Lecture 2
Complexity Analysis

EECS 281: Data Structures & Algorithms

Assignments

- First reading assignment (now)
  - CLRS chapter 1 (short)
  - Link to textbook available on CTools

What Affects Runtime?

- The algorithm
- Implementation details
  - Skills of the programmer
- CPU Speed / Memory Speed
- Compiler (Options used)

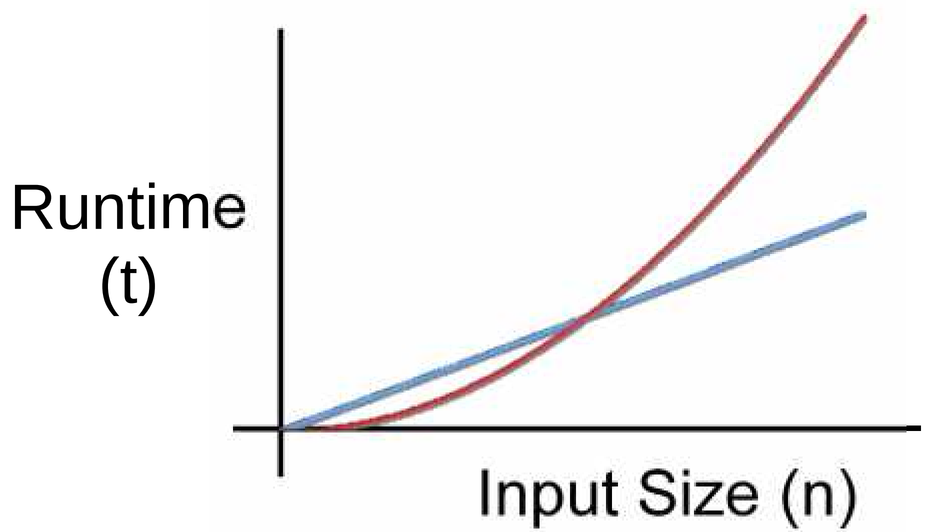
  `g++ -g` (for debu**g**ging)

  `g++ -O3` (**O**ptimization level **3** for speed)
- Other programs running in parallel

- Amount of data processed (Input size)

Input Size versus Runtime

- Rate of growth independent of most factors
  - CPU speed, compiler, etc.
- Does doubling input size mean doubling runtime?
- Will a “fast” algorithm still be “fast” on large inputs?



How do we measure input size?

Measuring & Using Input Size

- Number of bits
  - In an **int**, a **double**? (32? 64?)
- Number of items: what counts as an item?
  - Array of integers? One integer? One digit? …
  - One string? Several strings? A char?

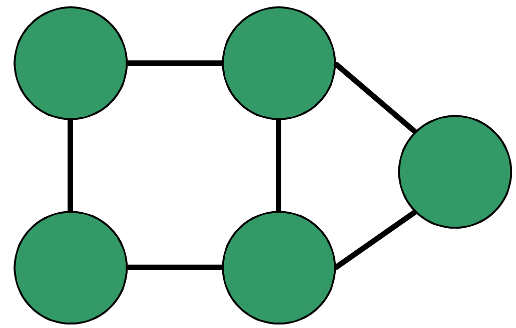
- Notation and terminology
  - *n* = input Size
  - *f(n)* = max number of steps taken by an algorithm when input has length *n*
  - *O(f(n))* or “Big-O of f(n)”
    - **upper bounds up to constants** (a whole topic in itself)

Input Size Example

Graph G = <V, E>:

V = 5 Vertices

E = 6 Edges



What should we measure?

- Vertices?
- Edges?
- Vertices and Edges?

