The DFT of Finite-Length Time-Reversed Sequences

by Richard Lyons [December 2019]

Recently I've been reading papers on underwater acoustic communications systems and this caused me to investigate the frequency-domain effects of *time-reversal of time-domain sequences*. I created this blog because there is so little coverage of this topic in the literature of DSP.

This blog reviews the two types of time-reversal of finite-length sequences and summarizes their discrete Fourier transform (DFT) frequency-domain characteristics.

The Two Types of Time-Reversal in DSP

In DSP the two types of finite-length sequence time-reversal are (1) what I call "Flip Time-Reversal", and (2) "Circular Time-Reversal." Figure 1 shows the two types of time-reversal of a six-sample $x[n]$ time sequence.

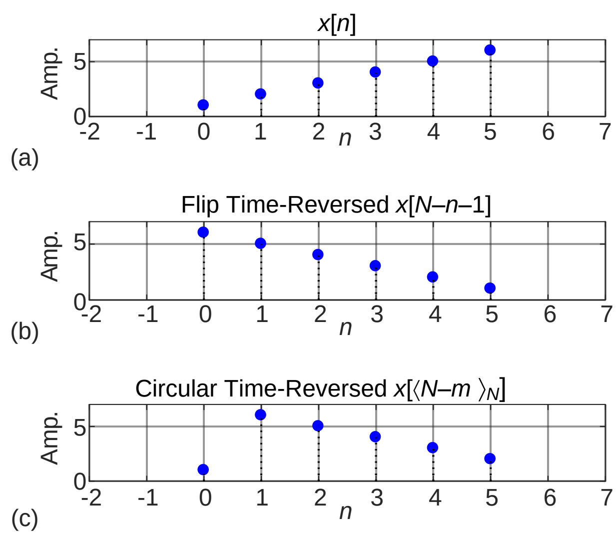

FIGURE 1. Two types of finite-length sequence time-reversal: (a) an original $N = 6$ sample $x[n]$ sequence; (b) a Flip Time-Reversed version of $x[n]$; (c) a Circular Time-Reversed version of $x[n]$.

Appendix A presents a review of Circular Time-Reversal and explains why the word *circular* is used.

The DFT of Finite-Length Time-Reversed Sequences

Here I present the principal information of this blog in the form of two tables, followed by the definitions of the algebraic notation used in those tables.

The DFTs of Flip Time-Reversed real- and complex-valued time sequences are listed in Table 1.

Table 1: Flip Time-Reversal

Length N time sequence:	**DFT:**
Real $\boldsymbol{x_R[n]}$	$\boldsymbol{X_R[m]}$
Flip time-reversed $x_R[n]$: $\boldsymbol{x_R[N-n-1]}$	$\boldsymbol{e^{j2\pi m/N} \cdot X_R[\langle N-m \rangle_N] = e^{j2\pi m/N} \cdot X_R^*[m]}$ (1)
Complex $\boldsymbol{x_C[n]}$	$\boldsymbol{X_C[m]}$
Flip time-reversed $x_C[n]$: $\boldsymbol{x_C[N-n-1]}$	$\boldsymbol{e^{j2\pi m/N} \cdot X_C[\langle N-m \rangle_N]}$ (2)
Conjugate of a Flip time-reversed $x_C[n]$: $\boldsymbol{x_C^*[N-n-1]}$	$\boldsymbol{e^{j2\pi m/N} \cdot X_C^*[m]}$ (3)

The key characteristic of Flip Time-Reversal is the $e^{j2\pi m/N}$ linear phase shift term in Eqs. (1), (2), and (3). The derivations of Eqs. (1) and (2) are given in Appendix B.

The DFTs of Circular Time-Reversed real- and complex-valued time sequences are listed in Table 2.

Table 2: Circular Time-Reversal

Length N time sequence:	**DFT:**
Real $\boldsymbol{x_R[n]}$	$\boldsymbol{X_R[m]}$
Circular time-reversed $x_R[n]$: $\boldsymbol{x_R[\langle N-n \rangle_N]}$	$\boldsymbol{X_R[\langle N-m \rangle_N] = X_R^*[m]}$ (4)
Complex $\boldsymbol{x_C[n]}$	$\boldsymbol{X_C[m]}$
Circular time-reversed $x_C[n]$: $\boldsymbol{x_C[\langle N-n \rangle_N]}$	$\boldsymbol{X_C[\langle N-m \rangle_N]}$ (5)
Conjugate of a Circular Time-Reversed $x_C[n]$: $\boldsymbol{x_C^*[\langle N-n \rangle_N]}$	$\boldsymbol{X_C^*[m]}$ (6)

The derivations of Table 1's Eq. (3) and Table 2's Eqs. (4), (5), and (6) are given in References [1-3].

