesigned, how are the new pass-band and stop-band ripples related to the original ones?

(c) If the weighting constants are changed to $K_p = 10$, $K_s = 1$ and the filter is redesigned, how are the new pass-band and stop-band ripples related to the original ones?

(d) If the length of the filter is increased to 81 and the filter is redesigned, how are the new pass-band and stop-band ripples related to the original ones (with $K_p = 1$, $K_s = 10$)?

7.4 The filter $h_1(n)$

$$h_1(n) = [1,\ 2,\ 2,\ 1]$$

has very simple coefficients (is simple to implement) and has zeros exactly on the unit circle that fall in the stop-band of a low-pass filter you are designing. You will use it in cascade with another filter $h_2(n)$ to reduce the total number of multiplications. Design $h_2(n)$ using the Remez algorithm so that the total response

$$A(\omega) = A_1(\omega)\, A_2(\omega)$$

of $h(n) = h_1(n) * h_2(n)$ minimizes the weighted Chebyshev error. The length of $h(n)$ should be 22. Use the following band edges and weightings. $\omega_p = 0.2\pi$, $\omega_s = 0.3\pi$, and $K_p = 1$, $K_s = 3$. Plot $A(\omega)$, $h(n)$ and the zero diagram.

You may want to use `fircheb` rather than `remez` because you will need to use a non-uniform weighting function $W(\omega)$ and desired response $D(\omega)$.

7.5 A lowpass filter $H(z)$ designed using the Remez algorithm will not in general have the property that the dc gain is 1 exactly. (That is, $H^f(\omega = 0)$ is not usually exactly 1.) However, suppose you want to design a filter according to the minimax criterion that does have unity dc gain ($H^f(\omega = 0) = 1$). How could you use the Remez algorithm to design a filter satisfying this constraint?

7.6 Modify the program `fircheb` so that it can design type II FIR filters that minimize the weighted Chebyshev error. The size of the reference set (R) should be equal to the *number of filter parameters plus 1.* Use your program to design a length 32 low-pass filter with $\omega_p = 0.26\pi$, $\omega_s = 0.34\pi$ and $K_p = 1$, $K_s = 4$.

7.7 **Linear Programming Method** In the second example of linear programming, we had an example where constraints were imposed on the step response of the filter. Notice that the step response $s(n)$ will equal a constant after a certain point.

$$s(n) = c, \quad \text{for all} \quad n \geq N - 1.$$

Show that this constant c is given by $c = A(0)$. Add the constraint $A(0) = 1$ to the design problem and redo the design. (Keep the other constraints on $s(n)$.) This will result in a filter where the final value of the step response is 1. Make plots of $A(\omega)$, $h(n)$, and $s(n)$. By adding the constraint $A(0) = 1$, what is the increase in the value of the peak-error in the pass-band?

Note: To indicate with the `lp` that the first `Ne` constraints are *equality* constraints, you can use the following syntax.

```
x = lp(c,Q,b,[],[],[],Ne);
```

This syntax means that the constraints are

```
Q(1:Ne,:) * x = b(1:Ne,:)
```

and

```
Q(Ne+1:end,:) * x <= b(Ne+1:end,:)
```

7.8 The following type of linear-phase FIR filter will be useful later on for signal interpolation.

$$h(n) = [h_0,\ h_1,\ 0, h_3,\ h_4,\ 1/3,\ h_4,\ h_3,\ 0,\ h_1,\ h_0]$$

This kind of filter is called a *Nyquist-3* filter.

Use the linear programming approach to design a length-11 lowpass digital of this form so as to minimize the maximum error. Use the following parameters,

$$K_p = K_s = 1,$$

$$\omega_o = \pi/3, \quad \omega_p = \omega_o - 0.1\,\pi, \quad \omega_s = \omega_o + 0.1\,\pi.$$

Write a Matlab program calling `lp` to implement the design procedure.

Display $h(n)$, $A(\omega)$, and the zero-diagram of the filter.

7.9 **Constrained Minimax Filter Design:** A Type II FIR filter has impulse response

$$g = [1, \quad 3, \quad 3, \quad 1]$$

for $n \leq 0 \leq 3$. Formulate as a linear program the design of 6-point Type II FIR filter $h(n)$ according to the minimax criteria subject to the side constraint, that the product filter $h(n) * g(n)$ is Nyquist-3. This type of filter design problem arises in communications.

The filter $h(n)$ is to be a lowpass filter with bandedges ω_p and ω_s. The weighting in the passband and stopband should be the same. The length of $h(n)$ is 6.

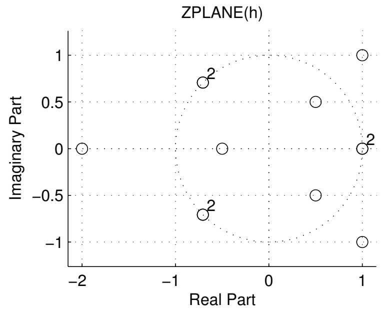
To formulate the design problem as a linear program, you should describe the 2 vectors and 1 matrix that describes a linear program. Also describe what the variables of the linear program are.

You should specify which of the constraints are *equality* constraints, and which of the constraints are *inequality* constraints.

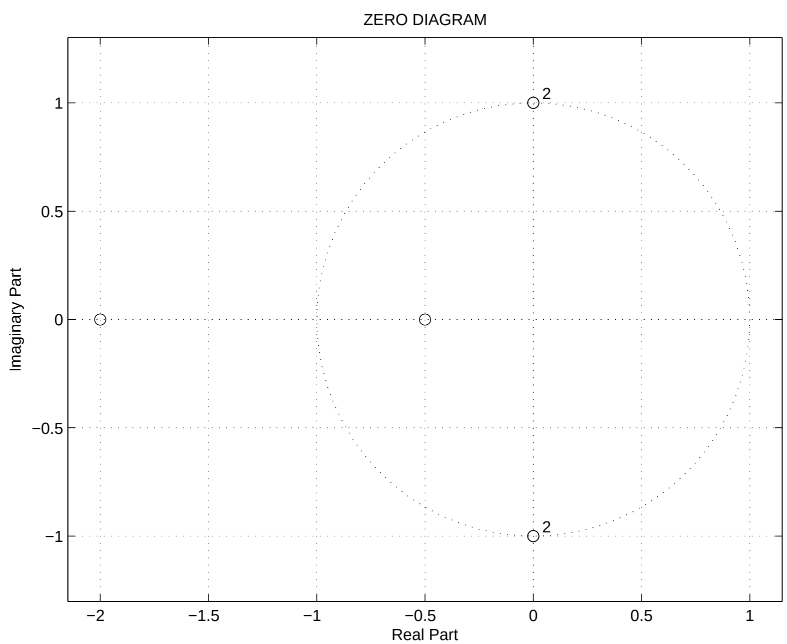
Also, could the Remez algorithm be used to solve this design problem?

8 Spectral Factorization

8.1 For the filter, $H(z)$, whose zeros are shown in the following diagram, make a sketch of the zero diagram of *each* spectral factorization of $H(z)$.



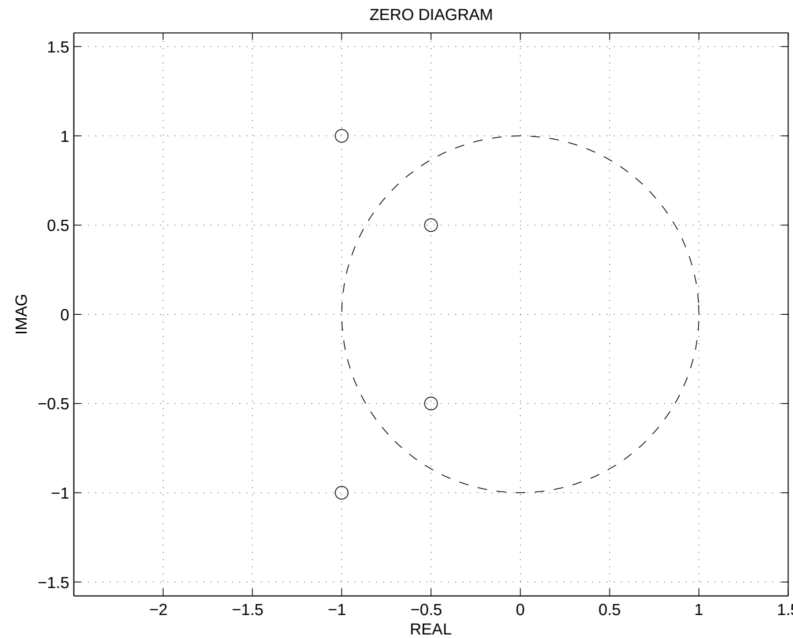
8.2 The sequence $p(n)$ is symmetric ($p(-n) = p(n)$). The zeros of $P(z)$ are shown in the following diagram. Also, $P(z)$ has unity dc gain (meaning $P(z = 1) = 1$).



Find and sketch, $h(n)$, a real-valued spectral factor of $P(z)$. That is, find $h(n)$ such that

$$P(z) = H(z)\,H(1/z).$$

8.3 The system $P(z)$ has the zeros shown in the following diagram and unity dc gain (meaning $P(z = 1) = 1$).



Find and sketch the impulse response $h(n)$ of a spectral factor of $P(z)$.

8.4 The transfer function of a discrete-time LTI system is given by

$$P(z) = (z + 2 + z^{-1})^2\,(-2\,z + 5 - 2\,z^{-1})$$

You should be able to do this problem by hand (not using MATLAB, etc.).

(a) Sketch the impulse response $p(n)$.

(b) Sketch the pole/zero diagram of the transfer function $P(z)$.

(c) Is $P(z)$ a lowpass, bandpass, bandstop, or highpass filter? Or none of these?

(d) Perform spectral factorization on $P(z)$. That means, find $h(n)$ (real-valued) so that

$$p(n) = \sum_k h(k)\, h(k-n)$$

Note: you can check that your answer is correct.

(e) Sktech the impulse response $h(n)$.

8.5 (a) The sequence p is

$$[1, \quad 5, \quad 10, \quad 10, \quad 5, \quad 1]$$

Does there exist a sequence h (real-valued) so that the convolution of h with its time-reverse h is equal to p? If yes, find h. If no, explain why not.

(b) The sequence p is

$$[-1, \quad -4, \quad -6, \quad -4, \quad -1]$$

Does there exist a sequence h (real-valued) so that the convolution of h with its time-reverse h is equal to p? If yes, find h. If no, explain why not.

8.6 (a) The sequence p is

$$p = [1, \quad -2, \quad 3, \quad -3, \quad 2, \quad -1]$$

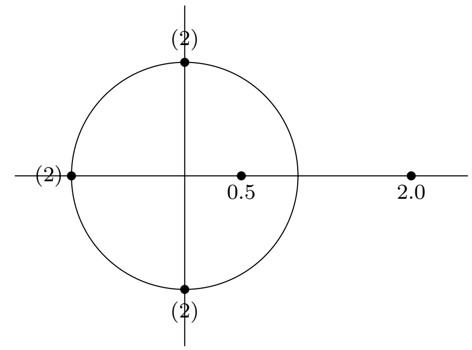
Does there exist a sequence h so that the convolution of h with its time-reverse h is equal to p? If yes, find h. If no, explain why not.

(b) The sequence p is

$$p = [1, \quad -2, \quad 3, \quad -2, \quad 1]$$

Does there exist a sequence h so that the convolution of h with its time-reverse h is equal to p? If yes, find h. If no, explain why not.

8.7 The transfer function $H(z)$ of a Type I FIR filter has zeros in the z-plane as illustrated.



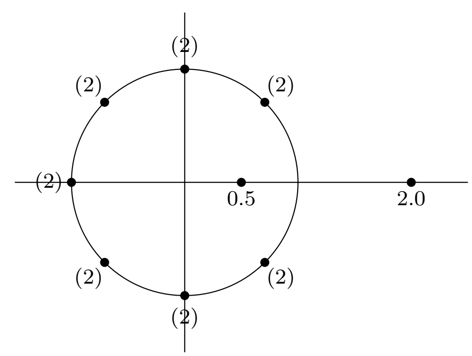
The dc-gain of the filter is unity. (Note that the roots on the unit circle are double zeros.)

(a) Accurately sketch the impulse response of the filter.

(b) Sketch the frequency response magnitude $|H(e^{j\omega})|$. Accurately indicate the nulls of the frequency response.

(c) Find a spectral factorization of $H(z)$. Accurately sketch the impulse response of the spectral factor and its zero-diagram.

8.8 **Spectral Factorization.** The transfer function $P(z)$ of a Type I FIR filter has zeros in the z-plane as illustrated.



The zeros on the unit-circle are double zeros located at powers of W_8. In addition, the impulse response $p(n)$ is centered at $n = 0$; that is, $p(-n) = p(n)$.

Find a spectral factorization of $P(z)$; that is, find $H(z)$ such that $P(z) = H(z)H(1/z)$. You may assume the dc-gain of $H(z)$ is unity. Accurately sketch the impulse response $h(n)$.

9 Minimum-Phase Filter Design

9.1 Use the lifting procedure to design a minimal-length minimum-phase low-pass FIR filter satisfying the specifications

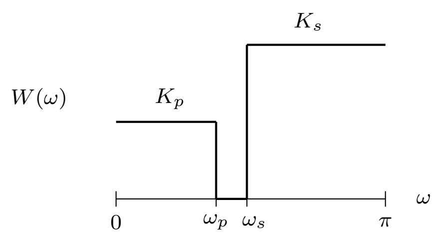
$$||H(e^{j\omega})| - 1| \le \Delta_p \quad \text{for} \quad 0 \le \omega \le \omega_p \tag{6}$$
$$|H(e^{j\omega})| \le \Delta_s \quad \text{for} \quad \omega_s \le \omega \le \pi \tag{7}$$

where

$$\Delta_p = 0.02, \quad \Delta_s = 0.01, \quad \text{and} \quad \omega_p = 0.72\pi, \quad \omega_s = 0.80\pi.$$

Also design a linear-phase filter of the same length with the weight function

$$W(\omega) = \begin{cases} K_p & 0 \le \omega \le \omega_p \\ 0 & \omega_p < \omega < \omega_s \\ K_s & \omega_s \le \omega \le \pi \end{cases}$$



where

$$K_p = \frac{1}{\Delta_p}, \qquad K_s = \frac{1}{\Delta_s},$$

(Note that the weight function will be different for the minimum-phase filter. For the minimum-phase filter use the weight function derived in the lecture notes.) Compare the minimum-phase and linear-phase filters to each other in terms of the maximum pass-band and stop-band error. For both filters, plot the impulse response, frequency response magnitude, and zero diagram (be sure to use `zplane` correctly!). Also plot the *unwrapped* phase and group delay; use the unwrap command.

9.2 Repeat the previous problem, but with

$$\Delta_p = 0.02, \quad \Delta_s = 0.01, \quad \text{and} \quad \omega_p = 0.12\pi, \quad \omega_s = 0.20\pi.$$

Comment on your observations.

9.3 Use the lifting procedure to design a minimal-length minimum-phase FIR *band*-pass filter satisfying the specifications

$$|H(e^{j\omega})| \le \Delta_{s1} \quad \text{for} \quad 0 \le \omega \le \omega_1 \tag{8}$$
$$||H(e^{j\omega})| - 1| \le \Delta_p \quad \text{for} \quad \omega_2 \le \omega \le \omega_3 \tag{9}$$
$$|H(e^{j\omega})| \le \Delta_{s2} \quad \text{for} \quad \omega_4 \le \omega \le \pi \tag{10}$$

where

$$\Delta_{s1} = \Delta_{s2} = 0.01, \quad \Delta_p = 0.02$$

and

$$\omega_1 = 0.30\pi, \quad \omega_2 = 0.35\pi, \quad \omega_3 = 0.60\pi, \quad \omega_4 = 0.65\pi.$$

Also design a linear-phase filter of the same length with the weight function

$$W(\omega) = \begin{cases} K_{s1} & 0 < \omega < \omega_1 \\ 0 & \omega_1 < \omega < \omega_2 \\ K_p & \omega_2 < \omega < \omega_3 \\ 0 & \omega_3 < \omega < \omega_4 \\ K_{s2} & \omega_4 < \omega < \pi \end{cases}$$

where

$$K_{s1} = \frac{1}{\Delta_{s1}}, \qquad K_p = \frac{1}{\Delta_p}, \qquad K_{s2} = \frac{1}{\Delta_{s2}},$$

(Note that the weight function will be different for the minimum-phase filter. For the minimum-phase filter use the weight function derived in the lecture notes.) Compare the minimum-phase and linear-phase filters to each other in terms of the maximum pass-band and stop-band error. For both filters, plot the impulse response, frequency response magnitude, and zero diagram. Also plot the *unwrapped* phase and the group delay.