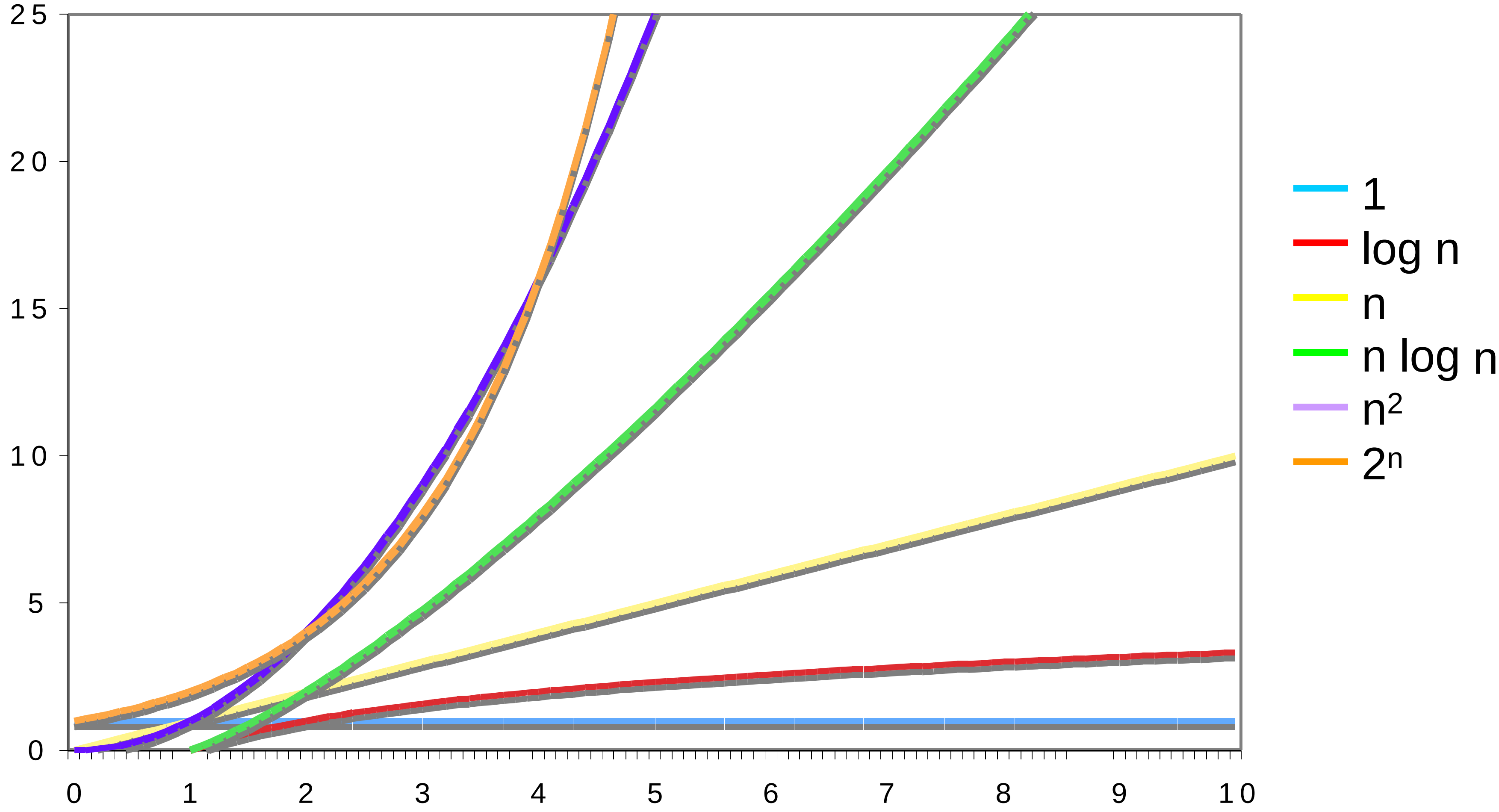
When in doubt, measure input size in bits

$$n = V + E$$

Using V and E tells which contributes more to the total number of steps

Examples: $E \log V$, EV, $V^2 \log E$

Examples of f(n) Runtime



Q: What counts as one step in a program ?

A: Primitive operations

- **a)** Variable assignment
- **b)** Arithmetic operation
- **c)** Comparison
- **d)** Array indexing or pointer reference
- **e)** Function call (not counting the data)
- **f)** Function return (not counting the data)

Runtime of 1 step is independent on input

Counting Steps

```
1  int func1(int n) {
2      int sum = 0;
3      for (int i = 0; i < n; i++) {
4          sum += i;
5      }
6      return sum;
7  }
```

1
2 1 step
3 1 + 1 + n * (2 steps)
4 1 step
5
6 1 step
7

Total steps: 4 + 3n

```
1  int func2(int n) {
2      int sum = 0;
3      for (int i = 0; i < n; i++) {
4          for (int j = 0; j < n; j++)
5              sum++;
6      }
7      for (int k = 0; k < n; k++) {
8          sum--;
9      }
10     return sum;
11 }
```