The Notation Used In Tables 1 and 2

The DFT of an arbitrary N-length $x[n]$ time sequence is

$$X[m] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi mn/N} \quad (7)$$

where n is the time-domain integer index and m is the frequency-domain integer index. Both n and m are in the range 0, 1, 2, ..., N-1.

The algebraic notation for Flip Time Reversal of an $x[n]$ time sequence is:

$$x[n] \rightarrow \text{Flip Time Reversal} \rightarrow x[N-n-1]. \qquad (8)$$

An example of Eq. (8) is the sequence in Figure 1(b).

In the literature of DSP I encountered four different algebraic notations for Circular Time Reversal. The notation I choose to use here is:

$$x[n] \rightarrow \text{Circular Time Reversal} \rightarrow x[\langle N-m \rangle_N] \qquad (9)$$

where

$$x[\langle N-n \rangle_N] = \begin{cases} x[0], & \text{for } n = 0 \\ x[N-n], & \text{for } 1 \le n \le N-1 \end{cases}. \qquad (10)$$

An example of Eq. (10) is the sequence in Figure 1(c). The $x[\langle N-m \rangle_N]$ notation in Eqs. (9) and (10) is *modulo N arithmetic* notation because typical algebraic derivations of time-reversed sequences require modulo N arithmetic as demonstrated in Appendix B.

Why We Care About Time-Reversal

Regarding the two types of finite-length sequence time-reversal, we encounter Flip Time-Reversal when we:

- Compare convolution to correlation
- Discuss the geometric symmetry of finite-length sequences
- Discuss multi-input/Multi-output (MIMO) radio and underwater communications, and sensing systems
- Discuss terrestrial radar and ground penetrating radar

We encounter Circular Time-Reversal when we:

- Discuss circular convolution
- Compute the DFT of a conjugated $x[n]$ time sequence
- Compute the inverse DFT using a forward DFT software routine
- Discuss the symmetry of finite-length sequences

Conclusion

To provide a more complete description than was currently available in one place, I've presented the two types of time-reversal of finite-length real- and complex-valued sequences, and tabulated their DFT frequency-domain characteristics in Table 1 and Table 2.

Finally, I listed the instances in which we may encounter time-reversal in the field of DSP.

References

[1] S. Mitra, "*Digital Signal Processing*", Fourth Edition, McGraw Hill Publishing, 2011, pp. 56, 211.

[2] R. Lyons, "*Understanding Digital Signal Processing*", Third Edition, Prentice Hall, Upper Saddle River, New Jersey, 1996, pp. 863-865.

[3] J. Proakis and D. Manolakis, "*Digital Signal Processing-Principles, Algorithms, and Applications*", Third Edition, Prentice Hall, Upper Saddle River, New Jersey, 1996, pp. 421.

Appendix A: A Brief Review of Circular Time-Reversal

In this appendix we explain why the word "circular" is used in the phrase Circular Time-Reversal. In the early days of DSP while studying the symmetric behavior of the FFT, the practitioners encountered sequences that we now call "circular time-reversed" sequences.

As an example, if

$x[n] = x[0], x[1], x[2], x[3], x[4], x[5]$

the DFT of $x[n]$ is a complex-valued $X[m]$. If we conjugate $X[m]$ and compute its inverse DFT we obtain the following circularly-reversed $y[n]$ time sequence:

Inverse DFT of $X[m]^* = y[n] = x[0], x[5], x[4], x[3], x[2], x[1]$.

Because all DFT operations are done over an indexing domain that is circular, we can depict the $x[n]$ and $y[n]$ sequences in a circular fashion as shown in Figure A-1. In Figure A-1(a), starting with $x[0]$, we *write* the $x[n]$ sequence counter clockwise (positive time) around a circle. In Figure A-1(b) we *read* the $y[n]$ values, starting with $x[0]$, going clockwise (negative time) around the circle.