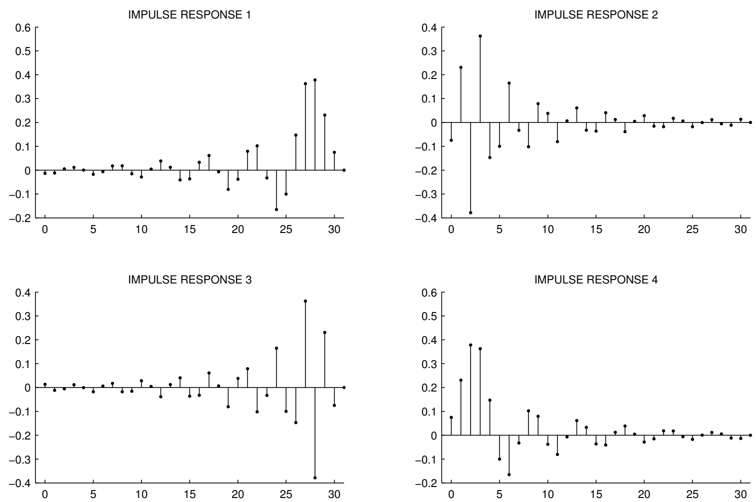
9.4 Consider the design of a minimum-phase FIR band-pass with specifications as in the previous problem but with

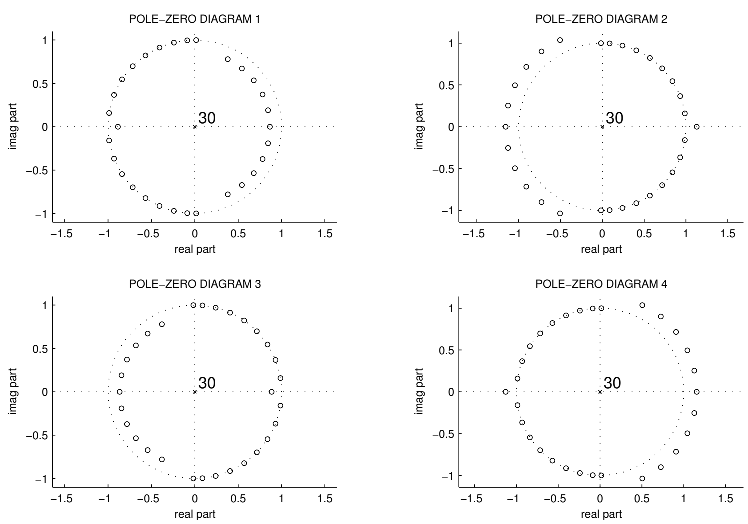
$$\Delta_{s1} = 0.06, \quad \text{and} \quad \Delta_{s2} = 0.03.$$

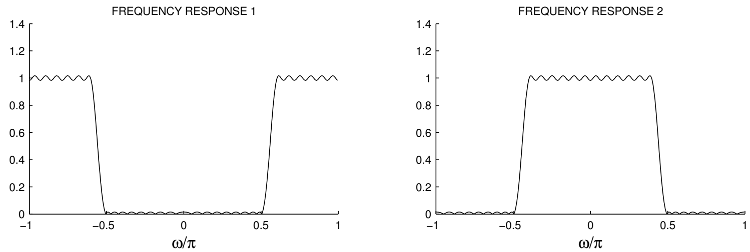
What problem arises in the lifting procedure?

9.5 **Matching.** The diagrams on the following three pages show the impulse responses, pole-zero diagrams, and frequency responses magnitudes of 4 discrete-time causal FIR filters. But the diagrams are out of order. Match each set of diagrams by filling out the following table. NOTE: There are only 2 distinct frequency response magnitudes among these 4 systems.

Impulse response	Pole-zero diagram	Frequency response
1		
2		
3		
4		

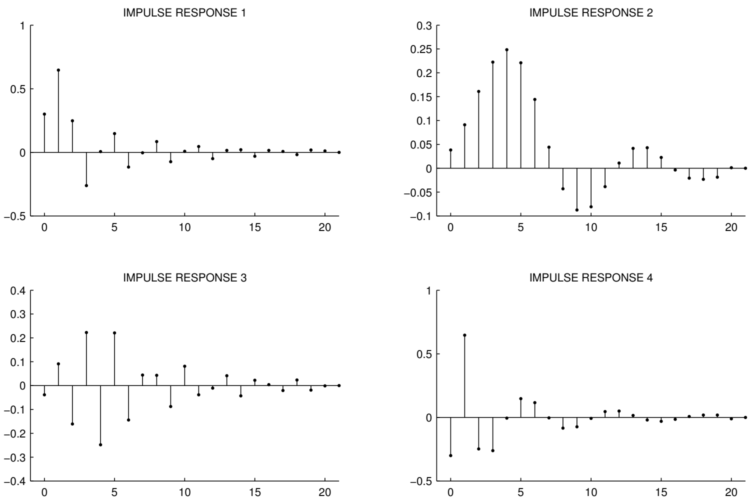


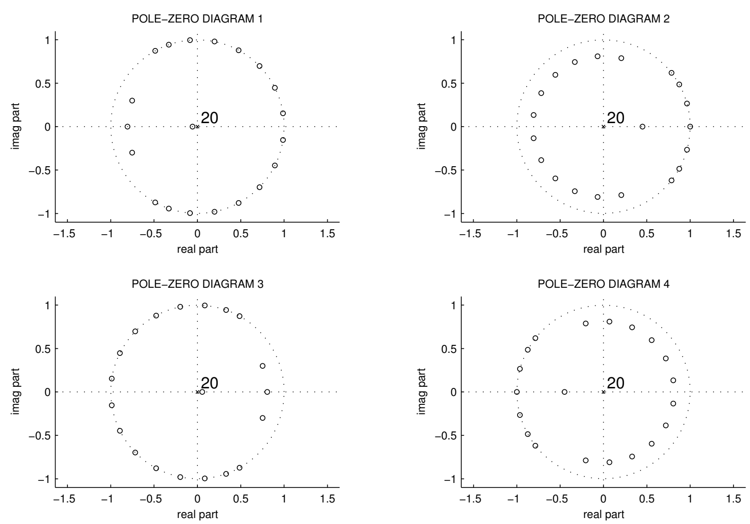


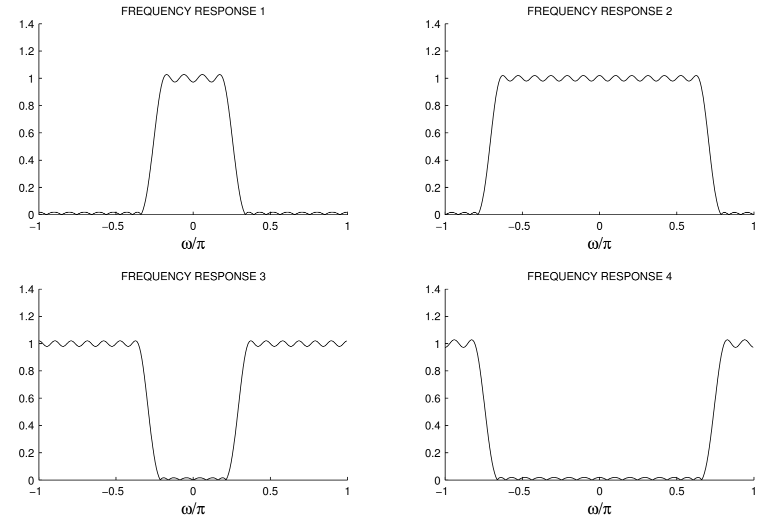


9.6 **Matching.** The diagrams on the following three pages show the impulse responses, pole-zero diagrams, and frequency responses magnitudes of 4 discrete-time causal FIR filters. But the diagrams are out of order. Match each set of diagrams by filling out the following table.

Impulse response	Pole-zero diagram	Frequency response
1		
2		
3		
4		

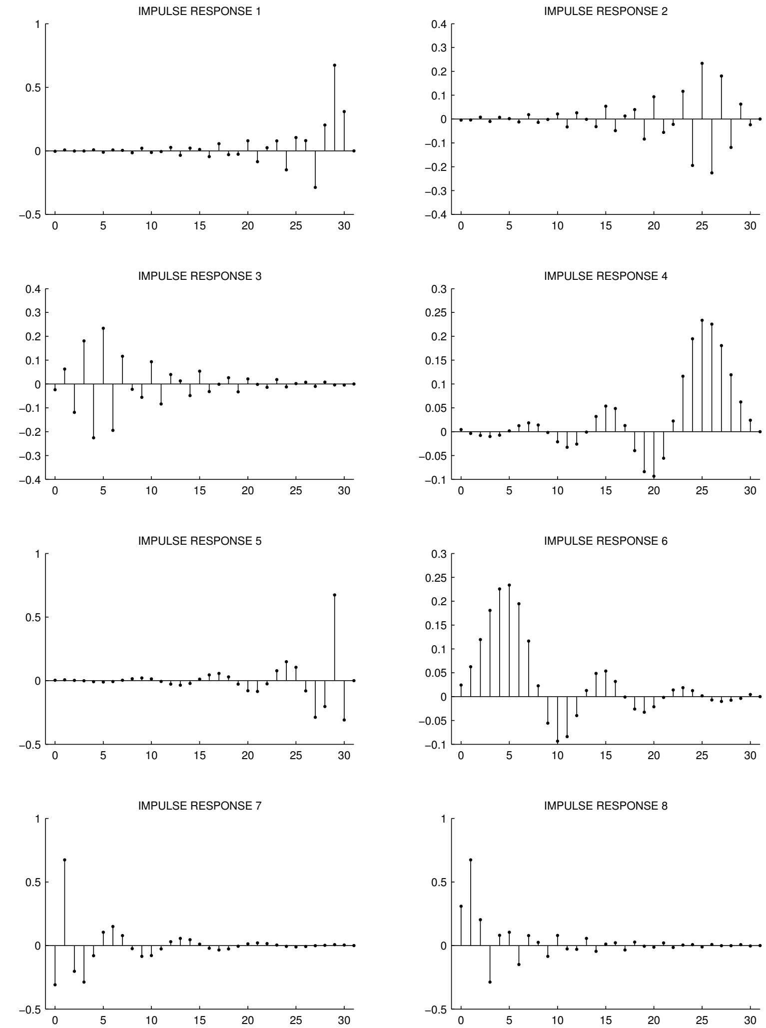


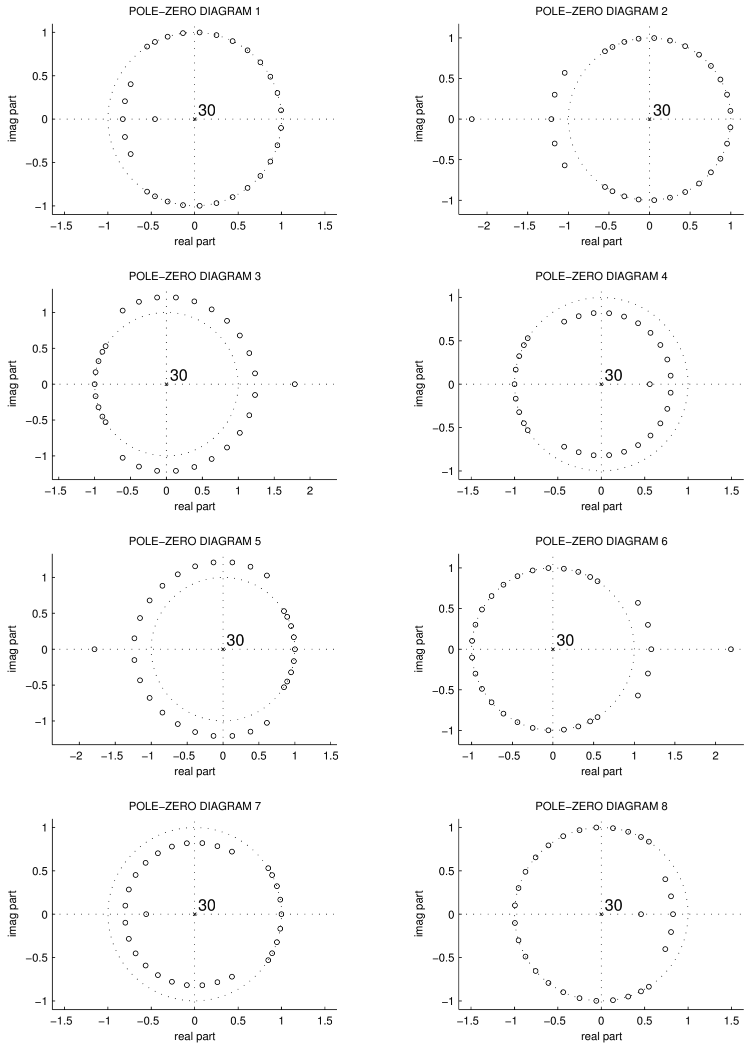


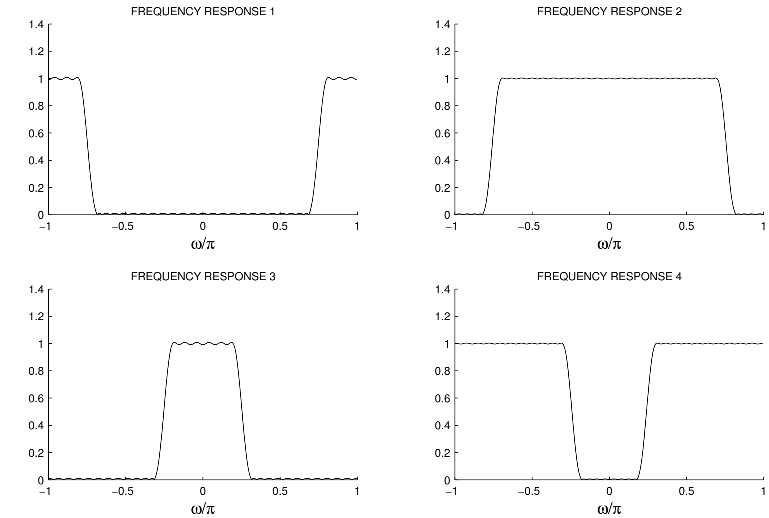


9.7 **Matching.** The diagrams on the following three pages show the impulse responses, pole-zero diagrams, and frequency responses magnitudes of 8 discrete-time causal FIR filters. But the diagrams are out of order. Match each set of diagrams by filling out the following table. NOTE: There are only four distinct frequency response magnitudes among these 8 systems.

Impulse response	Pole-zero	Frequency response
1		
2		
3		
4		
5		
6		
7		
8		

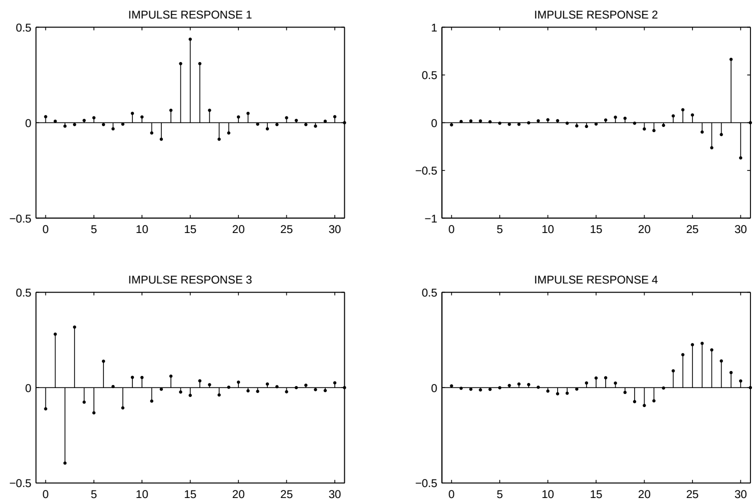


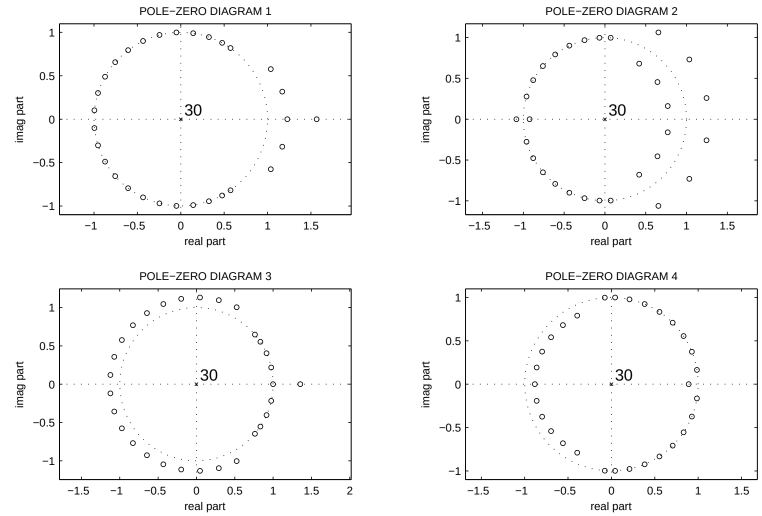

POLE–ZERO DIAGRAM 1
POLE–ZERO DIAGRAM 2
POLE–ZERO DIAGRAM 3
POLE–ZERO DIAGRAM 4
POLE–ZERO DIAGRAM 5
POLE–ZERO DIAGRAM 6
POLE–ZERO DIAGRAM 7
POLE–ZERO DIAGRAM 8
30
real part
imag part

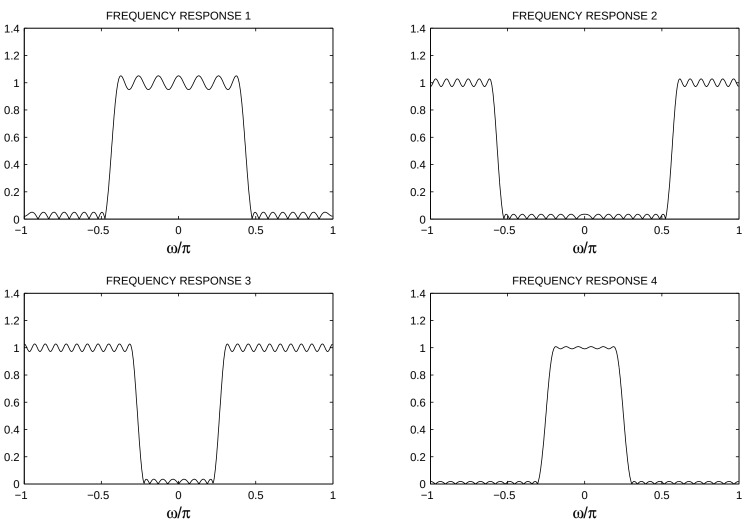

FREQUENCY RESPONSE 1
FREQUENCY RESPONSE 2
FREQUENCY RESPONSE 3
FREQUENCY RESPONSE 4
ω/π

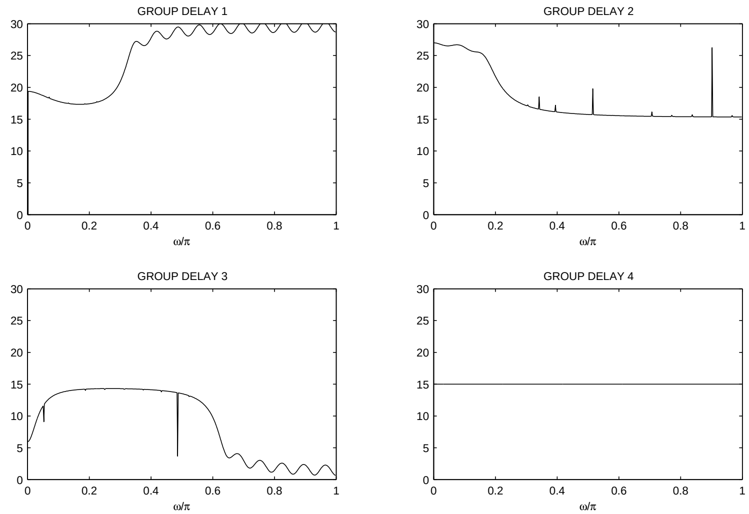
9.8 **Matching.** The following diagrams show the impulse responses, pole-zero diagrams, frequency responses magnitudes, and group delay responses of four discrete-time causal FIR filters. But the diagrams are out of order. Match each set of diagrams by filling out the following table. Each impulse response is 31 samples long.