1
2 1 step
3 1 + 1 + n * (2 steps)
4 1 + 1 + n * (2 steps)
5 1 step
6
7 1 + 1 + n * (2 steps)
8 1 step
9
10 1 step
11

Total steps: $3n^2 + 7n + 6$

Counting Steps: `for Loop`

- Remember the basic form of the loop:
  - for (initialization; test; update)
- The initialization is performed once (1)
- The test is performed every time the body of the loop runs, plus once for when the loop ends (n + 1)
- The update is performed every time the body of the loop runs (n)

Algorithm Exercise

How many multiplications, if size = ***n*?**

```
//REQUIRES: in and out are arrays with size elements
//MODIFIES: out
//EFFECTS:  out[i] = in[0] *...* in[i-1] *
//   * in[i+1] *...* in[size-1]
void f(int *out, const int *in, int size) {
  for (int i = 0; i < size; ++i) {
    out[i] = 1;
    for (int j = 0; j < size; ++j) {
      if (i != j)
        out[i] *= in[j];
    }
  }
}
```

Algorithm Exercise

How many multiplications and divisions, if size = n?

```
void f(int *out, const int *in, int size) {
  int product = 1;
  for (int i = 0; i < size; ++i)
    product *= in[i];

  for(int i = 0; i < size; ++i)
    out[i] = product / in[i];
}
```

Big-O - Definition 1

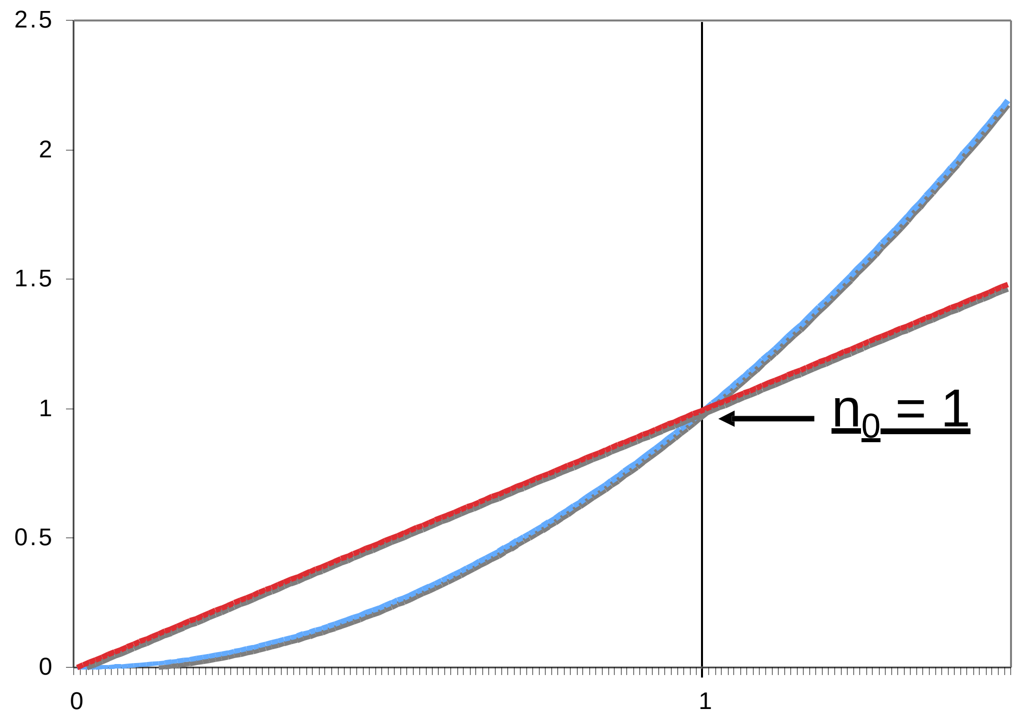
f(n) = O(g(n)) if and only if there are constants

$$\left.\begin{array}{l} c > 0 \\ n_0 \geq 0 \end{array}\right\} \text{ such that } f(n) \leq c\, g(n) \text{ whenever } n \geq n_0$$

Is $n = O(n^2)$?