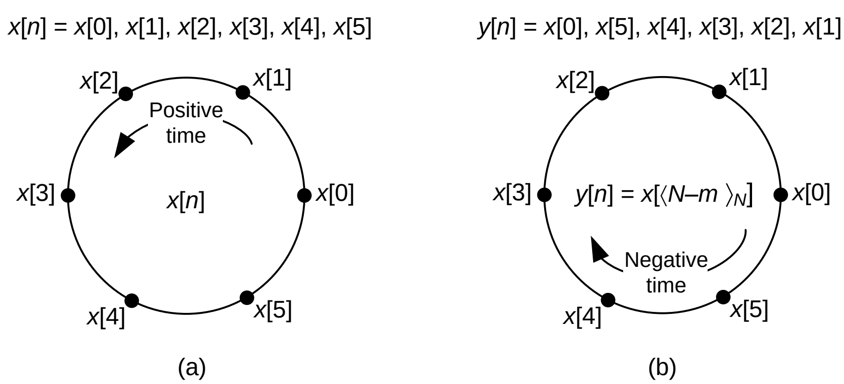

FIGURE A-1. Graphical description of circular time-reversal: (a) an $x[n]$ time sequence; (b) $y[n]$ sequence equal to a circular reversed $x[n]$.

Thus we refer to $y[n]$ as a *circular reversed* version of sequence $x[n]$. The sequences in Figure A-1 correspond to the sequences in this blog's Figures 1(a) and 1(c).

Appendix B: Derivations of Eqs. (1) and (2)

Here we derive this blog's Eqs. (1) and (2). First, consider an N-sample discrete signal $x[n]$, be it real- or complex-valued, having $X[m]$ as its DFT as:

$$x[n] \rightarrow \text{DFT} \rightarrow X[m]. \qquad \text{(B-1)}$$

Next let's think about a Flip Time-Reversed $x[N-n-1]$ sequence whose N-point DFT, $X_{\text{FTR}}[m]$, is:

$$X_{\mathrm{FTR}}[m] = \sum_{n=0}^{N-1} x[N-n-1]\, e^{-j2\pi mn/N} . \qquad \text{(B-2)}$$

We wish to find a closed-form expression for $X_{\mathrm{FTR}}[m]$ that does not use a summation. If we change Eq. (B-2)'s time indexing by letting $p = N-n-1$, then in terms of p, $n = N-p-1$. Substituting p for $N-n-1$, and $N-p-1$ for n, in Eq. (B-2) we now have:

$$X_{\mathrm{FTR}}[m] = \sum_{p=0}^{N-1} x[p]\, e^{-j2\pi m(N-p-1)/N} . \qquad \text{(B-3)}$$

Note that sequence $x[p]$, for $0 \le p \le N-1$, in Eq. (B-3) is equal to the original non-reversed $x[n]$ sequence. By factoring Eq. (B-3) we can write:

$$X_{\mathrm{FTR}}[m] = \sum_{p=0}^{N-1} x[p] \left[e^{-j2\pi mN/N}\right]\left[e^{-j2\pi m(-p)/N}\right]\left[e^{-j2\pi m(-1)/N}\right]. \qquad \text{(B-4)}$$

In Eq. (B-4) the first factor in brackets is equal to one for all m, and the last factor in brackets is not a function of p so we move it outside the summation to write:

$$X_{\mathrm{FTR}}[m] = e^{-j2\pi m(-1)/N} \cdot \sum_{p=0}^{N-1} x[p] e^{-j2\pi m(-p)/N}$$

$$= e^{j2\pi m/N} \cdot \sum_{p=0}^{N-1} x[p] e^{-j2\pi(-m)p/N} . \qquad \text{(B-5)}$$

Because the summation in Eq. (B-5) is an N-point DFT with negative frequency indexing ($-m$), we rewrite that equation as:

$$X_{\mathrm{FTR}}[m] = e^{j2\pi m/N} \cdot X[-m] . \qquad \text{(B-6)}$$

Recall that all DFT operations are done over an indexing domain that is circular. When the negative-m indexing in Eq. (B-6) is interpreted in a circular fashion using modulo N arithmetic, $X[-m]$ mod N in Eq. (B-6) is equal to $X[\langle N-m \rangle_N]$ enabling us to write:

$$X_{\mathrm{FTR}}[m] = e^{j2\pi m/N} \cdot X[\langle N-m \rangle_N] \qquad \text{(B-7)}$$

verifying Table 1's Eq. (2) when $x[n]$ is complex-valued.

Due to the symmetry of the DFT of real-valued sequences, if $x[n]$ is real-valued then Eq. (B-7) becomes:

$$X_{\text{Real-FTR}}[m] = e^{j2\pi m/N} \cdot X[\langle N-m \rangle_N] = e^{j2\pi m/N} \cdot X^*[m] . \qquad \text{(B-8)}$$

verifying Table 1's Eq. (1).