Impulse resp.	P/Z	Freq. resp.	Group delay
1			
2			
3			
4			









9.9 Consider the design of a minimum-phase FIR low-pass filter with a specified null in the stop-band, as we have seen in previous notes on linear-phase FIR filters.

(a) What problem arises when you try to use the lifting procedure to design this type of filter?

(b) (*Optional.*) An alternative to the lifting procedure is to use an approach based on a linear program formulation. Develop an approach to design minimum-phase FIR filters based again on the spectral factorization of a Type I FIR filter. Design a minimum-phase FIR filter with the same specification and length as the example described in the notes on Chebyshev linear-phase FIR design and compare your result to that filter.

9.10 In previous notes we designed linear-phase FIR notch filters with a specified notch frequency. For this problem, you will design *minimum-phase* FIR notch filters. This can be done by performing spectral factorization of a suitably designed Type I linear-phase FIR notch filter. It will not be necessary to 'lift' the frequency response of the notch filter because it already has $A(\omega) \geq 0$. Recall the examples in previous notes. Specifically, design a minimum-phase notch filter of length 51 with notch frequency $\omega_n = 0.6\pi$ and band-edges $\omega_0 = 0.55\pi$, $\omega_1 = 0.65\pi$ where

$$D(\omega) = 1, \qquad \text{and} \qquad W(\omega) = \begin{cases} 1 & 0 \leq \omega \leq \omega_0 \\ 0 & \omega_0 < \omega < \omega_1 \\ 1 & \omega_1 \leq \omega \leq \pi. \end{cases}$$

Design two notch filters:

(a) Design a notch filter with a single zero at the notch frequency (accordingly, the linear-phase filter must have a double zero at the notch frequency).

(b) Design a notch filter with a double zero at the notch frequency (accordingly, the linear-phase filter must have a zero at the notch frequency of multiplicity 4).

Compare your notch-filter with the linear-phase ones given in the earlier notes. For both filters, plot the impulse response, frequency response magnitude, and zero diagram. Also plot the *unwrapped* phase and the group delay.

10 IIR Filter Design

10.1 Consider the design of an analog filter to meet the following requirements.

$$0.96 \leq |H_a(j\omega)| \leq 1 \qquad \text{for} \quad |\omega| \leq 4$$

$$|H_a(j\omega)| \leq 0.03 \qquad \text{for} \quad |\omega| \geq 4.6$$

(a) Design an analog Chebyshev-I filter of minimal degree to meet the requirements, by scaling the prototype analog filter. Make a plot of $|H_a(j\omega)|$ and the pole-zero diagram.

(b) Repeat using a Chebyshev-II analog filter.

(c) Repeat using a elliptic analog filter.

(d) Compare the results. Which filter type meets the requirements with the lowest degree?

You should use the Matlab commands `cheb1ap, cheb2ap, ellipap`. You will need to modify the normalized transfer function because the desired band edges do not conform to the prototype filter.

$$G_a(s) = H_a(C\, s)$$

This can be done by rescaling the poles and zeros by a suitable scaling constant,

```
z = z/C;
p = p/C;
```

where `z` and `p` contain the zeros and poles of the transfer function $H_a(s)$.

You will need to determine the appropriate value of C based on the desired band-edges and the filter type.

The commands `cheb1ord, cheb2ord, ellipord` will be useful for this problem. For analog filters, the syntax is

`[N,wn]=cheb1ord(wp,ws,Rp,Rs,'s')`.

Also, look at the commands, `cheby1(...,'s')`, `cheby2(...,'s')`, `ellip(...,'s')` for the design of analog lowpass filters. You might check that the analog filters you design above by scaling the prototype filter are in agreement with the analog filters given by these commands.

10.2 You should do this problem by hand (no computer).

(a) Suppose you have a *stable* causal analog filter. Prove why the bilinear transformation is guaranteed to produce a *stable* causal digital IIR filter.

(b) Suppose you have an *unstable* causal analog filter. Is it possible that the bilinear transformation will produce a *stable* causal digital IIR filter?

(c) Suppose the transfer function of an analog filter is

$$H_a(s) = \frac{1}{(s+1)(s+2)}.$$

When you apply the bilinear transformation to convert it to a digital filter, what will the transfer function of the digital filter be?

10.3 Consider the design of a recursive digital filter whose frequency response satisfies:

$$H^f(\omega) \approx \begin{cases} 1 & 0 \le \omega \le 0.3\,\pi \\ 0 & 0.5\,\pi \le \omega \le \pi \end{cases}$$

That means, the passband edge is $\omega_p = 0.3\,\pi$ and the stopband edge is $\omega_s = 0.5\,\pi$. Suppose that the recursive digital filter is to be obtained from an analog filter $H_a(s)$ using the bilinear transformation with

$$s = \frac{z-1}{z+1}.$$

Then what should be the passband and stopband edges of the analog filter $H_a(s)$ so that after the bilinear transformation the resulting digital filter will have the desired band edges?

10.4 (From Mitra `M7.5`.) Design a digital Chebyshev-I lowpass filter operating at a sampling rate of 80 kHz with a passband edge frequency at 4 kHz, a passband ripple of 0.5 dB, and a minimum stopband attenuation of 45 dB at 20 kHz using the bilinear transformation method. Determine the order of the analog prototype using the command `cheb1ord` and then design the analog prototype using `cheb1ap`. Transform the analog filter into a digital one using the `bilinear` command. Plot the frequency response, pole-zero diagram and impulse response of the digital filter. Show the steps of the design procedure.

10.5 **BLT of the integrator.** Apply the bilinear transformation to the ideal continuous-time integrator

$$H_a(s) = \frac{1}{s}$$

to obtain a digital integrator. The bilinear transformation is described by

$$s = C \cdot \frac{z-1}{z+1}$$

(a) What is the transfer function $H(z)$ of the discrete-time system?

(b) Write the difference equation for $H(z)$.

(c) Find an expression for the frequency response $H^f(\omega)$, magnitude $|H^f(\omega)|$ and the phase $\angle H^f(\omega)$. Sketch $|H^f(\omega)|$. Compare with the frequency response of the ideal continuous-time integrator.

(d) Sketch the pole-zero plot of $H(z)$ and comment on its stability.

10.6 **Impulse invariance method.** The bilinear transformation (BLT), for converting an analog filter into a digital filter, uses a transformation of variables,

$$s = C\,\frac{z-1}{z+1}.$$

Let us use $C = 2/T$ where T is the sampling interval for the digital filter.

Another way to convert an analog filter into a digital filter is the *impulse invariant* method. This problem compares the two methods applied to the following analog filter,

$$H_{\mathrm{a}}(s) = \frac{1}{s+3}$$

having impulse response

$$h_{\mathrm{a}}(t) = e^{-3t}\,u(t)$$

where $u(t)$ is the step function. Let us use $T = 0.1$ seconds.

(a) Use the BLT, to transform the analog filter into a digital filter. Find the transfer function and sketch its pole/zero diagram.

(b) The impulse invariance method samples the analog impulse response, to obtain the impulse resonse of a digital filter,

$$h(n) = h_{\mathrm{a}}(nT),$$

Applying this method to the analog filter above, find the transfer function of the resulting digital filter and sketch its pole/zero diagram.

(c) For the two digital filters obtained in parts (a) and (b), what is the difference between the frequency responses?

(d) For a higher order analog system $H_{\mathrm{a}}(s)$, the impulse invariance method can be applied to first order terms in a partial fraction expansion of $H_{\mathrm{a}}(s)$. If the impulse invariance method is applied to a *stable* analog filter, is it guaranteed that the digital filter will be stable?

10.7 **BLT and FIR filters.** The bilinear transformation (BLT), for converting an analog filter into a digital filter, uses a transformation of variables,

$$s = \frac{z-1}{z+1}.$$

Usually, the BLT produces a *recursive* digital filter. This problem investigates this issue.

(a) Consider an FIR digital filter with impulse response

$$h = [\underline{1}, \quad \frac{5}{6}, \quad \frac{1}{6}]$$

where $\underline{1}$ is the value of the impulse response $h(n)$ at $n = 0$. Sketch the pole/zero diagram of the digital filter.

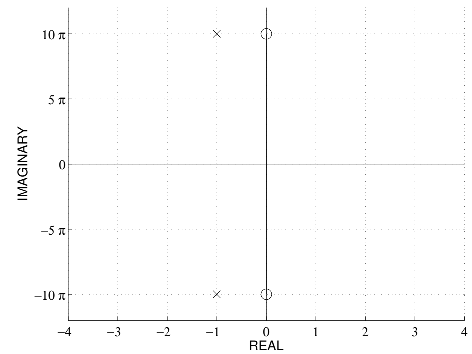
(b) Use the BLT in reverse, to transform the FIR digital filter into an analog filter. Find the transfer function $H_a(s)$ of the analog filter.

(c) Sketch the pole/zero diagram of the analog filter.

(d) In order that the BLT convert an analog filter into a digital filter with a *finite* impulse response, what condition must the analog filter satisfy? Is this condition generally satisfied by standard analog filters?

10.8 **Part 1.** A continuous-time LTI system with transfer function $H(s)$ has the pole-zero diagram shown. The system has a dc gain of unity.

No Matlab needed.



(a) Express in hertz the frequencies that are annihilated (blocked) by the system.

(b) Find the transfer function $H(s)$ of the system. Simplify.

(c) Sketch the frequency response magnitude $|H(\mathrm{j}\omega)|$. Explain how it can be sketched directly from the pole-zero diagram rather than direct numerical calculation. What kind of filter is the system?

Part 2. Design a second-order discrete-time system, operating at a sampling frequency of 30 samples/second, that approximates the continuous-time system in Part 1. Your discrete-time system should annihilate (block) the same frequencies as the continuous-time system. Clearly describe your approach and your steps.

(a) What are the poles and zeros of $H(z)$ of your discrete-time system? Sketch the pole-zero diagram.

(b) Sketch the frequency response $|H(\mathrm{e}^{\mathrm{j}\omega})|$.

(c) Give the difference equation that implements your discrete-time system.

(d) Name and briefly describe two general methods to convert an continuous-time LTI system to a discrete-time LTI system.

10.9 The following figure shows the pole-zero plots of 6 different digital IIR filters. By inspecting the pole-zero plots,

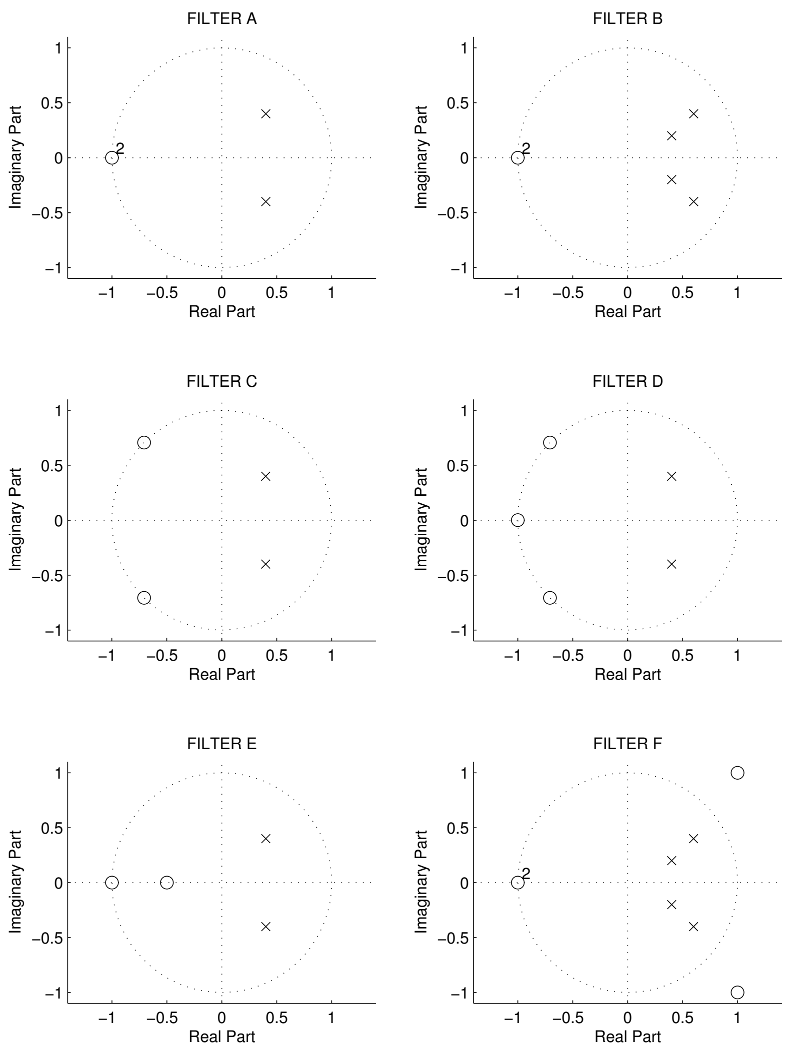
(a) which of the 6 filters could be a digital Butterworth filter?

(b) which of the 6 filters could be a digital Chebyshev (Chebyshev-I) filter?

(c) which of the 6 filters could be a digital inverse-Chebyshev (Chebyshev-II) filter?

(d) which of the 6 filters could be a digital Elliptic filter?

Give an explanation for your answers.


FILTER A
FILTER B
FILTER C
FILTER D
FILTER E
FILTER F
Imaginary Part
Real Part
2
1
0.5
0
−0.5
−1
−1
−0.5
0
0.5
1

10.10 **Filter design.** The spectrum of an analog signal is contained between 10 Hz and 25 Hz. The signal is corrupted by additive low-frequency noise band-limited to 0.5 Hz. (This is a 'baseline-drift'.) The noisy analog signal is sampled at 50 samples/second. Design a first-order recursive discrete-time filter, operating at 50 samples/second, to remove the noise from the signal. Give the difference equation; specify the poles and zeros, and sketch the pole-zero diagram. Plot the frequency response magnitude. Clearly explain your derivation and show your work.

10.11 **Forwards/backwards filtering.** A technique to perform filtering with no phase distortion (i.e. linear-phase filtering) is the following. ($h(n)$ is real-coefficient filter, not necessarily linear-phase.)

(1) Filter the data $x(n)$ with the filter $h(n)$.

(2) Reverse the filtered data; call it $g(n)$.

(3) Filter the data $g(n)$ with the filter $h(n)$.

(4) Reverse the resulting data again.

Show that this has the over all effect of filtering with a linear-phase filter. What is the total impulse response?

This method is especially useful for performing linear-phase filtering with IIR filters (which do not have linear phase)!

10.12 **The MATLAB `filtfilt` command.**

(a) What does the MATLAB `filtfilt` command do?

(b) If the causal filter

$$y(n) = x(n) + x(n-1) - 0.9y(n-1).$$

is used with the MATLAB `filtfilt` command, then what is the transfer function of the system being implemented? Sketch its pole-zero diagram.

(c) Sketch the frequency response magnitude for the system in (b).

10.13 If the causal filter

$$y(n) = 0.1\,x(n) + 0.2\,x(n-1) + 0.3\,x(n-2) + 0.4\,x(n-3)$$

is used in the MATLAB `filtfilt` command, then accurately sketch the total impulse response of the system being implemented.

10.14 **Zero-phase filtering.** Forwards-backwards filtering is a way to implement a zero-phase filter based on a causal filter.

(a) If the causal filter

$$y(n) = x(n) + \frac{2}{3}\,y(n-1)$$

is used in forwards-backwards filtering, then what is the transfer function of the total system being implemented? Sketch its pole-zero diagram.

(b) Sketch the frequency response magnitude for the system implemented using forwards-backwards filtering.

(c) Find the impulse response of the system implemented using forwards-backwards filtering.

(d) What are the benefits of implemented using forwards-backwards filtering?

(e) Why is forwards-backwards filtering rarely used with FIR filters?

(f) What MATLAB function implements forwards-backwards filtering?

11 Multirate Systems

11.1 The signal $x(n)$

$$x(n) = \{\ldots, 0,\ 0,\ \underline{1},\ 2,\ 3,\ 2,\ 1,\ 0,\ 0,\ \ldots\},$$

where $\underline{1}$ represents $x(0)$, is applied as the input to the following system.

$$x(n) \longrightarrow \boxed{\uparrow 2} \longrightarrow \boxed{H(z)} \longrightarrow \boxed{\downarrow 3} \longrightarrow y(n)$$

If the impulse response $h(n)$ is given by

$$h(n) = \{\underline{1},\ 2\}$$

then what is the output signal $y(n)$?

11.2 The signal $x(n)$

$$x(n) = \{\ldots, 0,\ 0,\ \underline{1},\ 2,\ -1,\ 0,\ 1,\ 0,\ 0,\ \ldots\}$$

where $\underline{1}$ represents $x(0)$ is applied as the input to the following system.

$$x(n) \longrightarrow \boxed{\uparrow 2} \longrightarrow \boxed{H(z^2)} \longrightarrow y(n)$$

If the impulse response $h(n)$ is given by

$$h(n) = \{\underline{1},\ 1\}$$

then what is the output signal $y(n)$?

11.3 A signal $x(n)$ is down-sampled by M, and the result is up-sampled by M to yield a signal $y(n)$. Express $Y(z)$ in terms of $X(z)$. Write the expression also in the special case when $M = 2$.

11.4 If the IIR filter $h(n)$ has the transfer function

$$H(z) = \frac{1}{1 - c\, z^{-1}}$$

find the polyphase components $H_0(z)$ and $H_1(z)$ so that

$$H(z) = H_0(z^2) + z^{-1}\, H_1(z^2).$$

How many poles does each polyphase component have?

