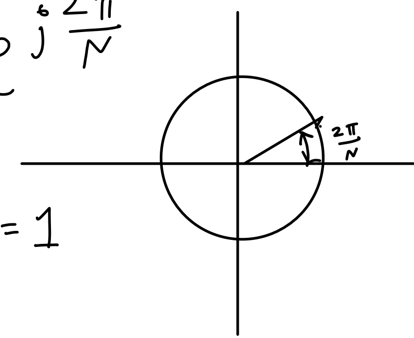
DFT

- DTFT
- N-point to N-point transform

  i.e. $N \times N$ matrix.

$$X^d(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{-nk}$$

$$W_N = e^{j\frac{2\pi}{N}}$$



$$W_N^N = 1$$

$$k = 0, \ldots, N-1$$

$$
\begin{array}{c}
k=0 \\ k=1 \\ \\ \\ k=N-1
\end{array}
\begin{bmatrix} X^d(0) \\ X^d(1) \\ \vdots \\ X^d(N-1) \end{bmatrix}
=
\underbrace{\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & W^{-1} & W^{-2} & W^{-3} \cdots & W^{-(N-1)} \\
1 & W^{-2} & W^{-4} & W^{-6} \cdots & W^{-2(N-1)} \\
\vdots & & & & \\
1 & W^{-(N-1)} & \cdots & & W^{-(N-1)(N-1)}
\end{bmatrix}}_{\text{DFT Matrix of size } N \times N}
\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}
\begin{array}{c}
n=0 \\ n=1 \\ \\ \\ n=N-1
\end{array}
$$

$$
\boxed{\underline{X}^d = \underbrace{F_N}_{\substack{N \times N \\ \text{matrix}}} \underline{x}}
$$

$$F = F^t$$

where F^t is the transpose (not conj transpose)

Inverse DFT

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X^d(k)\, W_N^{nk}$$

So

$$F^{-1} = \frac{1}{N} F^* = \frac{1}{N} F^H$$

F^H is "Hermitian transpose"
or conjugate transpose.

$$F^{-1} F = F F^{-1} = I$$

$$\boxed{F^{-1} = \frac{1}{N} F^H} \rightarrow F^H F = N \cdot I$$

$$F \cdot F^H = N \cdot I$$

Parseval's property:

$$E = \sum_{n=0}^{N-1} |x(n)|^2 \quad \text{"Energy" of the signal.}$$

$$= \sum_{n=0}^{N-1} \mathrm{conj}(x(n)) \cdot x(n)$$

$$= x^H x \quad \text{where } x = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$ signal as a column vector

$$\sum_{k=0}^{N-1} |X^d(k)|^2 = (X^d)^H X^d$$

$$= (Fx)^H (Fx) = x^H \underbrace{F^H F}_{N \cdot I} x$$

$$= x^H (N \cdot I) x = N x^H x = N \sum_{n=0}^{N-1} |x(n)|^2$$

$$E = \sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X^d(k)|^2$$

$$\boxed{\begin{array}{c}\text{key property for Parseval's Energy Identity} \\ F^{-1} = \alpha F^H\end{array}}$$

If a matrix F satisfies $F^{-1} = F^H$ then F is a "unitary" matrix.

If F is also real then $F^{-1} = F^t$ and F is called a "orthonormal" matrix. And F represents a transform with Parseval's property.

convolution as a matrix-vector multiplication

$y = h * x$ Linear convolution.

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \end{bmatrix} = \begin{bmatrix} h(0) & 0 & 0 \\ h(1) & h(0) & 0 \\ h(2) & h(1) & h(0) \\ 0 & h(2) & h(1) \\ 0 & 0 & h(2) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \end{bmatrix}$$

5×1 5×3 3×1

TOEPLITZ MATRIX [const along each diag]

$$\boxed{y = H x}$$

Circular convolution as mtx-vec mult....

$$y = h \circledast x$$

$$\underbrace{\begin{bmatrix} y(0) \\ y(1) \\ y(2) \end{bmatrix}}_{3\times 1} = \underbrace{\begin{bmatrix} h(0) & h(2) & h(1) \\ h(1) & h(0) & h(2) \\ h(2) & h(1) & h(0) \end{bmatrix}}_{3\times 3} \underbrace{\begin{bmatrix} x(0) \\ x(1) \\ x(2) \end{bmatrix}}_{3\times 1}$$

circulant matrix.

Convolution property of the DFT

$$X^d = Fx \qquad H^d = Fh \qquad Y^d = Fy$$

$$\Rightarrow Y^d(k) = H^d(k)\, X^d(k) \iff Y^d = \mathrm{diag}(H^d)\, X^d$$

$$\begin{bmatrix} Y^d(0) \\ Y^d(1) \\ \vdots \\ Y^d(N-1) \end{bmatrix} = \begin{bmatrix} H^d(0)\, X^d(0) \\ H^d(1)\, X^d(1) \\ \vdots \\ H^d(N-1)\, X^d(N-1) \end{bmatrix} = \begin{bmatrix} H^d(0) & & & O \\ & H^d(1) & & \\ & & \ddots & \\ O & & & H^d(N-1) \end{bmatrix} \begin{bmatrix} X^d(0) \\ X^d(1) \\ \vdots \\ X^d(N-1) \end{bmatrix}$$

$$= Fy = \mathrm{diag}(H^d)\, X^d$$

$$F^{-1}\left[Fy = \mathrm{diag}(Fh)\, Fx\right] \rightarrow y = F^{-1}\, \mathrm{diag}(Fh)\, Fx$$

$$y = h \circledast x \iff y = Hx$$

circulant matrix

So the DFT matrix diagonalizes any circulant matrix.

$$\boxed{H = F^{-1}\,\mathrm{diag}(F h)\, F}$$

circlcant matrix (h)

matrix factorization result.

$$H^f(\omega) = A(\omega)\, e^{j\theta(\omega)}$$

$$\begin{aligned}
|H^f(\omega)| &= |A(\omega)\, e^{j\theta(\omega)}| \\
&= |A(\omega)|\, \underbrace{|e^{j\theta(\omega)}|}_{1} \\
&= |A(\omega)|
\end{aligned}$$