$f(n) = n$

$g(n) = n^2$



Big-O: Sufficient (but not necessary) Condition

$$\text{If } \left[\lim_{n\to\infty} \left(\frac{f(n)}{g(n)} \right) = d < \infty \right] \text{ then } f(n) \text{ is } O(g(n))$$

$\boxed{\log_2 n = O(2n)?}$

$f(n) = \log_2 n$

$g(n) = 2n$

$$\lim_{n\to\infty} \left(\frac{\log n}{2n} \right)$$: ∞ / ∞

$$\lim_{n\to\infty} \left(\frac{1}{2n} \right)$$: Use L'Hopital's Rule

$$0 = d < \infty$$: $\log_2 n = O(2n)$

$\boxed{\sin\left(\frac{n}{100}\right) = O(100)?}$

$f(n) = \sin\left(\frac{n}{100}\right)$

$g(n) = 100$

$$\lim_{n\to\infty} \left(\frac{\sin\left(\frac{n}{100}\right)}{100} \right)$$: Condition does not hold but it is true that $f(n) = O(g(n))$

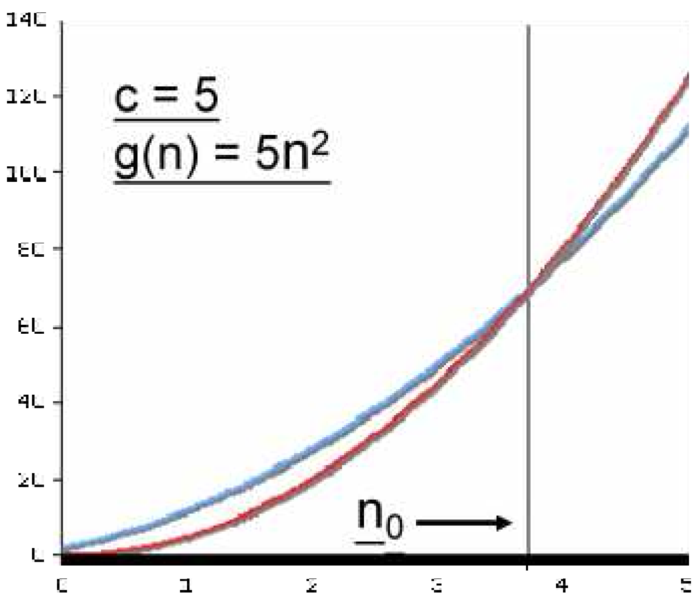
Big-O: Can We Drop Constants?

$3n^2 + 7n + 42 = O(n^2)$?

$f(n) = 3n^2 + 7n + 42$

$g(n) = n^2$

Definition	Sufficient Condition
$c > 0, n_0 \geq 0$ such that $f(n) \leq c \cdot g(n)$ whenever $n \geq n_0$	$\lim_{n\to\infty} \frac{f(n)}{g(n)} = d < \infty$



$$\lim_{n\to\infty}\left(\frac{3n^2+7n+42}{n^2}\right)$$

$$\lim_{n\to\infty}\left(\frac{6n+7}{2n}\right)$$

$$\lim_{n\to\infty}\left(\frac{6}{2}\right)$$

Common Orders of Functions

Notation	Name
O(1)	Constant
O(log n)	Logarithmic
O(n)	Linear
O(n log n)	Loglinear, Linearithmic
$O(n^2)$	Quadratic
O(n3), O(n4), …	Polynomial
$O(c^n)$	Exponential
O(n!)	Factorial
$O(2^{2^n})$	Doubly Exponential

Rules of Thumb

1. Lower-order terms can be ignored
   - $n^2 + n + 1 = O(n^2)$
   - $n^2 + \log(n) + 1 = O(n^2)$

2. Coefficient of the highest-order term can be ignored
   - $3n^2 + 7n + 42 = O(n^2)$

Log Identities

Identity	Example
$\log_a(xy) = \log_a x + \log_a y$	$\log_2(12) =$
$\log_a(x/y) = \log_a x - \log_a y$	$\log_2(4/3) =$
$\log_a(x^r) = r \log_a x$	$\log_2 8 =$
$\log_a(1/x) = -\log_a x$	$\log_2 1/3 =$
$\log_a x = \frac{\log x}{\log a} = \frac{\ln x}{\ln a}$	$\log_7 9 =$
$\log_a a = ?$ $\log_a 1 = ?$	

Power Identities

Identity	Example
$a^{(n+m)} = a^n a^m$	$2^5 =$
$a^{(n-m)} = a^n / a^m$	$2^{3-2} =$
$(a^{(n)})^m = a^{nm}$	$(2^2)^3 =$
$a^{-n} = \frac{1}{a^n}$	$2^{-4} =$
$a^{-1} = ?$ $a^0 = ?$ $a^1 = ?$	

Exercise

True or false?