11.5 If $h(n)$ is the impulse response of a linear-phase FIR filter, are the two polyphase components of $h(n)$ linear-phase as well? Consider separately the case when the length N is even and odd.

11.6 (From Mitra 10.20) The *running-sum* filter, also called the *boxcar filter*,

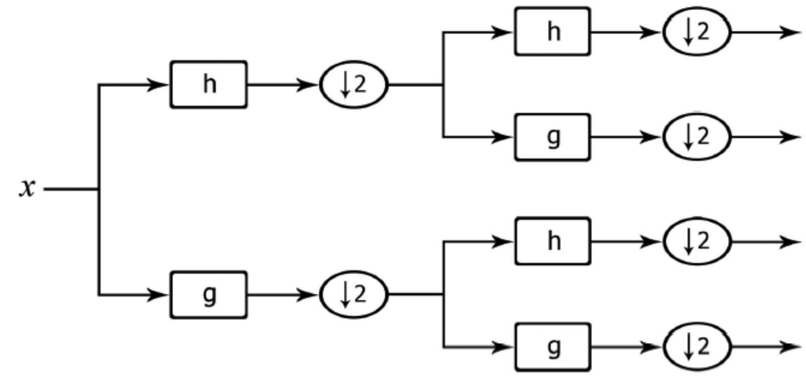
$$H(z) = \sum_{n=0}^{N-1} z^{-n}$$

can be expressed in the form

$$H(z) = (1 + z^{-1})\,(1 + z^{-2})\,(1 + z^{-4}) \cdots (1 + z^{-2^{K-1}})$$

where $N = 2^K$. Verify this for $N = 16$. What is the impulse response $h(n)$? Using a length 16 boxcar filter, develop a realization of a factor-16 decimator using a *cascade* of simple filters with downsampling by two between each filter. (Use the noble identities.)

11.7 In the following discrete-time multirate system the filter h is a lowpass filter, and the filter g is a highpass filter. The system produces four *subband* signals, we can call them $s_1(n), \ldots, s_4(n)$.



The path from the input signal to each of the four output signals can be rewritten using the Noble identities as

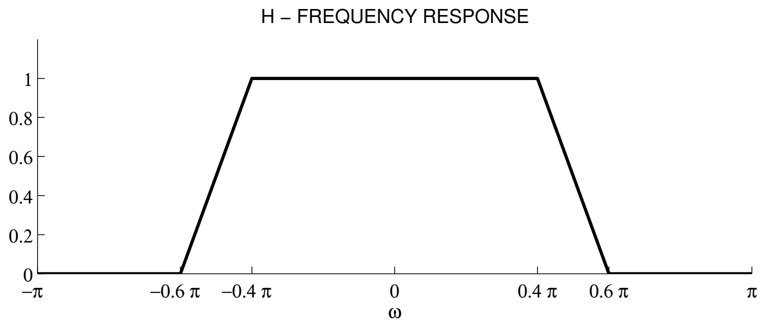
$$x(n) \longrightarrow \boxed{F_k(z)} \longrightarrow \boxed{\downarrow 4} \longrightarrow s_k(n)$$

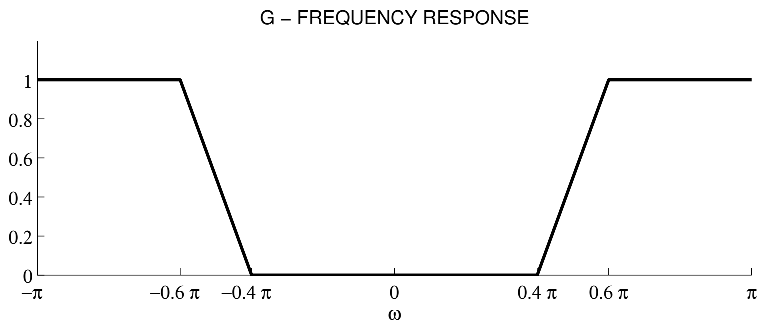
for $1 \le k \le 4$.

(a) What are the four transfer functions $F_k(z)$?

(b) Given the frequency response of h and g shown below, sketch the frequency responses of the four transfer functions $F_k(z)$.

(c) Classify each of the four transfer functions $F_k(z)$ as lowpass, bandpass, bandstop, or highpass.





11.8 Can you change the order of an up-sampler and a down-sampler with out change the total system? In other words, are the following two systems equivalent?

System A:

$$\longrightarrow \boxed{\downarrow M} \longrightarrow \boxed{\uparrow L} \longrightarrow$$

System B:

$$\longrightarrow \boxed{\uparrow L} \longrightarrow \boxed{\downarrow M} \longrightarrow$$

Consider the following cases:

(a) $M = 2$, $L = 2$.

(b) $M = 2$, $L = 3$.

(c) $M = 2$, $L = 4$.

Try an example in each case with a simple test input signal.

11.9 Simplify each of the following systems.

(a)

$$\longrightarrow \boxed{\downarrow 2} \longrightarrow \boxed{\downarrow 3} \longrightarrow$$

(b)

$$\longrightarrow \boxed{\uparrow 2} \longrightarrow \boxed{\uparrow 2} \longrightarrow$$

(c)

$$\longrightarrow \boxed{\uparrow 5} \longrightarrow \boxed{\downarrow 10} \longrightarrow \boxed{\uparrow 2} \longrightarrow$$

11.10 Rewrite the following multirate system

$$x(n) \longrightarrow \boxed{\downarrow 2} \longrightarrow \boxed{H(z)} \longrightarrow \boxed{\downarrow 5} \longrightarrow \boxed{G(z)} \longrightarrow \boxed{\uparrow 3} \longrightarrow y(n)$$

in the following form:

$$x(n) \longrightarrow \boxed{\uparrow M} \longrightarrow \boxed{\text{LTI}} \longrightarrow \boxed{\downarrow N} \longrightarrow y(n)$$

What is M, N, and what is the transfer function of the LTI system?

11.11 **Rate changing.** A discrete-time signal $x(n)$ having a rate of 100 samples per second must be converted into a new discrete-time signal $y(n)$ having a rate of 90 samples per second.

(a) Sketch a multirate system that performs the necessary sampling rate conversion. Sketch the frequency responses of the required filter(s).

(b) Because ideal 'brick-wall' filters can not be realized, how would you proceed with the design of filters(s) for this multirate system?

11.12 **Rate changing.** A discrete-time signal $x(n)$ having a rate of 12 samples per second must be converted into a new discrete-time signal $y(n)$ having a rate of 9 samples per second.

Sketch a multirate system that performs the necessary sampling rate conversion. Sketch the frequency responses of the required filter(s).

11.13 Suppose the DTFT of $x(n)$ is $X^f(\omega)$:

$$X^f(\omega) = 1 - \frac{|\omega|}{\pi}, \qquad \text{for } |\omega| \le \pi$$

Suppose we generate the sequences $y(n)$ and $s(n)$ from $x(n)$ with the following system

$$x(n) \longrightarrow \boxed{H(z)} \longrightarrow \boxed{\downarrow 2} \longrightarrow \boxed{H(z)} \longrightarrow \boxed{\downarrow 2} \xrightarrow{s(n)} \boxed{\uparrow 4} \longrightarrow y(n)$$

where

$$H^f(\omega) = \begin{cases} 1, & |\omega| < \pi/2 \\ 0, & \pi/2 \le |\omega| < \pi \end{cases}$$

Sketch $X^f(\omega)$, $H^f(\omega)$, $S^f(\omega)$ and $Y^f(\omega)$.

11.14 Suppose $x(n)$ is the following discrete-time signal,

$$x(n) = 2\cos(0.2\,\pi\,n) + 3\cos(0.4\,\pi\,n) + 4\cos(0.6\,\pi\,n).$$

Suppose we generate the sequences $y(n)$ and $s(n)$ from $x(n)$ with the following system

$$x(n) \longrightarrow \boxed{H(z)} \longrightarrow \boxed{\downarrow 2} \longrightarrow \boxed{H(z)} \longrightarrow \boxed{\downarrow 2} \xrightarrow{s(n)} \boxed{\uparrow 4} \longrightarrow y(n)$$

where

$$H^f(\omega) = \begin{cases} 1, & |\omega| < \pi/2 \\ 0, & \pi/2 \le |\omega| < \pi \end{cases}$$

Sketch $X^f(\omega)$, $H^f(\omega)$, $S^f(\omega)$ and $Y^f(\omega)$.

11.15 **Linear-phase FIR Nyquist-L filter design.** Using Matlab, design a Type I linear-phase *Nyquist*-3 FIR filter of length 29. A Nyquist-3 filter $h(n)$ is one for which every third coefficient is zero, except for one such value. (See the notes on multirate systems.) The cut-off frequency ω_o should be $\pi/3$. Use the following design methods.

(a) Spline method.
Why does the spline method produce a Nyquist filter?

(b) (Constrained) Least squares.
Use $K_p = K_s = 1$ and $\omega_p = \omega_o - \frac{\pi}{20}$, $\omega_s = \omega_o + \frac{\pi}{20}$.

(c) Constrained Chebyshev with linear programming.
Use $K_p = K_s = 1$ and $\omega_p = \omega_o - \frac{\pi}{20}$, $\omega_s = \omega_o + \frac{\pi}{20}$.

For (b) and (c), the Nyquist property can be included in the design by adding constraints on $h(n)$.

Demonstrate the use of one of the filters to do interpolation of a signal $x(n)$ by a factor of 3. First upsample $x(n)$ by 3, and then filter it to get $y(n)$. Verify that the values $x(n)$ are not changed by the process. A data set will be available on the course webpage.

11.16 Let $h(n)$ be a low-pass half-band filter with real coefficients. Then show that

$$H^f(\omega) + H^f(\pi - \omega) = 2.$$

(That means the ripple in the pass-band of $H^f(\omega)$ is the same as the ripple in the stop-band of $H^f(\omega)$). You can assume the Nyquist filter is centered at $n_o = 0$.

Here, the DC gain will be about 2, rather than 1 as is usual for a low-pass filter. What must the cut-off frequency ω_o be? ($H^f(\omega_o) = 1$) Does this explain the terminology *half-band*?

11.17 **Nyquist filter design.** Design a Type I linear-phase FIR Nyquist-3 filter of minimal length having a transfer function $H(z)$ of the form

$$H(z) = Q(z)\,(1 + z^{-1} + z^{-2})^3.$$

(a) Find the impulse response $h(n)$ of the filter.

(b) Sketch the zeros of $H(z)$ in the complex z-plane.

(c) What particular properties does the filter have, when it is used for interpolation?

11.18 **Half-band filter design.** Design a Type I linear-phase FIR digital half-band filter of minimal length having a transfer function $H(z)$ of the form

$$H(z) = Q(z)\,(1 + z^{-1})^2\,(1 + z^{-1} + z^{-2}).$$

(a) Find the impulse response $h(n)$ of the filter.

(b) Sketch the zeros of $H(z)$ in the complex z-plane.

(c) Roughly sketch the frequency response $|H(e^{j\omega})|$ based on the filter's zero-diagram and half-band property.

(d) What particular properties does the filter have, when it is used for interpolation?

11.19 **Half-band filter design.** Design a Type I linear-phase FIR digital half-band filter of minimal length having a transfer function $H(z)$ of the form

$$H(z) = Q(z)\,(1 + 2z^{-1} + z^{-2})\,(1 + 3z^{-1} + z^{-2}).$$

(a) Find the impulse response $h(n)$ of the filter.

(b) Sketch the zeros of $H(z)$ in the complex z-plane.

(c) Roughly sketch the frequency response $|H(e^{j\omega})|$ based on the filter's zero-diagram and half-band property.

(d) What particular properties does the filter have, when it is used for interpolation?

11.20 **Interpolation and half-band filters.** Show using multirate identities that the interpolation with a factor of 2 using a halfband filter preserves the samples of the input signal $x(n)$.

$$x(n) \longrightarrow \boxed{\uparrow 2} \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

Do this by downsampling the output $y(n)$ of the 2X-interpolator and showing that it is the same as $x(n)$ when a halfband filter is used.

11.21 **Interpolation.** Consider the interpolation of a discrete-time signal $x(n)$ by a factor of three. The new higher rate signal is denoted $y(n)$. It is required that the values $x(n)$ be preserved in $y(n)$, i.e. $y(3n) = x(n)$ for all $n \in \mathbb{Z}$.

(a) What condition should the impulse response of the interpolation filter satisfy?

(b) Verify that your condition satisfies the requirement.

11.22 **Nyquist filters.** For a linear-phase half-band filter (centered at $n = 0$), we have

$$H(\omega) + H(\omega - \pi) = 2.$$

What is the frequency-domain equivalent condition for a Nyquist-3 filter?

11.23 **Rate changing.** To increase the sampling rate of a discrete-time signal $x(n)$, to four thirds its original rate, the following system is used.

$$x(n) \longrightarrow \boxed{\uparrow 4} \longrightarrow \boxed{H(z)} \longrightarrow \boxed{\downarrow 3} \longrightarrow y(n)$$

where the filter $H(z)$ is a low-pass filter.

If the filter $H(z)$ is furthermore a Nyquist-4 filter, are any samples of the input signal $x(n)$ preserved in the output signal $y(n)$? In other words, does the output signal satisfy:

$$y(Kn) = x(Ln)$$

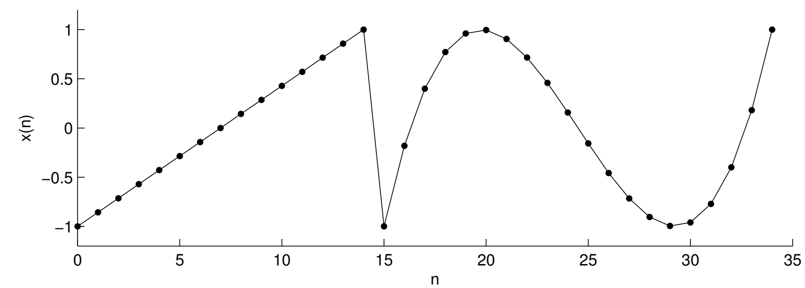
for some integers K and L? If so, show a derivation and identify K and L.

You can assume the Nyquist filter is centered at $n = 0$, i.e. $h(4n) = \delta(n)$.

Demonstrate your answer by a MATLAB example.

11.24 **Interpolation of polynomial signals**

The following test signal consists of two polynomial segments.



Interpolate this signal by 2 (upsample by 2 and filter) using each of the following filters $h(n)$ and plot the new signal. (You should obtain signals which are about twice as long as the test signal shown above.)

(a) $h(n) = [1 \quad 1]/2$

(b) $h(n) = [1 \quad 2 \quad 1]/4$

(c) $h(n) = [1 \quad 3 \quad 3 \quad 1]/8$

For each filter, give $H(z)$ in factored form.

Explain your observations using the results in the lecture notes concerning the interpolation of polynomial signals.

That Matlab code to genearate the test signal shown above is available on the course webpage.

11.25 An Interpolated FIR (IFIR) filter is an FIR filter implemented as a cascade of two filters in the following structure.

$$x(n) \longrightarrow \boxed{F(z)} \longrightarrow \boxed{G(z^2)} \longrightarrow y(n)$$

To satisfy some filter specifications, this kind of multistage filter can be more efficient than a single FIR filter. Suppose $G(z)$ is a lowpass filter with the transition band from 0.2π to 0.3π. (Although it is not realistic, suppose $G^f(\omega)$ is exactly 1 in the passband and exactly 0 in the stopband, and linear in between. Make the same assumption about $F(z)$ in part (c).)

(a) Sketch the frequency response of the transfer function $G(z)$.

(b) Sketch the frequency response of the transfer function $G(z^2)$.

(c) $F(z)$ is to be a lowpass filter so that the total system is a lowpass filter with transition band from 0.1π to 0.15π. What should be the transition band of the lowpass filter $F(z)$? Sketch the frequency response of the total system with your $F(z)$.

(d) Given an expression for the impulse response $h(n)$ of the total system in terms of $f(n)$ and $g(n)$. Explain why this system is called an *Interpolated* FIR filter.

11.26 An IFIR (Interpolated FIR) filter is an FIR filter of the form:

$$H(z) = H_1(z^M)\, H_2(z).$$

Sometimes an IFIR filter can meet given specifications with a lower implementation complexity than a generic FIR filter.

Consider the design of a lowpass filter $H(z)$ having cut-off frequency $0.125\,\pi = \pi/8$. Suppose

$$H(z) = H_1(z^4)\, H_2(z)$$

and that $H_1(z)$ is the ideal lowpass filter with cut-off frequency $0.5\,\pi$. That means, $H_1(z)$ is given, and $H_2(z)$ is to be designed so that the total filter $H(z)$ is a lowpass filter with cut-off frequency $0.125\,\pi = \pi/8$.

(a) Sketch the frequency response of $H_1(z)$ and of $H_1(z^4)$.

(b) What should be the passband edge and stopband edge of the filter $H_2(z)$? What is the widest transition band $H_2(z)$ can have? What is the advantage of choosing a wide transition band for the design of $H_2(z)$?

11.27 The following system consists of a rate changer in sequence with itself.

$$x(n) \longrightarrow \boxed{\uparrow 2} \longrightarrow \boxed{H(z)} \longrightarrow \boxed{\downarrow 3} \longrightarrow \boxed{\uparrow 2} \longrightarrow \boxed{H(z)} \longrightarrow \boxed{\downarrow 3} \longrightarrow y(n)$$

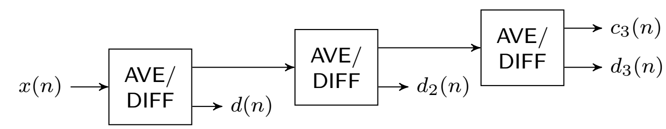
(a) The total system can be rewritten as a simpler one:

$$x(n) \longrightarrow \boxed{\uparrow p} \longrightarrow \boxed{G(z)} \longrightarrow \boxed{\downarrow r} \longrightarrow y(n)$$

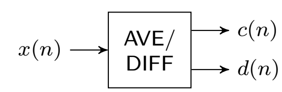
What is p, r, and the transfer function $G(z)$?

(b) Let $H(z)$ be a LPF with cut-off frequency $\omega_c = \pi/3$ with a transition band of width $\Delta\omega$. Let the frequency response of $H(z)$ be exactly unity and zero in the pass-band and stop-band respectively. Let the transition-band be linear over the frequency interval $[\omega_c - \Delta\omega/2,\ \omega_c + \Delta\omega/2]$. Then sketch the frequency response of $G(z)$.

11.28 **The wavelet transform.** The Haar wavelet transform is implemented using the system



where the sub-system



is described by the two equations:

$$c(n) = 0.5\,x(2n) + 0.5\,x(2n+1)$$
$$d(n) = 0.5\,x(2n) - 0.5\,x(2n+1).$$

(a) Draw a block diagram for obtaining $c(n)$ from $x(n)$. The block diagram may combine an up-sampler, a down-sampler, and an LTI filter; but not more than one of each. What is the impulse response of the LTI filter? Sketch the zeros of the filter.
Similarly, draw a block diagram for obtaining $d(n)$ from $x(n)$.

(b) Draw a block diagram expressing the system between $x(n)$ and $d_2(n)$ in the 3-level system above. The block diagram may combine up-samplers, down-samplers, and LTI filters; but it should not have more than one of each. (Hint: use part a and the Noble Identities.) Sketch the impulse response and the zeros of the LTI system. Based on the zero diagram, roughly sketch the frequency response of the LTI system. Show your work.

(c) Repeat part (b), but for $d_3(n)$ instead of $d_2(n)$.

12 Quantization

12.1 **Scaling.** Suppose the signal $x(n)$ is bounded by 0.5, i.e. $|x(n)| < 0.5$ for all n. The impulse response of an FIR digital filter is

$$h(n) = \{\underline{1.5},\ 2.0,\ -0.5,\ 0.3,\ 0,\ 0,\ \dots\}$$

To ensure that the output signal $y(n)$ does not exceed 1 in absolute value, how should $x(n)$ be scaled before filtering? Show your work.

12.2 **Scaling.** The DTFT $X^f(\omega)$ of signal $x(n)$ is bounded by 2, i.e. $|X^f(\omega)| < 2$ for all ω. It is filtered with an LTI system with frequency response

$$H^f(\omega) = \begin{cases} 1, & |\omega| \le 0.3\pi \\ 0, & 0.3\pi < |\omega| \le \pi \end{cases}$$

To ensure that the output signal $y(n)$ does not exceed 1 in absolute value, how should $x(n)$ be scaled before filtering? Show your work.

12.3 You have developed a system using an 8 bit uniform quantizer. For this system, the SNR (ratio of signal power to quantization noise power) turns out to be 10 dB, although the target SNR is 20 dB. How many additional bits do you expect is needed in the uniform quantizer in order to meet that goal?

12.4 *SNR of a quantizer.* In this problem we assume the original signal $x(n)$ is Gaussian (normally distributed). We will measure the SNR

$$\text{SNR} = 10\ \log_{10} \frac{\sigma_x^2}{\sigma_e^2}$$

under different conditions. We will use the Matlab command `randn` to generate random numbers according to the Gaussian bell shape distribution.

To simulate the quantization, we will use the Matlab program `fxquant` (available on the webpage). This program allows various options for how overflows are treated. We will use the saturation mode, in which overflows are clipped to the maximum and minimum.

```
y = fxquant(x,b,'round','sat');
```

The program `fxquant` returns y with $-1 \le y < 1$.

(a) Choose $\sigma_x^2 = 0.01$ and use 10000 samples. Measure and plot the SNR in dB as a function of the number of bits b for $1 \le b \le 12$. Check to see if the plot you obtain agrees with the expression obtained in the lecture notes,

$$\text{SNR} = 10\ \log_{10} \sigma_x^2 + 20\, b\ \log_{10} 2 - 10\ \log_{10} \left(\frac{R_{fs}^2}{12} \right)$$

by plotting both curves on the same graph. (Note that `fxquant` uses $R_{fs} = 2$.) Does the SNR improve by 6 dB per bit?

Repeat with $\sigma_x^2 = 0.1$. Explain any deviation you find from the SNR formula. What assumptions are violated?

(b) The following experiment will be done with an 8 bit wordlength. Measure and plot the SNR in dB as a function of σ_x^2 in dB for $-70\text{dB} \le \sigma_x^2 \le 10\text{dB}$. In a certain range, the SNR increases linearly with σ_x^2 in dB. Check to see if the plot you obtain agrees with the expression obtained in the lecture notes,

$$\text{SNR} = 10\ \log_{10} \sigma_x^2 - 10\ \log_{10} \sigma_e^2$$

by plotting both curves on the same graph. Explain any deviation you find from SNR formula. What assumptions are violated?

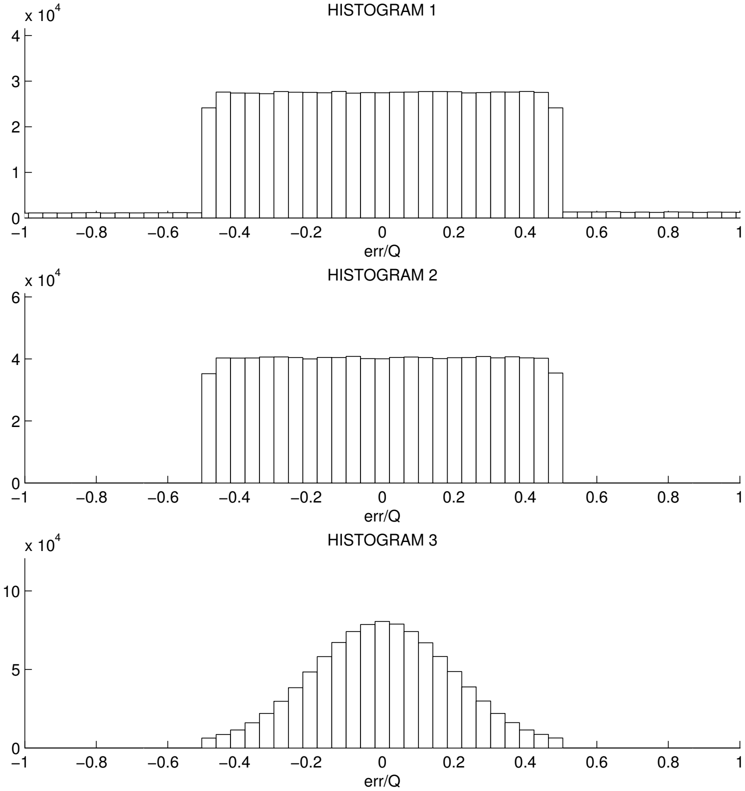
12.5 **Quantization.** If a white Gaussian random discrete-time signal with variance σ^2 is quantized using a uniform quantizer with N bits, then under standard assumptions, the quantization error is uniformly distributed.

The following figure shows quantization error histograms for three different combinations of N and σ. The quantizer provides an output in the range $[-1, 1)$. Identity which histogram results from each of the three combinations by completing the table.

Explain in each case how the shape of the histogram is influenced by N and σ.

In each case, is the true SNR likely to be higher than, lower than, or in agreement, with the SNR predicted using the assumption that the quantization error is uniformly distributed?

Number of bits	σ	Histogram
2	0.1	
4	0.1	
4	1.0	



13 Spectral Estimation

13.1 A zero-mean white stationary random signal $x(n)$ has a variance of 3,

$$E[|x(n)|^2] = 3.$$

The signal $x(n)$ is passed through an FIR filter having the impulse response

$$h(0) = 1,\ h(1) = 0.7,\ h(2) = 0.3,\ h(n) = 0 \text{ for other } n,$$

to produce an output signal $y(n)$.

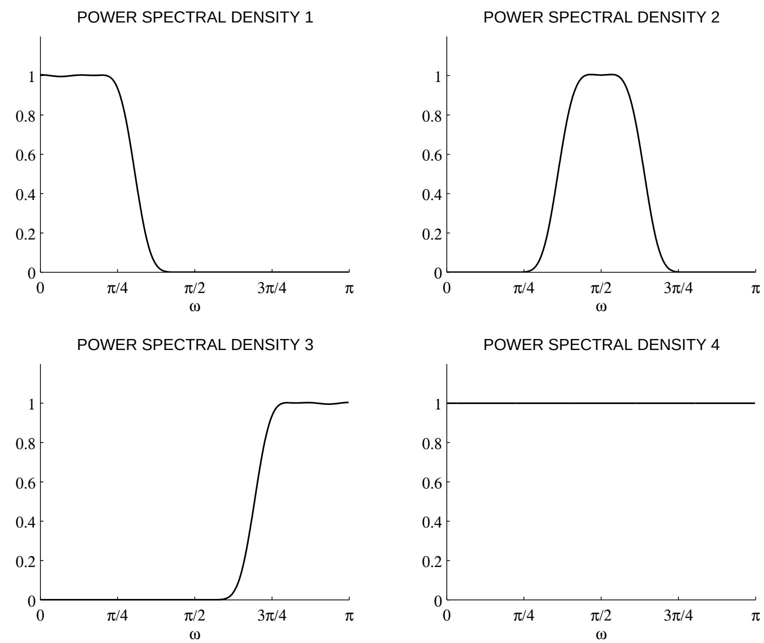
$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

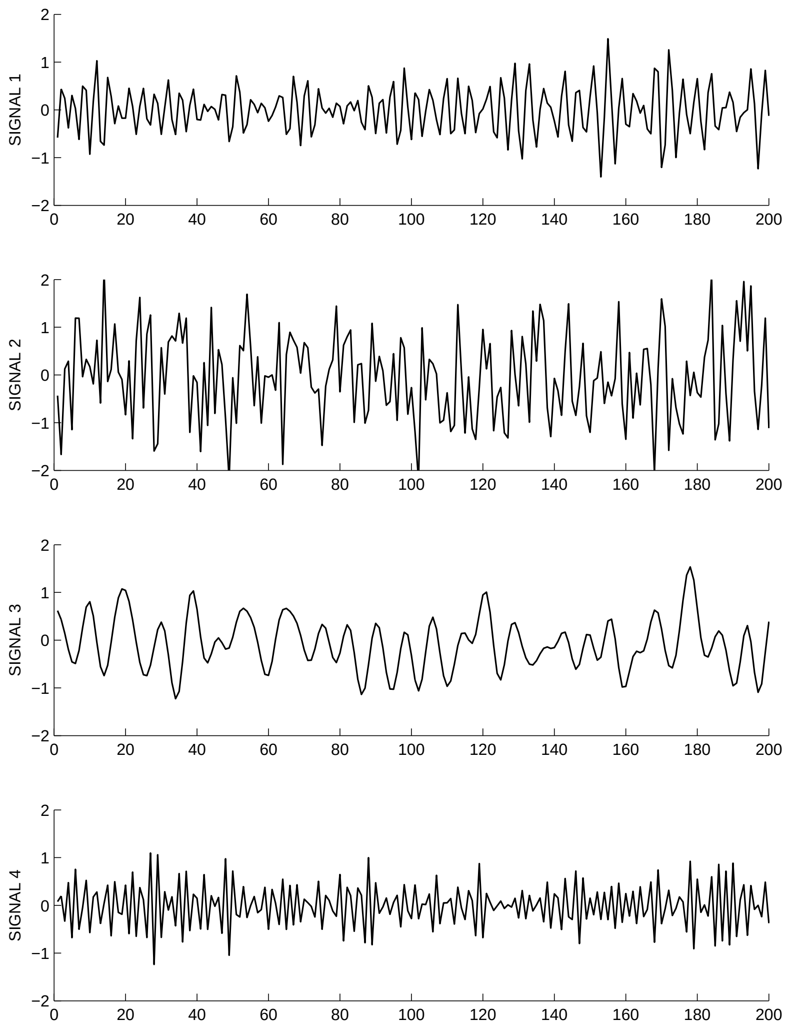
(a) Find the variance of the output signal, $E[|y(n)|^2]$.

(b) Accurately sketch the autocorrelation function $r_y(k)$ of $y(n)$.

13.2 **Power Spectral Densities.** The figure below shows the power spectral density (PSD) of four stationary distrectre-time random processes. The following figures also shows a realization (signal) generated using each of the four PSDs, but they are out of order. Match each signal to its most likely PSD by completing the table.

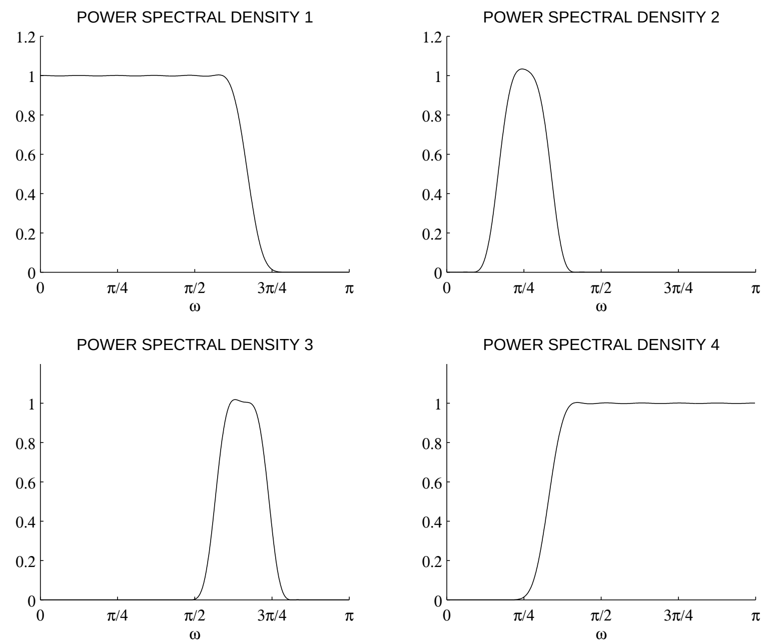
PSD	SIGNAL
1	
2	
3	
4	





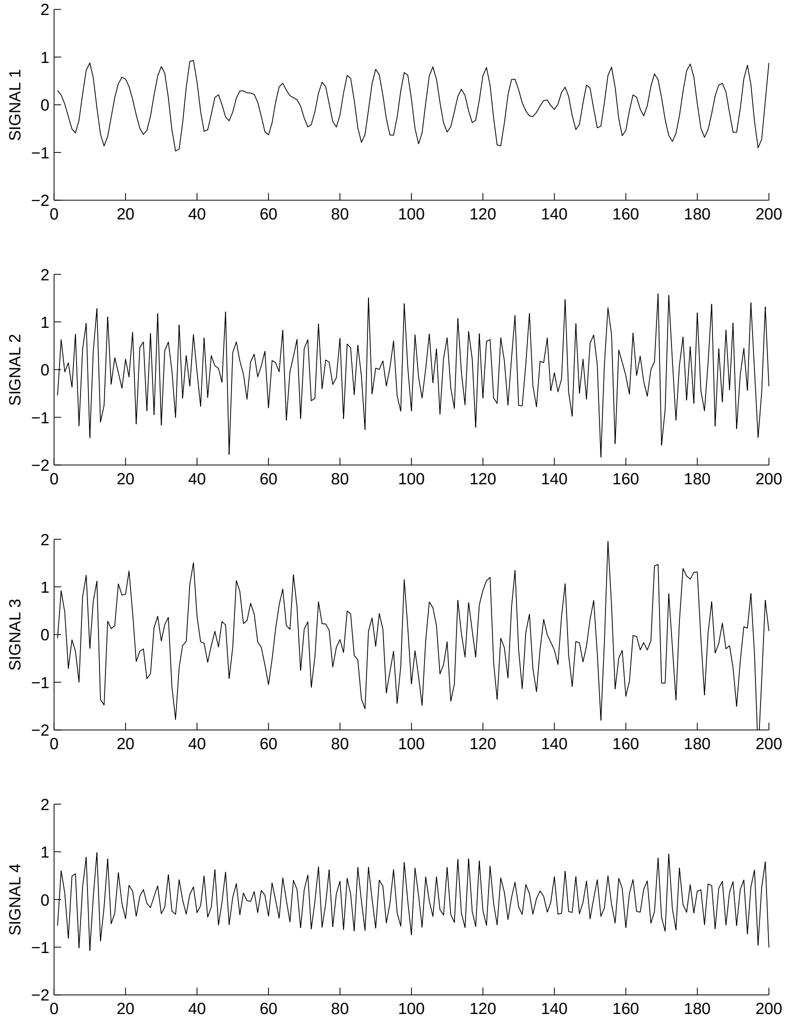
13.3 **Power Spectral Densities.** The figure below shows the power spectral density (PSD) of four different stationary discrete-time random processes. The following figure also shows a realization (signal) generated using each of the four PSDs, but they are out of order. Match each signal to its most likely PSD by completing the table. Explain your answers.