$10^{100} = O(1)$

$3n^4 + 45n^3 = O(n^4)$

$3^n = O(2^n)$

$2^n = O(3^n)$

$45 \log(n) + 45n = O(\log(n))$

$\log(n^2) = O(\log(n))$

$[\log(n)]^2 = O(\log(n))$

Find f(n) and g(n), such that $f(n) \neq O(g(n))$ and $g(n) \neq O(f(n))$

Big-O, Big-Theta, and Big-Omega

	Big-O (O)	Big-Theta (Q)	Big-Omega (Ω)
Defines	Asymptotic upper bound	Asymptotic tight bound	Asymptotic lower bound
Definition	f(n) = O(g(n)) if and only if there exists an integer n_0 and a real number c such that for all n ≥ n_0, f(n) ≤ c·g(n)	f(n) = Θ(g(n)) if and only if there exists an integer n_0 and real constants c1 and c2 such that for all n ≥ n_0: c1·g(n) ≤ f(n) ≤ c2·g(n)	f(n) = Ω(g(n)) if and only if there exists an integer n_0 and a real number c such that for all n ≥ n_0, f(n) ≥ c·g(n)
Mathematical Definition	\$$n_0$ Î Z, \$c Î R : " n ³ n_0, f(n) £ c × g(n)	$\Theta(f(n)) = O(f(n)) \cap \Omega(f(n))$	\$$n_0$ Î Z, \$c Î R : " n ³ n_0, f(n) ³ c × [redacted]
$f_1(n)=2n+1$	O(n) or O(n^2) or O(n^3)…	Q(n)	Ω(n) or Ω(1)
$f_2(n)=n^2+n+5$	O(n^2) or O(n^3) …	Q(n^2)	Ω(n^2) or Ω(n) or Ω(1)

Example: $O(\log n)$ Time

```
1  int func3(int n) {
2      int sum = 0;
3      for(int i = n; i > 1; i = i / 2) {
4          sum += i;
5      }
6      return sum;
7  }
```

Line	Steps
1	
2	1 step
3	1 + 1 + ~log n * (2 steps)
4	1 step
5	
6	1 step
7	

Total: 4 + 3 log n = O(log n)

k times

n items = 1*…*2*2*2

n/2 items = 1*...*2*2

n/4 items = 1*…*2

.
.
.

1 item = 1

$1*2^k = 2^k$

Remember that $k = \log_2 n$ iff $2^k = n$

Additional Examples of $O(\log n)$ Time

```
unsigned ctz(unsigned n) {
  // count trailing zero bits if (n
  == 0) return 0; unsigned r = 0;
  while (n % 2 == 0) {
    n /= 2,
    r++;
  }
  return r;
}
```

```
unsigned logB(unsigned n) {
  // find binary log, round up
  unsigned r = 0;
  while (n > 1) {
    n /= 2
    r++;
  }
  return r;
}
```

```
unsigned onecount(unsigned n) {
  // count nonzero bits
  unsigned r = 0;
  for(; n; n &= (n - 1))
    r++;
  return r;
}
```

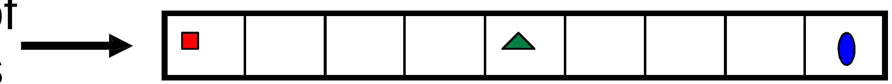
```
int* bsearch(int* lo, int* hi, int val) {
  // find position of val between lo,hi
  while (hi >= lo) {
    int* mid = lo + (hi - lo) / 2;
    if (*mid < val) lo = mid + 1;
    else if (*mid > val) hi = mid - 1;
    else return mid;
  }
  return NULL;
}
  // Q: how can this code be optimized ?
```

Complexity Analysis

- What is it?
  - Each step should take O(1) time
  - Given an algorithm and input size n,
    how many steps are needed?
  - As input size grows, how does number of steps change?
    - Focus is on TREND
- How do we measure it?
  - Express the rate of growth as a function f(n)
  - Use the big-O notation
- Why do we care?
  - Tells us how well an algorithm scales to larger inputs
  - Given two algorithms, we can compare performance before implementation

Metrics of Algorithm Complexity

Array of n items



Using a linear search over n items,
how many steps will it take to find item x?