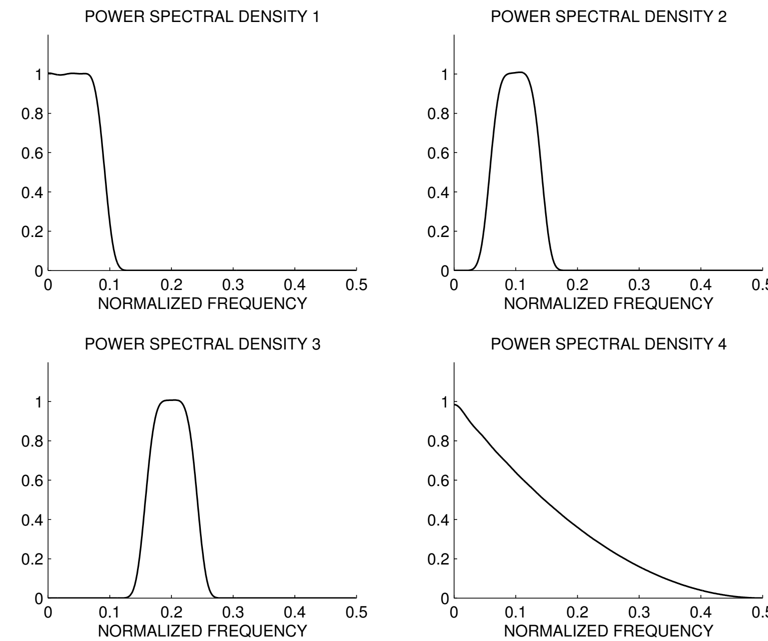
PSD	SIGNAL
1	
2	
3	
4	

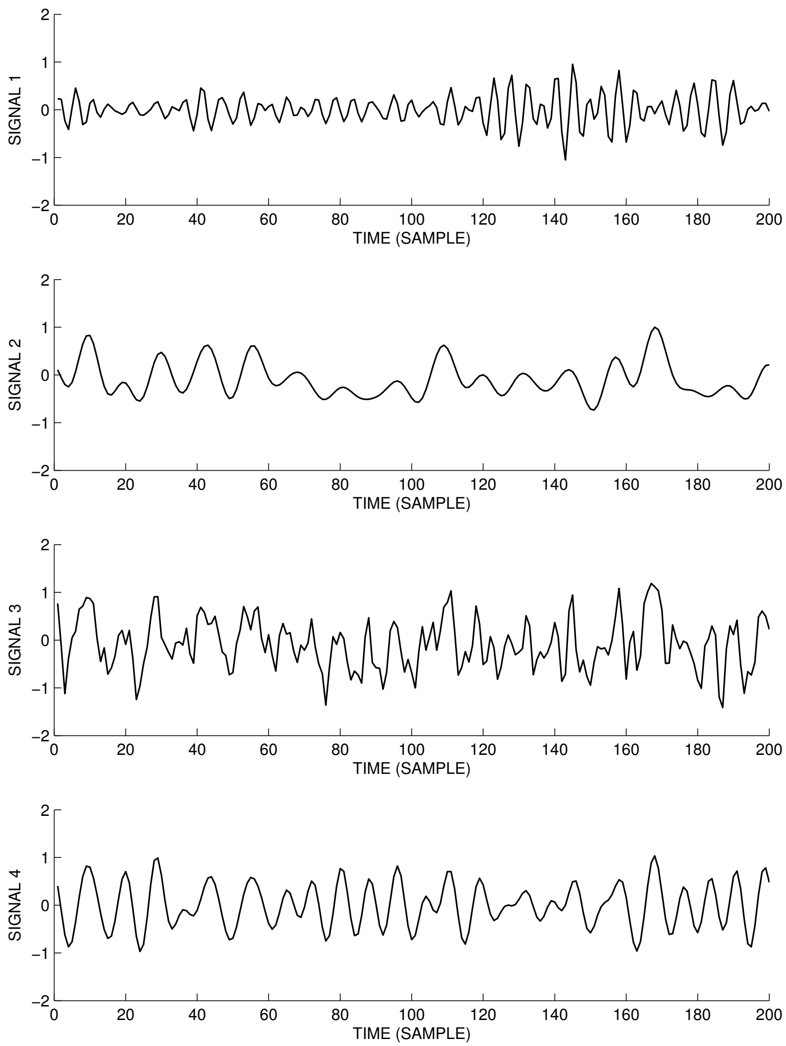


13.4 **Power spectral densities.** The figures below show the power spectral density (PSD) of four stationary distrectre-time random processes. The following figures also show a realization (signal) generated using each of the four PSDs, but they are out of order. Match each signal to its most likely PSD by completing the table.

PSD	Signal
1	
2	
3	
4	

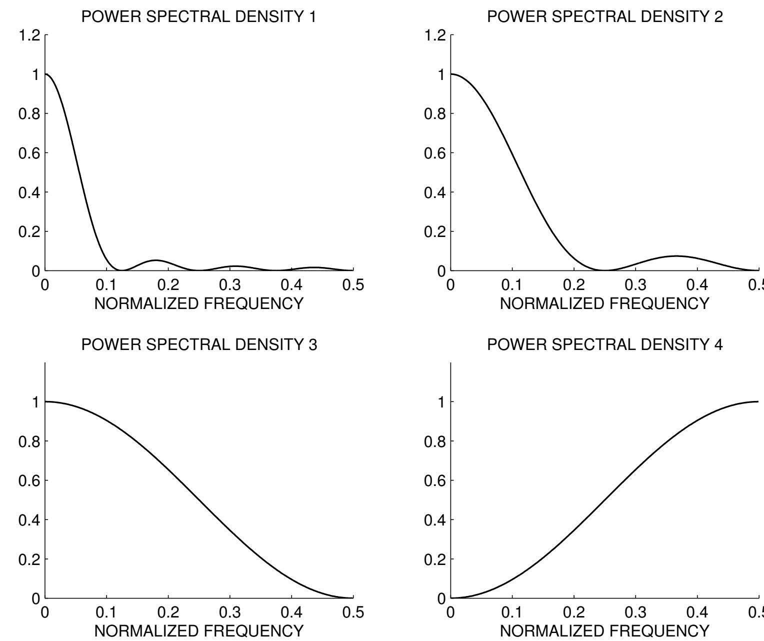


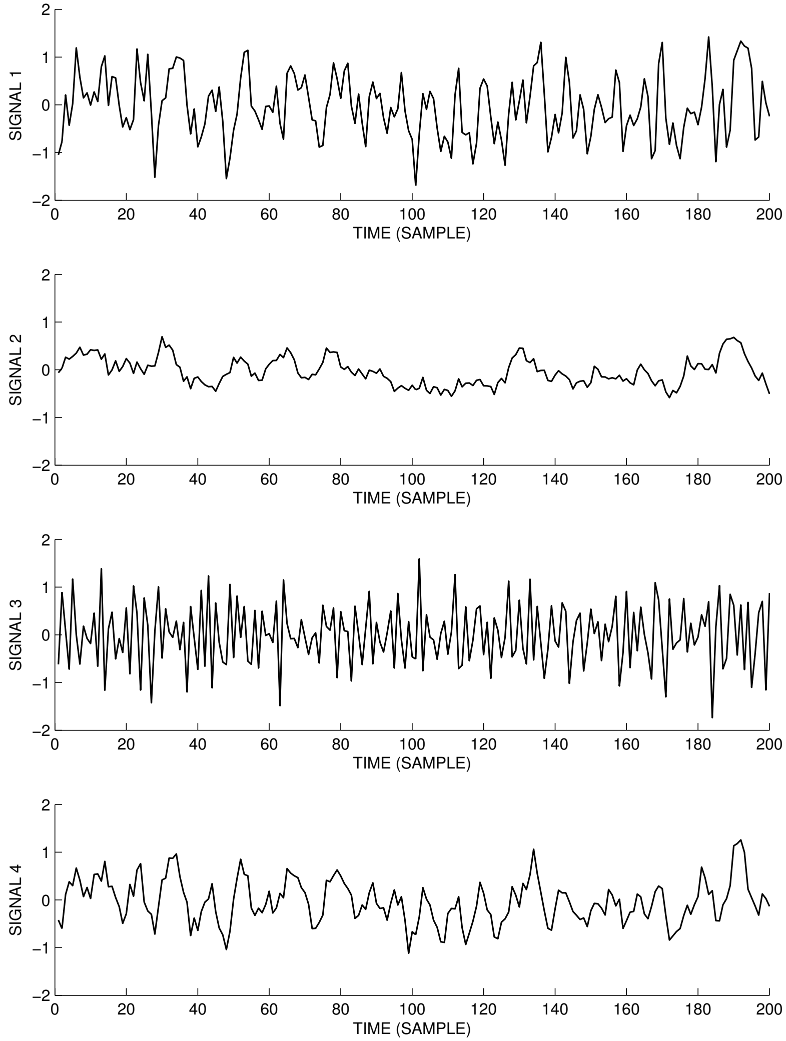




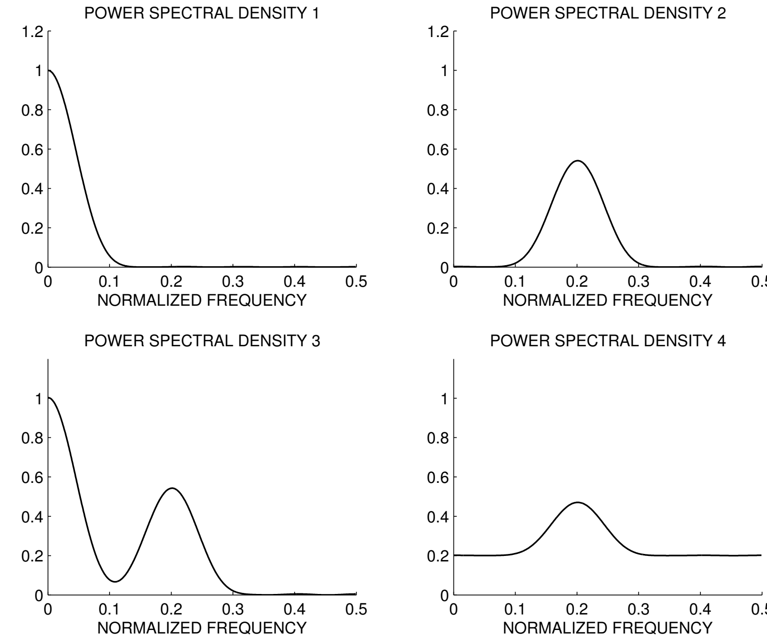
13.5 **Power spectral densities.** The figures below show the power spectral density (PSD) of four stationary distrectre-time random processes. The following figures also show a realization (signal) generated using each of the four PSDs, but they are out of order. Match each signal to its most likely PSD by completing the table.

PSD	Signal
1	
2	
3	
4	



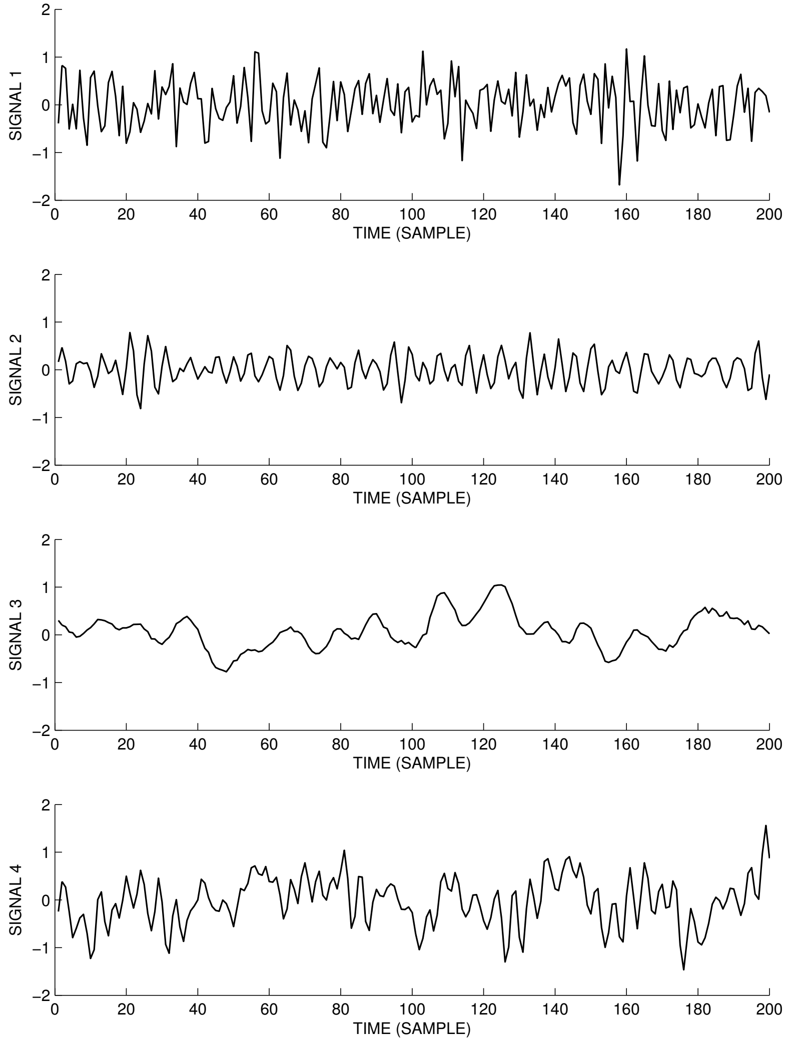


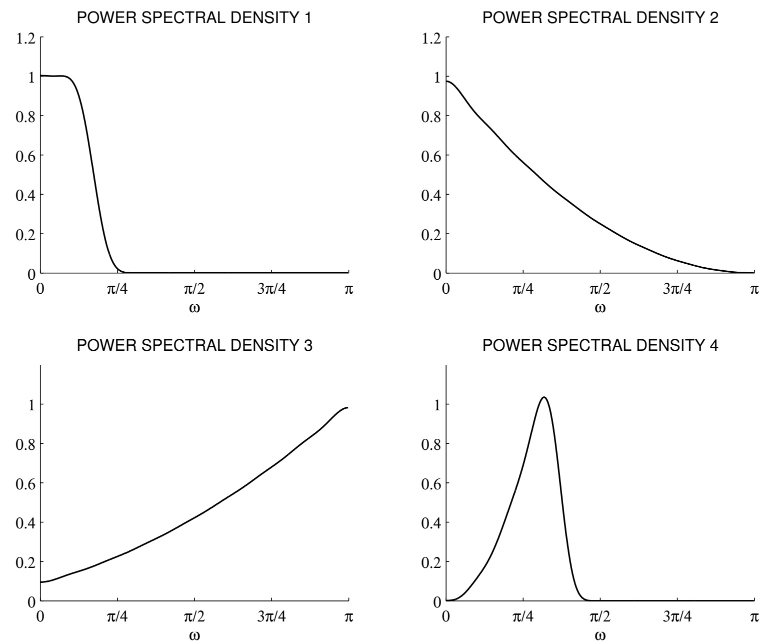
13.6 **Power spectral densities.** The figures below show the power spectral density (PSD) of four stationary discrete-time random processes and a realization (signal) generated using each of the four PSDs, but they are out of order. Match each signal to its most likely PSD.

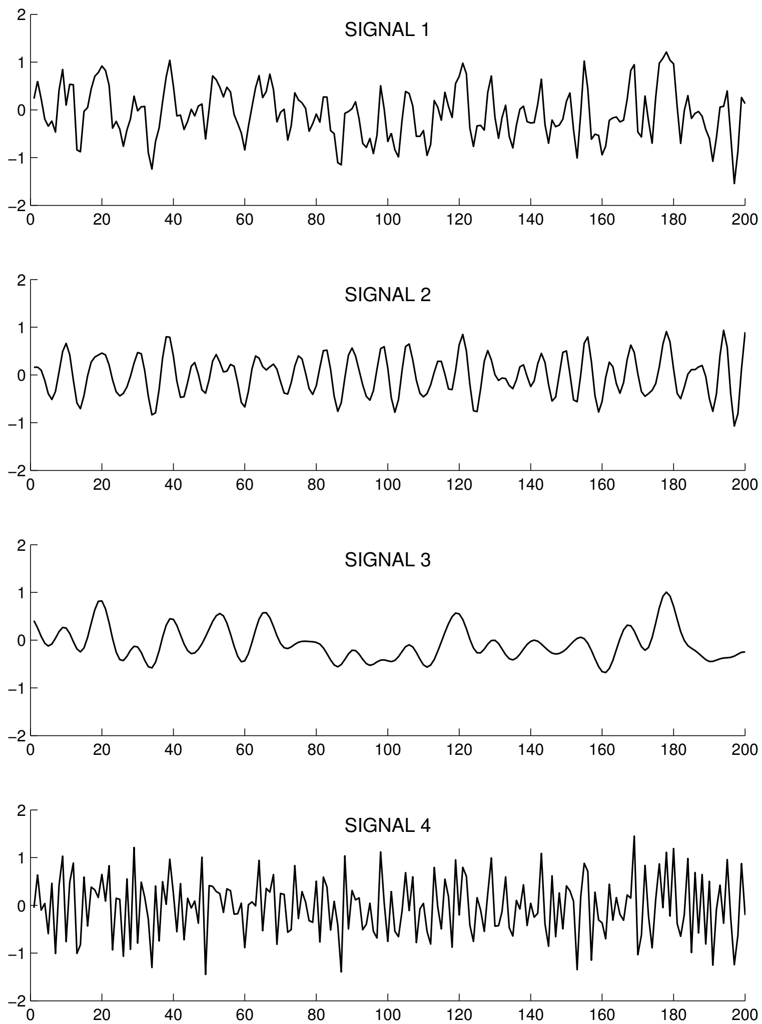


13.7 **Power spectral densities.** The figures below show the power spectral density (PSD) of four stationary discrete-time random processes and a realization (signal) generated using each of the four PSDs, but they are out of order. Match each signal to its most likely PSD by completing the table.

PSD	Signal
1	
2	
3	
4	







13.8 **Power spectral densities.** A stationary signal $y_1(n)$ is produced by LTI filtering a zero-mean white unit-variance random sequence with impulse response $h_1(n)$. Similarly, $y_2(n)$ is produced by filtering a white unit-variance random sequence with impulse response $h_2(n)$. The PSD of $y_i(n)$ is denoted $S_i(\omega)$.

True or false: To generate a stationary signal $y_3(n)$ with PSD $S_3(\omega) = S_1(\omega) + S_2(\omega)$ we can filter a white random sequence with impulse response $h_3(n) = h_1(n) + h_2(n)$. Explain, use equations as appropriate.

13.9 **Power spectral densities.** The discrete-time signal $x(n)$ is a white stationary random sequence with zero-mean and unit variance. The signal $x(n)$ is filtered with a discrete-time LTI system to produce output signal $y(n)$. What is the variance of $y(n)$ in each of the following cases?

(a) The frequency response of the LTI system is

$$H(\omega) = 1, \quad |\omega| \le \pi.$$

(b) The frequency response of the LTI system is

$$H(\omega) = \begin{cases} 1, & |\omega| \le 0.4\pi \\ 0, & 0.4\pi < |\omega| \le \pi. \end{cases}$$

(c) The frequency response of the LTI system is

$$H(\omega) = \begin{cases} 1, & |\omega| \le 0.4\pi \\ 0.5, & 0.4\pi < |\omega| \le \pi. \end{cases}$$

13.10 **AR modeling.** Consider a stationary random signal generated by the causal difference equation

$$x(n) = d(n) - a_1\, x(n-1) - a_2\, x(n-2)$$

where the driving signal $d(n)$ is a stationary white zero-mean noise signal with variance σ^2.

$$d(n) \longrightarrow \boxed{H(z)} \longrightarrow x(n)$$

The autocorrelation function $r_x(k)$ of the generated random signal $x(n)$ has the values:

$$r_x(0) = 3, \quad r_x(1) = 2, \quad r_x(2) = 1.$$

(a) Find a_1, a_2, and σ^2.

(b) Find the autocorrelation values of $x(n)$ for the next two lags, i.e. find $r_x(3)$ and $r_x(4)$.

13.11 Create a non-white WSS random discrete-time signal by filtering white noise with a filter of your choice. You may use MATLAB commands to compute the estimated PSD.

(a) Display the random signal you generate.

(b) Display the PSD of the signal (computed using the filter you use).

(c) Estimate the PSD from the signal using the periodogram.

(d) Estimate the PSD from the signal using Welch's periodogram.

(e) Estimate the PSD from the signal using AR spectral estimation.

13.12 The signal in the file `cat_8KHz.wav` is recorded using a laptop microphone. There is noise present in the signal. Using the first 10,000 samples of the signal (during which there is no speech) estimate the PSD of the noise. Use the periodogram, Welch's periodogram, and AR spectral estimation. Display the estimated PSD on linear and on logarithmic scales. Comment on your observations.

14 Speech Filtering

14.1 **Speech Recording.** In this exercise, you are to record yourself saying 'yes' and 'no'. You can create your recordings using a computer or other digital recording device (some mp3 players have a recording function). In case it is not available in the operating system, there are many utilities for both MS and Mac systems. For example, 'Audacity' is a free recording and audio editing software system for both Mac and MS.

Audacity: a free cross-platform audio recorder and editor.
http://audacity.sourceforge.net/

Using the recording utility, you can create a `wav` file; most utilities can export to a `wav` file. You can then read the `wav` file into MATLAB using the `wavread` command. Your recordings should be at 8000 samples per second. If they are not at that sampling rate, then you can change them after you load them into MATLAB using the `resample` command.

Using MATLAB, you can form one vector of 4000 samples (half second) for the 'yes'. You should form a second vector also of 4000 samples of the 'no'. Note that the recording you originally make does not need to be a half second in duration; you can trim the signal down to 4000 samples after you read the signal into MATLAB.

To turn in: Plots of your two speech signals (like below).

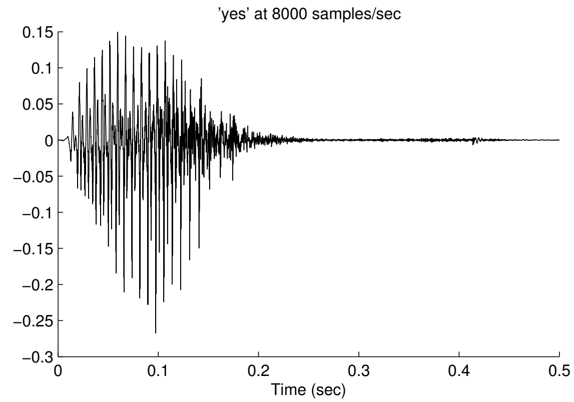
You can save your 4000-point signal as a simple data file using the `save` command. For example:

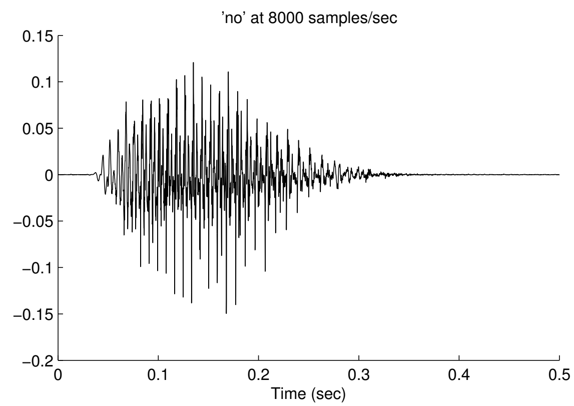
```
>> save -ascii no_data.txt x
```

To verify that your data file works correctly, you can use the following commands to listen to the audio and plot the signal.

```
>> clear
>> load no_data.txt
>> whos
  Name            Size             Bytes  Class     Attributes
  no_data      4000x1              32000  double

>> fs = 8000;                          % sampling rate
>> plot( (1:4000)/fs, no_data );
>> xlabel( 'Time (sec)' )
>> sound( no_data, fs )
```

Everyone's signal will be different. To give you a sense of what you might get, my data is shown in the figure.





14.2 **Frequency Analysis of Speech Segments.** In the previous speech exercise, you recorded yourself saying 'yes' and 'no'. In this exercise you will perform frequency analysis on your two recordings. Specifically, you will compute and display the spectrum of one segment of each of your two signals.

(a) Extract a 50 millisecond segment of voiced speech from your 'yes' signal. You should select the segment during the 'e' sound of 'yes'. The segment should be roughly periodic.

Using the MATLAB `fft` command, compute and display the spectrum (discrete-time Fourier transform, DTFT) of your 50 millisecond speech segment. The frequency units should be in Herz and should be computed according the sampling rate (8000 samples/second).

Plot the spectrum on a linear scale and on a log scale. (For the log scale, you should use $20\log_{10}(|X(f)|)$ for dB).

Based on your spectrum, what is the fundamental frequency present? Can you recognize the harmonics in the spectrum?

(b) Repeat (1) for your 'no' signal.

(c) Based on the spectra that you compute, what is the pitch frequency of your speech? (The pitch frequency is the fundamental frequency of the quasi-periodic voiced speech signal.)

To turn in: For each of your 'yes' and 'no' signals, you should turn in plots of your speech signal, the selected 50 millisecond segment, and the spectrum on linear and dB scales. For example, see the following figures for my speech signal.

14.3 **Notch filter.** In the previous speech exercises, you have seen that the spectrum of a voiced segment of your speech signal contains peaks which are approximately equally spaced. In this exercise you will design a notch filter and apply it to your speech signal so as to eliminate one of the prominent peaks in the spectrum (without affecting the adjacent peaks).

(a) In a previous exercise, you saw roughly equally-spaced peaks in the spectrum of a short segment of your speech signal. Select one of the prominent peaks to eliminate. Determine the frequency (in cycles/second) of the peak from the spectrum you computed. Using the sampling frequency at which your speech signal was recorded, convert the frequency of the peak to normalized frequency (cycles/sample); this should be a number f_n between zero and one half.

Determine the transfer function

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

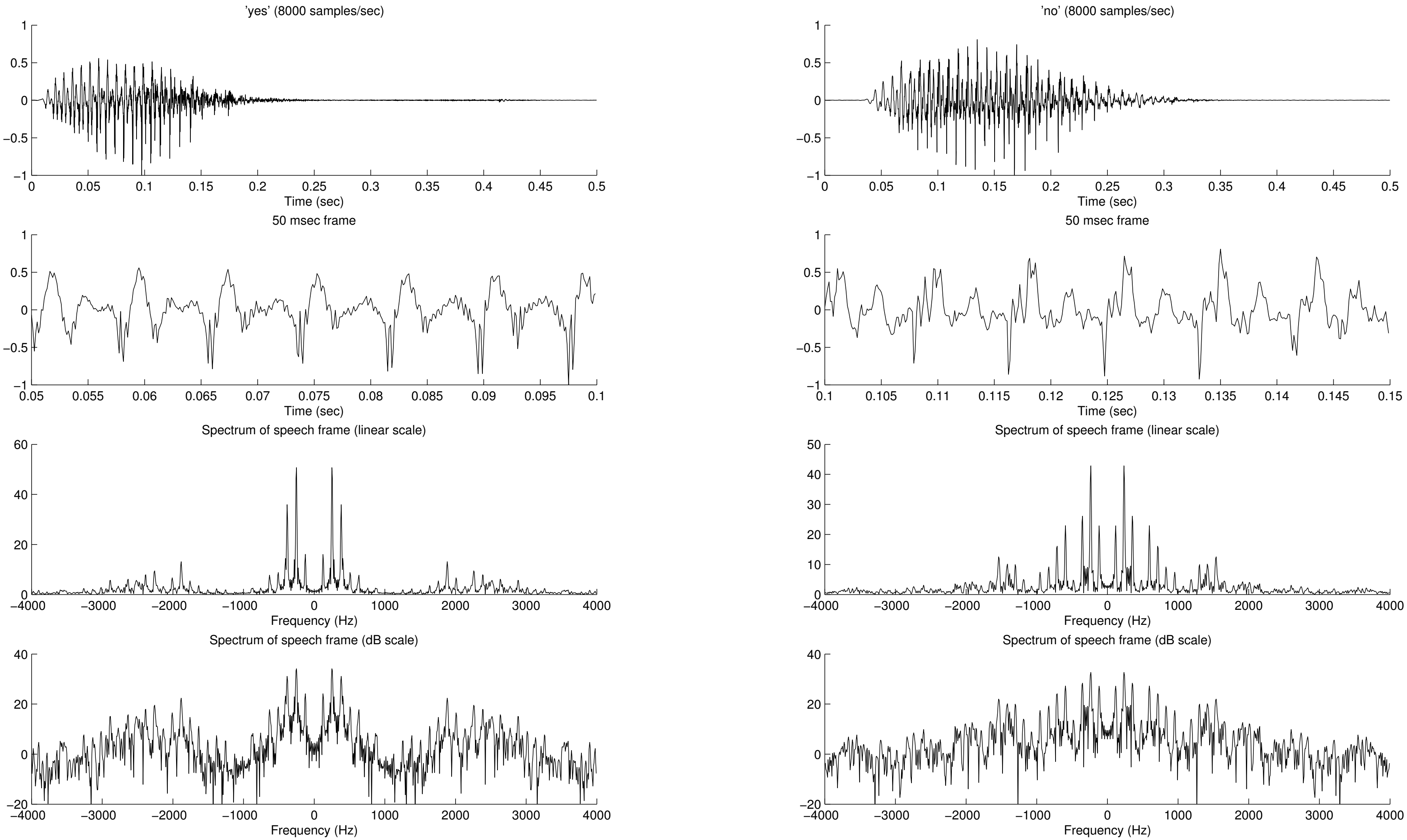
of a second order digital notch filter, that has zeros at the selected frequencies and corresponding poles. The zeros should be at $e^{\pm j 2\pi f_n}$ and the poles should be at $r\, e^{\pm j 2\pi f_n}$ where r is slightly less than 1. You can try different values of r.

(b) Plot the pole-zero diagram of your filter (use the Matlab command `zplane`) in Matlab. Verify that the poles and zero match where they were designed to be.

(c) Plot the frequency response magnitude of your filter $|H^f(f)|$ versus physical frequency (the frequency axis should go from zero to half the sampling frequency). You can use the Matlab command `freqz` to compute the frequency response. Verify that the frequency response has a null at the intended frequency.

(d) Plot the impulse response $h(n)$ of your filter. You can create an impulse signal and then use the Matlab command `filter` to apply your filter to the impulse signal.

(e) Apply your filter to your speech signal (use Matlab command `filter`). Extract a short segment of your speech signal before and after filtering. Plot both the original speech waveform $x(n)$ and filtered speech


'yes' (8000 samples/sec)
Time (sec)
50 msec frame
Time (sec)
Spectrum of speech frame (linear scale)
Frequency (Hz)
Spectrum of speech frame (dB scale)
Frequency (Hz)
'no' (8000 samples/sec)
Time (sec)
50 msec frame
Time (sec)
Spectrum of speech frame (linear scale)
Frequency (Hz)
Spectrum of speech frame (dB scale)
Frequency (Hz)

waveform $y(n)$ (you might try to plot them on the same axis). Also plot the difference between these two waveforms $d(n) = y(n) - x(n)$. What do you expect the signal $d(n)$ to look like?

For the short segment you extract from the original and filtered speech waveforms, compute and plot the spectrum. Were you able to eliminate the spectral peak that you intended to?

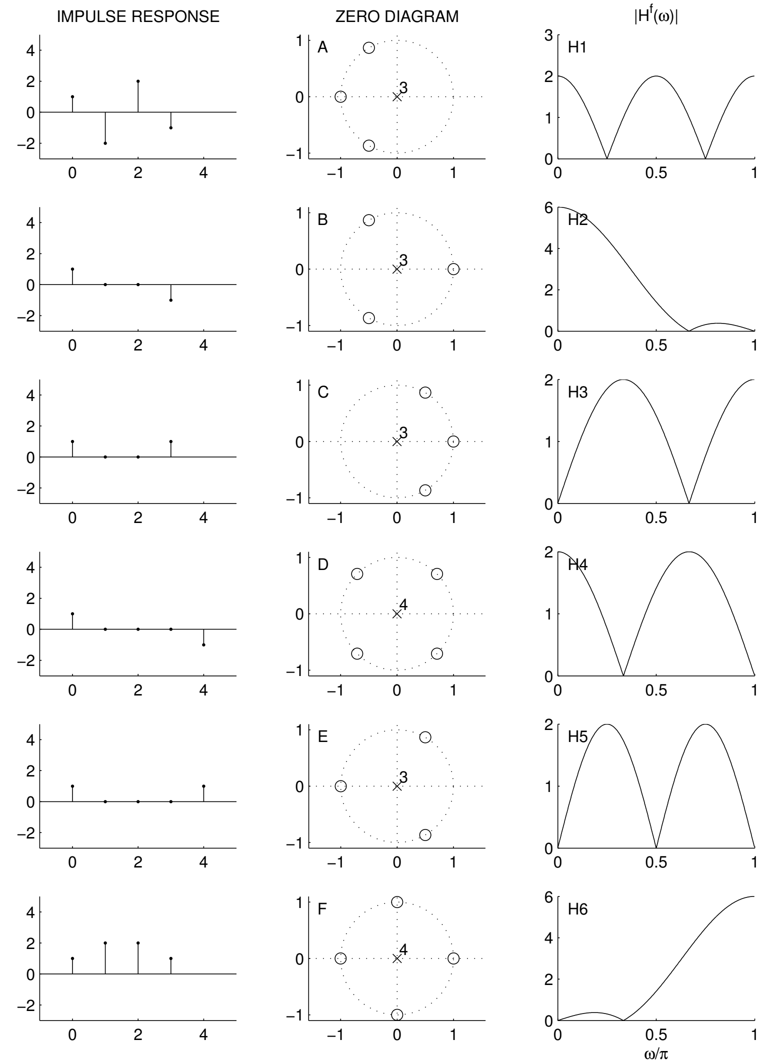
To turn in: plots of your filter (frequency response, impulse response, pole-zero diagram), plots of your speech signal (short segment) before and after filtering, plots of the spectrum (of the short segment) before and after filtering, discussion.

14.4 **Zero-phase notch filter.** In a previous exercise you designed a second order recursive notch filter and you used the MATLAB command `filter` to apply the filter to a speech signal. In that case, the filter was a causal filter and the frequency response did not have linear phase. The phase distortion became apparent when the filtered signal was compared with the original signal.

In this exercise, use the same second order recursive filter and the same speech waveform, but this time apply the filter to the data using 'forward-backward' filtering with the MATLAB command `filtfilt`. This implements the filter as a non-causal zero-phase filter. As before, to examine the phase-distortion, compare the filtered signal with the original signal (subtract one from the other). Also, plot the impulse response of the filter. Because the filter is non-causal, to compute the impulse response, you should use as the input to the filter an impulse signal of the form `[zeros(1,L) 1 zeros(1,L)]` or such (the '1' should not be at the beginning or end of the input signal because in that case you will not see both sides of the two-sided impulse response).

15 More Exercises

15.1 Several linear-phase FIR filters are shown in the following figure. Match each impulse response with its zero diagram and frequency response. Also classify each as Type I, II, III, IV.