Best-Case: 1 step Worst-Case: n steps Average-Case: n/2 steps

- Best-Case
  - Least number of steps required, given ideal input
  - Analysis performed over inputs of a given size
  - Example: Data is found in the first place you look
- Worst-Case
  - Most number of steps required, given hard input
  - Analysis performed over inputs of a given size
  - Example: Data is found in the last place you could possibly look
- Average-Case
  - Average number of steps required, given any input
  - Average performed over all possible inputs of a given size

Amortized Complexity

- A type of worst-case complexity
- Analysis performed over a sequence of inputs of a given size
  - The sequence is selected to be a worst case
- Considers the average cost of one step over a sequence of operations
  - Best/Worst/Average-case only consider a single operation
  - Different from average-case complexity!
- Key to understanding expandable arrays and STL's vector class, STL's implementations of stacks, queues, priority queues, hash tables

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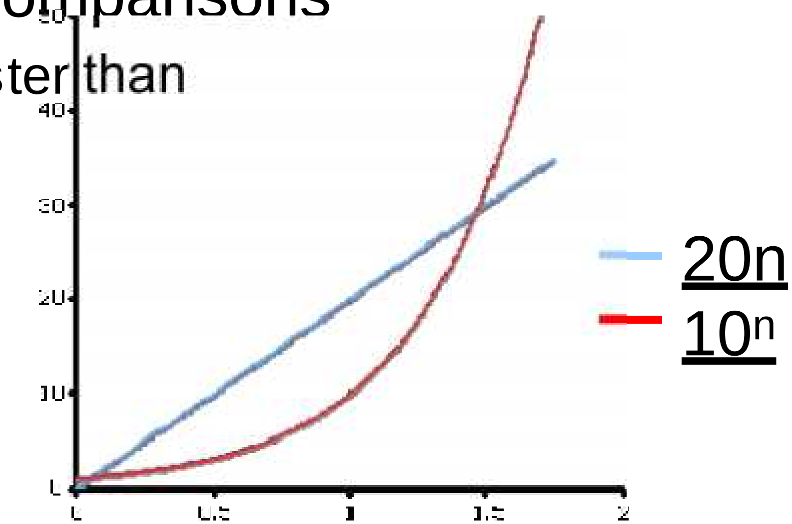
- Example: pre-paid telephone cards
  - Pay $20 upfront and call many times, until $20 is exhausted
  - Amortizes to, say, 10c per minute (then recharge with another $20)
  - Better than paying for each call at international rates
  - Worst-case sequences of calls: any sequence that exhausts $20
  - Sequences that do not require recharge are not worst-case sequences

From Analysis to Application

- Algorithm comparisons are independent of hardware, compilers and implementation tweaks
- Predict which algorithms will eventually be faster
  - *For large enough inputs*
  - O(n^2) time algorithms will take longer than O(n) algorithms
- Constants can often be ignored because they do not affect asymptotic comparisons
  - Algorithm with 20n steps runs faster than algorithm with 10^n steps. Why?



Exercise

- You have *n* balls. All have equal weight, except for one which is heavier. Find the heavy ball using only a balance.
- Describe an $O(n^2)$ algorithm
- Describe an $O(n)$ algorithm
- Describe an $O(\log n)$ algorithm
- Describe another $O(\log n)$ algorithm

Two O(log n) solutions

- True or false? Why?
- $\log_3(n) = O(\log_2 n)$
- $\log_2(n) = O(\log_3 n)$

Job Interview Question

- Implement this function

```
//returns x^y
int power(int x, unsigned int y);
```

- The obvious solution uses y - 1 multiplications
  - $2^8 = 2*2* ... *2$
- Less obvious: O(log **n**) multiplications
  - Hint: $2^8 = ((2^2)^2)^2$
  - Does it work for 2^7 ?
- Write two solutions