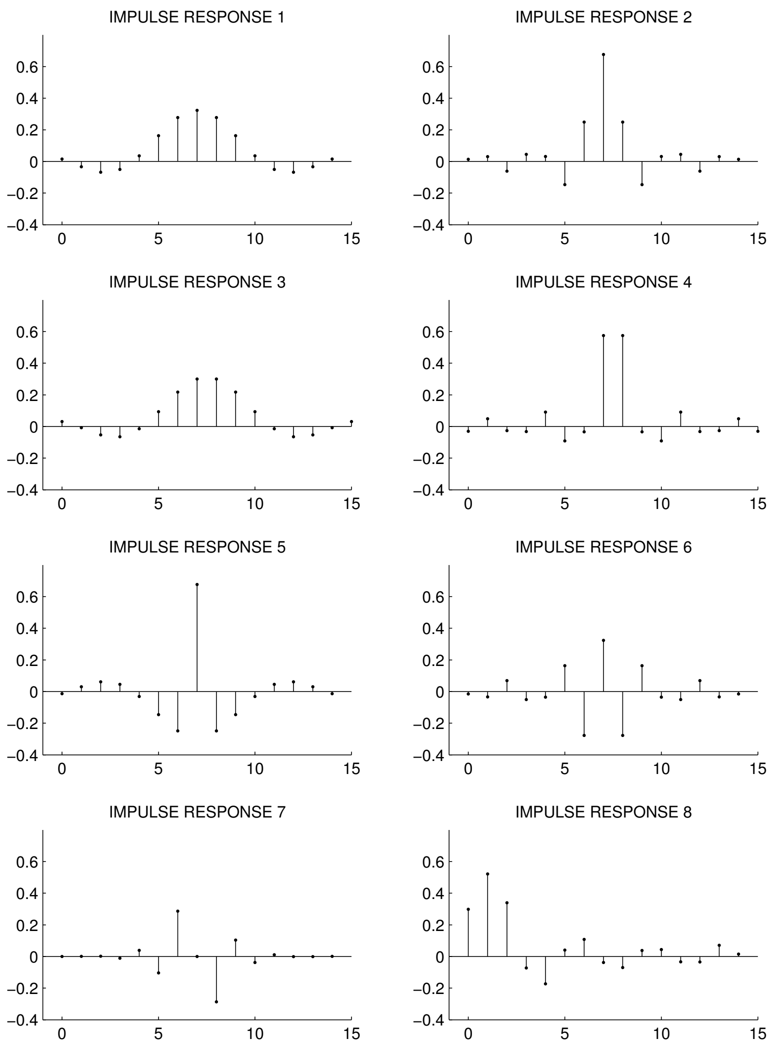


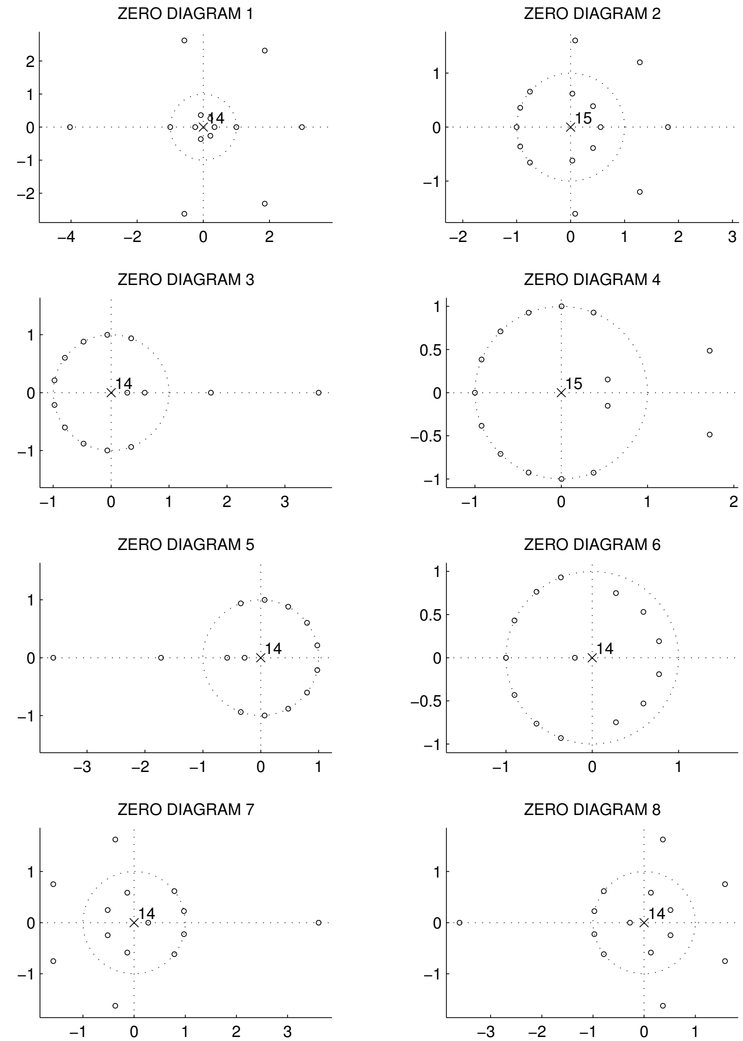
15.2 **DTFT.** Consider the infinite-length periodic discrete-time impulse train with period 4:

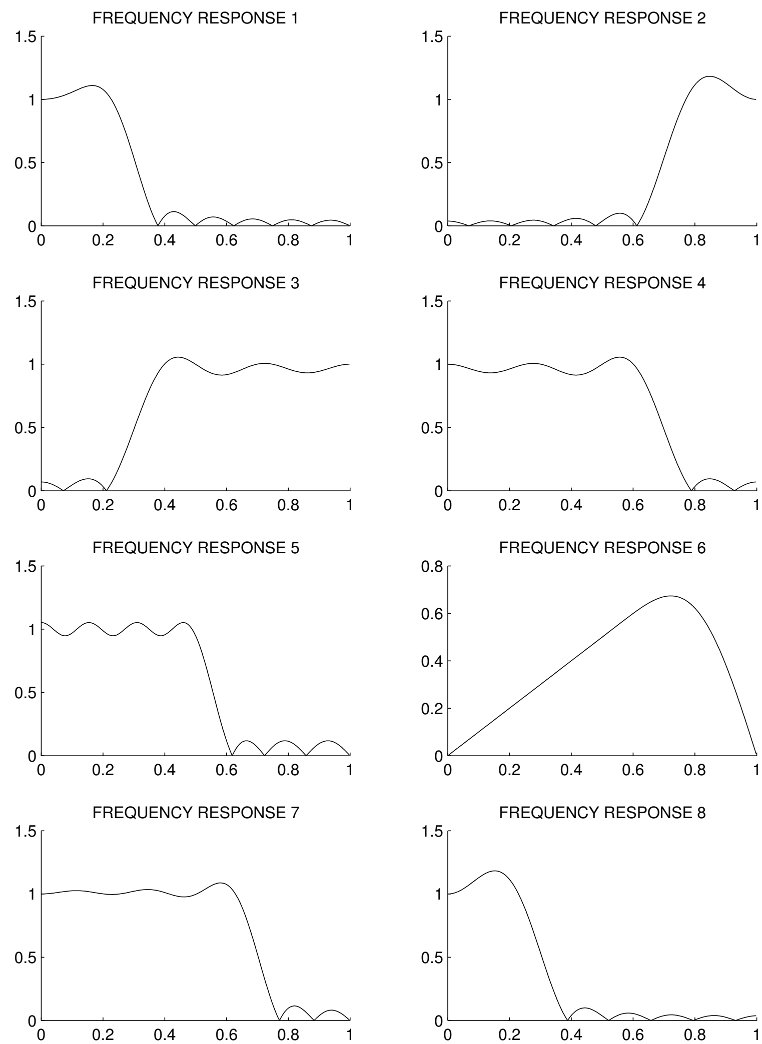
$$x(n) = \begin{cases} 1, & n = 0 \pmod 4 \\ 0, & \text{otherwise} \end{cases}$$

Find and sketch the DTFT $X^f(\omega)$. Hint: use multirate concept.

15.3 Match each impulse response with its zero diagram and frequency response.







15.4 **Filter Design.** An analog signal $x(t)$, bandlimited to 5 Hz, is sampled at a rate of 10 samples per second to give a discrete-time signal $x[n] = x(0.1n)$. A new signal $y[n]$ is obtained by the system

$$y[n] = \sum_{k=0}^{K} h[k]\, x[n-k].$$

Find the shortest sequence $h[k]$ so that

(a) The frequencies 5 Hz and 2.5 Hz are completely rejected.

(b) The DC gain of the system is 1.

(c) The system has linear-phase.

Sketch the frequency response magnitude of the system.

15.5 **Fractional-delay filters.** The ideal discrete-time delay system has the frequency response

$$D(\omega) = \mathrm{e}^{-\mathrm{j}\tau\omega}, \qquad |\omega| < \pi$$

where $\tau \in \mathbb{R}$ is the delay in samples, which need not be an integer.

Consider the ideal discrete-time delay system for a delay of a half-sample, $\tau = 0.5$.

(a) Sketch $|H(\mathrm{e}^{\mathrm{j}\omega})|$ for $-2\pi \le \omega \le 2\pi$.

(b) Sketch the phase response, $\angle H(\mathrm{e}^{\mathrm{j}\omega})$, for $-2\pi \le \omega \le 2\pi$.

(c) Roughly sketch the impulse response $h(n)$ for $-3 \le n \le 4$.

15.6 **Fractional delay system.** The ideal fractional delay system has the frequency response

$$D(\omega) = \mathrm{e}^{-\mathrm{j}d\omega}, \qquad |\omega| < \pi$$

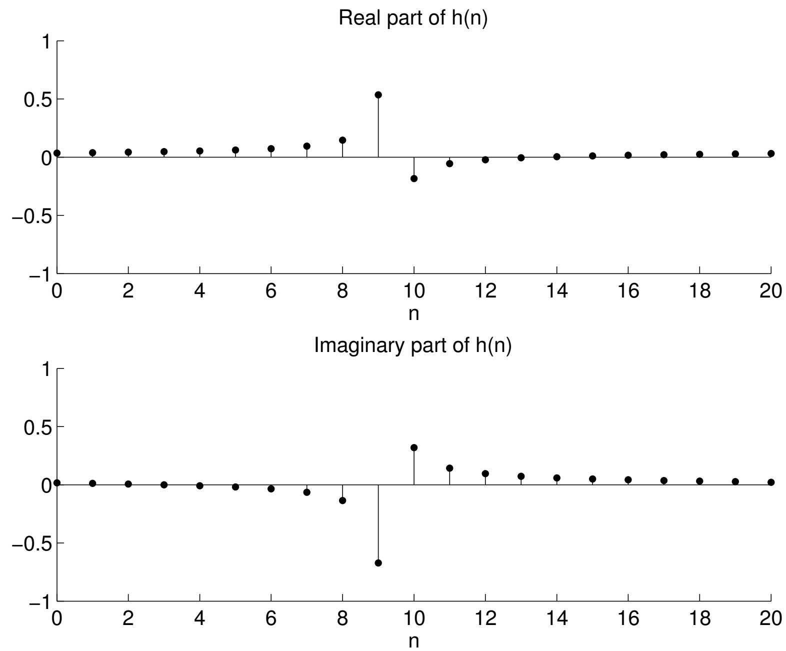
where d is the delay in samples, where d can be a fraction.

Consider the design of a real FIR filter to implement an approximate fractional delay system with delay $d = 9.3$ samples and with impulse response length $N = 21$. For a real FIR filter, the impulse response must be real-valued. Consider the design of such a filter by interpolation using the DFT. In this case, the N-point impulse response $h(n)$ is given by the inverse DFT of an N-point sequence, the later sequence representing N equally spaced values of the frequency response between $\omega = 0$ and 2π.

Therefore, it appears suitable to use the following Matlab code fragment to generate the impulse response $h(n)$.

```
N = 21;
j = sqrt(-1);
d = 9.3;
H = exp(-j*d*2*pi*(0:N-1)/N);
h = ifft(H);
```

The code almost works. However, the impulse response produced by this code is not real-valued. The impulse response (complex-valued) is shown in the following figure:



(a) Explain why the resulting impulse response is not real valued.

(b) Explain how to modify the approach to produce a real-valued N-point impulse response that approximates the fractional delay system with a delay of d samples.

(c) Provide a correction of the code based on your modification.

15.7 Suppose the impulse response of a digital filter is:

$$h(n) = [-1,\ 2,\ 3,\ 3,\ 2,\ -1] \qquad 0 \le n \le 5.$$

If the input

$$x(n) = C\,\cos(\omega_o\, n)$$

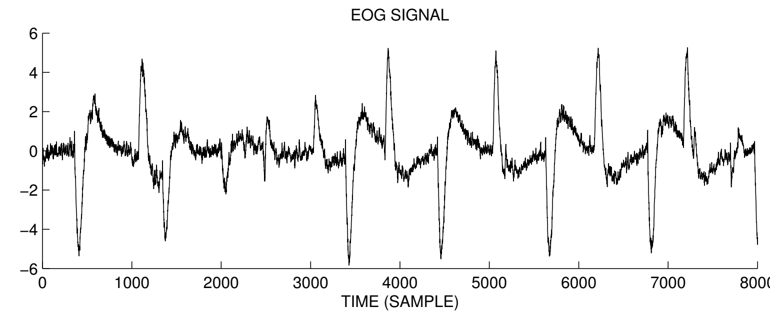
produces the output

$$y(n) = D\,\cos(\omega_o\,(n - n_o)).$$

What is n_o?

15.8 **Lowpass Filtering an EOG signal**

The following is an EOG signal measured from the head. The EOG signal measures an electrical signal related to eye movement. The positive (negative) spikes correspond to a sudden movement of the eye to the left (right).



For this exercise, design a lowpass filter of your own choosing to remove as much of the noise as possible, while maintaining the spikes in the signal. This signal is available on the course webpage (`data.m` or `data.mat` — they contain the same data).

Provide plots of signals and the filters you used and comment on your observations.

15.9 **Differentiating an EOG signal**

In some bio-signal processing tasks it is necessary to automatically determine when an event occurs. Many algorithms for this detection problem begin by differentiating the signal (and proceed by thresholding). Because differentiation is a linear time-invariant process, it can be viewed as an LTI system (as convolution). See the text book for a description of the design of filters for digital differentiation. The Matlab `remez` program can be used for example. Note that a linear-phase FIR differentiator must by either a Type III or Type IV (why?).

For the EOG signal in the previous problem, perform digital differentiation using an appropriate linear-phase FIR filter. How well does this work? (It should work poorly.)

One problem with differentiation in practice is that any noise becomes amplified. Therefore, a lowpass filter should be used first, followed by a differentiator, to avoid this noise amplification. Equivalently, one can use a lowpass differentiator.

Perform lowpass differentiation on the EOG signal above. How does it compare with the (full-band) differentiator?

Provide plots of signals and the filters you used and comment on your observations.

15.10 You are asked to design a digital filter to perform *integration* (as opposed to differentiation).

(a) What is the ideal frequency response of the integrator?

(b) What problem may arise when attempting to design an integrator?

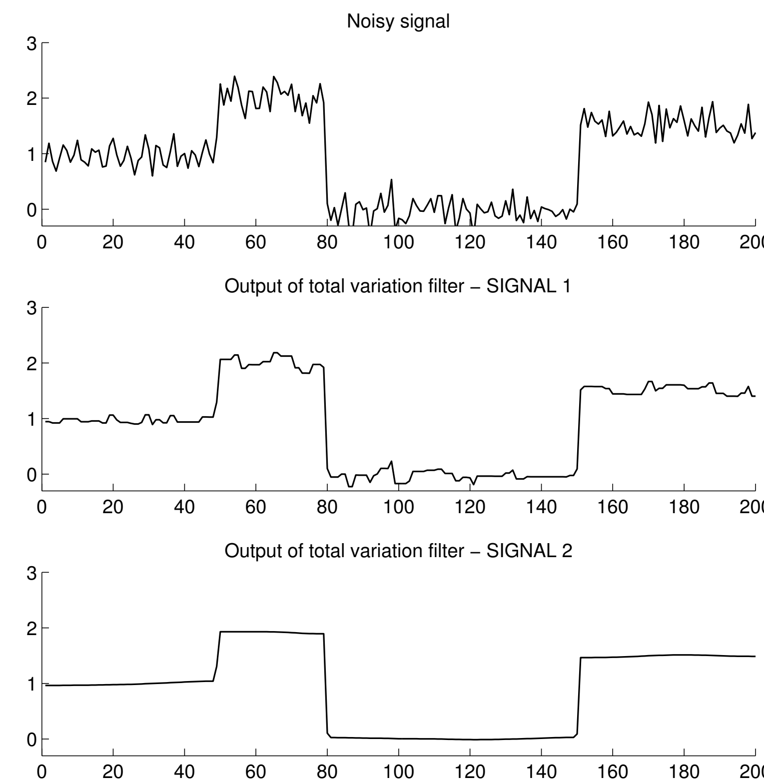
(c) Describe a method to design a digital integrator. How might you choose the weighting function $W(\omega)$ so as to deal with the problem in (b)?

15.11 **Total variation filtering.** Given a noisy signal $y = [y(1), \ldots, y(N)]$, the method of total variation (TV) filtering attempts to remove the noise by finding the signal $x = [x(1), \ldots, x(N)]$ that minimizes the cost function $F(x)$:

$$F(x) = \sum_{n=1}^{N} |y(n) - x(n)|^2 + \lambda \sum_{n=2}^{N} |x(n) - x(n-1)|.$$

The result of TV filtering depends on λ which must be specified by the user.

The figure below shows a noisy signal and the result of TV filtering using two different values of λ: $\lambda = 0.3$ and $\lambda = 3$. Which signal was produced by each of the two values of λ? Explain the reason.



15.12 **STFT.** Assume the window function is used in both the forward and inverse STFT. Which of the following 8-point window functions (if any) satisfy the perfect reconstruction condition for the STFT implemented with 50% overlapping? Explain.

(a) `w = [1    1    1    1    1    1    1    1]/sqrt(2)`

(b) `w = [0.6    0.6    0.8    0.8    0.8    0.8    0.6    0.6]`

(c) `w = [0.2    0.4    0.6    0.8    0.8    0.6    0.4    0.2]`

15.13 **Short-Time Fourier Transform.** The inverse of the STFT can be computed using the same window that is used in the forward STFT provided the window satisfies appropriate properties.

(a) When 50% overlapping is used, what property should the window satisfy?

(b) Given an example of such a window (other than the one used in the class handout).

(c) When the overlapping is by 2/3 ($66.\overline{6}$%) what property should the window satisfy?

15.14 **The short-time Fourier transform.**

(a) Consider the short-time Fourier transform implemented with 25% overlapping. Assuming the same N-point window $w(n)$ is used in both the forward STFT and inverse STFT, what is the perfect reconstruction condition on the window? (That is, what condition should the window satisfy so that the inverse STFT correctly recovers the original signal in the absence of STFT-domain processing?)

(b) If a perfect-reconstruction window is designed for 50% overlapping, does it also provide perfect reconstruction for the case of 25% overlapping? Explain your answer.

15.15 Consider the following *recursive* difference equation.

$$y(n) = x(n) - x(n-4) + y(n-1)$$

(a) Draw the pole-zero plot of this system. Hint: think about roots of unity.

(b) Sketch the frequency response $|H(\omega)|$.

(c) Find the impulse response response $h(n)$.

(d) If a filter is implemented with a recursive difference equation is it necessarily an IIR filter?

15.16 The impulse response $h(n)$ is 1 for $0 \le n \le 5$, and zero otherwise.

(a) Sketch the frequency response $|H^f(\omega)|$ by hand.

(b) Define $G(z) := H(z^3)$. Sketch the impulse response $g(n)$ and sketch $|G^f(\omega)|$.

15.17 Prove that the frequency response of a discrete-time LTI system,

$$H(e^{j\omega}) = \sum_n h(n)\, e^{-jn\omega},$$

where $h(n)$ is the impulse response, is a 2π periodic function of ω.

16 Old Exercises

16.1 An analog signal is bandlimited to 50 Hz.

$$X(f) = 0, \quad \text{for} \quad |f| > 50 \text{ Hz}$$

You need to filter the signal with a lowpass filter with a cut-off frequency at 30 Hz. You will do this by sampling the analog signal and applying a digital lowpass filter. You will sample the signal with a sampling rate of 150 Hz. In terms of normalized frequency, what should be the cut-off frequency ω_o of the digital filter? ($0 < \omega_o < \pi$).